Applications of Mathematics

Kobra Rabiei; Yadollah Ordokhani

Boubaker hybrid functions and their application to solve fractional optimal control and fractional variational problems

Applications of Mathematics, Vol. 63 (2018), No. 5, 541–567

Persistent URL: http://dml.cz/dmlcz/147413

BOUBAKER HYBRID FUNCTIONS AND THEIR APPLICATION TO SOLVE FRACTIONAL OPTIMAL CONTROL AND FRACTIONAL VARIATIONAL PROBLEMS

Kobra Rabiei, Yadollah Ordokhani, Tehran

Received March 27, 2018. Published online October 11, 2018.

Abstract. A new hybrid of block-pulse functions and Boubaker polynomials is constructed to solve the inequality constrained fractional optimal control problems (FOCPs) with quadratic performance index and fractional variational problems (FVPs). First, the general formulation of the Riemann-Liouville integral operator for Boubaker hybrid function is presented for the first time. Then it is applied to reduce the problems to optimization problems, which can be solved by the existing method. In this way we find the extremum value of FOCPs without adding slack variables to inequality trajectories. Also we show that if the number of bases is increased, the used approximations in this method are convergent. The applicability and validity of the method are shown by numerical results of some examples, moreover, a comparison with the existing results shows the preference of this method.

Keywords: fractional optimal control problems; fractional variational problems; Riemann-Liouville fractional integration; hybrid functions; Boubaker polynomials; Laplace transform; convergence analysis

MSC 2010: 49J15, 49J40

1. Introduction

A branch of optimization theory is the calculus of variations and the queen Dido's problem has been considered as an important example. Some scientists such as Newton and Galileo have worked on this problem and the calculus of variations of a functional was proposed by some researchers such as Bernoulli brothers, Leibniz, Euler and Lagrange to solve those problems. In a large number of problems arising in physics, mechanics, geometry, control theory and others, it is necessary to determine the maximum or minimum of a certain functional. Optimal control and variational problems are two classes of these kinds of problems and the importance of these

DOI: 10.21136/AM.2018.0083-18

fields motivated many researchers to consider them. The problems of integer order dynamic system have occurred in engineering, science, geometry and many other fields and the researchers have widely worked on this topic [36], [43], [64], however they have considered the area of fractional problems during last few decades. The application of fractional optimal control problems can be found in engineering and physics. For example, it has been shown that materials with memory and hereditary effects, and dynamical processes, including gas diffusion and heat conduction, in fractal porous media have more accurate models by fractional-order models than integer-order models [23], [49], [57].

The general definition of an optimal control problem requires extremizing of a performance index over an admissible set of control and state functions. The system should be solved subject to constrained dynamics and state and control variables. A great number of numerical methods for solving optimal control problems (OCPs) have been considered and classified as direct methods or indirect ones. In the most direct methods, OCPs are transformed to a nonlinear programming problem (NLP), which can be solved numerically via collocation method. In fact the state and control variables are approximated by a set of trial functions and then the dynamical systems and constraints are collocated at a specified set of points in the solution domain. Among direct collocation nonlinear programming methods, pseudospectral methods are more popular due to their simple structures and globally interpolated scheme. The used collocation points are generally based on Gaussian quadrature rules in the most pseudospectral methods while basic functions are commonly Chebyshev [19] or Legendre [13]. Since costate variables for most direct methods are not estimated, it is pretty hard to determine whether the obtained numerical solutions satisfy the necessary conditions for optimal control problem. In contrast to direct methods, an optimal control problem is transferred into a two-point boundary value problem in indirect methods using variational principle or Pontryagain's maximum principle [11]. Some different methods such as shooting methods, generating function methods and finite difference methods are proposed to solve the corresponding two-point boundary value problem. To find further details refer to [22], [38], [48].

However, these suggested methods can be applied to solve nonlinear optimal control problems too, but the problem can also be converted into a sequence of quadratic programming problems with help of quasilinearization techniques [31]. The convergence of quasilinearization techniques depends on the initial guesses, and for most practical engineering problems finding the good guesses is not easy. Penalty function methods and Lagrangian multiplier methods can be mentioned as two most efficient methods to solve constrained optimal control problems. In penalty function methods [24], the product of the penalty index and the penalty function is added to the cost functional and a good choice requires lots of practical experience, so improper selection may make it difficult to converge. In Lagrangian multiplier methods, constraints are added into the Hamiltonian function with help of Lagrangian multipliers. Thus constraints can be strictly satisfied and can be taken as an advantage over penalty function methods [31]. These methods can be expanded to solve the fractional optimal control problems as well. The optimality conditions for fractional optimal control problems have been under development by now, for example Agrawal presented a general formulation for this problem with Riemann-Liouville derivative in [2] and also a numerical algorithm to solve it in [6]. Since the dynamic constraints of this problem involve fractional differential equations, finding exact analytic solutions of the Hamiltonian system is difficult. Therefore, finding accurate numerical methods to solve different types of fractional optimal control problems has gained much attention recently. Some are listed as follow:

- ▷ Quadratic numerical scheme (Agrawal, 2007 [3]).
- ▷ Eigen functions method (Agrawal, 2008 [5]).
- ▷ Bernstein polynomials operational matrices (Alipour et al., 2013 [7]).
- ▷ A discrete method (Almeida and Torres, 2015 [8]).
- ▷ Legendre operational technique (Bhrawy and Ezz-Eldien, 2016 [10]).
- ▷ Bernoulli polynomials method (Keshavarz et al., 2016 [26]).
- ▷ Legendre orthonormal basis method (Lotfi et al., 2013 [35]).
- ▷ Hybrid of block-pulse functions and Bernoulli polynomials (Mashayekhi and Razzaghi, 2018 [41]).
- ▷ Bernstein operational matrix method (Nemati et al., 2016 [45]).
- ▷ Boubaker polynomials method (Rabiei et al., 2017 [50]).
- ▷ Fractional Boubaker method (Rabiei et al., 2018 [51]).
- ▷ Bessel collocation method (Tohidi and Nik, 2015 [58]).
- ▷ Rational approximation method (Tricaud and Chen, 2010 [59]).
- ▷ The hybrid of block-pulse functions and Taylor polynomials method (Yonthanthum et al., 2018 [65]).
- ▷ Legendre multiwavelet collocation method (Yousefi et al., 2011 [67]).
- ▷ A Legendre collocation method (Zaky, 2018 [68]).

However, few works are devoted to optimal control problems with only the inequality conditions and these sorts of problems are considered as examples of general form in the most papers listed above, but the theory of trajectory inequality constraints was introduced by Dreus [14] and in [42] authors considered the difficulties of the presence of an inequality constraint. These problems arise in many fields of engineering such as human-operated bridge crane [29], design of robust nonlinear controllers based on both the conventional and hierarchical sliding mode techniques for double-pendulum overhead crane systems [60], optimal control of feedback linearizable dynamical systems [21], Van der Pol oscillator problem [34] and Breakwell problem [62].

Regarding FVPs, Euler-Lagrange equations for the variational problems with fractional derivatives were introduced by Riewe [53], [54] for the first time. In [1], the fractional Euler-Lagrange equation has been obtained for fractional variational problems. In [44] the formulation of Hamiltonian equations for fractional variational problems is proposed. A general finite element formulation for a class of FVPs is proposed by Agrawal. The necessary and sufficient optimality conditions for problems of the fractional calculus of variations with a Lagrangian depending on the free-end-points are achieved by Malinowska and Torres [37]. A discrete-time fractional calculus of variations on the time scale is presented for solving FVPs in [9]. Necessary conditions for fractional variational problems with completely free boundary conditions are proposed by Yousefi et al. [66]. The fractional discrete Euler-Lagrange equation and the fractional variational integrators for a class of fractional variational problems are presented in [63]. A general fractional Chebyshev finite difference formulation for solving FVPs is used in [28]. A wide classes of FVPs are solved via an approximate formula for the Caputo fractional derivative using the Rayleigh-Ritz method and the chain rule. In [18], the operational matrix of shifted Legendre orthonormal polynomials is applied for solving FVPs.

In the present paper, we have considered our previous works [50] and [51], in which Boubaker and fractional order Boubaker polynomials give a very good approximate solution for fractional optimal control problems in comparison to the other polynomials. This advantage results from the presence of fewer terms in each Boubaker polynomial compared with the other polynomials and also the sparse operational matrices of these polynomials. In addition we have noticed that these polynomials are not a good choice for solving some problems and piecewise polynomial approximation will give better results than the smooth one and this fact has motivated us to construct them and their operational matrices to show the effectiveness of piecewise version of these polynomials. The organization of the current article is as follows.

We introduce some basic definitions needed later. Section 3 is devoted to the Boubaker polynomials, constructing the new hybrid functions of block pulse, and to Boubaker polynomials and their application to approximate functions. In Section 4 we find the Riemann-Liouville fractional integral operator on these functions without using any approximation and this is the power point of this work. The problems statement is presented in Section 5 and the analysis of the approximation error for solving is stated in Section 6. Our numerical results and numerical examples are included in Section 7. A conclusion is given in Section 8.

2. Preliminaries and notation

In this section, some necessary definitions and mathematical preliminaries are given.

Definition 2.1. The Laplace transform $F(s)$ of a locally integrable function $f(t)$ is defined by [55]

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt, \tag{2.1}$$

where s is a complex number, and this operator has the following properties:

(1) $L[\lambda_1 f_1(t) + \lambda_2 f_2(t)] = \lambda_1 L[f_1(t)] + \lambda_2 L[f_2(t)]$,
(2) $L[f(t-a)u(t-a)] = e^{-as}F(s)$,
(3) $L[f \star g] = L[f(t)]L[g(t)]$,

where λ_1, λ_2, and a are constants, $u(t)$ is the Heaviside step function and $f \star g$ is the convolution of two functions f and g.

Definition 2.2. The Riemann-Liouville fractional integral of order α is defined as [30]

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{f(s)}{(t-s)^{1-\alpha}}\,ds = \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1} \star f(t), & \alpha > 0,\ t > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.2}$$

Definition 2.3. Caputo's fractional derivative of order α is defined as [30]

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,ds, \quad n-1 < \alpha \leqslant n,\ n \in \mathbb{N}, \tag{2.3}$$

with the following properties:

(1) $I^\alpha D^\alpha f(t) = f(t) - \sum\limits_{i=0}^{n-1} f^{(i)}(0) t^i / i!$,
(2) $D^\alpha c = 0$,
(3) $D^\alpha(\lambda_1 f_1(t) + \lambda_2 f_2(t)) = \lambda_1 D^\alpha f_1(t) + \lambda_2 D^\alpha f_2(t)$,

where c, λ_1, and λ_2 are constants.

3. Hybrid of block-pulse functions and Boubaker polynomials

In this section we define the hybrid of block-pulse functions and Boubaker polynomials. Hybrid functions $b_{nm}(t)$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M$, are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} B_m\Big(\dfrac{N}{t_f}t - n + 1\Big), & t \in \Big[\dfrac{n-1}{N}t_f, \dfrac{n}{N}t_f\Big), \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

where n and m are the orders of the block-pulse functions and the Boubaker polynomials, respectively.

In equation (3.1), $B_m(t)$ is the Boubaker polynomial of order m, which can be defined by [25]

$$B_m(t) = \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \binom{m-r}{r} \frac{m-4r}{m-r} t^{m-2r}, \quad m \geqslant 1, \tag{3.2}$$

where $\lfloor \cdot \rfloor$ is the floor function.

The Boubaker polynomials also satisfy the recursive relation

$$B_m(t) = tB_{m-1}(t) - B_{m-2}(t), \quad m > 2,$$

and the first few Boubaker polynomials are

$$B_0(t) = 1,\ B_1(t) = t,\ B_2(t) = t^2 + 2,\ B_3(t) = t^3 + t,\ \ldots$$

Theorem 3.1. *Suppose that $\Theta(t) = [B_0(t), B_1(t), \ldots, B_M(t)]^\top$ is the vector of Boubaker polynomials. Then there is an $(M+1) \times (M+1)$ matrix Ω such that*

$$\Theta(t+1) = \Omega\Theta(t).$$

Proof. First we write

$$\Theta(t) = \Lambda T_M(t), \tag{3.3}$$

where

$$T_M(t) = [1, t, \ldots, t^M]^\top,$$

and $\Lambda = (\Upsilon_{i,j})_{i,j=0}^M$ is an $(M+1) \times (M+1)$ matrix.

In view of

$$B_i(t) = \sum_{j=i-2\lfloor i/2\rfloor}^{i} (-1)^{(i-j)/2} \binom{\frac{i+j}{2}}{\frac{i-j}{2}} \frac{2j-i}{\frac{1}{2}(i+j)} t^j = \sum_{j=0}^{M} \Upsilon_{i,j} t^j,$$

we can obtain the entries of the matrix Λ for $i \geqslant 1$, $j = i - 2\lfloor i/2\rfloor, \dots, i$, as in [51]

$$\Upsilon_{i,j} = \begin{cases} 0, & \text{if } (i-j) \text{ is odd,} \\ (-1)^{(i-j)/2} \binom{\frac{i+j}{2}}{\frac{i-j}{2}} \dfrac{2j-i}{\frac{1}{2}(i+j)}, & \text{if } (i-j) \text{ is even.} \end{cases}$$

For $B_0(t)$ we have

$$\Upsilon_{0,0} = 1, \quad \Upsilon_{0,j} = 0, \quad j = 1, \dots, M.$$

Also we have

$$T_M(t+1) = \Phi T_M(t),$$

where Φ is

$$\Phi = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ 1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \binom{M}{M-1} & \dots & 1 \end{bmatrix}.$$

Hence, using equation (3.3), we can write

$$\Theta(t+1) = \Lambda\Phi T_M(t) = \Lambda\Phi\Lambda^{-1}\Theta(t).$$

□

Now it is enough to set $\Omega = \Lambda\Phi\Lambda^{-1}$.

3.1. Function approximation.

$$\{b_{10}(t), b_{20}(t), \dots, b_{NM}(t)\} \subset L^2[0,1]$$

is a set of hybrid block-pulse functions and Boubaker polynomials and

$$Y = \mathrm{Span}\{b_{10}(t), b_{20}(t), \dots, b_{NM}(t)\}.$$

It is clear that Y is a complete subspace, so for an arbitrary element $f(t)$ in $L^2[0,1]$, there is a unique best approximation $f_0(t)$ in Y such that (see [32])

$$\forall\, y \in Y, \quad \|f - f_0\| \leqslant \|f - y\|.$$

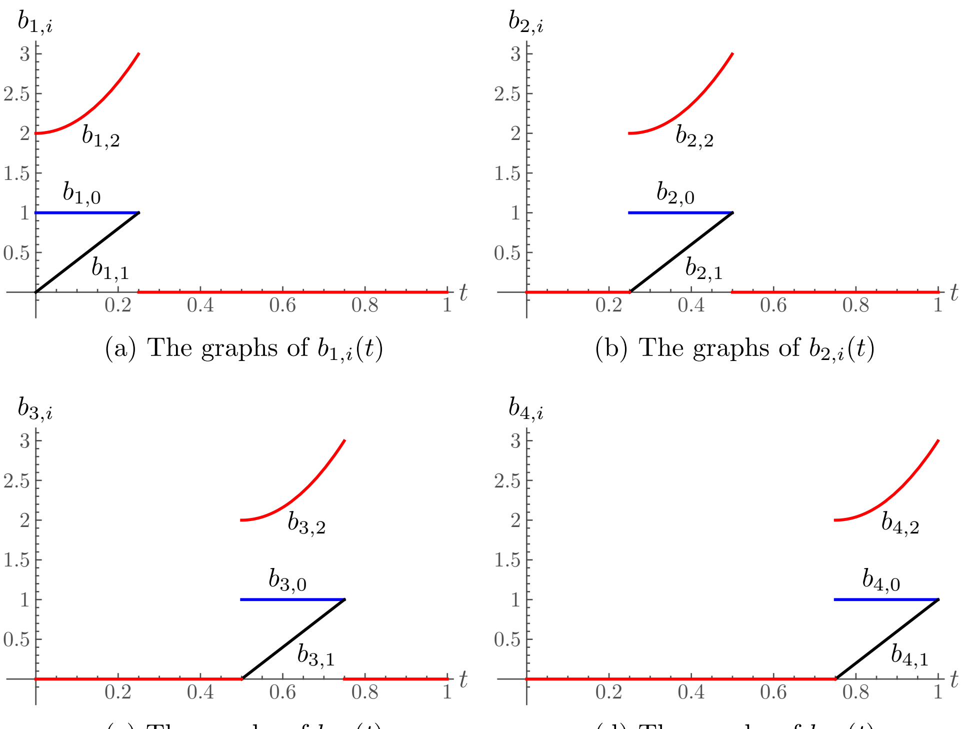
Since $f_0 \in Y$, there exist unique coefficients $a_{10}, a_{20}, \ldots, a_{NM}$ such that

$$f(t) \simeq f_0(t) = \sum_{m=0}^{M} \sum_{n=1}^{N} a_{nm} b_{nm}(t) = A^\top \Psi(t), \tag{3.4}$$

where A and $\Psi(t)$ are $N(M+1) \times 1$ vectors given by

$$\begin{aligned} A^\top &= [a_{10}, a_{20}, \ldots, a_{N0}, a_{11}, \ldots, a_{N1}, \ldots, a_{1M}, a_{2M}, \ldots, a_{NM}], \\ \Psi(t)^\top &= [b_{10}(t), b_{20}(t), \ldots, b_{N0}(t), b_{11}(t), \ldots, b_{N1}(t), \ldots, \\ &\qquad b_{1M}(t), b_{2M}(t), \ldots, b_{NM}(t)]. \end{aligned} \tag{3.5}$$

Figure 1 shows graphs of Boubaker hybrid functions of the vector $\Psi(t)$ for $N = 4$, $M = 2$.



(a) The graphs of $b_{1,i}(t)$ (b) The graphs of $b_{2,i}(t)$

(c) The graphs of $b_{3,i}(t)$ (d) The graphs of $b_{4,i}(t)$

Figure 1. Curves of functions of $\Psi(t)$ for $N = 4$, $M = 2$.

4. Riemann-Liouville fractional integral for hybrid of block-pulse functions and Boubaker polynomials

The Riemann-Liouville fractional integration of the vector $\Psi(t)$ given in equation (3.5) is obtained by

$$I^{\alpha}\Psi(t) = \bar{\Psi}(t), \tag{4.1}$$

where $\bar{\Psi}(t)$ is the $N(M+1)\times 1$ vector

$$\begin{aligned}\bar{\Psi}(t)^T = [&I^{\alpha}b_{10}(t), I^{\alpha}b_{20}(t), \dots, I^{\alpha}b_{N0}(t), I^{\alpha}b_{11}(t), I^{\alpha}b_{21}(t), \dots,\\ &I^{\alpha}b_{N1}(t), \dots, I^{\alpha}b_{1M}(t), I^{\alpha}b_{2M}(t), \dots, I^{\alpha}b_{NM}(t)].\end{aligned}$$

First, we write

$$b_{nm}(t) = u_{(n-1)t_f/N}(t)B_m\Big(\frac{N}{t_f}t - n + 1\Big) - u_{nt_f/N}(t)B_m\Big(\frac{N}{t_f}t - n + 1\Big),$$

where

$$u_c(t) = u(t-c) = \begin{cases} 1, & t \geqslant c,\\ 0, & t < c.\end{cases}$$

Therefore,

$$L[b_{nm}(t)] = \mathrm{e}^{-(n-1)t_f s/N} L\Big[B_m\Big(\frac{N}{t_f}t\Big)\Big] - \mathrm{e}^{-nt_f s/N} L\Big[B_m\Big(\frac{N}{t_f}t + 1\Big)\Big]. \tag{4.2}$$

Since the formula of Boubaker polynomials given in equation (3.2) is valid for $m \geqslant 1$, we consider the following cases.

For $m = 0$,

$$\begin{aligned}L[b_{n0}(t)] &= \mathrm{e}^{-(n-1)t_f s/N} L\Big[B_0\Big(\frac{N}{t_f}t\Big)\Big] - \mathrm{e}^{-nt_f s/N} L\Big[B_0\Big(\frac{N}{t_f}t + 1\Big)\Big]\\ &= \frac{1}{s}(\mathrm{e}^{-(n-1)t_f s/N} - \mathrm{e}^{-nt_f s/N}),\end{aligned}$$

and we have

$$L[I^{\alpha}b_{n0}(t)] = \frac{1}{s^{\alpha+1}}(\mathrm{e}^{-(n-1)t_f s/N} - \mathrm{e}^{-nt_f s/N});$$

taking the Laplace inverse transform yields

$$\begin{aligned}I^{\alpha}b_{n0}(t) &= L^{-1}\Big[\frac{1}{s^{\alpha+1}}(\mathrm{e}^{-(n-1)t_f s/N} - \mathrm{e}^{-nt_f s/N})\Big]\\ &= \frac{(t-(n-1)t_f/N)^{\alpha}}{\Gamma(\alpha+1)}u_{(n-1)t_f/N}(t) - \frac{(t-nt_f/N)^{\alpha}}{\Gamma(\alpha+1)}u_{nt_f/N}(t),\end{aligned}$$

which can be written as

$$I^{\alpha}b_{n0}(t)=\begin{cases}0, & t\in\Big(-\infty,\dfrac{n-1}{N}t_f\Big),\\ \dfrac{(t-(n-1)t_f/N)^{\alpha}}{\Gamma(\alpha+1)}, & t\in\Big[\dfrac{n-1}{N}t_f,\dfrac{n}{N}t_f\Big],\\ \dfrac{(t-(n-1)t_f/N)^{\alpha}}{\Gamma(\alpha+1)}-\dfrac{(t-nt_f/N)^{\alpha}}{\Gamma(\alpha+1)}, & t\in\Big[\dfrac{n}{N}t_f,\infty\Big).\end{cases}$$

For $m\geqslant 1$, and the first term of the right-hand side of equation (4.2) we have

$$\begin{aligned}&\mathrm{e}^{-(n-1)t_fs/N}L\Big[B_m\Big(\frac{N}{t_f}t\Big)\Big]\\&\quad=\mathrm{e}^{-(n-1)t_fs/N}L\Big[\sum_{r=0}^{\lfloor m/2\rfloor}(-1)^r\binom{m-r}{r}\frac{m-4r}{m-r}\Big(\frac{N}{t_f}\Big)^{m-2r}t^{m-2r}\Big]\\&\quad=\mathrm{e}^{-(n-1)t_fs/N}\sum_{r=0}^{\lfloor m/2\rfloor}(-1)^r\binom{m-r}{r}\frac{m-4r}{m-r}\Big(\frac{N}{t_f}\Big)^{m-2r}\frac{\Gamma(m-2r+1)}{s^{m-2r+1}}.\end{aligned}$$

For the second term of the right-hand side of equation (4.2), it should be noticed that we can write

$$B_m\Big(\frac{N}{t_f}t+1\Big)=\sum_{j=0}^{M}c_{mj}B_j\Big(\frac{N}{t_f}t\Big),$$

where c_{mj} are the elements of the mth row of the matrix Ω obtained in Theorem 3.1.

Thus, we have

$$\begin{aligned}&\mathrm{e}^{-nt_fs/N}L\Big[B_m\Big(\frac{N}{t_f}t+1\Big)\Big]=\mathrm{e}^{-nt_fs/N}L\Big[c_{m0}+\sum_{j=1}^{M}c_{mj}B_j\Big(\frac{N}{t_f}t\Big)\Big]\\&\quad=\mathrm{e}^{-nt_fs/N}\Big(\frac{c_{m0}}{s}+\sum_{j=1}^{M}c_{mj}\sum_{r=0}^{\lfloor j/2\rfloor}(-1)^r\binom{j-r}{r}\frac{j-4r}{j-r}\Big(\frac{N}{t_f}\Big)^{j-2r}\frac{\Gamma(j-2r+1)}{s^{j-2r+1}}\Big).\end{aligned}$$

It is noticeable that c_{m0} cannot be combined with other coefficients c_{mj} because of the definition of equation (3.2), so for $m\geqslant 1$,

$$\begin{aligned}L[b_{nm}(t)]&=\mathrm{e}^{-(n-1)t_fs/N}\sum_{r=0}^{\lfloor m/2\rfloor}(-1)^r\binom{m-r}{r}\frac{m-4r}{m-r}\Big(\frac{N}{t_f}\Big)^{m-2r}\frac{\Gamma(m-2r+1)}{s^{m-2r+1}}\\&\quad-\mathrm{e}^{-nt_fs/N}\frac{c_{m0}}{s}\\&\quad-\mathrm{e}^{-nt_fs/N}\sum_{j=1}^{M}c_{mj}\sum_{r=0}^{\lfloor j/2\rfloor}(-1)^r\binom{j-r}{r}\frac{j-4r}{j-r}\Big(\frac{N}{t_f}\Big)^{j-2r}\frac{\Gamma(j-2r+1)}{s^{j-2r+1}},\end{aligned}$$

and also,

$$
\begin{aligned}
L[I^{\alpha} b_{nm}(t)] = \mathrm{e}^{-(n-1)t_f s/N} & \sum_{r=0}^{\lfloor m/2\rfloor} (-1)^r \binom{m-r}{r} \frac{m-4r}{m-r} \left(\frac{N}{t_f}\right)^{m-2r} \frac{\Gamma(m-2r+1)}{s^{m-2r+1+\alpha}} \\
& - \mathrm{e}^{-nt_f s/N} \frac{c_{m0}}{s^{\alpha+1}} \\
& - \mathrm{e}^{-nt_f s/N} \sum_{j=1}^{M} c_{mj} \sum_{r=0}^{\lfloor j/2\rfloor} (-1)^r \binom{j-r}{r} \frac{j-4r}{j-r} \left(\frac{N}{t_f}\right)^{j-2r} \frac{\Gamma(j-2r+1)}{s^{j-2r+1+\alpha}}.
\end{aligned}
$$

Hence, using the Laplace inverse operator we have

$$
\begin{aligned}
I^{\alpha} b_{nm}(t) = & \sum_{r=0}^{\lfloor m/2\rfloor} (-1)^r \binom{m-r}{r} \frac{m-4r}{m-r} \left(\frac{N}{t_f}\right)^{m-2r} \frac{\Gamma(m-2r+1)}{\Gamma(m-2r+1+\alpha)} \\
& \qquad \times \left(t - \frac{(n-1)t_f}{N}\right)^{m-2r+\alpha} u_{(n-1)t_f/N}(t) \\
& - \frac{c_{m0}}{\Gamma(\alpha+1)} \left(t - \frac{nt_f}{N}\right)^{\alpha} u_{nt_f/N}(t) \\
& - \sum_{j=1}^{M} \sum_{r=0}^{\lfloor j/2\rfloor} c_{mj} (-1)^r \binom{j-r}{r} \frac{j-4r}{j-r} \left(\frac{N}{t_f}\right)^{j-2r} \\
& \qquad \times \frac{\Gamma(j-2r+1)}{\Gamma(j-2r+1+\alpha)} \left(t - \frac{nt_f}{N}\right)^{j-2r+\alpha} u_{nt_f/N}(t),
\end{aligned}
$$

which can be written as

$$
I^{\alpha} b_{nm}(t) = \begin{cases}
0, & t \in \left(-\infty, \frac{n-1}{N} t_f\right), \\
\left(t - \frac{(n-1)t_f}{N}\right)^{\alpha} d_{nm}, & t \in \left[\frac{n-1}{N} t_f, \frac{n}{N} t_f\right], \\
\left(t - \frac{(n-1)t_f}{N}\right)^{\alpha} d_{nm} - \left(t - \frac{nt_f}{N}\right)^{\alpha} \hat{d}_{nm}, & t \in \left[\frac{n}{N} t_f, \infty\right),
\end{cases}
$$

where

$$
d_{nm} = \sum_{r=0}^{\lfloor m/2\rfloor} (-1)^r \frac{(m-r-1)!}{r!} \frac{m-4r}{\Gamma(m-2r+1+\alpha)} \left(\frac{N}{t_f}\right)^{m-2r} \left(t - \frac{(n-1)t_f}{N}\right)^{m-2r},
$$

and

$$
\begin{aligned}
\hat{d}_{nm} = & \frac{c_{m0}}{\Gamma(\alpha+1)} \\
& + \sum_{j=1}^{M} \sum_{r=0}^{\lfloor j/2\rfloor} c_{mj} (-1)^r \frac{(j-r-1)!}{r!} \frac{j-4r}{\Gamma(j-2r+1+\alpha)} \left(\frac{N}{t_f}\right)^{j-2r} \left(t - \frac{nt_f}{N}\right)^{j-2r}.
\end{aligned}
$$

Now, the obtained terms $I^{\alpha} b_{nm}(t)$ and $I^{\alpha} b_{n0}(t)$ are replaced in the vector introduced in equation (4.1).

5. Problem statement

In the current section, we solve a class of fractional optimal control problems (a) and the fractional variational problems (b).

(a) Consider the following class of nonlinear fractional systems with inequality constraints

$$D^{\alpha} x(t) = F(t, x(t), u(t)), \quad 0 \leqslant \alpha < 1,$$
$$S_j(t, x(t), u(t)) \leqslant 0, \quad j = 1, 2, \ldots, k,$$
$$x(0) = x_0,$$

where

$$x(t) = [x_1(t), x_2(t), \ldots, x_l(t)]^{\top}, \quad u(t) = [u_1(t), u_2(t), \ldots, u_q(t)]^{\top},$$

are state and control vectors, respectively. The aim is to find the optimal control vector $u(t)$ and the corresponding state functions satisfying this system and minimizing the quadratic performance index, i.e.,

$$\text{minimize} \quad J(x, u) = \int_0^1 (x^{\top}(t) R(t) x(t) + u^{\top}(t) Q(t) u(t)) \, \mathrm{d}t,$$

where $R(t)$ is a symmetric positive-semidefinite matrix and $Q(t)$ is a symmetric positive-definite matrix.

By considering $t_f = 1$ and expanding the fractional derivative of the elements of the state vectors and every component of the control vector in terms of hybrid of block-pulse and Boubaker polynomials, we have

$$D^{\alpha} x_i(t) \simeq \sum_{m=0}^{M} \sum_{n=1}^{N} a_{nm}^i b_{nm}(t) = A_i^{\top} \Psi(t), \quad i = 1, \ldots, l, \tag{5.1}$$

$$u_j(t) \simeq \sum_{m=0}^{M} \sum_{n=1}^{N} b_{nm}^j b_{nm}(t) = B_j^{\top} \Psi(t), \quad j = 1, \ldots, q, \tag{5.2}$$

where A_i and B_j are the unknown coefficients vectors

$$A_i = [a_{10}^i, a_{20}^i, \ldots a_{N0}^i, a_{11}^i \ldots a_{N1}^i \ldots a_{1M}^i \ldots, a_{NM}^i]^{\top},$$
$$B_j = [b_{10}^j, b_{20}^j, \ldots b_{N0}^j, b_{11}^j \ldots b_{N1}^j \ldots b_{1M}^j \ldots, b_{NM}^j]^{\top}.$$

Also considering Property 1 of Definition 2.3 we can write

$$x_i(t) = I^{\alpha} D^{\alpha} x_i(t) + x_i(0) \simeq A_i^{\top} \bar{\Psi}(t) + x_i(0). \tag{5.3}$$

Now the fractional derivative of the state variable, the control and the state vectors can be represented as

$$\begin{aligned} D^{\alpha} x(t) &\simeq [A_1^{\top}\Psi(t), A_2^{\top}\Psi(t), \ldots, A_l^{\top}\Psi(t)]^{\top} \simeq \hat{\Psi}^{\top}(t)\hat{A}, \\ u(t) &\simeq [B_1^{\top}\Psi(t), B_2^{\top}\Psi(t), \ldots, B_q^{\top}\Psi(t)]^{\top} \simeq \hat{\Psi}^{\top *}(t)\hat{B}, \\ x(t) &\simeq [A_1^{\top}\bar{\Psi}(t), A_2^{\top}\bar{\Psi}(t), \ldots, A_l^{\top}\bar{\Psi}(t)]^{\top} + x(0) \simeq \hat{\bar{\Psi}}^{\top}(t)\hat{A} + x(0), \end{aligned}$$

where $\hat{\Psi}(t)$,$\hat{\bar{\Psi}}(t)$ and $\hat{\Psi}^{*}(t)$ are the following $lN(M+1) \times l$, $lN(M+1) \times l$ and $qN(M+1) \times q$ matrices, respectively:

$$\hat{\Psi}(t) = I_l \otimes \Psi(t), \quad \hat{\bar{\Psi}}(t) = I_l \otimes \bar{\Psi}(t), \quad \hat{\Psi}^{*}(t) = I_q \otimes \Psi(t), \tag{5.4}$$

where I_l and I_q are $l \times l$ and $q \times q$ identity matrices, respectively, and $\otimes$ denotes the Kronecker product [33] and $\hat{A}$, $\hat{B}$ are vectors of order $lN(M+1)\times 1$ and $qN(M+1)\times 1$ respectively, given by

$$\hat{A} = [A_1, A_2, \ldots, A_l]^{\top}, \quad \hat{B} = [B_1, B_2, \ldots, B_q]^{\top}.$$

Substituting these approximations into the problem yields

$$\begin{aligned} J[\hat{A}, \hat{B}] = \int_0^1 & [(\hat{\bar{\Psi}}(t)^{\top}\hat{A} + x(0))^{\top} R(t) (\hat{\bar{\Psi}}(t)^{\top}\hat{A} + x(0)) \\ & + (\hat{\Psi}^{\top *}(t)\hat{B})^{\top} Q(t) (\hat{\Psi}^{\top *}(t)\hat{B})] \, \mathrm{d}t, \end{aligned} \tag{5.5}$$

which can be solved numerically by the Gauss-Legendre integration method, subject to

$$\hat{\Psi}^{\top}(t)\hat{A} - F(t, \hat{\bar{\Psi}}^{\top}(t)\hat{A} + x(0), \hat{\Psi}^{\top *}(t)\hat{B}) = 0, \tag{5.6}$$

$$S_j(t, \hat{\bar{\Psi}}^{\top}(t)\hat{A} + x(0), \hat{\Psi}^{\top *}(t)\hat{B}) \leqslant 0, \quad j = 1, 2, \ldots, k. \tag{5.7}$$

In the present method adding slack variables to converting inequality constraints is not needed and the optimal control problem has now been reduced to an optimization problem and we need to find $\hat{A}$ and $\hat{B}$ satisfying equations (5.6) and (5.7) and in addition minimizing the functional stated in equation (5.5).

For this purpose we collocate equations (5.6) and (5.7) at Newton-Cotes nodes t_i defined by

$$t_i = \frac{i+1}{2N(M+1)}, \quad i = 0, 1, \ldots, 2NM.$$

Many well-developed nonlinear programming techniques such as the SQP method can be used to solve this extremum problem (see [16], [56]).

(b) Consider the fractional variational problem:

$$\text{minimize} \quad J[y(t)] = \int_0^1 F(t, y(t), D^\alpha y(t))\,\mathrm{d}t, \quad 0 \leqslant \alpha < 1,$$

with the boundary conditions

$$y(0) = y_0, \quad y(1) = y_1.$$

Here F is a linear or nonlinear function. For solving this problem we set

$$D^\alpha y(t) \simeq \sum_{m=0}^{M} \sum_{n=1}^{N} c_{nm} b_{nm}(t) = C^\top \Psi(t),$$

where C is the following unknown coefficients vector:

$$C = [c_{10}, c_{20}, \ldots, c_{N0}, c_{11}, \ldots, c_{N1}, \ldots, c_{1M}, \ldots, c_{NM}]^\top,$$

Also considering property (1) of Definition 2.3, we can write

$$y(t) = I^\alpha D^\alpha y(t) + y(0) \simeq C^\top \bar{\Psi}(t) + y_0.$$

Substituting these approximations into the functional yields

$$J[C] = \int_0^1 F[(t, C^\top \bar{\Psi}(t) + y_0, C^\top \Psi(t))]\,\mathrm{d}t,$$

which should be solved numerically and minimized subject to the condition.

$$C^\top \bar{\Psi}(1) - y_1 = 0.$$

The rest of the method for solving this problem is similar to problem (a).

6. Approximation error

In this section we focus on equations (5.1), (5.2), and (5.3) and obtain error bounds for these approximations in terms of Sobolev norms and then show that the numerical value of the cost function converges to the exact value.

Sobolev norm of integer order μ in the interval (a,b) is defined by

$$\|f\|_{H^\mu(a,b)} = \Big(\sum_{k=0}^{\mu}\int_a^b |f^{(k)}(x)|^2\,\mathrm{d}x\Big)^2 = \Big(\sum_{k=0}^{\mu}\|f^{(k)}\|^2_{L^2(a,b)}\Big)^2,$$

where $f^{(k)}$ denotes the kth derivative of f. Furthermore, $|f|_{H^{\mu;M}(0,1)}$ given in [12] is defined as

$$|f|_{H^{\mu;M}(0,1)} = \Big(\sum_{k=\min(\mu,M+1)}^{\mu}\|f^{(k)}\|^2_{L^2(0,1)}\Big)^2.$$

It is convenient to recall the following seminorm introduced in [40] for $f\in H^\mu(0,1)$, $0\leqslant r\leqslant\mu$, $M\geqslant 0$, and $N\geqslant 1$:

$$|f|_{H^{r;\mu;M;N}(0,1)} = \Big(\sum_{k=\min(\mu,M+1)}^{\mu} N^{2r-2k}\|f^{(k)}\|^2_{L^2(0,1)}\Big)^2.$$

Obviously, whenever $M\geqslant\mu-1$, we have

$$|f|_{H^{r;\mu;M;N}(0,1)} = N^{r-\mu}\|f^{(\mu)}\|_{L^2(0,1)}. \tag{6.1}$$

Theorem 6.1. *Suppose $f\in H^\mu(0,1)$ with $\mu\geqslant 1$, and $M\geqslant 0$, while $f^{(M,N)}$ is the best approximation of f as*

$$f(t)\simeq f^{(M,N)}(t) = \sum_{m=0}^{M}\sum_{n=1}^{N} a_{nm}b_{nm}(t).$$

Then

$$\|f-f^{(M,N)}\|_{L^2(0,1)}\leqslant cM^{-\mu}|f|_{H^{0;\mu;M;N}(0,1)},$$

and for $1\leqslant r\leqslant\mu$,

$$\|f-f^{(M,N)}\|_{H^r(0,1)}\leqslant cM^{2r-\mu-1/2}|f|_{H^{r;\mu;M;N}(0,1)},$$

where c depends on μ.

P r o o f. This theorem was proved in [40] for hybrid functions and since the best approximation is unique, we can have the same results for Boubaker hybrid. The proof in this case is straightforward and similar to [40]. □

R e m a r k 6.1. Suppose $f \in H^{\mu}(0,1)$ and $\mu \geqslant 1$. By setting $M \geqslant \mu - 1$, and considering equation (6.1), we get

$$\|f - f^{(M,N)}\|_{L^2(0,1)} \leqslant cM^{-\mu}N^{-\mu}\|f^{(\mu)}\|_{L^2(0,1)},$$

and for $r \geqslant 1$,

$$\|f - f^{(M,N)}\|_{H^r(0,1)} \leqslant cM^{2r-\mu-1/2}N^{r-\mu}\|f^{(\mu)}\|_{L^2(0,1)}.$$

This theorem shows that the rate of convergence of $f^{(M,N)}$ to f is faster than $1/N$ to the power of $M+1-r$ and $1/M$ to the power of $M+\frac{3}{2}-r$, which is superior to the classical spectral method [12].

Corollary 6.1. *In the relation*

$$\begin{aligned} D^{\alpha}x_i(t) &\simeq \sum_{m=0}^{M}\sum_{n=1}^{N} a^i_{nm}b_{nm}(t) = A_i^{\top}\Psi(t), \\ u_j(t) &\simeq \sum_{m=0}^{M}\sum_{n=1}^{N} b^j_{nm}b_{nm}(t) = B_j^{\top}\Psi(t), \end{aligned}$$

$A_i^{\top}\Psi(t)$ and $B_j^{\top}\Psi(t)$ are the best approximations of $D^{\alpha}x_i(t)$ and $A_i^{\top}\Psi(t)$ so by considering Theorem 6.1 we can conclude that if we increase M or N then $(D^{\alpha}x_i(t) - A_i^{\top}\Psi(t))$ and $(u_j(t) - B_j^{\top}\Psi(t))$ tend to zero.

Since

$$x_i(t) = I^{\alpha}D^{\alpha}x_i(t) + x_i(0) \simeq I^{\alpha}A_i^{\top}\Psi(t) + x_i(0) = A_i^{\top}\bar{\Psi}(t) + x_i(0),$$

the following theorem shows that the error in $x_i(t)$ tends to zero when the dimension of the basis functions is increased.

Theorem 6.2. *Suppose $f \in H^{\mu}(0,1)$ with $\alpha \geqslant 0$, while $f^{(M,N)}$ is the best approximation of f. Then*

$$\|I^{\alpha}f - I^{\alpha}f^{(M,N)}\|_{L^2(0,1)} \leqslant \frac{1}{\Gamma(\alpha)}cM^{-\mu}N^{-\mu}\|f^{(\mu)}\|_{L^2(0,1)},$$

P r o o f.

$$\begin{aligned} \|I^{\alpha}f - I^{\alpha}f^{(M,N)}\|_{L^2(0,1)} &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s) - f^{(M,N)}(s))\,\mathrm{d}s\right\|_{L^2(0,1)} \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t \|f(s) - f^{(M,N)}(s)\|_{L^2(0,1)}\,\mathrm{d}s; \end{aligned}$$

now Theorem 6.1 completes the proof. □

It should be noticed that problem (b) is a simpler case of problem (a) and so the used approximations in problem (b) are convergent too.

7. Numerical examples

In this section, numerical examples are presented to demonstrate the applicability and accuracy of the proposed technique of Section 5. Since many physical phenomena do not follow a continuous pattern, smooth approximation does not give acceptable numerical results, so piecewise approximation like using hybrid functions should be applied to achieve more accurate numerical findings (Example 7.1). In addition, however the exact solutions of some test problems are polynomial the previous works on these examples have not obtained the exact values because of the fact that some approximations and operational matrices are used in their methods, while in our current work we have achieved the integration operator without any approximation and the exact solutions are calculated using this proposed method (Examples 7.4, 7.5). All numerical computations have been done using Mathematica software.

Example 7.1. Consider the performance index [15], i.e.,

$$\text{minimize} \quad J = \int_0^1 [x^2(t) + u^2(t)]\,\mathrm{d}t,$$

subject to

$$D^\alpha x(t) = u(t), \quad 0 < \alpha \leqslant 1, \quad u(t) \leqslant 1, \quad x(0) = \frac{1+3\mathrm{e}}{2-2\mathrm{e}}.$$

The problem for $\alpha = 1$ has the optimal solution

$$u(t) = \begin{cases} 1, & 0 \leqslant t \leqslant \frac{1}{2}, \\ \dfrac{\mathrm{e}^t - \mathrm{e}^{2-t}}{\sqrt{\mathrm{e}}(1-\mathrm{e})}, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

and

$$J = \frac{55\mathrm{e}^2 - 2\mathrm{e} - 5}{24(\mathrm{e}-1)^2} = 5.587955.$$

This problem was solved in [15] by the Chebyshev finite difference method for $\alpha = 1$, and the best result reported is $J = 5.58797$, for $M = 13$, with 16 iterations and CPU time = 52.78.

Table 1 shows our results for different values of M and N and we can see the same accuracy of J is obtained by 6 hybrid functions with less calculation and time. Since the exact solution of this example is a piecewise function, it can be expected that the hybrid approximation will give better results than smooth approximation.

N	M	J	CPU time
1	2	5.60276	0.405
1	3	5.59196	0.422
2	2	5.58796	0.608

Table 1. The estimated value of J for $\alpha = 1$, for Example 7.1.

Figure 2 shows the approximation curve of the control function obtained by the present method and the exact values of $u(t)$ over $[0, 1]$ for $N = 2$, $M = 2$ and Table 2 shows the convergence between the values of J for different α as α approaches 1.

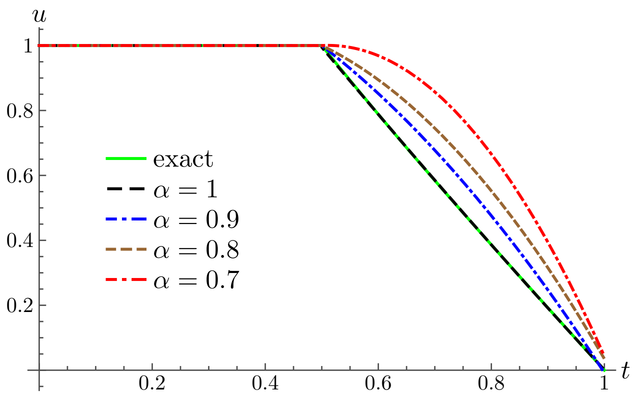


Figure 2. Exact and numerical values of control function for Example 7.1.

	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
$N = 1$, $M = 2$	5.0808	5.2610	5.4359	5.6027
$N = 1$, $M = 3$	5.0496	5.2340	5.4160	5.5919
$N = 2$, $M = 2$	5.0566	5.2487	5.4174	5.5879

Table 2. The estimated value of J for different α for Example 7.1.

Example 7.2. Consider the problem [7], i.e.,

$$\text{minimize} \quad J = \frac{1}{2}\int_0^1 [x_1^2(t) + u^2(t)]\,\mathrm{d}t, \tag{7.1}$$

s.t.

$$D^\alpha x_1(t) = x_2(t), \quad 0 \leqslant \alpha < 1,$$
$$D^\alpha x_2(t) = -x_2(t) + u(t), \quad |u(t)| \leqslant 1,$$

and the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 10.$$

Table 3, shows the values of J obtained by the hybrid functions [39], the Rationalized Haar Functions [47] and the method proposed in [7] for $\alpha = 1$, together with the present method. Comparing the values of J shows that our approach can solve the problem effectively.

methods		J
Hybrid functions [39]	$N = 4,\ M = 3$	8.07059
	$N = 4,\ M = 4$	8.07056
Rationalized Haar functions [47]	$K = 4$	8.07473
	$K = 8$	8.07065
Bernstein polynomials [7]	$M = 7$	8.07061
	$M = 9$	8.07059
Presented method	$N = 3,\ M = 2$	8.07417
	$N = 3,\ M = 3$	8.07073
	$N = 4,\ M = 2$	8.07272
	$N = 4,\ M = 3$	8.07055
Exact		8.07054

Table 3. The values of J with $\alpha = 1$, for Example 7.2.

Table 4 shows the convergence between the values of J for different α as α approaches to 1 for $N = 2$, $M = 2$.

	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
J	9.8634	9.2632	8.9045	8.0769

Table 4. The estimated value of J for different α for Example 7.2.

Moreover, Figure 3 shows the graphs of state functions for different values of α and $N = 2$, $M = 2$.

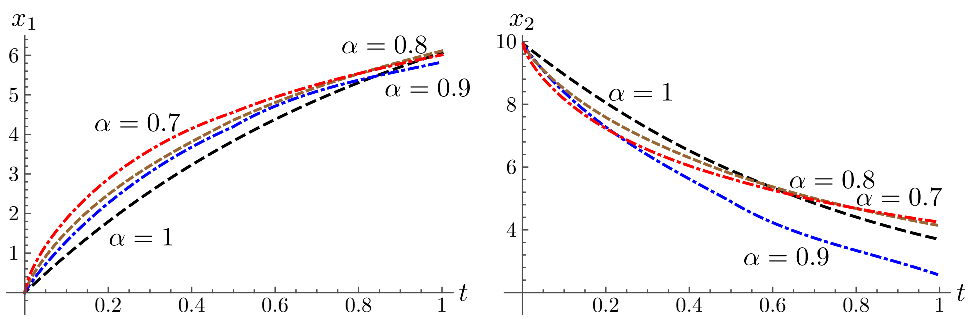


Figure 3. Curves for $\alpha = 0.7, 0.8, 0.9, 1$, Example 7.2.

E x a m p l e 7.3. Consider the two dimensional fractional optimal control problem [17], i.e.,

$$\text{minimize} \quad J = \int_0^1 (x_1^2(t) + x_2^2(t) + 0.005u^2(t))\,\mathrm{d}t, \tag{7.2}$$

s.t.

$$D^\alpha x_1(t) = x_2(t), \quad 0 < \alpha \leqslant 1, \quad D^\alpha x_2(t) = -x_2(t) + u(t),$$
$$x_1(0) = 0, \quad x_2(0) = -1,$$

subject to inequality conditions

$$x_1(t) \leqslant 8(t - 0.5)^2 - 0.5.$$

The resulting values of J together with the solutions obtained by [61] using Chebyshev polynomials, results reported in [20] using interpolating scaling functions and the method presented in [17] are summarized in Table 5.

methods		J
Chebyshev polynomials [61]	$N = 5$, $K = 12$	0.766
	$N = 10$, $K = 20$	0.748
	$N = 13$, $K = 28$	0.740
The Pseudospectral Legendre Method [17]	$M = 5$	0.743013
	$M = 9$	0.740962
Interpolating scaling functions [20]	$N = 5$, $r = 4$	0.746
	$N = 3$, $r = 4$	0.738
	$N = 5$, $r = 5$	0.737
Presented method	$N = 1$, $M = 2$	0.864076
	$N = 2$, $M = 2$	0.712148
	$N = 3$, $M = 2$	0.707853
	$N = 5$, $M = 2$	0.696027

Table 5. The values of J with $\alpha = 1$, for Example 7.3.

One can observe that our method gives state and control functions satisfying the constraints while achieving a lower value of J in comparison with the other methods.

Also the values of the estimated J for various α are presented in Table 6 for $N = 1$, $M = 2$ to show convergence of the estimated values of J as α approaches 1.

	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
J	0.4913	0.72703	0.8459	0.8640

Table 6. The estimated values of J for different α for Example 7.3.

Example 7.4. Consider the FVP [28], i.e.,

$$\text{minimize} \quad J[y(t)] = \frac{1}{2}\int_0^1 (D^\alpha y(t))^2\,\mathrm{d}t, \quad 0 \leqslant \alpha < 1,$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

A closed-form solution for this problem is given by

$$\frac{1}{2\alpha - 1}\int_0^t \frac{\mathrm{d}x}{[(1-x)(t-x)]^{1-\alpha}},$$

and so for $\alpha = 1$ the exact solution is $y(t) = t$. This problem was solved in [28], [46], [4], and [27]. The best results for the absolute error of $y(t)$ are of order 10^{-17} with $m = 4$ and $\alpha = 1$, reported in [46], while by using our proposed method for $M \geqslant 1$ and arbitrary N we have obtained $y(t) = t$, and $J = \frac{1}{2}$, which is the exact solution of this problem.

The validity of this method for different values of α is demonstrated by Figure 4. It shows the approximate and exact curves of $y(t)$ obtained by the present method over $[0, 1]$ for $N = 2$, $M = 3$ and $\alpha = 0.7$, 0.8, 0.9, 1.

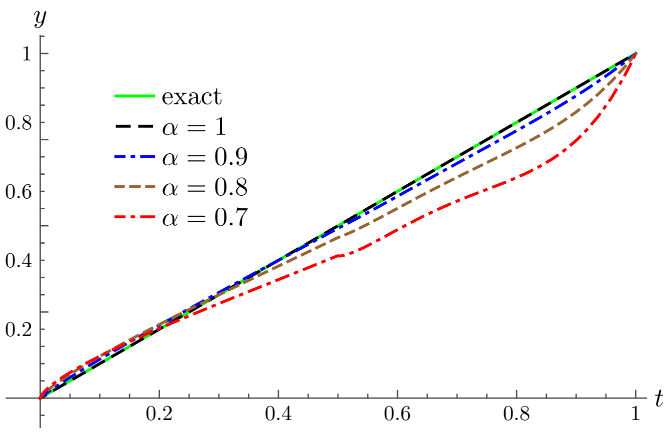


Figure 4. Exact and numerical values of $y(t)$ for Example 7.4.

Example 7.5. Consider the FVP [46], i.e.,

$$\text{minimize} \quad J[y(t)] = \int_0^1 [\tfrac{1}{2}(D^\alpha y(t))^2 - y(t)]\,\mathrm{d}t, \quad 0 \leqslant \alpha < 1,$$

with the boundary conditions

$$y(0) = y(1) = 0.$$

For $\alpha = 1$ the exact solution is $y(t) = \frac{1}{2}(1-t)t$. This problem was solved in [46], [4] and [63], and [46] has presented the best results for the absolute error of $y(t)$ which are of order 10^{-17} with $m = 2$ and $\alpha = 1$. In our method for $M \geqslant 2$ and any N we have obtained $y(t) = \frac{1}{2}(1-t)t$ and $J = -\frac{1}{24}$, which is the exact solution of this problem. The exactness of our numerical solution for this problem results from the accurateness of suggested Riemann-Liouville fractional integration operator.

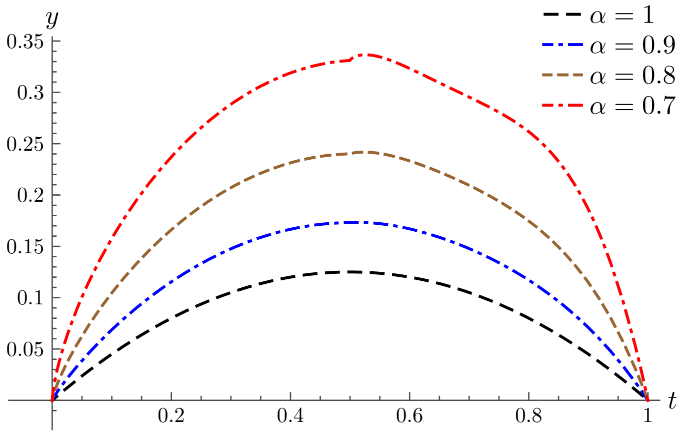


Figure 5. Exact and numerical values of $y(t)$ for Example 7.5.

Since the exact solution is not known for fractional case, we have measured the convergence of solutions by their curves. Figure 5 shows the approximate and exact curves of $y(t)$ obtained by the present method over $[0,1]$ for $N = 2, M = 3$ and $\alpha = 0.7,\ 0.8,\ 0.9,\ 1$. We can see that approximate solutions by the present method approach the exact solution, when α approaches 1.

E x a m p l e 7.6. Consider the FVP [46], i.e.,

$$\text{minimize} \quad J[y(t)] = \int_0^1 [(D^\alpha y(t))^2 + tD^\alpha y(t) + y(t)^2]\,\mathrm{d}t, \quad 0 \leqslant \alpha < 1,$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = \frac{1}{4}.$$

Table 7 gives the approximate values of $y(t)$ using the Legendre wavelet method of [52], for $M = 3$, $k = 3$, Müntz Legendre method [46] for $m = 5$, and the present technique for $M = 3$, $N = 2$, together with the exact solution.

t	Legendre wavelet $k=3, M=3$	Müntz Legendre $m=5$	our method $M=3,\ N=2$	exact
0	0.000000	0.000000	0.000000	0.000000
0.1	0.041949	0.041950	0.041950	0.041950
0.2	0.079315	0.079317	0.079317	0.079316
0.3	0.112471	0.112473	0.112473	0.112472
0.4	0.141749	0.141751	0.141750	0.141750
0.5	0.167443	0.167443	0.167442	0.167442
0.6	0.189807	0.189807	0.189806	0.189806
0.7	0.209064	0.209066	0.209066	0.209065
0.8	0.225411	0.225414	0.225413	0.225412
0.9	0.239010	0.239013	0.239012	0.239011
1	0.249999	0.250000	0.250000	0.250000

Table 7. Comparison of estimated and exact values of $y(t)$ for Example 7.6.

In Figure 6, we present the behavior of the numerical solutions of the problem at $M=3$, $N=2$, for different values of $\alpha = 0.7,\ 0.8,\ 0.9,\ 1$.

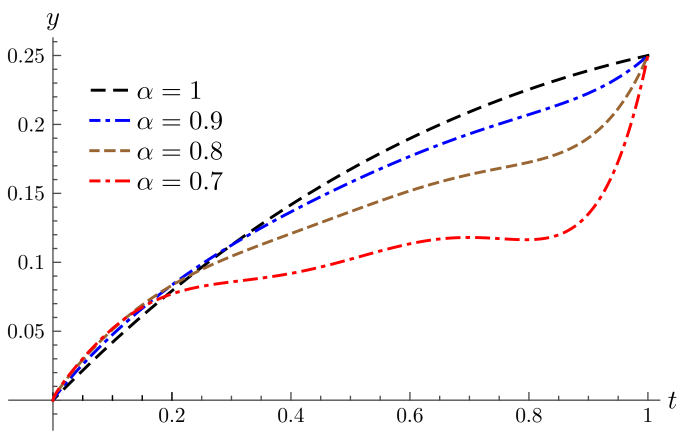


Figure 6. Exact and numerical values of $y(t)$ for Example 7.6.

8. Conclusion

In this article, we introduce an efficient and accurate method to solve a class of fractional optimal control problems and fractional variational problems. First we have constructed the hybrid functions of block-pulse and Boubaker polynomials for the first time and then the general formulation of Riemann-Liouville fractional integral operator for these functions is presented and used to convert the mentioned problems to an optimization problem. To solve the optimization problem we find the minimum of the functional under a system of algebraic equations and inequalities which are collocated at Newton-Cotes points. Our numerical results are compared with exact solution and the findings of other methods.

Acknowledgements. We would like to thank the referees for their valuable comments and helpful suggestions improving the earlier version of this article.

References

[1] *O. P. Agrawal*: Formulation of Euler-Lagrange equations for fractional variational problems. J. Math. Anal. Appl. *272* (2002), 368–379. zbl MR doi

[2] *O. P. Agrawal*: A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dyn. *38* (2004), 323–337. zbl MR doi

[3] *O. P. Agrawal*: A quadratic numerical scheme for fractional optimal control problems. J. Dyn. Syst. Meas. Control *130* (2007), 011010, 6 pages. doi

[4] *O. P. Agrawal*: A general finite formulation for fractional variational problems. J. Math. Anal. Appl. *337* (2008), 1–12. zbl MR doi

[5] *O. P. Agrawal*: Fractional optimal control of a distributed system using eigenfunctions. J. Comput. Nonlinear Dyn. *3* (2008), 021204, 6 pages. doi

[6] *O. P. Agrawal, D. Baleanu*: A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems. J. Vib. Control *13* (2007), 1269–1281. zbl MR doi

[7] *M. Alipour, D. Rostamy, D. Baleanu*: Solving multi-dimensional fractional optimal control problems with inequality constraint by Bernstein polynomials operational matrices. J. Vib. Control *19* (2013), 2523–2540. zbl MR doi

[8] *R. Almeida, D. F. M. Torres*: A discrete method to solve fractional optimal control problems. Nonlinear Dyn. *80* (2015), 1811–1816. zbl MR doi

[9] *N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres*: Discrete-time fractional variational problems. Signal Process. *91* (2011), 513–524. zbl doi

[10] *A. H. Bhrawy, S. S. Ezz-Eldien*: A new Legendre operational technique for delay fractional optimal control problems. Calcolo *53* (2016), 521–543. zbl MR doi

[11] *A. E. Bryson, Y. C. Ho*: Applied Optimal Control. Optimization, Estimation, and Control. Hemisphere, Washington, 1975. MR

[12] *C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang*: Spectral Methods. Fundamentals in Single Domains. Scientific Computation, Springer, Berlin, 2006. zbl MR doi

[13] *C. L. Darby, W. W. Hager, A. V. Rao*: An *hp*-adaptive pseudospectral method for solving optimal control problems. Optim. Control Appl. Methods *32* (2011), 476–502. zbl MR doi

[14] *S. F. Dreyfus*: Variational problems with inequality constraints. J. Math. Anal. Appl. *4* (1962), 297–308. zbl MR doi

[15] *M. El-Kady*: A Chebyshev finite difference method for solving a class of optimal control problems. Int. J. Comput. Math. *80* (2003), 883–895. zbl MR doi

[16] *G. N. Elnagar, M. A. Kazemi*: Pseudospectral Chebyshev optimal control of constrained nonlinear dynamical systems. Comput. Optim. Appl. *11* (1998), 195–217. zbl MR doi

[17] *G. Elnagar, M. A. Kazemi, M. Razzaghi*: The Pseudospectral Legendre method for discretizing optimal control problems. IEEE Trans. Autom. Control *40* (1995), 1793–1796. zbl MR doi

[18] *S. S. Ezz-Eldien*: New quadrature approach based on operational matrix for solving a class of fractional variational problems. J. Comput. Phys. *317* (2016), 362–381. zbl MR doi

[19] *F. Fahroo, I. M. Roos*: Direct trajectory optimization by a Chebyshev pseudospectral method. J. Guid Control Dyn. *25* (2002), 160–166. doi

[20] *Z. Foroozandeh, M. Shamsi*: Solution of nonlinear optimal control problems by the interpolating scaling functions. Acta Astronautica *72* (2012), 21–26. doi

[21] *Q. Gong, W. Kang, I. M. Ross*: A pseudospectral method for the optimal control of constrained feedback linearizable systems. IEEE Trans. Autom. Control *51* (2006), 1115–1129. zbl MR doi

[22] *R. Holsapple, R. Venkataraman, D. Doman*: A modified simple shooting method for solving two-point boundary value problems. IEEE Aerospace Conf. Proc. *6* (2003), 2783–2790. doi

[23] *I. S. Jesus, J. A. Tenreiro Machado*: Fractional control of heat diffusion systems. Nonlinear Dyn. *54* (2008), 263–282. zbl MR doi

[24] *C. Jiang, Q. Lin, C. Yu, K. L. Teo, G.-R. Duan*: An exact penalty method for free terminal time optimal control problem with continuous inequality constraints. J. Optim. Theory Appl. *154* (2012), 30–53. zbl MR doi

[25] *B. Kafash, A. Delavarkhalafi, S. M. Karbassi, K. Boubaker*: A numerical approach for solving optimal control problems using the Boubaker polynomials expansion scheme. J. Interpolat. Approx. Sci. Comput. *2014* (2014), Article ID 00033, 18 pages. MR doi

[26] *E. Keshavarz, Y. Ordokhani, Y. Razzaghi*: A numerical solution for fractional optimal control problems via Bernoulli polynomials. J. Vib. Control *22* (2016), 3889–3903. zbl MR doi

[27] *M. M. Khader*: An efficient approximate method for solving fractional variational problems. Appl. Math. Model. *39* (2015), 1643–1649. MR doi

[28] *M. M. Khader, A. S. Hendy*: A numerical technique for solving fractional variational problems. Math. Methods Appl. Sci. *36* (2013), 1281–1289. zbl MR doi

[29] *A. Khalid, J. Huey, W. Singhose, J. Lawrence, D. Frakes*: Human operator performance testing using an input-shaped bridge crane. J. Dyn. Syst. Meas. Control *128* (2006), 835–841. doi

[30] *A. A. Kilbas, H. M. Srivastava, J. J. Trujillo*: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006. zbl MR doi

[31] *D. E. Kirk*: Optimal Control Theory: An Introduction. Dover Publication, New York, 2004.

[32] *E. Kreyszig*: Introductory Functional Analysis with Applications. John Wiley & Sons, New York, 1978. zbl MR

[33] *P. Lancaster*: Theory of Matrices. Academic Press, New York, 1969. zbl MR

[34] *M. Li, H. Peng*: Solutions of nonlinear constrained optimal control problems using quasilinearization and variational pseudospectral methods. ISA Trans. *62* (2016), 177–192. doi

[35] *A. Lotfi, S. A. Yousefi, M. Dehghan*: Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule. J. Comput. Appl. Math. *250* (2013), 143–160. zbl MR doi

[36] *Z. Lu*: New a posteriori $L^\infty(L^2)$ and $L^2(L^2)$-error estimates of mixed finite element methods for general nonlinear parabolic optimal control problems. Appl. Math., Praha *61* (2016), 135–163. zbl MR doi

[37] *A. B. Malinowska, D. F. M. Torres*: Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative. Comput. Math. Appl. *59* (2010), 3110–3116. zbl MR doi

[38] *H. R. Marzban, S. M. Hoseini*: A composite Chebyshev finite difference method for nonlinear optimal control problems. Commun. Nonlinear Sci. Numer. Simul. *18* (2013), 1347–1361. zbl MR doi

[39] *H. R. Marzban, M. Razzaghi*: Hybrid functions approach for linearly constrained quadratic optimal control problems. Appl. Math. Modelling *27* (2003), 471–485. zbl doi

[40] *H. R. Marzban, H. R. Tabrizidooz, M. Razzaghi*: A composite collocation method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations. Commun. Nonlinear Sci. Numer. Simul. *16* (2011), 1186–1194. zbl MR doi
[41] *S. Mashayekhi, M. Razzaghi*: An approximate method for solving fractional optimal control problems by hybrid functions. J. Vib. Control *24* (2018), 1621–1631. MR doi
[42] *R. K. Mehra, R. E. Davis*: A generalized gradient method for optimal control problems with inequality constraints and singular arcs. IEEE Trans. Autom. Control *17* (1972), 69–79. zbl doi
[43] *M. Motta, C. Sartori*: The value function of an asymptotic exit-time optimal control problem. NoDEA, Nonlinear Differ. Equ. Appl. *22* (2015), 21–44. zbl MR doi
[44] *S. I. Muslih, D. Baleanu*: Formulation of Hamiltonian equations for fractional variational problems. Czech J. Phys. *55* (2005), 633–642. zbl MR doi
[45] *A. Nemati, S. Yousefi, F. Soltanian, J. S. Ardabili*: An efficient numerical solution of fractional optimal control problems by using the Ritz method and Bernstein operational matrix. Asian J. Control *18* (2016), 2272–2282. zbl MR doi
[46] *Y. Ordokhani, P. Rahimkhani*: A numerical technique for solving fractional variational problems by Müntz-Legendre polynomials. J. Appl. Math. Comput. *58* (2018), 75–94. zbl MR doi
[47] *Y. Ordokhani, M. Razzaghi*: Linear quadratic optimal control problems with inequality constraints via rationalized Haar functions. Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms *12* (2005), 761–773. zbl MR
[48] *C. Park, D. J. Scheeres*: Determination of optimal feedback terminal controllers for general boundary conditions using generating functions. Automatica *42* (2006), 869–875. zbl MR doi
[49] *Y.-F. Pu, P. Siarry, J.-L. Zhou, N. Zhang*: A fractional partial differential equation based multiscale denoising model for texture image. Math. Methods Appl. Sci. *37* (2014), 1784–1806. zbl MR doi
[50] *K. Rabiei, Y. Ordokhani, E. Babolian*: The Boubaker polynomials and their application to solve fractional optimal control problems. Nonlinear Dyn. *88* (2017), 1013–1026. zbl MR doi
[51] *K. Rabiei, Y. Ordokhani, E. Babolian*: Fractional-order Boubaker functions and their applications in solving delay fractional optimal control problems. J. Vib. Control *24* (2018), 3370–3383. MR doi
[52] *M. Razzaghi, S. Yousefi*: Legendre wavelets direct method for variational problems. Math. Comput. Simul. *53* (2000), 185–192. MR doi
[53] *F. Riewe*: Nonconservative Lagrangian and Hamiltonian mechanics. Phys. Rev. E. *53* (1996), 1890–1899. MR doi
[54] *F. Riewe*: Mechanics with fractional derivatives. Phys. Rev. E. *55* (1997), 3581–3592. MR doi
[55] *J. L. Schiff*: The Laplace Transform: Theory and Applications. Undergraduate Texts in Mathematics, Springer, New York, 1999. zbl MR doi
[56] *K. Schittkowski*: NLPQL: A FORTRAN subroutine solving constrained nonlinear programming problems. Ann. Oper. Res. *5* (1986), 485–500. MR doi
[57] *I. J. Suárez, B. M. Vinagre, Y. Q. Chen*: A fractional adaptation scheme for lateral control of an AGV. J. Vib. Control *14* (2008), 1499–1511. zbl MR doi
[58] *E. Tohidi, H. Saberi Nik*: A Bessel collocation method for solving fractional optimal control problems. Appl. Math. Model. *39* (2015), 455–465. MR doi
[59] *C. Tricaud, Y. Chen*: An approximate method for numerically solving fractional order optimal control problems of general form. Comput. Math. Appl. *59* (2010), 1644–1655. zbl MR doi
[60] *L. A. Tuan, S. G. Lee*: Sliding mode controls of double-pendulum crane systems. J. Mech. Sci. Technol. *27* (2013), 1863–1873. doi
[61] *J. Vlassenbroeck*: A Chebyshev polynomial method for optimal control with state constraints. Automatica *24* (1988), 499–506. zbl MR doi

[62] *X. Wang, H. Peng, S. Zhang, B. Chen, W. Zhong*: A symplectic pseudospectral method for nonlinear optimal control problems with inequality constraints. ISA Trans. *68* (2017), 335–352. doi

[63] *D. Wang, A. Xiao*: Fractional variational integrators for fractional variational problems. Commun. Nonlinear Sci. Numer. Simul. *17* (2012), 602–610. zbl MR doi

[64] *N. Yan*: Superconvergence analysis and a posteriori error estimation of a finite element method for an optimal control problem governed by integral equations. Appl. Math., Praha *54* (2003), 267–283. zbl MR doi

[65] *W. Yonthanthum, A. Rattana, M. Razzaghi*: An approximate method for solving fractional optimal control problems by the hybrid of block-pulse functions and Taylor polynomials. Optim. Control Appl. Methods *39* (2018), 873–887. zbl MR doi

[66] *S. A. Yousefi, M. Dehghan, A. Lotfi*: Generalized Euler-Lagrange equations for fractional variational problems with free boundary conditions. Comput. Math. Appl. *62* (2011), 987–995. zbl MR doi

[67] *S. A. Yousefi, A. Lotfi, M. Dehghan*: The use of Legendre multiwavelet collocation method for solving the fractional optimal control problems. J. Vib. Control *17* (2011), 2059–2065. zbl MR doi

[68] *M. A. Zaky*: A Legendre collocation method for distributed-order fractional optimal control problems. Nonlinear Dyn. *91* (2018), 2667–2681. zbl doi

Authors' address: *Kobra Rabiei*, *Yadollah Ordokhani* (corresponding author), Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, P. Code: 1993893973, Iran, e-mail: `k.rabiee@alzahra.ac.ir`, `ordokhani@alzahra.ac.ir`.