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Bouncing loop quantum cosmology from $F(T)$ gravity

Jaume Amorós,[1,*] Jaume de Haro,[1,†] and Sergei D. Odintsov[2,3,4,‡]

[1]*Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain*
[2]*Institució Catalana de Recerca i Estudis Avançats (ICREA), Barcelona, Spain*
[3]*Institut de Ciències de l'Espai (CSIC-IEEC), Campus UAB, Facultat de Ciències, Torre C5-Par-2a pl, E-08193 Bellaterra (Barcelona), Spain*
[4]*Tomsk State Pedagogical University, Tomsk, Russia and Eurasian National University, Astana, Kazakhstan*
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The big bang singularity could be understood as a breakdown of Einstein's general relativity at very high energies. By adopting this viewpoint, other theories that implement Einstein cosmology at high energies might solve the problem of the primeval singularity. One of them is loop quantum cosmology (LQC) with a small cosmological constant that models a universe moving along an ellipse, which prevents singularities like the big bang or the big rip, in the phase space (H, ρ), where H is the Hubble parameter and ρ the energy density of the universe. Using LQC one considers a model universe filled by radiation and matter where, due to the cosmological constant, there are a de Sitter and an anti–de Sitter solution. This means that one obtains a bouncing nonsingular universe which is in the contracting phase at early times. After leaving this phase, i.e., after bouncing, it passes trough a radiation- and matter-dominated phase and finally at late times it expands in an accelerated way (current cosmic acceleration). This model does not suffer from the horizon and flatness problems as in big bang cosmology, where a period of inflation that increases the size of our universe in more than 60 e-folds is needed in order to solve both problems. The model has two mechanisms to avoid these problems: the evolution of the universe through a contracting phase and a period of super inflation ($\dot{H} > 0$).

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I. INTRODUCTION

When one considers a universe filled by radiation and matter expanding following the standard Einstein cosmology (EC), i.e., when the dynamics of the universe is dictated by the equations of general relativity, coming back in time, one concludes that there exists, at very early times, a primeval singularity named the big bang.

The big bang singularity could be seen as a deficiency of EC at high energies, because there is not any objective reason which supports the same physics at high rather than at low energies. In fact, one can claim that the big bang signals the breakdown of general relativity at high energy-density scales. However, there is observational evidence—such as the discovery of the cosmic microwave background (CMB) by Arno Penzias and Robert Wilson in 1964—that the ''big bang model'' works correctly at scales lower than the Planck scale. At those scales, the universe is filled by a hot photon-baryon plasma that could be modelled by a radiation fluid which cools as the universe expands, and nonrelativistic matter starts to dominate, allowing the formation of structures.

A possible solution to the big bang singularity could come from a modification, at high energies, of Einstein's general relativity. Since this theory could be understood as a linear teleparallel theory (recall that Einstein used teleparallelism in an unsuccessful attempt to unify gravitation with electromagnetism [1]), because since its Lagrangian is a linear function of the spacetime *scalar torsion*, namely T, one can assume that our universe could be described by nonlinear teleparallel theories [$F(T)$ theories] [2–5] that become nearly linear at low energies.

It is known that $F(T)$ gravity can realize both inflation [6] and the late-time cosmic acceleration [7–9], revealed by recent observations of, for example, type Ia supernovae [10], baryon acoustic oscillations (BAO) [11], large scale structure (LSS) [12], cosmic microwave background (CMB) radiation [13], and the effects of weak lensing [14] (see Ref. [15] for a recent review of current cosmic acceleration). In fact, a very large number of recent papers were devoted to investigating diverse properties of $F(T)$ gravity in order to check whether it could be a veritable alternative to general relativity [16]. Moreover, models of $F(T)$ gravity in which the finite-time future singularities appear have been reconstructed [17].

When one considers a homogeneous and isotropic spacetime, i.e., when one considers the Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, the scalar torsion is given by $T = -6H^2$, where H is the Hubble parameter [7]. As a very remarkable consequence, $F(T)$ cosmologies entail that the modified Friedmann equation depicts a curve in the plane (H, ρ), where ρ denotes the energy density of the universe. That is, the universe moves along this curve and its dynamics is given by the so-called *modified Raychaudhuri*

*jaume.amoros@upc.edu
†jaime.haro@upc.edu
‡odintsov@ieec.uab.es

equation and the conservation equation. This opens the possibility to build nonsingular models of universes with a cosmological constant and filled by radiation and matter. Moreover, $F(T)$ theories could be used to reconstruct cosmologies in two ways. i) Given the scale factor $a(t)$ and the equation of state (EOS), one can build the corresponding $F(T)$. ii) Given the scale factor $a(t)$ and the $F(T)$ theory one can build the corresponding EOS.

Our main result is to show that, for the flat FLRW geometry, choosing as an $F(T)$ theory the effective formulation of loop quantum cosmology (see Ref. [18] for papers on effective LQC), the modified Friedmann equation that includes holonomy corrections gives, at early times, a universe in an anti–de Sitter phase, which after leaving this phase starts to accelerate, leaving the contracting phase and entering the expanding one (it bounces), and then starts to decelerate and passes trough a radiation- and matter-dominated phase. Finally, at late times it enters in a de Sitter phase (late time cosmic acceleration). Our model does not suffer the flatness and horizon problems that appear in big bang cosmology, because it has a contracting phase and a super-inflationary period ($\dot{H} > 0$), which—in principle—makes an inflationary epoch such as that of big bang cosmology unnecessary, where the scale factor increases more that 60 e-folds in order to solve these problems. Moreover, the evolution of the universe at early times, in a contracting matter-dominated phase, could produce an scale-invariant spectrum of cosmological perturbations that agrees with current observations. Finally, it is important to stress that our viewpoint of LQC as an $F(T)$ theory opens the possibility to study perturbations in LQC using the perturbation equations in $F(T)$ gravity, recently deduced in Ref. [19]. We believe that this fact could be very important because perturbations with holonomy corrections in LQC were introduced on a phenomenological level by replacing the Ashtekar connection $\gamma\bar{k}$ by $\frac{\sin(m\bar{\mu}\gamma\bar{k})}{m\bar{\mu}}$, m being a number (see for example Ref. [20]). Moreover, to obtain an anomaly-free perturbation theory some counterterms must be introduced in the Hamiltonian constraint [21], which for vector perturbations gives rise to counterterms depending on $\bar{k}$, i.e., they are no longer almost periodic functions of $\bar{k}$, which seems to be in contradiction with the spirit of LQC.

The paper is organized as follows. In Sec. II we study EC and we discuss the different ways to deal with the avoidance of the big bang singularity. Section III is devoted to the study of LQC, showing that its effective formulation gives a bouncing nonsingular model where the universe evolves from a contracting phase to our current cosmic acceleration. We also show that this model does not suffer the flatness and horizon problems. Finally, in Sec. IV the reconstruction of cosmologies is considered in both, via a scalar field and via $F(T)$ theories.

The units used throughout the paper are $c = \hbar = 8\pi G = 1$.

II. EINSTEIN COSMOLOGY: RADIATION PLUS MATTER PLUS COSMOLOGICAL CONSTANT

Assuming that, at large scales, our universe is homogeneous and isotropic leads us to consider a flat FLRW spacetime, whose metric is given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{1}$$

where a is the *scale factor*: the quantity that "measures" the distance between points along time.

For this metric we consider a universe filled by radiation plus dust matter, which means that the energy density is given by $\rho = \rho_r + \rho_m$, where ρ_r is the energy density of the radiation and ρ_m is the energy density of the matter. Here, as usual, we assume that matter is dust (cold dark matter).

The pressures of these kinds of fluids satisfy $P_r = \frac{1}{3}\rho_r$ and $P_m = 0$. From the first principle of thermodynamics or the conservation equation $d(\rho V) = -P dV$ ($V = a^3$ being the volume), one obtains the following solutions:

$$\rho_r = \rho_{r,0} V^{-4/3} \quad \text{and} \quad \rho_m = \rho_{m,0} V^{-1}, \tag{2}$$

where the subindex 0 means that the quantity is evaluated at the present time, and where we have taken $V_0 \equiv 1$.

Now we consider the so-called Benchmark model, where EC with a small cosmological constant Λ is used to study our universe filled by radiation plus matter $\rho = \rho_r + \rho_m$.

Note that EC can be seen as a linear teleparallel theory with the Lagrangian [22]

$$\mathcal{L}_E(T) = \frac{1}{2} TV - (\rho + \Lambda)V, \tag{3}$$

where $T = -6H^2$ is the so-called *scalar torsion*. Or, in its more conventional form,

$$\mathcal{L}_E(R) = \frac{1}{2} RV - (\rho + \Lambda)V, \tag{4}$$

where $R = 6(\dot{H} + 2H^2)$ is the *scalar curvature*.

In spite of the fact that both formulations are equivalent, it is important to recall that teleparallel theories are constructed from the Weitzenbök connection, obtaining a spacetime with vanishing curvature (the Riemann tensor vanishes) but that is not torsion free, in contrast with the standard Levi-Civita connection which gives a curved torsion-free spacetime.

From these Lagrangians one easily obtains the Hamiltonian constraint that leads to the basic equation in cosmology—the so-called *Friedmann equation*—which in EC is given by

$$H^2 = (\rho + \Lambda)/3, \tag{5}$$

depicting a parabola in the plane (H, ρ), i.e., the evolution of the universe follows this parabola, and its dynamics is given by the system (which could be easily obtained from the conservation and Friedmann equations)

$$\dot{H} = -\frac{2\rho_r}{3} - \frac{\rho_m}{2}, \qquad \dot{\rho} = -4H\rho_r - 3H\rho_m, \tag{6}$$

provided that the universe moves along the parabola $H^2 = (\rho + \Lambda)/3$, and that ρ_r and ρ_m satisfy Eq. (2). In Eq. (6) the first equation is the so-called Raychaudhuri equation, and the second one is an equivalent form of the conservation equation $d(\rho V) = -PdV$.

Equation (6) is a first-order two-dimensional dynamical system. These kinds of systems have a very simple dynamics that could be easily understood by calculating their *critical points* [points in the phase space (H, ρ) satisfying $\dot{H} = \dot{\rho} = 0$], which are stationary solutions.

The system (6), in the expanding phase ($H > 0$), has a unique critical point ($H = \sqrt{\Lambda/3}, \rho = 0$) which is a global attractor [from the second equation of (6) one easily deduces that ρ decreases with time]. This means that the universe enters into a de Sitter phase at late times (the late-time cosmic acceleration).

On the other hand, at early times the universe is dominated by ρ_r and ρ_m. Since we are in the expanding phase $H > 0$, the volume V is an increasing function of time. As a consequence, one deduces from Eq. (2) that at very early times the universe is radiation dominated, and when the energy density reaches the value $\rho = 2\rho_{m,0}^4/\rho_{r,0}^3$ it changes to a matter-dominated phase.

Note also that, since the parabola is an unbounded curve, there is only one critical point of the system and ρ is a decreasing function. The interesting point is to know if the universe reaches the singularity ($\rho = \infty$) in a finite or infinite time. Solving the system (6), using that at early times the universe is in the radiation-dominated phase, gives the solution

$$H(t) = \frac{H_0}{1 + 2H_0(t - t_0)}. \tag{7}$$

From this solution one concludes that the time from the big bang to the present is $t_0 - t_{\text{big bang}} = \frac{1}{2H_0}$. See Fig. 1.

A. What exactly does the big bang mean?

At $t_{\text{big bang}}$ the energy density diverges ($\rho \to \infty$). This could be understood as a deficiency of Einstein cosmology and not as the beginning of our universe, because Einstein's general theory of relativity is, in principle, a low-energy theory. Thus, there is no objective reason to use this theory at high energies.

The big bang was discussed during the 1970s, when the idea emerged that quantum effects could be important at very high energies, leading to a universe without a primeval singularity [23]. Efforts in this direction gave rise to the so-called *semiclassical gravity*, where quantum effects due to fields coupled with gravity are taken into account at early times (see for instance Ref. [24]). The most successful model was the so-called "Starobinsky model" [25], where an unstable nonsingular model was obtained in which the universe starts in the de Sitter phase and ends in a matter-dominated phase (the accelerated expansion of the universe had not yet been discovered at that moment).

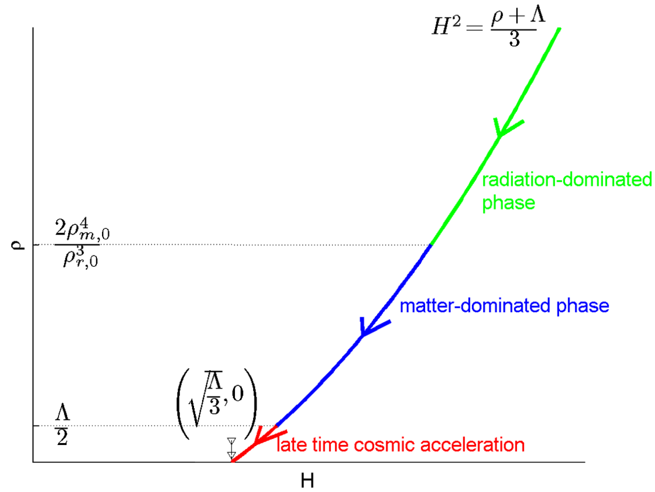

FIG. 1 (color online). The different epochs of the universe in Einstein cosmology: a radiation-dominated expansion phase following the big bang, a matter-dominated expansion phase following it, and a final phase describing the current accelerating expansion.

Another step in order to deal with the universe at early times was the "inflation theory" [26]. A beginning of the universe seems incompatible with its homogeneity and isotropy (the horizon problem), and it is also very difficult, from a beginning, to understand the present spatial flatness of the universe (the flatness problem). The underlying idea behind inflation in EC is that, at early times, the universe had a period where the quantity aH increased considerably. Since in EC, when the universe is not phantom dominated, H is a decreasing function: to achieve the increase of aH one looks for a mechanism so that our universe remains, for a brief period of time, in a quasi–de Sitter phase. Then H is nearly constant and the increase of the scale factor is exponential. The best way to achieve this quasi–de Sitter period is due to a field called the *inflaton*, which rolls very slowly according to a potential at very early times [the Planck epoch or later, for example, at the grand unified theories (GUTs) epoch], producing the accelerated expansion of the universe. At the end of this inflationary epoch the inflaton field decays creating the matter of the universe which thermalizes, being the universe in the radiation-dominated phase. Finally, at that epoch, the model matches with the standard big bang theory. (It is always said that the inflationary paradigm is not a theory itself but an implementation of the standard big bang theory). Here, it is important to realize that the inflationary theory does not deal with the problem of an initial singularity of our universe because the theory starts at the Planck epoch or, in some models, later. (Sometimes it is argued that before the Planck epoch there is no classical description of the universe, and only a quantum description of it is possible).

However, although the inflationary paradigm is the most popular among and most used by the majority of cosmologists, it has some problems. i) Inflation deals with the *singularity problem* in an unconventional way; it effaces all the early history of our universe by being itself a beginning of the universe. In this sense, one could understand the beginning of the inflation as the beginning of our universe, and it seems impossible to form a previous idea of the universe before inflation. ii) The *amplitude problem* is related with the power spectrum of the cosmological perturbation. In a wide class of inflationary models, the potential of the inflation field and the change of the inflation field during inflation, namely $\Delta\phi$, must satisfy the relation $V(\phi)/(\Delta\phi)^4 \leq 10^{-12}$, which imposes a hierarchy in energy scales. iii) There is also the trans-Planckian problem, which could be formulated as follows: inflation provides a mechanism to produce structure formation based on the fact that scales currently observable were originated by wavelengths smaller than the Hubble radius at the beginning of inflation. This typically requires that inflation lasts past the scale factor increase of 60 e-folds. However, if the period of inflation was longer, which happens in the majority of current models, then the wavelengths of all observable scales would be smaller than the Planck length at the beginning of inflation, but we do not know what kind of physics operates at such scales (see, for instance, Ref. [27]).

Another completely different way to deal with the initial singularity problem is to assume EC is only right at low energies, and then in the Lagrangian (3) T has to be changed by $F(T)$ or in Eq. (4) R has to be changed by $F(R)$, where F must be a nearly linear function for small values of its argument to understand this theory as an implementation of EC at high energies.

The field equations of the teleparallel Lagrangian are of second order, which is a great advantage compared to the Lagrangian constructed with the scalar curvature R, whose fourth-order equations lead to pathologies like instabilities or large corrections to Newton's law [8].

This is a good reason to use $F(T)$ teleparallel theories instead of the $F(R)$ ones [see Ref. [28] for a recent review of $F(R)$ gravity], because their simplicity gives rise to modified Friedmann equations depicting curves in the plane (H, ρ). According to these theories the universe moves along a curve, and its dynamics is given by the so-called "modified Raychaudhuri equation" and the conservation equation.

III. LOOP QUANTUM COSMOLOGY: RADIATION PLUS MATTER PLUS COSMOLOGICAL CONSTANT

The standard viewpoint of LQC assumes, at the quantum level, a discrete nature of space which leads to a quadratic modification (ρ^2) in its effective Friedmann equation at high energies [29]. This modified Friedmann equation depicts the following ellipse in the plane (H, ρ) (see Ref. [30] for details):

$$\frac{H^2}{\rho_c/12} + \frac{(\rho + \Lambda - \frac{\rho_c}{2})^2}{\rho_c^2/4} = 1, \tag{8}$$

where $\rho_c \equiv \frac{2\sqrt{3}}{\gamma^3} \cong 258.51$ is the so-called critical density, with $\gamma \cong 0.2375$ being the so-called Barbero-Immirzi parameter [31]. Note that, in the units used through this paper, the Planck density has the numeric value $\rho_{\text{Planck}} = 64\pi^2 \cong 631.61$, which is greater than ρ_c, and thus a classical description of the universe seems possible because its energy density will never exceed the Planck scale's.

Here an important remark is in order. Equation (8) could be obtained by considering the regularized Hamiltonian

$$H_{\text{LQC}} \equiv -\frac{2V}{\gamma^3\lambda^3}\sum_{i,j,k}\varepsilon^{ijk}\,\text{Tr}[h_i(\lambda)h_j(\lambda)h_i^{-1}(\lambda)h_j^{-1}(\lambda)h_k(\lambda) \times \{h_k^{-1}(\lambda), V\}] + \rho V, \tag{9}$$

where $h_j(\lambda) \equiv e^{-i\frac{\lambda\beta}{2}\sigma_j}$ are holonomies, $\lambda = \sqrt{\frac{\sqrt{3}}{4}\gamma}$ is a paramater with dimensions of length [29], and β is the canonically conjugate variable to the volume V satisfying $\{\beta, V\} = \frac{\gamma}{2}$.

An explicit calculation of this hamiltonian was done in Ref. [32], giving as a result

$$H_{\text{LQC}} = -3V\frac{\sin^2(\lambda\beta)}{\gamma^2\lambda^2} + \rho V. \tag{10}$$

Then the Hamilton equation $\dot{V} = \{V, \mathcal{H}_{\text{LQC}}\}$ is equivalent to the identity $H = \frac{\sin(2\lambda\beta)}{2\gamma\lambda}$ that, combined with the Hamiltonian constraint $H_{\text{LQC}} = 0$, gives rise to the modified Friedmann equation (8) (see, for instance, Ref. [30]).

The dynamics is now given in LQC by the system

$$\begin{aligned} \dot{H} &= -\frac{4\rho_r + 3\rho_m}{6}\left(1 - \frac{2(\rho + \Lambda)}{\rho_c}\right), \\ \dot{\rho} &= -4H\rho_r - 3H\rho_m. \end{aligned} \tag{11}$$

In order to understand the dynamics of the system it is very useful to introduce the parameter $\omega_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2}$, which in LQC becomes

$$\omega_{\text{eff}} = -1 + \frac{4\rho_r + 3\rho_m}{3(\rho + \Lambda)}\frac{\rho_c - 2(\rho + \Lambda)}{\rho_c - (\rho + \Lambda)}. \tag{12}$$

This quantity is related to the expansion of the universe. Actually, when $\omega_{\text{eff}} < -1/3$ ($\omega_{\text{eff}} > -1/3$) the universe accelerates (decelerates). In fact, one can see the universe filled by an effective fluid that drives its dynamics, and whose pressure and energy density are related by $\omega_{\text{eff}} = P/\rho$.

Coming back to the system (11), note first that at low energies ($\rho \ll \rho_c$) it coincides with the system (6), which means that at low energies LQC coincides with EC, and it could be understood as an implementation of EC at high energies. In fact, by writing Eq. (11) in its more usual form,

$$H^2 = \frac{\rho + \Lambda}{3}\left(1 - \frac{\rho + \Lambda}{\rho_c}\right), \quad (13)$$

one can see that for the current value of the energy density ρ_0, which satisfies $\frac{\rho_0}{\rho_c} \sim 10^{-120}$, one has $H^2 = \frac{\rho+\Lambda}{3}$, which means that nowadays there is no visible difference with standard ΛCDM cosmology.

By studying Eq. (11) as a dynamical system we can see that it has two critical points: $p_f \equiv (\sqrt{\frac{\Lambda}{3}}\sqrt{(1 - \frac{\Lambda}{\rho_c})}, 0)$ and $p_i \equiv (-\sqrt{\frac{\Lambda}{3}}\sqrt{(1 - \frac{\Lambda}{\rho_c})}, 0)$. The first one is a de Sitter solution and the second one is an anti–de Sitter solution. The universe moves along the ellipse from p_i to p_f in a clockwise sense [this comes from the second equation of (11), because in the contracting phase the energy density is an increasing function and in the expanding one it is decreasing]. At very early times the size of the universe was very large and it contracts with positive acceleration because for $\rho \sim 0$ one has $\omega_{\text{eff}} \sim -1 < -1/3$. When the cosmological constant Λ stops its domination, the universe enters a contracting matter-dominated phase ($\omega_{\text{eff}} \sim 0$) because the volume is still big enough. Then the volume decreases and the universe enters the contracting radiation-dominated phase ($\omega_{\text{eff}} \sim 1/3$). In the contracting phase, as we have already shown, ρ is an increasing function and when $\rho \sim \rho_c/3$ one has $\omega_{\text{eff}} \sim -1/3$, which means that the universe accelerates (that is, it contracts in a decelerating way). This behavior is due to the form of the ellipse and it could be understood as a sort of dark energy that drives our universe to this accelerated phase. In this phase, when it arrives at the point $p_1 = (\rho_c - \Lambda, 0)$ (the top of the ellipse) it bounces, leaving the contracting phase and entering the expanding one where the energy density starts to decrease. At that moment one has $\omega_{\text{eff}} \ll -1/3$ and thus the universe expands in an accelerating way; it is in a *super-inflationary phase* that only increases the size of the universe by a small number of e-folds, which is not enough to solve the flatness and horizon problems that appear in EC. However, as we will show in the next section, our model does not suffer from these problems. This accelerating period finishes when the universe arrives at $p_2 \cong (\rho_c/3, \rho_c/3)$. At that moment, it starts to decelerate and when the density satisfies $\rho_c \ll \rho \ll \Lambda$ the universe enters first a radiation-dominated phase—which it leaves when $\rho = 2\rho_{m,0}^4/\rho_{r,0}^3$—and then a matter-dominated one ($\omega_{\text{eff}} \cong 0$). Finally, after leaving this phase, it enters an accelerated phase when $\rho = \Lambda/2$ ($\omega_{\text{eff}} < -1/3$) and goes asymptotically, at late times, to the point p_f (de Sitter phase that mimics the late-time accelerated cosmic expansion). See Fig. 2.

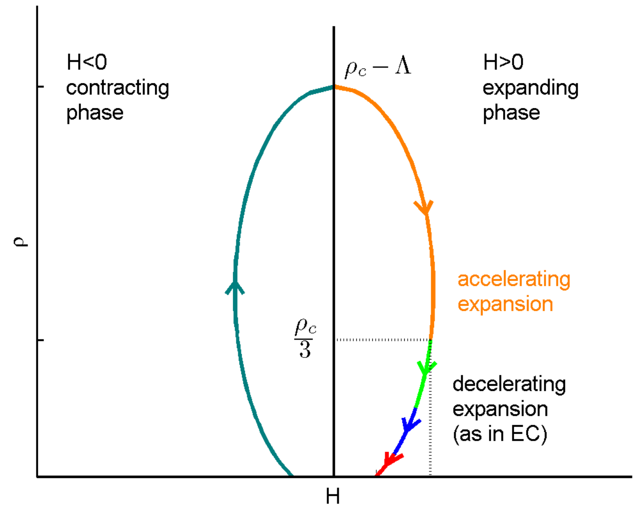

FIG. 2 (color online). The different epochs of the universe in loop quantum cosmology: after a contracting phase and an accelerated expansion phase, the universe enters a decelerating expansion phase as in Einstein cosmology.

A. Does this model need an inflationary epoch as in EC?

In our model the inflationary epoch in the expanding phase starts at the bouncing time, namely t_i, when the universe has an energy density $\rho_c - \Lambda \cong \rho_c$. At that moment the scale factor is a minimum, and thus we can assume that the universe is radiation dominated. As we have seen the universe stops accelerating when the energy density is approximately equal to $\rho_c/3$. Let t_f be the time when inflation ends; then, from Eq. (2) one deduces $a(t_f) = 3^{1/4}a(t_i)$. But in EC, to solve the horizon and flatness problem one needs a scale factor greater than 60 e-folds [26], which clearly does not happen in our model.

In fact, for a fluid with a linear EOS $P = \omega\rho$ where $\omega > -2/3$, the accelerated expansion ends when $\rho \cong \frac{\rho_c(1+3\omega)}{2(2+3\omega)}$. Then, a simple calculation yields

$$a(t_f) = \left(\frac{2(2 + 3\omega)}{1 + 3\omega}\right)^{\frac{1}{3(1+\omega)}} a(t_i), \quad (14)$$

which means that to obtain 60 e-folds one needs a value of ω very close to $-2/3$. Note also that for fluids with $\omega > 0$ one obtains a "*bad inflation*" (an inflation with a small increase of the scale factor).

However, it is important to realize that in our model these problems do not appear. First, we start with the horizon problem. To simplify the calculation we assume a matter-dominated universe. The particle horizon in the contracting phase is

$$d_{\text{hor}} = a(t_c)\int_{-\infty}^{t_c} \frac{dt}{a(t)}, \quad (15)$$

where t_c is the bouncing time.

Using the identity $\rho(t)V(t) = \rho(t_c)V(t_c)$ and the conservation equation $\dot{\rho} = -3H\rho$ one obtains

$$d_{\rm hor} = \frac{\rho_c}{\rho^{1/3}(t_c)} \int_0^{\rho_c-\Lambda} \frac{d\rho}{(\rho+\Lambda)\rho^{2/3}(\rho_c-(\rho+\Lambda))} \sim \frac{1}{\rho_c}\int^{\rho_c-\Lambda} \frac{d\rho}{(\rho_c-(\rho+\Lambda))} = +\infty, \quad (16)$$

which means that, when the universe enters the expanding phase, all the points of it are in causal contact and thus the universe has had enough time to be homogeneous and isotropic when it bounces. Note that the same result was deduced in Ref. [6] where the authors studied the teleparallel version of the Born-Infeld Lagrangian, $\mathcal{L}_{\rm BI} = \frac{1}{2}V\lambda[\sqrt{1+\frac{2R}{\lambda}}-1]$, with λ being a parameter introduced with the aim of smoothing singularities.

The flatness problem in EC goes as follows. For a spatially curved FLRW spacetime the Friedmann equation, in EC, can be written as

$$\Omega - 1 = \frac{1}{\dot{a}^2} = \frac{1}{a^2H^2}, \quad (17)$$

where $\Omega = \frac{\rho+\Lambda}{3H^2}$. In EC cosmology $\dot{a}^2$ is a decreasing function because $\frac{d}{dt}\dot{a}^2 = 2\ddot{a}\,\dot{a} < 0$. Since nowadays one has $|\Omega - 1| \leq 0.2$ one easily deduces that at Planck scales $|\Omega - 1| \sim 10^{-60}$. From this result it seems that it would be far better to find a physical mechanism for flattening the universe, instead of relying on contrived initial conditions at the Planck epoch. In EC this problem is solved with a brief period of inflation ($\ddot{a} > 0$) after the Planck epoch. If the number of e-folds is large enough, then assuming that $|\Omega - 1| \sim 1$ at the Planck epoch one obtains, for the majority of current inflationary models, $|\Omega - 1| \ll 10^{-60}$. However, our model contains its own mechanism to solve that problem. Namely, in order to simplify we consider a matter-dominated universe without a cosmological constant (although our reasoning is completely general). Then the solution of the system (11) is given by [33]

$$H(t) = \frac{\rho_c t/2}{3\rho_c t^2/4 + 1}, \qquad \rho(t) = \frac{\rho_c}{3\rho_c t^2/4 + 1}, \quad (18)$$

where here we have chosen as a bouncing time $t = 0$. From these values one easily finds the scale factor $a(t) = a(0) \times (\rho(t)/\rho_c)^{-1/3}$.

Near the bouncing time $t \sim 0$, $a(t) \cong a(0)$ and $H(t) \cong \rho_c t/2$, and consequently

$$\Omega - 1 \cong \frac{4}{a^2(0)\rho_c^2 t^2} \gg 1, \quad (19)$$

that is, the fine-tuning of $\Omega - 1$ is not needed at any scale.

As one can easily see, this situation is very different from inflation in EC. Since in EC H decreases for nonphantom universes, one needs a brief period of time where the Hubble parameter is nearly constant and the scale factor sustains a huge increase. In LQC, at high energies the universe is in a super-inflationary phase ($\dot{H} > 0$). Then to solve the flatness (and also the horizon) problems one only needs a huge increase of aH. In fact, to solve these problems one needs that $\bar{N} \equiv \ln\frac{a(t_f)H(t_f)}{a(t_i)H(t_i)} \sim 60$, where t_i and t_f are, respectively, the beginning and end of the inflationary period [34,35]. And, since in LQC $H \cong 0$ near the bounce, one always obtains $\bar{N} \gg 1$. Finally, note that if inflation was produced in a quasi–de Sitter phase, $\bar{N}$ would coincide with the standard quantity that measures the number of e-folds in inflationary EC, i.e., $\bar{N}$ will coincide with $N \equiv \ln\frac{a(t_f)}{a(t_i)}$.

Dealing with the problem of the origin of density perturbations is a different subject. One can assume initial conditions, at very early times, for the density perturbations; then, one shows that—at late times—they evolve into a scale-invariant spectrum, or one has to look for a mechanism that produces an almost scale-invariant spectrum of cosmological perturbations. In this second case, one may consider a condensate scalar field (the inflaton field), and use its quantum fluctuations at high-energy scales in order to explain the generation of large-scale perturbations. The fact that $H(t)$ is almost constant during the slow-roll period means that it is possible to generate scale-invariant density perturbations on large scales.

The alternative possibility we propose is to consider initial perturbations, for example given by quantum fluctuations due to a very light field minimally coupled with gravity (the quantum fluctuations of the inflaton and the quantum fluctuations of a massless minimally coupled field satisfy the same Klein-Gordon equation), in our model.

Cosmological perturbations in a contracting, matter-dominated phase of a bouncing universe have been studied in the last decade, showing analytically and numerically in some toy models that they evolve into a scale-invariant spectrum of cosmological perturbations at late times (after the bounce) [36–39]. In our model we can consider, at very early times, quantum fluctuations that at the contracting matter-dominated phase would produce on long wavelengths (at scales larger than the Hubble radius) a scale-invariant spectrum which would survive after the bounce. This is, of course, a topic that needs future detailed investigation, but in principle—from previous works—it seems plausible that our model provides a scale-invariant spectrum after the bounce.

All these reasons indicate that models—such as nonsingular bouncing cosmologies, where inflation is not needed—should be taken into account in order to explain the evolution of our universe.

IV. RECONSTRUCTING COSMOLOGIES

In this section we take another viewpoint: given the evolution of our universe, i.e., choosing the evolution of the scale factor, we will construct the Lagrangian whose dynamical equations have as a solution the chosen scale factor.

A. Reconstruction via a scalar field

First of all, we consider in EC a scalar field ϕ with energy density and pressure given by

$$\rho = \frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi), \qquad P = \frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi), \tag{20}$$

where ω and V are functions of the field ϕ. After some algebra one obtains the relations

$$\omega(\phi)\dot{\phi}^2 = -2\dot{H}, \qquad V(\phi) = 3H^2 + \dot{H}. \tag{21}$$

Equation (21) has two different solutions. i) If one takes $\omega(\phi) \equiv 1$ then one has [40]

$$V(t) = 3H^2 + \dot{H}, \qquad \phi(t) = \int dt\sqrt{-2\dot{H}}. \tag{22}$$

These equations determine $\phi(t)$ and $V(t)$ in terms of the scale factor, thereby implicitly determining $V(\phi)$. ii) Taking $\phi = t$ [41] gives

$$V(t) = 3H^2 + \dot{H}, \qquad \omega(t) = -2\dot{H}, \tag{23}$$

where once again any cosmology with scale factor $a(t)$ is realized by the potential V.

As an example, a power-law expansion $a(t) = a_0|t/t_0|^p$ is obtained using formulas (21) from an exponential potential of the form

$$V(\phi) = e^{-\sqrt{\frac{2}{p}}\phi}. \tag{24}$$

However, realistic cosmologies require very complicated potentials that in general do not have a minimum like the potentials used in inflation. Then, in general, the scalar field does not oscillate around the minimum and consequently does not release its energy by producing light particles that thermalize our universe, as occurs in inflationary cosmologies. In order to obtain a realistic reheating theory, one has to use gravitational particle production. Gravitational particle production due to a transition from a de Sitter to a radiation phase has been studied extensively in the past. Given a consistent reheating temperature [42–44], it then seems mandatory that—by reconstructing models via a scalar field—this transition occurs.

Different examples of reconstructing the history of our universe are given in Ref. [45]. Here we study one of them in order to show the complicated potentials obtained.

The dynamics $H(t) = \frac{H_i + \lambda e^{\alpha t}}{1+e^{\alpha t}}$—where λ, H_i, and α are constants satisfying $\alpha, \lambda \ll H_i$ so that slow-roll conditions can be satisfied—describes a universe which at early times is dominated by an effective cosmological constant with value $3H_i^2$ that is driving inflation, and at late times is dominated by another cosmological constant with value $3\lambda^2$ given the current accelerated expansion of our universe. Then, by using Eq. (23) one obtains the following complicated quantities:

$$\begin{aligned} \omega(\phi) &= \frac{\alpha(H_i - \lambda)e^{\alpha\phi}}{(1 + e^{\alpha\phi})^2}, \\ V(\phi) &= \frac{3H_i^2 + [6H_i\lambda - \alpha(H_i - \lambda)]e^{\alpha\phi} + \lambda^2 e^{2\alpha\phi}}{(1 + e^{\alpha\phi})^2}. \end{aligned} \tag{25}$$

B. Reconstruction via $f(T)$ gravity

In a flat FLRW spacetime filled by a perfect fluid with energy density ρ, general teleparallel theories are obtained from the Lagrangian

$$\mathcal{L} = VF(T) - V\rho. \tag{26}$$

The conjugate momentum is then given by $p_V = \frac{\partial \mathcal{L}}{\partial V} = -4HF'(T)$, and thus the Hamiltonian is

$$\mathcal{H} = \dot{V}p_V - \mathcal{L} = (2TF'(T) - F(T) + \rho)V. \tag{27}$$

In general relativity the Hamiltonian is constrained to be zero, which leads to the modified Friedmann equation

$$\rho = -2F'(T)T + F(T) \equiv G(T), \tag{28}$$

which is a curve in the plane (H, ρ).

Then, given a curve of the form $\rho = G(T)$ for some function G, a first way to reconstruct the Lagrangian (26) consists of integrating the modified Friedmann equation (28), obtaining as a result

$$F(T) = -\frac{\sqrt{-T}}{2}\int \frac{G(T)}{T\sqrt{-T}}dT. \tag{29}$$

The simplest example is to take as a curve a parabola, for example

$$\rho = \bar{\rho}\left(1 - \frac{3H^2}{\Lambda}\right), \tag{30}$$

which models a nonphantom universe, i.e., for $\frac{P}{\rho} \geq -1$, a universe that moves clockwise from $(-\sqrt{\Lambda/3}, 0)$ to $(\sqrt{\Lambda/3}, 0)$, bouncing when $(0, \bar{\rho})$. Using the formula (29) one obtains

$$F(T) = \bar{\rho}\left(1 - \frac{T}{2\Lambda}\right). \tag{31}$$

In this case, if one considers a matter-dominated universe and inserts into the conservation equation $\dot{\rho} = -3H\rho$

(the value of H as a function of ρ), one obtains a solvable differential equation whose solution is

$$\rho(t) = \bar{\rho}\frac{4e^{-\sqrt{3\Lambda t^2}}}{\left(1+e^{-\sqrt{3\Lambda t^2}}\right)^2}, \quad H_\pm(t) = \pm\sqrt{\frac{\Lambda}{3}}\frac{1-e^{-\sqrt{3\Lambda t^2}}}{1+e^{-\sqrt{3\Lambda t^2}}}, \tag{32}$$

where we have chosen as a bouncing time $t = 0$.

As a second example we consider LQC, where the curve (8) can be written in two pieces: $\rho = G_-(T)$ (which corresponds to energy densities below $\rho_c/2 - \Lambda$) and $\rho = G_+(T)$ (which corresponds to energy densities between $\rho_c/2 - \Lambda$ and $\rho_c - \Lambda$), where

$$G_\pm(T) = -\Lambda + \frac{\rho_c}{2}\left(1 \pm \sqrt{1+\frac{2T}{\rho_c}}\right). \tag{33}$$

Then, by using Eq. (29) one gets

$$F_\pm(T) = \mp\sqrt{-\frac{T\rho_c}{2}}\arcsin\left(\sqrt{-\frac{2T}{\rho_c}}\right) + \frac{\rho_c}{2}\left(1 \pm \sqrt{1+\frac{2T}{\rho_c}}\right) - \Lambda. \tag{34}$$

From this formula one obtains, in LQC, the Lagrangian that models a universe with a cosmological constant filled by radiation and matter,

$$\mathcal{L}(V,\dot{V}) = \begin{cases} F_-(T)V - \rho_{r,0}V^{-1/3} - \rho_{m,0} & \text{for } 0 \le \rho_{r,0}V^{-4/3} + \rho_{m,0}V^{-1} \le \rho_c/2 - \Lambda, \\ F_+(T)V - \rho_{r,0}V^{-1/3} - \rho_{m,0} & \text{for } \rho_c/2 - \Lambda < \rho_{r,0}V^{-4/3} + \rho_{m,0}V^{-1} \le \rho_c - \Lambda, \end{cases} \tag{35}$$

which shows that the effective formulation of LQC can be considered as a teleparallel theory.

Coming back to Eq. (29), it seems very useful to construct simple bouncing models. One only has to consider a closed curve in the phase space (H, ρ). This curve has to be symmetric with respect to the axis $H = 0$. By splitting the curve into some points (as we have done in LQC) one will easily obtain an $F(T)$ theory for each part of the curve.

A second way to reconstruct a model using $f(T)$ theories is as follows. Given the scale factor $a(t)$, the conservation equation $d(\rho V) = -Pd(V)$ and the equation of state $P = P(\rho)$, one obtains the energy density as a function of time $\rho(t)$. From the scale factor $a(t)$ one also obtains the scalar torsion as a function of time, $T = T(t) = -6(\dot{a}(t)/a(t))^2$. Then, performing the change of variable $T = T(t)$ in Eq. (29), one obtains

$$F(T) = -\frac{\sqrt{-T}}{2}\int^{t(T)}\frac{\rho(s)\dot{T}(s)}{T(s)\sqrt{-T(s)}}ds, \tag{36}$$

where the time t as a function of T, i.e., $t(T)$ is obtained by inverting the equation $T = T(t)$.

Finally, note that—as in the case of a scalar field—Eq. (36) shows that realistic cosmologies, i.e., a realistic $a(t)$ will require a very complicated $f(T)$ theory.

The final way to construct such models has recently been introduced in Ref. [30]. The idea is that given a scale factor $a(t)$, from the modified Friedmann and Raychaudhuri equations of an $F(T)$ theory one can build the corresponding EOS, which we will assume has the form $P(\rho) = -\rho - f(\rho)$. To be precise, taking the derivative with respect to time in Eq. (28) and using the conservation equation one obtains the Raychaudhuri equation,

$$\dot{H} = -\frac{f(\rho)}{4}(G^{-1})'(\rho). \tag{37}$$

Then, from Eq. (28) one obtains the time $t(\rho)$ as a function of the energy density. Inserting this expression into Eq. (37) one finally obtains $f(\rho)$, and thus one has built an EOS that gives the dynamics $a(t)$ in the corresponding $F(T)$ theory.

As an example, we consider in EC [$F(T) = T/2$] the dynamics

$$H(t) = H_i + H_1e^{-\gamma(t-t_i)} \quad \text{for } t_i \le t \le 60H_i^{-1} + t_i, \tag{38}$$

where we assume $H_1 \ll H_i$ and $\gamma H_i^{-1} \ll 1/60$, which means that $H(t)$ is nearly constant during this period of time, and consequently the scale factor increases the required 60 e-folds to solve the horizon and flatness problems.

From Eqs. (28) and (37) one easily obtains the following nonlinear EOS:

$$f(\rho) = 2\gamma H_i\left(1 - \sqrt{\rho/(3H_i^2)}\right), \tag{39}$$

when $\rho \in [3(H_i + H_1)^2, 3(H_i + H_1e^{-60\gamma H_i^{-1}})^2]$.

This opens the possibility to consider models where the EOS is nonlinear. One of these models was studied in Ref. [22], where in EC with a small cosmological constant Λ, the following EOS was considered: $f(\rho) = -\rho(1 - \rho/\rho_i)$. In this case the point $(\sqrt{(\rho_i + \Lambda)/3}, \rho_i)$ is a de Sitter solution, and the universe evolves from it, passing through a matter-dominated phase to the point $(\sqrt{\Lambda/3}, 0)$, which mimics the late-time cosmic acceleration. In this case $\omega_{\rm eff} = P(\rho)/\rho = -\rho/\rho_i$, which means that the universe accelerates when $\rho \in [\rho_i/3, \rho_i]$ and decelerates when $\rho \in [0, \rho_i/3]$. Finally, note that this model does not contain the horizon and flatness problems. The first one is avoided because at the end of the accelerating phase all the points of the universe are in causal contact ($d_{\rm hor} = +\infty$), and the second is avoided due to the accelerated period that reduces the value of $|\Omega - 1|$ at early times.

To finish, we consider once again the dynamics $H(t) = \frac{H_i + \lambda e^{\alpha t}}{1+e^{\alpha t}}$ in EC, and we try to find the EOS. From the Friedmann equation one obtains

$$e^{\alpha t} = \frac{\frac{H_i}{\lambda} - \frac{\rho}{3\lambda^2} + \sqrt{\frac{\rho}{3\lambda^2}}\left(\frac{H_i}{\lambda} - 1\right)}{\frac{\rho}{3\lambda^2} - 1}. \quad (40)$$

Then, inserting this value into the Raychaudhuri equation $\dot{H} = \frac{f(\rho)}{2}$ one obtains the function $f(\rho)$. The calculation is easy but cumbersome, and the final result is a nonlinear EOS given by

$$P(\rho) = -\rho + 2\alpha\left(\sqrt{\frac{\rho}{3}} - \lambda\right)\frac{H_i - \sqrt{\frac{\rho}{3}}}{H_i - \lambda}$$
$$\text{for } 3\lambda^2 \leq \rho \leq 3H_i^2. \quad (41)$$

V. CONCLUSIONS

A large number of models describing nonsingular universes could be constructed in $F(T)$ gravity. In this paper we have chosen LQC [an $F(T)$ theory, as we have already showed] with a small cosmological constant to propose a nonsingular bouncing universe filled by radiation and matter, which at late times mimics the current cosmic acceleration. Our model does not suffer the horizon and flatness problems, so it does not need a quasi–de Sitter phase producing a huge increase in the scale factor, as must happen in EC. Moreover, since at early times our model passes through a contracting matter-dominated phase it could (although this is a complicated point that deserves future investigation) be possible to generate a scale-invariant spectrum of perturbations.

The development of LQC as an $F(T)$ theory allows for the study of LQC perturbations using the perturbation equations in $F(T)$ gravity. This is an alternative to the study of perturbations in LQC up to the present, which is based on phenomenological corrections. The authors will pursue this topic in a subsequent work.

However, teleparallel theories are based on an arbitrary choice of an orthonormal basis, namely $\{\mathbf{e}_j\colon j = 0, 1, 2, 3\}$, in each point of the spacetime. For example, the particular choice of the basis $\{\mathbf{e}_0 = \partial_t, \mathbf{e}_1 = a^{-1}(t)\partial_x, \mathbf{e}_2 = a^{-1}(t)\partial_y, \mathbf{e}_3 = a^{-1}(t)\partial_z\}$, where $\partial_t, \ldots, \partial_z$ are the vectors corresponding to the Cartesian axis in coordinates (t, x, y, z), gives as a result the scalar torsion $T = -6H^2$, but different choices (local choices) give a different scalar torsion [46], and thus completely different cosmologies.

Fortunately, cosmology based in $F(T)$ gravity does not need this selection. Effectively, in cosmology one assumes, at large scales, a homogeneous spacetime, which means that the basis $\{\mathbf{e}_j\colon j = 0, 1, 2, 3\}$ could only have a time dependence, because the scalar torsion must be only a function of time. As a consequence, all admissible bases are related by time-dependent Lorentz transformations, i.e., by transformations of the form $\Lambda_j^k(t)$, and for these admissible bases it is easy to show that T is invariant, with the value $T = -6H^2$.

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