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Bound State Solutions of the Klein-Gordon Equation with Manning-Rosen Plus Yukawa Potential Using Pekeris-Like Approximation of the Coulomb Term and Parametric Nikiforov-Uvarov

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Authors' contributions

This work was carried out in collaboration between all authors. Author BII designed and supervised the study. Author HL performed the Mathematical analyses, wrote the protocol and the first draft of the manuscript. Authors PIA and TOM reviewed the mathematical and statistical analyses of the study. Author NAN managed the literature searches. All authors read and approved the final manuscript.

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ABSTRACT

The solutions of the klein-gordon equation with Manning-Rosen plus Yukawa potential (MRYP) has been presented using the Pekeris-like approximation of the coulomb term and parametric Nikiforov-Uvarov (NU) method. The bound state energy eigenvalues and the corresponding un-normalized eigen functions were obtained in terms of Jacobi polynomials. So also, Yukawa, Manning-Rosen

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and coulomb potentials have been recovered from the mixed potentials and their eigen values obtained.

Keywords: Klein-gordon equation; Manning-Rosen potential; Yukawa potential; Pekeris-like approximation; parametric Nikiforov-Uvarov method; Jacobi polynomials.

1. INTRODUCTION

In recent years, the study of the relativistic wave equation, particularly the Klein–Gordon equation, has attracted the attention of many authors because the solutions to this equation plays an important role in obtaining relativistic effect.It is well known that when a particle moves in a strong potential field, the relativistic effect yields the correction for non-relativistic quantum mechanics [1-3]. Taking the relativistic effect into account, one could apply the Klein– Gordon equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle [3,4]. This therefore, contains two major parameters which are, the vector potential V(r) and the scalar potential S(r). The Klein-Gordon equation with the vector and scalar potentials can be written as follows:

$$\left[-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+(S(r)+M)^2\right]\psi(r,\theta,\phi)=0$$

Where M is the rest mass, $i\frac{\partial}{\partial t}$ = energy eigen value, $V(r)$ and $S(r)$ are the vector and scalar potentials respectively [5-7]. However, the analytical solutions of the Klein-Gordon equation are possible only in the s-wave case with the angular momentum l = 0 for some well-known potentials. Conversely, when l ≠ 0, one can only solve approximately, the Klein-Gordon equation for some potential using a suitable approximation scheme [8-10]. Some of the potentials studied with this techniques are; Manning-Rosen Potential, [11-14] Hulthen Potential, [15,16] Kratzer Potential, [11,17] Wood-Saxon Potential, [18,19] and Poschl-Teller Potential [20]. Different methods have been employed to obtain the bound state Klein-Gordon equation for these exponential-type potentials which includes; the supersymmetric (SUSY) and shape invariance method [21,22], the asymptotic iteration method (AIM) [23,24], and the Nikiforov-Uvarov (NU) Method [25]. The Klein-Gordon equation for the potential under studies is solved by using the parametric NU method to obtain the energy eigenvalues and eigen functions of the bound state. Recently our group made some attempts to study the bound state solutions of Klein-Gordon, Dirac and Schrodinger equations using a combined or mixed potentials. Some of which includes Woods-Saxon plus Attractive Inversely Quadratic potential (WSAIQP) [26], Manning-Rosen plus a class of Yukawa potential (MRYP) [27], generalized Woods-Saxon plus Mie-type potential (GWSMP) [28], and finally, the Kratzer plus Reduced Pseudoharmonic Oscillator potential (KRPHOP) [29]. The purpose of the present paper is to solve the Klein-Gordon equation for the mixed potential MRYP defined as,

$$V(r)=-\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]-\frac{V_0e^{-\alpha r}}{r} \tag{1}$$

using the parametric NU method. The paper is organized as follows: After a brief introduction in section 1, the NU method was reviewed in section 2, the radial Klein-Gordon equation was solved using the NU method in section 3, the result obtained was discussed in section 4, and finally, a brief conclusion was given in section 5.

2. REVIEW OF PARAMETRIC NIKIFAROV-UVAROV METHOD

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi''_n(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\Psi'_n(s)+\frac{\overline{\sigma}(s)}{\sigma^2(s)}\Psi_n(s)=0 \tag{2}$$

Where σ(s) and $\overline{\sigma}$(s) are polynomials at most second degree and $\tilde{\tau}$ (s) is first degree polynomials. The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi''(s)+\frac{c_1-c_2s}{s(1-c_3s)}\Psi'(s)+\frac{1}{s^2(1-c_3s)^2}[-\epsilon_1s^2+\epsilon_2s-\epsilon3\Psi(s)=0 \tag{3}$$

Thus eqn. (2) can be solved by comparing it with equation (3) and the following polynomials are obtained

$$\tilde{\tau}(s)=(c_1-c_2s)\ ,\quad \sigma(s)=s(1-c_3s)\ ,\quad \overline{\sigma}(s)=-\epsilon_1s^2+\epsilon_2s-\epsilon_3 \tag{4}$$

The parameters obtainable from equation (4) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equations respectively

$c_2 n-(2n+1)c_5+(2n+1)(\sqrt{c_9}+c_3\sqrt{c_8})+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0$ (5)

$(c_2-c_3)n+c_3n^2-(2n+1)c_5+(2n+1)(\sqrt{c_9}+c_3\sqrt{c_8})+c_7+2c_3c_8+2\sqrt{c_8c_9}=0$ (6)

While the wave function is given as:

$$\Psi_n(s)=N_{n,l}S^{c_{12}}(1-c_3s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3s) \quad (7)$$

Where

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\epsilon_1,\ c_7=2c_4c_5-\epsilon_2,\ c_8=c_4^2+\epsilon_3,$$
$$c_9=c_3c_7+c_3^2c_8+c_6,\quad c_{10}=c_1+2c_4+2\sqrt{c_8},$$
$$c_{11}=c_2-2c_5+2(\sqrt{c_9}+c3\sqrt{c_8})$$
$$c_{12}=c_4+\sqrt{c_8},\ c_{13}=c_5-(\sqrt{c_9}+c3\sqrt{c_8}) \quad (8)$$

and P_n, is the orthogonal polynomials.

Given that $P_n^{(\alpha,\beta)}=\sum_{r=0}^{n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\beta-r+1)(n-r)!r!}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}$ (9)

This can also be expressed in terms of the Rodriguez's formula

$$P_n^{(\alpha,\beta)}(x)=\frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\left(\frac{d}{dx}\right)^n\left((x-1)^{n+\alpha}(x+1)^{n+\beta}\right) \quad (10)$$

3. SOLUTIONS OF THE RADIAL PART OF THE KLEIN-GORDON EQUATION WITH MRYP POTENTIAL

The radial part of the Klein-Gordon Equation with vector V(r) potential = scalar S(r) potential in atomic units (ħ = c = 1) is given as

$$\frac{d^2R(r)}{dr^2}+[(E^2-M^2)-2(E+M)V(r)]R(r)=0 \quad (11)$$

Substituting potential of Eq. (1) into the Klein-Gordon equation of eq. (11), we obtain

$$\frac{d^2R(r)}{dr^2}+\left[(E^2-M^2)-2(E+M)\left(-\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]-\frac{V_0e^{-\alpha r}}{r}\right)\right]R(r)=0 \quad (12)$$

Where $\lambda=l(l+1)$ and $V(r)$ is the Mixed potential energy function

Since the Klein-Gordon equation with the above combined potentials rarely has exact analytical solution, an approximation to the centrifugal term has to be made. The good approximation for $1/r^2$ in the centrifugal barrier is taken as

$$\frac{1}{r^2}=\frac{4\alpha^2}{(1+e^{2\alpha r})^2}, \quad (13)$$

Similar to other related work,

Making the transformation $s=e^{-\alpha r}$ equation (1) becomes

$$V(s)=-\left[\frac{CS+DS^2}{(1-S)^2}\right]-\frac{\alpha V_0S}{1-S} \quad (14)$$

To solve Eq.(12) by the present method, we need to recast Eq. (13) and apply the transformation given as $s=-e^{2\alpha r}$

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{(1-s)s}\frac{dR(s)}{ds}+\frac{1}{(1-s)^2s^2}[-(\beta^2-F+B)s^2+(2\beta^2+A+B)s-(\beta^2)]R(s)=0, \quad (15)$$

Where,

$$-\beta^2=\frac{E^2-M^2}{4\alpha^2};\ B=2\left(\frac{E+M}{\alpha}\right)V_0;\ A=2\left(\frac{E+M}{\alpha^2}\right)C;\ F=2\left(\frac{E+M}{\alpha^2}\right)D \quad (16)$$

Comparing equation (12) with equation (3) yields the following parameters

$$c_1=c_2=c_3=1, c_4=0, c_5=-\frac{1}{2}, c_6=\frac{1}{4}+\beta^2+B-F, c_7=-2\beta^2-A-B, c_8=\beta^2, c_9=\frac{1}{4}-(A+F), c_{10}=1+2\sqrt{\beta^2}, c_{11}=2+2\left(\sqrt{\frac{1}{4}-A-F}+\sqrt{\beta^2}\right), c_{12}=\sqrt{\beta^2}, c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}-A-F}+\sqrt{\beta^2}\right), \epsilon_1=\beta^2+B-F, \epsilon_2=2\beta^2+A+B, \epsilon_3=\beta^2, \quad (17)$$

Now using equations (5), (13) and (14) we obtain the energy eigen spectrum of the MRYP as

$$\beta^2=\left[\frac{A+B-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-A-F}}{(2n+1)+2\sqrt{\frac{1}{4}-A-F}}\right]^2 \quad (18)$$

Equation (15) can be solved explicitly and the energy eigen spectrum of MRYP becomes

$$E^2 - M^2 = -4 \propto^2 \left[\frac{2\left(\frac{E+M}{\alpha^2}\right)C + 2\left(\frac{E+M}{\alpha}\right)V_0 - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D}}{(2n+1) + 2\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D}} \right]^2, \quad (19)$$

We now calculate the radial wave function of the MRYP as follows:

The weight function $\rho(s)$ is given as

$$\rho(s) = s^{c_{10}-1}(1-c_3 s)^{\frac{c_{11}}{c_3} - c_{10} - 1}, \quad (20)$$

Using equation (14) we obtain the weight function as

$$\rho(s) = s^U (1-s)^V, \quad (21)$$

Where $U = 2\sqrt{\beta^2}$ and $V = 2\sqrt{\frac{1}{4} - A - F}$

So also, we obtain the wave function $\chi(s)$ as

$$\chi(s) = P_n^{c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1}(1 - 2c_3 s), \quad (22)$$

Using equation (14) we got the function χ(s) as

$$\chi(s) = P_n^{(U,V)}(1-2s), \quad (23)$$

Where $P_n^{(U,V)}$ are Jacobi polynomials

And lastly,

$$\varphi(s) = s^{c_{12}}(1-c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}}, \quad (24)$$

And using equation (14) we obtain

$$\varphi(s) = s^{U/2}(1-s)^{V-1/2}, \quad (25)$$

We then obtain the radial wave function from the equation

$$R_n(s) = N_n \varphi(s) \chi_n(s), \quad \text{as,} \quad (26)$$

$$R_n(s) = N_n s^{U/2}(1-s)^{(V-1)/2} P_n^{(U,V)}(1-2s) \quad (27)$$

Where n is a positive integer and N_n is the normalization constant.

4. DISCUSSION

We have solved the radial Schrӧdinger equation and obtained the energy eigen values for the Manning-Rosen plus Yukawa potential (MRYP) in equation (16).

The following cases are considered:

Case 1: If $C = D = 0$ in equation (10), the potential turns back into the Yukawa potential and equation (16) yields the energy eigen values of the Yukawa potential as,

$$E^2 - M^2 = -4 \propto^2 \left[\frac{2\left(\frac{E+M}{\alpha}\right)V_0 - (n+1)^2}{2(n+1)} \right]^2,$$

$$E^2 - M^2 = -4\frac{(E+M)}{(n+1)^2} + 4\alpha(E+M)V_0 - \propto^2 (n+1)^2 \quad (28)$$

Case 2: If $\propto \rightarrow 0$ in equation (28), the energy eigen values for Coulomb potential becomes

$$E^2 - M^2 = -4\frac{(E+M)}{(n+1)^2} \quad (29)$$

Case 3: If $V_0 = 0$ the potential in equation (10) yields the Manning-Rosen potential with energy eigen values given as

$$E^2 - M^2 = -4 \propto^2 \left[\frac{2\left(\frac{E+M}{\alpha^2}\right)C - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D}}{(2n+1) + 2\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D}} \right]^2 \quad (30)$$

5. CONCLUSION

We have obtained the energy eigen values and the corresponding un-normalized wave function using the parametric NU method for the Schrӧdinger equation with MRYP. Special cases of the potential have also been considered. The approximate analytical bound state energy eigenvalues and the corresponding un-normalized wave functions have been obtained. Interestingly, the Schrödinger and Dirac equation with the arbitrary angular momentum values for this potential can be solved by this method. The resulting eigen energy equations can be used to study the spectroscopy of some selected diatomic atoms and molecules.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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