Bounce Problem with Weak Hypotheses of Regularity (*).

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Summary. – *In this paper the elastic bounce problem is formulated in very general hypotheses. More precisely we consider the motion of a material point constrained to move in a domain* $\Omega \subset \boldsymbol{R}^n$, *bouncing against its boundary, and we suppose that* Ω *is neither regular nor convex. Assuming that* Ω *is in the class of p-convex sets introduced in* [4] *and* $\partial\Omega \in C^{0,1}$, *an existence theorem is stated.*

Introduction.

In several recent papers the elastic bounce problem for a material point constrained to move in a bounded regular domain $\Omega \subseteq \boldsymbol{R}^n$ elastically reflected by the boundary of Ω (see [1], [6], [7]) has been studied. We remark explicitly that a function $x\colon [0, 1] \to \boldsymbol{R}^n$, $x \in \mathrm{Lip}\,([0, 1])$ is said to be a solution of the elastic bounce problem if, setting $\Omega = \{x\colon f(x) \leqslant 0\}$ with $f \in C^2$ and $df(x) \neq 0$ on $\partial\Omega$, we have

i) $f(x(t)) \leqslant 0$;

ii) there exists a positive Radon measure μ on $[0, 1]$ such that

$$\text{(P)} \qquad \operatorname{spt}\mu \subseteq \{t\colon f(x(t)) = 0\} \quad \text{and} \quad \ddot{x} = -\mu\nabla f(x(t))\ ;$$

iii) the function $\mathcal{E}\colon t \to |\dot{x}(t)|^2$ is continuous on $[0, 1]$.

Indeed such a formulation applies only when Ω is regular enough (namely $\partial\Omega \in C^1$) and we have existence results only when f is at least of class $C^{1,1}$.

Another formulation was given by M. SCHATZMAN assuming the convexity of Ω and neglecting the regularity of $\partial\Omega$ (see [8]).

In both cases we cannot consider—for example—those domains of $\boldsymbol{R}^n$ which are piecewise « regular » and piecewise « convex » (see Fig. 1).

The aim of this paper is to find an adequate formulation of the elastic bounce problem (which is equivalent to the preceding ones in the « regular » and in the « convex » case) and a larger class of domains (in general neither regular nor convex) to which such a formulation can be applied.

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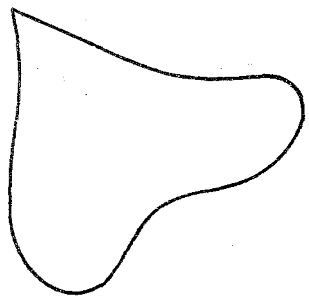

Fig. 1.

Here is an outline of the paper.

In Section I we give some definitions and results which are useful tools in proving our main theorems. The notion of (p, q)-convexity is defined and its connection with Γ-convergence is pointed out.

In Section II we reformulate the elastic bounce problem in non convex and non regular domains and we state an « equivalence » theorem between the « classical » and this new formulation of the elastic bounce problem; moreover an existence theorem in p-convex domains having Lipschitz continuous boundary is stated.

Sections III and IV are devoted to the proof of the previous theorems; as for the latter case we remark that the method consists in defining suitable approximating problems whose limit (in the sense of Γ-convergence) is the generalized elastic bounce problem itself.

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I. – Notations and preliminary results.

Let (X, d) be a metric space and $f_h\colon X \to \boldsymbol{R}$. We set

$$\Gamma(X^-) \liminf_{h\to\infty} f_h(x) = f_\infty^-(x) = \sup_{A\in\mathfrak{F}_x} \liminf_{h\to\infty} \inf_{y\in A} f_h(y)$$

$$\Gamma(X^-) \limsup_{h\to\infty} f_h(x) = f_\infty^+(x) = \sup_{A\in\mathfrak{F}_x} \limsup_{h\to\infty} \inf_{y\in A} f_h(y)$$

and we say that

$$\Gamma(X^-) \lim_{h\to\infty} f_h(x) = f_\infty(x)$$

if and only if $f_\infty^+(x) = f_\infty^-(x) = f_\infty(x)$.

The following properties will be useful in the sequel

PROPOSITION I.1. – Let (X, d) be a metric space, $f_h, f_\infty\colon X \to \boldsymbol{R}$ such that

$$\Gamma(X^-)\lim_{h\to\infty} f_h = f_\infty$$

then

$$\inf_A f_\infty \geqslant \limsup_{h\to\infty} \inf_A f_h$$

for every open set $A \subseteq X$.

PROOF. – From the definition of Γ-limit we argue

$$f_\infty(x) \geqslant \limsup_{h\to\infty} \inf_{y\in A} f_h(y)$$

whence the thesis.

DEFINITION I.2. – Let $x_0 \in X$, $\varrho > 0$; we put

$$(\mathrm{Dm}\, f)(x_0, \varrho) = f(x_0) - \inf_{B_\varrho(x_0)} f(x)$$

with the convention $+\infty - \infty = 0$.
$(\mathrm{Dm}\, f)(x_0, \varrho)$ is called « lack of minimality » of x_0 in $B_\varrho(x_0)$.

PROPOSITION I.3. – If $\varrho > 0$, $\Gamma(X^-)\lim f_h = f_\infty$, $x_h \to x_0 \in X^-$ then

$$(\mathrm{Dm}\, f_\infty)(x_0, \varrho) \leqslant \liminf_{h\to\infty} (\mathrm{Dm}\, f_h)(x_h, \varrho)\,.$$

PROOF. – The proof follows easily from Proposition I.1.

REMARK I.4. – x_0 is a minimum point for f_∞ in $B_\varrho(x_0)$ if and only if $(\mathrm{Dm}\, f_\infty)(x_0, \varrho) = 0$; therefore if $(\mathrm{Dm}\, f_h)(x_0, \varrho) = 0$ we argue $(\mathrm{Dm}\, f_\infty)(x_0, \varrho) = 0$ and x_0 is a minimum point for f_∞ in $B_\varrho(x_0)$.

In addition to the previous hypothesis we suppose now that X is a Hilbert space, $\|\cdot\|$ its norm, $\langle\cdot,\cdot\rangle$ its inner product and $f\colon X \to \overline{\boldsymbol{R}}$ is a proper lower semicontinuous function. We set $D(f) = \{x \in X\colon f(x) \in \boldsymbol{R}\}$ and for each $y \in D(f)$ we define

$$\partial^- f(y) = \left\{\alpha \in X\colon \liminf_{z\to y} \frac{f(z) - f(y) - \langle \alpha, z - y\rangle}{\|z - y\|} \geqslant 0\right\}$$

the set of subdifferentials of f at y.

REMARK I.5. – Suppose that $(\mathrm{Dm}\, f)(x_0, \varrho) = 0$ and $f(x_0) < +\infty$ then $0 \in \partial^- f(x_0)$.
We now need some further definitions.

DEFINITION I.6. – Let $f\colon X \to \mathbf{R} \cup \{+\infty\}$ a l.s.c. function. We say that f is a (p, q)-convex function if for every $x, y \in X$ there exists $z \in X$ such that

i) $\left|z - \dfrac{x+y}{z}\right| \leqslant p|x-y|^2$;

ii) $2f(z) \leqslant f(x) + f(y) - 2q|x-y|^2$.

REMARK I.7. – If f has Lipschitz continuous gradient and M is the Lipschitz constant of grad f then f is $(0, -M/8)$-convex.

We recall now some properties of (p, q)-convex functions (see [4], [5]).

PROPOSITION I.7. – If f is (p, q)-convex then for every $x, y \in X$, $16p\|y\| \leqslant 1$ and for every $\alpha \in \partial^- f(x)$ we have

$$f(x+y) \geqslant f(x) + \langle \alpha, y \rangle - 8(q - |\alpha|p)^- \|y\|^2 . \tag{1}$$

PROPOSITION I.8. – Let $\{x_h\}_{h\in N}$ be a sequence of points such that $x_h \to x_0$ and $f(x_h) + f(x_0) < +\infty$ and suppose that there exists a sequence $\{\alpha_h\}_{h\in N}$ such that $\alpha_h \in \partial^- f(x_h)$ and $\alpha_h \xrightarrow{w} \alpha \in X$. Then $\alpha \in \partial^- f(x_0)$ and $f(x_h) \to f(x_0)$.

PROOF. – Let $y \in D(f)$ and let y_h be a sequence such that $f(y) = \lim_h f(y_h)$ and $y_h \to y$. Since $f(x_0) \leqslant \liminf_{h\to\infty} f(x_h)$ from (1) we argue

$$f(y_h) \geqslant f(x_n) + \langle \alpha_h, y_h - x_h \rangle - 8(q - |\alpha_h|p)^- \|y_h - x_h\|^2$$

and therefore

$$f(y) \geqslant f(x_0) + \langle \alpha, y - x \rangle - 8(q - |\alpha|p)^- \|y - x_0\|^2$$

at least when $\|y - x_0\| \leqslant 1/16p$ and therefore $\alpha \in \partial^- f(x_0)$. Let now $x'_h \to x_0$ such that $f(x'_h) \to f(x_0)$. By passing to the lim sup in the following inequality

$$f(x'_h) \geqslant f(x_h) + \langle \alpha_h, x'_h - x_h \rangle - 8(q - |\alpha_h|p)^- \|x'_h - x_h\|^2$$

we obtain $f(x_0) \geqslant \limsup_h f(x_h)$ and the proof is over.

It is not difficult to generalize this proof to obtain the following result (see [5]).

PROPOSITION I.9. – Let $(f_h)_h$ be a sequence of (p, q)-convex functions such that $\Gamma(X^-) \lim f_h = f\colon X \to \mathbf{R} \cup \{+\infty\}$. Then if $x_h \to x_\infty$ in X with $2pd(u, D(f)) < 1$ and $\alpha_h \to \alpha$ with $\alpha_h \in \partial^- f_h(x_h)$, then $\subseteq \in \partial^- f(x_\infty)$ and $f_h(x_h) \to f(x_\infty)$.

Let now $\Omega \subseteq X$ be a closed set; we say that Ω is p-convex if the function

$$J(x) = \begin{cases} 0 & \text{if } x \in \Omega \\ +\infty & \text{if } x \notin \Omega \end{cases}$$

is $(p, 0)$-convex.

We remark explicitly that this is equivalent to saying that for every $x, y \in \Omega$ there exists $z \in \Omega$ such that

$$\left\| z - \frac{x+y}{2} \right\| \leqslant p\|x-y\|^2 .$$

Let us recall here some properties of p-convex sets (see [4]).

PROPOSITION I.10. – If $\Omega \subseteq X$ is p-convex then for every $x \in X$ such that $8pd(x, \Omega) \leqslant 1$ there exists a unique $T(x) \in \Omega$ such that

$$d(x, \Omega) = \|x - T(x)\| .$$

PROPOSITION I.11. – Let $E = \{x \in X\colon 64pd(x, \Omega) \leqslant 1\}$. The map $T\colon E \to \Omega$ defined by $T(x)$ in the above theorem is a Lipschitz map.

PROOF. – Let $x, y \in E$, then

$$\|T(x) - T(y)\| \leqslant \|T(x) - x\| + \|T(y) - y\| + \|x - y\| \leqslant \frac{1}{32p} + \|x - y\|$$

and therefore two cases may occur

i) $\|x-y\| \geqslant 1/32p$ which yields $\|T(x) - T(y)\| \leqslant 2\|x-y\|$ and the thesis is proven;

ii) $\|x-y\| \leqslant 1/32p$ yields $\|T(x) - T(y)\| \leqslant 1/16p$ and therefore the following inequalities hold.

$$(*) \qquad \langle T(x) - T(y), y - T(y)\rangle - 8p\|T(x) - T(y)\|\|y - T(y)\| \leqslant 0$$

$$(**) \qquad \langle T(y) - T(x), x - T(x)\rangle - 8p\|T(x) - T(y)\|\|x - T(x)\| \leqslant 0 .$$

Adding $(*)$ and $(**)$ we get

$$4\langle T(x) - T(y), y - x + T(x) - T(y)\rangle \leqslant \|T(x) - T(y)\|^2;$$

since we may suppose $T(x) \neq T(y)$ the above inequality yields

$$\|T(x) - T(y)\| \leqslant \tfrac{4}{3}\|x - y\| .$$

Combining this inequality with i) we get

$$\|T(x) - T(y)\| \leqslant 2\|x - y\|$$

for every $x, y \in E$. ■

From the above result we argue that the map $\psi: E \to \boldsymbol{R}_+$ defined by $\psi(x) = \|x - T(x)\|^2$ is a C^1 function with Lipschitz continuous gradient $\operatorname{grad} \psi(x) = 2(x - T(x))$.

From now on we will denote by Ω a closed p-convex subset of $\boldsymbol{R}^n$ such that $\overset{\circ}{\Omega} = \Omega$ and we put $\Omega_\varepsilon = \{x \in \boldsymbol{R}^n : d(x, \Omega) \leqslant \varepsilon\}$. We prove the following

PROPOSITION I.12. – If Ω is p-convex then Ω_ε is $4p$-convex.

PROOF. – We have to show that for every $x, y \in \Omega_\varepsilon$ there exists $z \in \Omega_\varepsilon$ such that

$$\left| z - \frac{x+y}{2} \right| \leqslant 4|x - y|^2 .$$

Since Ω is p-convex there exists $z \in \Omega$ such that

$$\left| z - \frac{T(x) + T(y)}{2} \right| \leqslant p|T(x) - T(y)|^2$$

for every $x, y \in \Omega_\varepsilon$ and therefore

$$\left| z + \frac{x+y}{2} - \frac{T(x) + T(y)}{2} - \frac{x+y}{2} \right| \leqslant p|T(x) - T(y)|^2 \leqslant 4p|x - y|^2 .$$

Put $z' = z + (x + y)/2 - (T(x) + T(y))/2$; we claim that $z' \in \Omega_\varepsilon$; in fact

$$d(z', \Omega) \leqslant d(z', z) = \left| \frac{x+y}{2} - \frac{T(x) + T(y)}{2} \right| \leqslant \frac{1}{2} \{|x - T(x)| + |y - T(y)|\} \leqslant \varepsilon$$

and the theorem is completely proven.

Let now $g: \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be a Lipschitz function such that $\operatorname{epi} g = \{(x, y) \in \boldsymbol{R}^{n-1} \times \boldsymbol{R} : g(x) \leqslant y\}$ is a p-convex set; if $A \subset \boldsymbol{R}^{n-1}$ is any open bounded set, we prove the following

PROPOSITION I.13. – For every ε small enough there exists $\psi_\varepsilon : \bar{A} \to \boldsymbol{R}$, $\psi_\varepsilon \in C^{1,1}(\bar{A})$ wuch that if we set $E_\varepsilon = \{(x, y) \in \bar{A} \times \boldsymbol{R} : d((x, y), \operatorname{epi} g|_{\bar{A}})\}$ then $\partial E_\varepsilon = \{(x, y) : y = \psi_\varepsilon(x)\}$.

PROOF. – We argue by contradiction and suppose that there exist $\varepsilon_k \searrow 0$ and $(x_k, y_k) \in \partial E_{\varepsilon_k}$, $x_k \in A$ such that $(\alpha_k, 0) \in \partial^- J_{E_{\varepsilon_k}}(x_k, y_k)$ with $|\alpha_k| = 1$. It is easy to see that $\Gamma((\bar{A} \times \boldsymbol{R})^-) \lim J_{E_{\varepsilon_k}} = J_{\operatorname{epi} g|\bar{A}}$ and moreover we may suppose that $(x_k, y_k) \to (x_0, g(x_0)) \in \bar{A} \times \boldsymbol{R}$ and $\alpha_k \to \alpha$ with $|\alpha| = 1$; since $J_{\operatorname{epi} g|\bar{A}}(x_0, g(x_0)) = 0$ we get

$\alpha \in \partial^- J_{\operatorname{epi} g|\bar{A}}(x_0, g(x_0))$. From the last inclusion we argue

$$\limsup_{t\to 0} \frac{t\langle \alpha, \alpha\rangle}{\sqrt{t^2|\alpha|^2 + |g(x_0 + t\alpha) - g(x_0)|^2}} \leqslant 0$$

but g is a Lipschitz function and then

$$\frac{t\langle \alpha, \alpha\rangle}{\sqrt{t^2|\alpha|^2 + |g(x_0 + t\alpha) - g(x_0)|^2}} \geqslant \frac{t|\alpha|^2}{|\alpha| t\sqrt{1 + M^2}} = \frac{1}{\sqrt{1 + M^2}}$$

which is a contradiction.

We conclude this section with the following

PROPOSITION I.13. – Let $g: \mathbf{R}^n \to \mathbf{R}$, $g \in C^{1,1}(\mathbf{R}^n)$; set $\Omega = \{x \in \mathbf{R}^n : g(x) \leqslant 0\}$ and suppose that $\Omega \neq 0$ and $|dg(x)| \geqslant \lambda > 0$ on $\partial\Omega$. Then Ω is $3M/8\lambda$-convex, where M is the Lipschitz constant of grad g.

PROOF. – By Remark I.7 (see also [4], Thm. 1.8) we have that g is $(0, -M/8)$-convex and therefore for every $x, y \in \Omega$, $g((x+y)/2) \leqslant (M/8)|x-y|^2$. Take $\bar{z} \in \Omega$ such that

$$\left|\bar{z} - \frac{x+y}{2}\right| = \min\left\{\left|z - \frac{x+y}{2}\right| : g(z) = 0\right\};$$

by a standard argument there exists $\lambda_0 > 0$ such that $(x+y)/2 = \bar{z} + \lambda_0 \nabla g(\bar{z})$ and then the following inequality holds

$$g(\bar{z} + \lambda_0 \nabla g(\bar{z})) \geqslant g(\bar{z}) + \lambda_0|\nabla g(\bar{z})|^2 - \lambda_0^2 M|\nabla g(z)|^2 = \lambda_0|\nabla g(\bar{z})|^2 - \lambda_0^2 M|\nabla g(\bar{z})|^2$$

and then

$$\lambda_0|\nabla g(\bar{z})|^2 \leqslant \frac{M}{8}|x-y|^2 + M\lambda_0^2|\nabla g(\bar{z})|^2$$

i.e.

$$\lambda_0|\nabla g(\bar{z})| \leqslant \frac{M}{8|\nabla g(\bar{z})|}|x-y|^2 + \frac{M\lambda_0^2}{\lambda}|\nabla g(\bar{z})|^2 \leqslant \frac{M}{8\lambda}|x-y|^2 + \frac{M\lambda_0^2}{\lambda}|\nabla g(\bar{z})|^2$$

which is equivalent to

$$\left|\bar{z} - \frac{x+y}{2}\right| \leqslant \frac{M}{8\lambda}|x-y|^2 + \frac{M}{\lambda}\left|\bar{z} - \frac{x+y}{2}\right|^2 \leqslant \frac{M}{8\lambda}|x-y|^2 + \frac{M}{4\lambda}|x-y|^2 \leqslant \frac{3M}{8\lambda}|x-y|^2 .$$

This inequality completes the proof. ∎

II. – Statement of the problem.

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ be a $C^{1,1}$ function such that $df(x) \neq 0$ on the set $\Omega = \{x: f(x) \leqslant 0\}$. In [1], [6], [7] we have studied the elastic bounce problem for a material point constrained to move in the region $\Omega = \{x: f(x) \leqslant 0\}$ and elastically reflected by the « wall » $\partial\Omega = \{x: f(x) = 0\}$. Without loss of generality in this paper we assume that no external forces are acting on the point.

Denoting by $x(t)$ the position of the material point at time t, we say, in accordance with [1], [6], [7], that $x \in \text{Lip}([0, 1]; \boldsymbol{R}^n)$ solves the elastic bounce problem, if

i) $f(x(t)) \geqslant 0$;

ii) there exists a positive Radon measure μ on $[0, 1]$ such that $\text{spt}\, \mu \subseteq \subseteq \{t: f(x(t)) = 0\}$ and

$$\ddot{x} = -\mu \nabla f(x(t)) ;$$

iii) the function $\mathcal{E}: t \to |\dot{x}(t)|^2$ is continuous in $[0, 1]$

where $\dot{x}_+$, $\dot{x}_-$ respectively denote the right and left derivatives of x since x is a BV function (see [1] and [6]). Following [1] we introduce the set

$$E = \{x \in \text{Lip}([0, 1]; \boldsymbol{R}^n): x \text{ solves } P\}$$

and define the initial trace $\mathcal{E}: [0, 1] \times \boldsymbol{R}^n \to \boldsymbol{R}^{3n+1}$

$$\mathcal{E}(t, x) = \left(\frac{|\dot{x}(t)|^2}{2},\ x(t), \dot{x}_\tau(t), f(x(t))\dot{x}(t)\right)$$

where $\dot{x}_\tau(t) = |\nabla f(x)|^2 \dot{x} - \langle \dot{x}, \nabla f(x)\rangle \nabla f(x)$.

As observed in [1] when the initial trace b is assigned then the initial Cauchy data are given; without writing complicated formulae we will denote $\dot{x}_+(0)$ by $\tilde{b}$ (which is a completely determined function of b).

Now, let $t_0 \in [0, 1]$ and $b \in \mathcal{C}(\{t_0\} \times E) = \mathcal{B}$ be fixed; we set

$$G(t_0, b) = \{x \in \text{Lip}([0, 1]; \boldsymbol{R}^n): x \in E,\ \mathcal{C}(t_0, x) = b\}.$$

We recall (see [ ]) that the set $G(t_0, b)$ is non empty at least when $f \in C^1{,}^1(\boldsymbol{R}^n)$.

Let now Ω be a closed subset of $\boldsymbol{R}^n$ satisfying the following conditions

(H_0) $\bar{\mathring{\Omega}} = \Omega$;

(H_1) Ω is p-convex;

(H_2) $\partial\Omega$ is locally the graph of a Lipschitz function.

Set

$$JI(x) = \begin{cases} 0 & \text{if } x \in \Omega \\ +\infty & \text{if } x \notin \Omega \end{cases}$$

and fixed $t_0 = 0$, $b \in \mathfrak{B}$, we introduce the spaces

$$X = \{x \in \text{Lip}, \dot{x} \in BV: x(0) = b_1, \dot{x}_+(0) = \tilde{b}, |\dot{x}| = |b| = \sqrt{2b_1}, J_\Omega(x(t)) = 0\},$$

$$Y = \{y \in (H^1(0, 1))^n: y(1) = 0\}$$

and consider the problem

(P*) Find $\bar{x} \in X$ and a constant $K(\bar{x}, \Omega, b)$ such that

$$\min_{\substack{y \in Y \\ 64p\|y\| \leqslant 1}} I(\bar{x}, y) + K(\bar{x}, \Omega, b)\|y\|^2_{H^1} = I(\bar{x}, 0) = 0$$

and

$$I(x, y) = \int_0^1 [\langle \dot{x}_+(0), y(0)\rangle - \langle \dot{x}, \dot{y}\rangle + J_\Omega(x + y)]\, dt\,.$$

The aim of this paper is to show that problems (P) and (P*) are—in a suitable sense—equivalent problems and there exists a large class of domains (indeed those verifying (H_0), (H_1), (H_2)) in which problem (P*) has a solution. More precisely we will prove the following

THEOREM 1. – If $f \in C^{1,1}(\boldsymbol{R}^n)$ and $\bar{x} \in G(0, b)$ then

$$I(x, 0) = \min_{\substack{y \in Y \\ 64p\|y\| \leqslant 1}} \{I(x, y) + K\|y\|^2_{H^1}\} = 0$$

with $K = M \cdot \mu([0, 1](2 + 1/\lambda_0)$ and $\lambda_0 = \min\{|\nabla f(\zeta)|: \zeta \in \partial\Omega \cap S(b_2, \sqrt{2b_1})\}$.
Conservely, if there exist $\bar{x} \in X$ and $\bar{K}(\bar{x}, \Omega, b)$ such that

$$0 = I(\bar{x}, 0) = \min_{\substack{y \in Y \\ 64p\|y\| \leqslant 1}} \{I(\bar{x}, y) + \bar{K}\|y\|^2_{H^1}\}$$

then $\bar{x} \in G(0, b)$. ■

THEOREM 2. – If Ω satisfies hypothesis (H_0), (H_1), (H_2) then problem (P*) admits a solution. ■

III. – Proof of Theorem 1.

Before beginning the proof of Thm. 1 it is useful to make some simplifying assumptions:

(I_1) the set $\partial\Omega \cap S(b_2, 2|\tilde{b}|)$ is supposed to be non empty;

(I_2) by virtue of the previous assumption we may suppose that $\partial\Omega$ is compact so that setting $\lambda = \min_{\partial\Omega} |\nabla f| > 0$ the set $\Omega = \{x: f(x) \leqslant 0\}$ becomes $3M/8\lambda$-convex as soon as f is a $C^{1,1}$-function and M is the Lipschitz constant of ∇f.

Now we may state the following

LEMMA III. 1.– If $\ddot{x} = -\mu\nabla f(\bar{x})$ and $J_\Omega(\bar{x}(t)) = 0$ $\big($i.e., $f(\bar{x}(t)) \leqslant 0\big)$ then setting $G(\zeta) = \int_0^1 J_\Omega(\zeta(t))\, dt$ we have

$$G(\bar{x} + y) \geqslant \int_0^1 \langle y, \nabla f(\bar{x})\rangle\, d\mu - M\frac{3}{\lambda}\mu([0,1])\|y\|^2_{H^1} \tag{3.1}$$

for every $y \in Y$ such that $\|y\|_{H^1} \leqslant 3M/\lambda$.

PROOF. – We first observe that $\int_0^1 J_\Omega(\bar{x} + y)\, d\mu \leqslant \int_0^1 J_\Omega(\bar{x} + y)\, dt$ and therefore

$$\int_0^1 J_\Omega(\bar{x} + y)\, dt - \int_0^1 \langle y, \nabla f(\bar{x})\rangle\, d\mu \geqslant \int_0^1 [J_\Omega(\bar{x} + y) - \langle y, f(\bar{x})\rangle]\, d\mu\,; \tag{3.2}$$

since Ω is $3M/8\lambda$-convex we get

$$J_\Omega(\bar{x} + y) - \langle y, \nabla f(\bar{x})\rangle \geqslant -M\frac{3}{\lambda}|y|^2\,, \qquad \forall t \in [0,1] \tag{3.3}$$

for every $y \in Y$ such that $\|y\| \leqslant \lambda_0/2M(2\lambda_0 + 1)$.

The last inequality yields:

$$G(\bar{x} + y) \geqslant \int_0^1 \langle y, \nabla f(\bar{x})\rangle\, d\mu - M\frac{3}{\lambda}\int_0^1 |y|^2\, d\mu\,, \tag{3.4}$$

and we observe that

$$(3.5)\qquad \int_0^1 |y(s)|^2\,d\mu = -2\int_0^1 \langle y, \dot y\rangle \mu([0,t])\,dt \leqslant \int_0^1 (|y|^2 + |\dot y|^2)\mu([0,t])\,dt \leqslant$$

$$\leqslant \mu([0,1])\int_0^1 (|y|^2 + |\dot y|^2)\,dt = \mu([0,1])\|y\|^2_{H^1};$$

combining the last two inequalities we get (3.1). ■

From the previous Lemma we argue

$$(3.6)\qquad \int_0^1 J_\Omega(\bar x + y)\,dt + \langle \ddot x, y\rangle \geqslant - M\frac{3}{\lambda}\mu([0,1])\|y\|^2_{H^1}$$

and then

$$(3.7)\qquad \int_0^1 \{J_\Omega(\bar x + y) - \langle \dot x, y\rangle\}\,dt + \langle \tilde b, y(0)\rangle \geqslant - M\frac{3}{\lambda}\mu([0,1])\|y\|^2_{H^1}$$

i.e.

$$(3.8)\qquad I(\bar x, y) + M\frac{3}{\lambda}\mu([0,1])\|y\|^2_{H^1} \geqslant 0$$

and being $I(\bar x, 0) = 0$ we obtain

$$(3.9)\qquad \max_{x\in X}\min_{\substack{y\in Y\\ 64p\|y\|\leqslant 1}} \left\{I(\bar x, y) + M\frac{3}{\lambda}\mu([0,1])\|y\|^2_{H^1}\right\} = 0\,,$$

and the first part of Thm. 1 is proven.

Suppose now that there exists $\bar x \in X$ and $K > 0$ such that

$$\min_{\substack{y\in Y\\ 64p\|y\|\leqslant 1}} \{I(\bar x, y) + K\|y\|^2\} = 0$$

then there exists $\bar x \in X$ such that for every $y \in Y$ we have

$$(3.10)\qquad \int_0^1 - \langle \dot{\bar x}, \dot y\rangle + J_\Omega(\bar x + y) + \langle b, y(0)\rangle\}\,dt + K\|y\|^2_{H^1} \geqslant 0\,.$$

Choose $\varphi \in \left(C_0^\infty([0,1])\right)^n \subset Y$ such that $\operatorname{spt}\varphi \subset \{t\colon f(\bar x(t)) < 0\}$; since

$$f(\bar x + \varepsilon\varphi) = f(\bar x) + \varepsilon\langle \varphi, \nabla f(\bar x + \varepsilon'\varphi)\rangle$$

an easy calculation shows that if ε is small enough (independently of t) then $f(x(t) + \varepsilon\varphi(t)) < 0$ for every $t \in [0, 1]$.

Therefore from (3.10) we argue

$$\int_0^1 \{-\langle \bar{x}, \dot{\varphi}\rangle + \langle b, \varphi(0)\rangle\}\, dt + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^1 J_\Omega(\bar{x} + \varepsilon\varphi) \geqslant 0$$

but $J_\Omega(\bar{x} + \varepsilon\varphi) = 0$ if ε is small enough and then

$$\int_0^1 \{-\langle \dot{\bar{x}}, \dot{\varphi}\rangle + \langle b, \varphi(0)\rangle\}\, dt \geqslant 0\,.$$

Since the same arguments hold when φ is changed into $-\varphi$ we argue

$$\int_0^1 \{-\langle \dot{\bar{x}}, \dot{\varphi}\rangle + \langle b, \varphi(0)\rangle\}\, dt = 0$$

and then $(\ddot{\bar{x}}, \varphi) = 0$ for every φ such that $\operatorname{spt}\varphi \subset \{t\colon f(\bar{x}(t)) < 0\}$ (1).

Choose now $\theta \in C_0^\infty([0, 1])$, $\theta \geqslant 0$ and set $y_\theta = -\theta\nabla f(\bar{x})$, $y_\theta \in Y$; since $\ddot{\bar{x}}$ is the derivative of a L^2-function, the duality between $\ddot{\bar{x}}$ and y_θ makes sense and we get

$$(3.11)\qquad ((\ddot{\bar{x}}, y_\theta)) = \sum_{i=1}^n \left(\ddot{\bar{x}}_i, -\theta\frac{\partial f}{\partial x_i}\right) = \sum_{i=1}^n \left(\ddot{\bar{x}}\frac{\partial f}{\partial x}, -\theta\right) = -(\langle \ddot{\bar{x}}, \nabla f\rangle, \theta)$$

so that from (3.10) we argue

$$(3.12)\qquad \liminf_{\varepsilon\leftarrow 0}\left[(\langle \ddot{\bar{x}}, \nabla f(\bar{x})\rangle, \theta) + \frac{1}{\varepsilon}\int_0^1 J_\Omega(\bar{x} + \varepsilon y_\theta)\, dt\right] \geqslant 0\,;$$

since

$$f(\bar{x}(t)) - \varepsilon\theta(t)\nabla f(\bar{x}(t)) \leqslant f(\bar{x}(t)) - \varepsilon\theta(\bar{t})|\nabla f(x)|^2(1 - \varepsilon\theta(t)M)$$

taking $\varepsilon < 1/M\|\theta\|_\infty$ we get $f(\bar{x} - \varepsilon\theta\nabla f(\bar{x})) \leqslant 0$ for every $t \in [0, 1]$ and therefore (3.12) yields

$$(3.13)\qquad (\langle \ddot{\bar{x}}, \nabla f(\bar{x})\rangle, \theta) \leqslant 0$$

and then $\langle \ddot{\bar{x}}, \nabla f(\bar{x})\rangle = -\nu$ with $\nu \in \mathcal{M}^+([0, 1])$.

---

(1) From now on we denote with $(\ ,\ )$ the duality between $\mathfrak{D}'(0, 1)$ and $\mathfrak{D}(0, 1)$, and with $((\ ,\ ))$ the duality between $(\mathfrak{D}'(0, 1))^n$ and $(\mathfrak{D}(0, 1))^n$.

Now, take $\psi \in \left(C_0^\infty([0,1])\right)^n$ and set

$$\psi_\varepsilon = |\nabla f|^2 \psi - \langle \psi, \nabla f(\bar{x})\rangle \nabla f(\bar{x}) - \varepsilon |\psi| \nabla f(\bar{x})\,;$$

then $\varepsilon\psi_\varepsilon \in Y$ as ε is small enough and—as in the previous case—$J_\Omega(\bar{x} + \varepsilon\psi_\varepsilon) = 0$ when ε goes to 0; since the same arguments hold when ψ is changed into $-\psi$ we get

$$\left(\!\left(\ddot{\bar{x}}, |\nabla f|^2 \psi\right)\!\right) = \left(\langle \ddot{\bar{x}}, \nabla f\rangle, \langle \psi, \nabla f\rangle\right) = -\left(\nu, \langle \psi, \nabla f(\bar{x})\rangle\right)$$

and then

$$\left(\!\left(|\nabla f|^2 \ddot{\bar{x}}, \psi\right)\!\right) = -\left(\!\left(\nu \nabla f(\bar{x}), \psi\right)\!\right)$$

for every $\psi \in \left(C_0^\infty([0,1])\right)^n$ so that

$$|\nabla f(\bar{x})|^2 \ddot{\bar{x}} = -\nu \nabla f(\bar{x})\,;$$

since $\nabla f(\bar{x}) \neq 0$ on $\operatorname{spt} \bar{x}$ we may write

$$\ddot{\bar{x}} = -\mu \nabla f(\bar{x})$$

with $\mu \in \mathcal{M}^+([0,1])$, $\mu = \nu/|\nabla f|^2$ and $\operatorname{spt} \mu \subseteq \{t\colon f(\bar{x}(t)) = 0\}$ and Theorem 1 is completely proven.

REMARK 1. – As we can see in a moment, if the functional contains terms of the type $U(t, x)$ which are L^1 in the time-variable and C^2 in the space-variable then Theorem 1 can be easily generalized and the proof remains the same.

REMARK 2. – It is not difficult to see that the theorem remains true if f is supposed to be only a Lipschitz map which is $C^{1,1}$ only in a neighborhood V of $\partial\Omega$.

IV. – Proof of Theorem 2.

Let $\Omega_\varepsilon = \{x \in \mathbf{R}^n\colon d(x, \Omega) \leqslant \varepsilon\}$; as we have seen in Proposition I.12, Ω_ε is $(16p + 3)p$-convex and $\partial\Omega_\varepsilon \in C^{1,1}$. Since $\mathcal{F}_\varepsilon = S(b_2, 2|\tilde{b}|) \cap \partial\Omega_\varepsilon \neq \emptyset$, by using Proposition I.13, we argue that $\mathcal{F}_\varepsilon$ is a finite union of graphs of $C^{1,1}$ functions, so that, without loss of generality, we may suppose $\mathcal{F}_\varepsilon = \{x\colon x_n = g_\varepsilon(x_1, \ldots, x_{n-1})\}$ and therefore $\partial d/\partial x_n \neq 0$ on $\mathcal{F}_\varepsilon$. Since $\mathcal{F}_\varepsilon$ is compact we may suppose $0 < m_1 \leqslant \partial d/\partial x_n \leqslant m_2$ on $\mathcal{F}_\varepsilon$.

Let now u_ε be a solution of the problem

i) $d(u_\varepsilon, \Omega) - \varepsilon \leqslant 0$;

ii) there exists a Radon measure $\mu_\varepsilon \geqslant 0$ such that

$$(\mathrm{P}_\varepsilon) \qquad \operatorname{spt} \mu_\varepsilon \leqslant \{t\colon d(u_\varepsilon(t), \Omega) = \varepsilon\}$$

and

$$\ddot{u}_\varepsilon = -\mu_\varepsilon \nabla d(u_\varepsilon);$$

iii) $|\dot{u}_\varepsilon| = |\tilde{b}|$.

Since Ω_ε is $4p$-convex we have

$$J_{\Omega_\varepsilon}(u_\varepsilon(t) + v(t)) \geqslant \langle v(t), \nabla f(u_\varepsilon(t))\rangle - 32p|v(t)|^2$$

for every $t \in \operatorname{spt} \mu_\varepsilon$ and for every $v \in Y$ with $\|v\| \leqslant 64p$. The last inequality yields

$$\int_0^1 J_\Omega(u_\varepsilon + v)\, d\mu_\varepsilon \geqslant \int_0^1 \langle v(t), \nabla f(u_\varepsilon)\rangle\, d\mu_\varepsilon - 62p + 3)\int_0^1 |v|^2\, d\mu_\varepsilon \geqslant \int_0^1 \langle v(t), \nabla f(u_\varepsilon)\rangle\, d\mu_\varepsilon -$$

$$- 62p\; \mu_\varepsilon([0,1])\|v\|^2_{H^1} = (\ddot{u}_\varepsilon, v) - 32p\, \mu_\varepsilon([0,1])\|v\|^2_{H^1} =$$

$$= \int_0^1 \langle \dot{u}_\varepsilon, \dot{v}\rangle\, dt - \langle b, v(0)\rangle - 32p\; \mu_\varepsilon([0,1])\|v\|^2_{H^1}$$

and then

$$\min_{\substack{v\in Y\\ \|v\|\leqslant 1/16c^2p}} \left\{ \int_0^1 \big(\langle b, v(0)\rangle - \langle \dot{u}, \dot{v}\rangle + 32p\mu_\varepsilon([0,1])(|v|^2 + |\dot{v}|^2) + J_{\Omega_\varepsilon}(u_\varepsilon + v)\big)\, dt \right\}$$

We observe that

$$-\ddot{u}_{n,\varepsilon} = +\mu_\varepsilon \frac{\partial d}{\partial x_n}$$

and then

$$m_1\mu_\varepsilon \leqslant -\ddot{u}_{n,\varepsilon} \leqslant m_2\mu_\varepsilon$$

which yields

$$0 \leqslant \mu_\varepsilon([0,1]) \leqslant \frac{\dot{u}^+_{n,\varepsilon}(0) - \dot{u}^-_{n,\varepsilon}(1)}{m_1} \leqslant \text{const}.$$

Then there exists a subsequence $u_{\varepsilon_k} \rightharpoonup \mu$, and then $\mu_{\varepsilon_k}([0,1]) \to \mu([0,1])$.

We consider on V the topology induced by H^1 and we observe that

(∗) $\|\dot{u}_\varepsilon\|_\infty \leqslant K$ and therefore we may suppose that $u_\varepsilon \xrightarrow{L^\infty} u_\infty$ and $\dot{u}_\varepsilon \xrightarrow{L^2} \dot{u}_\infty$ so that $\|\dot{u}_\varepsilon - \dot{u}_\infty\|_{L^2} \to 0$ because $\|\dot{u}_\varepsilon - \dot{u}_\infty\|^2_{L^2} = 2\|\dot{u}\|^2_{L^2} - 2\int_0^1 \dot{u}_\varepsilon \dot{u}\, dt \to 0$. Assuming that $\mu_\varepsilon \rightharpoonup \mu$, $u_\varepsilon \xrightarrow{L^\infty} u_\infty$, $\dot{u}_\varepsilon \rightharpoonup \dot{u}_\infty$ in L^2, we define $F_\varepsilon\colon Y \to \mathbf{R} \cup \{+\infty\}$

$$F_\varepsilon(v) = \int_0^1 \{\langle b, v(0)\rangle - \langle \dot{u}_\varepsilon, \dot{v}\rangle + J_{\Omega_\varepsilon}(u_\varepsilon + v)\}\, dt + K_\varepsilon \|v\|^2_{H^1}$$

where $K_\varepsilon = 32p\mu_\varepsilon([0,1])$ and we claim that

$$\Gamma(V^-)\lim_{\varepsilon\to 0} F_\varepsilon(v) = F_\infty(v) = I(u,v) + 32p\,p\mu([0,1])\|v\|^2 .$$

In fact it is easy to see that for every $v_\varepsilon \to$ in H^1 we have $\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \geq F_\infty(v)$; moreover if we choose $v_\varepsilon = v + (u_\infty - u_\varepsilon)$ we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(a_\varepsilon) = \limsup_{\varepsilon\to 0}\Big[\int_0^1 \{\langle b, v_\varepsilon(0)\rangle - \langle \dot u_\varepsilon, \dot v_\varepsilon\rangle + J_{\Omega_\varepsilon}(u_\varepsilon + v_\varepsilon)\}\,dt + K_\varepsilon \int_0^1 |v_\varepsilon|^2 + |\dot v_\varepsilon|^2\Big] =$$
$$= \limsup \int_0^1 [\langle b, v_\varepsilon(0)\rangle - \langle \dot u_\varepsilon, \dot v_\varepsilon\rangle + J_{\Omega_\varepsilon}(u_\varepsilon + v_\varepsilon)]\,dt + K_\infty \|v\|^2_{H^1}$$

Since $d(u_\varepsilon, \Omega) \leq \varepsilon$ and $u_\varepsilon = u_\infty$ we have $u_\infty \in \Omega$ so that $J_\Omega(u_\infty) = 0$ and $u_\infty \in X$; $\dot v_\varepsilon \to \dot v$ in L^2 and then

$$\int_0^1 \langle \dot u_\varepsilon, \dot v_\varepsilon\rangle \to \int_0^1 \langle \dot u_\infty, \dot v\rangle \qquad \text{and} \qquad v_\varepsilon(0) \to v(0) .$$

It remains to prove that

$$(**) \qquad \limsup_{\varepsilon\to 0} \int_0^1 J_{\Omega_\varepsilon}(u_\varepsilon + v_\varepsilon)\,dt = \int_0^1 J_\Omega(u_\infty + v) ;$$

two cases may occur

A) There exists $\bar t \in [0,1]$ such that $u_\infty(\bar t) + v(\bar t) \notin \Omega$. In this case for ε small enough we have $J_{\Omega_\varepsilon}(u_\infty + v) = +\infty$ and observing that $u_\varepsilon + v_\varepsilon = u_\varepsilon + v + (u_\infty - u_\varepsilon) = u_\infty + v$ (**) easily follows.

B) $u_\infty(t) + v(t) \in \Omega$ for every $t \in [0,1]$. This fact yields $J_{\Omega_\varepsilon}(u_\varepsilon + v_\varepsilon) = J_{\Omega_\varepsilon}(u_\infty + v) = 0$ for ε small enough and (**) is proven.

We recall now that

$$\mathrm{Dm}\, F_\varepsilon\left(0, \frac{1}{64p}\right) = 0$$

and by using Proposition I.11 we argue

$$0 \leq \mathrm{Dm}\, F_\infty\left(0, \frac{1}{64p}\right) \leq \liminf_{\varepsilon\leftarrow 0} \mathrm{Dm}\, F_\varepsilon\left(0, \frac{1}{64p}\right) = 0$$

and Theorem 2 is completely proven. ∎

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