Boundary behaviour of the complex Monge-Ampère equation

by

JOHN LEE[1] and RICHARD MELROSE[1]

Harvard University, Cambridge, MA, U.S.A.

M.I.T., Cambridge, MA, U.S.A.

Table of Contents

1. Introduction

Following work by Yau [5] on the Calabi conjecture, Cheng and Yau [1] have shown that each smoothly bounded strictly pseudoconvex open set $\Omega\subset\mathbf{C}^n$, $n\geqslant 2$, admits a unique Kähler-Einstein metric equivalent to the Bergman metric. The condition that the metric be Einstein can be expressed as

$$R_{j\bar{k}} = -\partial_j\partial_{\bar{k}}(\log\det(G_{p\bar{q}})) = -(n+1)\,G_{j\bar{k}} \tag{1.1}$$

where $R_{j\bar{k}}$, $G_{j\bar{k}}$ are the components of the Ricci tensor and metric tensor respectively. The constant on the right-hand side could be any negative number; $-(n+1)$ is chosen for convenience.

One can search for such a metric by requiring that the potential $G\in C^\infty(\Omega)$ satisfy the following complex Monge-Ampère equation:

$$\det(\partial_j\partial_{\bar{k}}G) = e^{(n+1)G}. \tag{1.2}$$

(1) Research supported in part by the National Science Foundation under grant number MCS 8006521.

Indeed, if G satisfies (1.2), then $G_{j\bar{k}}=\partial_j\partial_{\bar{k}}G$ satisfies (1.1). This condition can be reexpressed in the form given by Fefferman [2]:

$$J(v)=(-1)^n \det\begin{pmatrix} v & v_{\bar{k}} \\ v_j & v_{j\bar{k}} \end{pmatrix}=1 \tag{1.3}$$

where $v=e^{-G}$, or else in the form given by Cheng and Yau [1]:

$$M(u)=\det(g_{j\bar{k}}+u_{j\bar{k}})\cdot(\det(g_{j\bar{k}}))^{-1}e^{(n+1)u}=e^F. \tag{1.4}$$

Here $g=-\log(-\varphi)$ is obtained from a smooth defining function φ for Ω, $G=g+u$ satisfies (1.2), and $F\in C^\infty(\bar{\Omega})$, defined by

$$e^F=J(-\varphi)^{-1}=e^{(n+1)g}(\det(g_{j\bar{k}}))^{-1}, \tag{1.5}$$

measures the failure of g to be a solution of (1.2). The condition that $G_{j\bar{k}}$ be equivalent to the Bergman metric is expressed as

$$C^{-1}g_{j\bar{k}}\leq(g_{j\bar{k}}+u_{j\bar{k}})\leq Cg_{j\bar{k}}. \tag{1.6}$$

In this paper it is shown that the solution $G=g+u$ to (1.4), (1.6) is a graded Lagrangian distribution associated to the conormal bundle $N^*(\partial\Omega)$. More particularly, there are functions $\psi_j\in C^\infty(\bar{\Omega})$ and a defining function φ_0 for Ω, with

$$\psi_j=\varphi_0^{(n+1)j}\alpha_j,\quad j\geq 1,\ \alpha_j\in C^\infty(\bar{\Omega})$$

such that, for all $N\in\mathbf{N}$,

$$u-\sum_{j=0}^{N}\psi_j(\log(-\varphi_0))^j\in C^{(n+1)N-1}(\bar{\Omega}) \tag{1.7}$$

vanishes to order $(n+1)N-1$ at the boundary. This asymptotic expansion completely determines the form of the singularity of G at the boundary, in the C^∞ sence. Conversely, the Taylor series of the coefficient functions ψ_j are completely determined by (1.7). The optimal Hölder regularity of the potential is easily seen to be

$$G-(-\log(-\varphi_0))\in C^{n,\delta}(\bar{\Omega})\quad\text{for all } 0<\delta<1, \tag{1.8}$$

or equivalently $v\in C^{n+1,\delta}(\bar{\Omega})$ for the solution to equation (1.3), unless the leading logarithmic term ψ_1 vanishes at the boundary. The regularity result (1.8) improves that obtained by Cheng and Yau; the expansion (1.7), up to the first logarithmic term, was obtained formally by Fefferman in [2].

The proof of (1.7) consists of a careful study of the nonlinear elliptic system (1.4), (1.6), and in particular the form of its degeneracy at the boundary. The linearization of the complex Monge-Ampère operator M, in (1.4), is $-(\Delta_g+n+1)$, where Δ_g is the Laplace-Beltrami operator of the metric $g_{j\bar{k}}$ associated to a smooth defining function φ for Ω. First it is shown that the linearized operator is an isomorphism between certain Hölder spaces, defined by estimates degenerating at $\partial\Omega$ in a way that reflects the strictly pseudoconvex geometry of the boundary. These results are strengthened by showing how to commute vector fields through Δ_g. Next, similar results are obtained for the nonlinear operator M, by regarding it as a perturbation of $1-(\Delta_g+n+1)$. These results show that the solution u to (1.4), (1.6) is a Lagrangian distribution. Finally, the asymptotic expansion (1.7) is derived by symbolic methods familiar from the theory of linear differential operators.

Certain of the results and methods of [4] are used, most especially the characterization of the space of Lagrangian, or conormal, distributions associated to the boundary in terms of the action of vector fields tangent to $\partial\Omega$. The space $L_b^m(\bar{\Omega})$ of totally characteristic pseudodifferential operators, discussed in [4], is used in a less essential way in the proof of degenerate Schauder estimates.

The first step is the analysis of the Laplace-Beltrami operator Δ_g. In Section 2 a detailed study of the form of Δ_g near the boundary is made, showing its relation to the boundary Laplacian $\Box_b$ of Kohn (see (2.30)). The Hölder spaces $\Lambda^{k,\alpha}(\Omega)$ are defined in Section 3 in terms of singular coordinate charts near the boundary, and are shown to be the same as the spaces used by Cheng and Yau in [1].

In Section 4, it is shown that for $\varkappa>0$,

$$\Delta_g+\varkappa\colon \varphi^r\Lambda^{k+2,\alpha}(\Omega)\to\varphi^r\Lambda^{k,\alpha}(\Omega),\quad \left(0\leq r<\tfrac{1}{2}\left(n+\sqrt{n^2+4\varkappa}\right)\right) \tag{1.9}$$

is an isomorphism. This can be shown by applying standard Schauder theory to suitable coordinate charts which send the boundary to infinity; we prefer, however, to apply the theory of totally characteristic pseudodifferential operators. While not shorter, the proof given here is more in the totally characteristic spirit which is fundamental to this paper.

In Section 5, the commutation properties of vector fields are used to improve the estimates (1.9) significantly. This is closely related to the invariant Cauchy-Riemann (CR) structure on the boundary. The maximal complex subbundle $H\subset \mathbf{C}T(\partial\Omega)$ is defined as the annihilator of the contact line bundle in $T^*(\partial\Omega)$, given by $\mathbf{R}(i\partial\varphi)$. Suppose that $V\in C^\infty(T\bar{\Omega})$ is a vector field on $\bar{\Omega}$ which is tangent to $\partial\Omega$. V is assigned weight (at most) 1 if it restricts to H over $\partial\Omega$; otherwise it is assigned weight 2. Any vector field vanishing on $\partial\Omega$ is given weight 0, and more generally, if V has weight s then $\varphi^r V$ is given weight $s-2r$. This weighting can be extended to a filtration of the space of totally characteristic linear differential operators on $\bar{\Omega}$, i.e. differential operators generated by vector fields tangent to $\partial\Omega$. In particular, formula (2.30) shows that Δ_g is totally characteristic and of weight 0. The spaces $\Lambda^{k,\alpha;s}(\Omega)$ for $0<\alpha<1$, $s\leq 2k\in\mathbf{N}$ are defined as consisting of those functions in $\Lambda^{k,\alpha}(\Omega)$ which are mapped into $\Lambda^{k-p,\alpha}(\Omega)$ by any differential operator of order p and weight at most s. Then it is shown that

$$\Delta_g+\varkappa\colon \varphi^r\Lambda^{k+2,\alpha;s}(\Omega)\to\varphi^r\Lambda^{k,\alpha;s}(\Omega), \quad \left(0<r<\tfrac{1}{2}\left(n+\sqrt{n^2+4\varkappa}\right),\ s\leq 2k\right) \tag{1.10}$$

is an isomorphism.

The complex Monge-Ampère operator M is considered in Section 6 and shown to be a totally characteristic, nonlinear, differential operator of order 2 and weight 0 with linearization $-(\Delta_g+n+1)$. This sets the value of $\varkappa$ in (1.9), and these estimates can be applied to strengthen the results of Cheng and Yau on the regularity of the solution to (1.4), (1.6) by using the inverse function theorem in the spaces $\varphi^r\Lambda^{k,\alpha}(\Omega)$. More importantly, the same method can be applied using the mapping properties (1.10) giving the much more refined regularity

$$u\in\bigcap_{k,s}\varphi^r\Lambda^{k,\alpha;s}(\Omega), \quad 0<\alpha<1,\ 0<r<n+1 \tag{1.11}$$

for the solution to (1.4), (1.6).

Section 7 is devoted to a discussion of the properties of distributions satisfying (1.11). Indeed, it follows from an argument in [4] that

$$\bigcap_{k,s}\varphi^r\Lambda^{k,\alpha;s}(\Omega)\subset\mathscr{A}(\bar{\Omega}) \tag{1.12}$$

is a subspace of the space of Lagrangian, or conormal, distributions associated to $\partial\Omega$. A filtration $\mathscr{A}^{(s)}(\bar{\Omega})\subset\mathscr{A}(\bar{\Omega})$ is introduced with the multiplicative properties

$$\varphi^r\mathscr{A}^{(s)}(\bar{\Omega})=\mathscr{A}^{(s+r)}(\bar{\Omega}), \quad \mathscr{A}^{(s)}(\bar{\Omega})\cdot\mathscr{A}^{(r)}(\bar{\Omega})\subset\mathscr{A}^{(s+r)}(\bar{\Omega}). \tag{1.13}$$

Finally, in Section 8, this filtration is used to derive the asymptotic expansion (1.7). Provided $s>0$,

$$M\colon\mathscr{A}^{(s)}(\bar{\Omega})\to\mathscr{A}^{(s)}(\bar{\Omega}). \tag{1.14}$$

If $N_+\,\partial\Omega$ is the inward pointing half of the normal bundle to $\partial\Omega$, local coordinates in $\bar{\Omega}$

induce a map: $N_+\partial\Omega\to\bar{\Omega}$ near the boundary, which in turn induces a well-defined symbol isomorphism for conormal distributions:

$$\mathscr{A}^{(s)}(\bar{\Omega})/\mathscr{A}^{(s+1)}(\bar{\Omega})\to\mathscr{A}^{(s)}(N_+\partial\Omega)/\mathscr{A}^{(s+1)}(N_+\partial\Omega). \tag{1.15}$$

This reduces M to an ordinary differential operator:

$$[M(u)-1]_s=E[u]_s,$$

where the indical operator is

$$E=(xD_x)^2+inxD_x+n+1,$$

well-defined on $N\partial\Omega$. This, and related higher order identities, allow (1.7) to be obtained by induction. Similar results are also given for solutions to the linear problem

$$(\Delta_g+\varkappa)\,w\in\varphi^rC^\infty(\bar{\Omega}),\quad w\in\mathscr{A}^{(r)}(\bar{\Omega})\quad\text{for some } r>0.$$

2. Laplace-Beltrami operator

Let $\Omega\subset\mathbf{C}^n$ be a bounded C^∞ strictly pseudoconvex domain, and suppose $\varphi\in C^\infty(\bar{\Omega})$ is any smooth defining function for Ω, with $\varphi<0$ exactly in the interior of Ω and $d\varphi\neq0$ on $M=\partial\Omega$. It is well-known (see Cheng and Yau [1]) that, provided φ is strictly plurisubharmonic, the real function $g=-\log(-\varphi)$ defines a complete Kähler metric on Ω:

$$ds^2=g_{j\bar{k}}\,dz^j\,dz^{\bar{k}}=\frac{\partial^2 g}{\partial z^j\partial z^{\bar{k}}}\,dz^j\,dz^{\bar{k}}. \tag{2.1}$$

In fact, given any defining function φ, it is possible to modify φ away from a neighborhood of $\partial\Omega$ so that (2.1) defines a complete Kähler metric. To see this, observe first that

$$g_{j\bar{k}}=\frac{\varphi_{j\bar{k}}}{-\varphi}+\frac{\varphi_j\varphi_{\bar{k}}}{\varphi^2}. \tag{2.2}$$

The condition that Ω is strictly pseudoconvex means that $\varphi_{j\bar{k}}$ is positive definite when restricted to the annihilator of $\partial\varphi$ on the boundary. Because of this, there is associated to φ a distinguished $(1,0)$ vector field ξ on $\bar{\Omega}$ near $\partial\Omega$, defined by

$$\xi\,\lrcorner\,\partial\varphi=1,\quad \xi\,\lrcorner\,\partial\bar{\partial}\varphi\equiv0 \bmod \bar{\partial}\varphi. \tag{2.3}$$

We define a function $r \in C^\infty(\bar{\Omega})$ near $\partial\Omega$ by

$$r = \varphi_{j\bar{k}}\xi^j\xi^{\bar{k}}. \tag{2.4}$$

Then we have, from (2.3),

$$\varphi_{j\bar{k}}\xi^j = r\varphi_{\bar{k}}. \tag{2.5}$$

The matrix $\psi_{j\bar{k}}$, defined by

$$\psi_{j\bar{k}} = \varphi_{j\bar{k}} + (1-r)\,\varphi_j\varphi_{\bar{k}} \tag{2.6}$$

is nondegenerate near $\partial\Omega$, since $\psi_{j\bar{k}}$ agrees with $\varphi_{j\bar{k}}$ when restricted to the annihilator of $\partial\varphi$, and $\psi_{j\bar{k}}\xi^j\xi^{\bar{k}} = 1$. If $g^{j\bar{k}}$ is the matrix of the dual metric on $T^*\Omega$, so that $g^{j\bar{k}}g_{m\bar{k}} = \delta^j_m$, one verifies directly that

$$g^{j\bar{k}} = (-\varphi)\left(\psi^{j\bar{k}} + \frac{1+\varphi-r\varphi}{r\varphi-1}\xi^j\xi^{\bar{k}}\right), \tag{2.7}$$

which shows that $g_{j\bar{k}}$ is indeed nondegenerate near $\partial\Omega$. It is then possible to modify g away from a neighborhood of $\partial\Omega$ so that it is strictly plurisubharmonic everywhere on Ω.

The non-negative Hermitian matrix

$$h^{j\bar{k}} = \frac{g^{j\bar{k}}}{-\varphi} \tag{2.8}$$

has corank one on $\partial\Omega$. To see this just note that

$$g^{j\bar{k}}\varphi_j = \frac{-\varphi^2}{r\varphi-1}\xi^{\bar{k}} \tag{2.9}$$

so, since $\psi^{j\bar{k}}$ is nondegenerate and the matrix $(\xi^j\xi^{\bar{k}})$ has rank one,

$$\ker(h^{j\bar{k}}) = \operatorname{span}(\varphi_j\,dz^j) \tag{2.10}$$

as a bilinear form on $T^*_M\bar{\Omega}$.

Observe that this differential form $\partial\varphi = \varphi_j\,dz^j$ restricted to the boundary is a pure imaginary form, and

$$\theta = \iota^*(i\partial\varphi), \quad \iota: M = \partial\Omega \hookrightarrow \bar{\Omega}$$

spans the contact line bundle of the CR-structure induced on M. The Levi form of θ is the Hermitian form L defined on the complex vector bundle $H = \theta^\perp \subset \mathbb{C}TM$ by $d\theta$:

$$L(v,w)=d\theta(Jv,w) \tag{2.11}$$

where J is the complex structure on $T\Omega \supset TM$.

In view of (2.10) the bilinear form defined by $h^{j\bar{k}}$ on $T^*_M\bar{\Omega}$ descends to a degenerate bilinear form, h, on T^*M given by

$$h(\alpha,\beta)=h^{j\bar{k}}\alpha_j\beta_{\bar{k}} \tag{2.12}$$

if $\alpha=\alpha_j dz^j+\alpha_{\bar{k}}dz^{\bar{k}}|_{\partial\Omega}$, $\beta=\beta_j dz^j+\beta_{\bar{k}}dz^{\bar{k}}|_{\partial\Omega}$.

LEMMA 2.13. *The bilinear form h is equal to the dual of L on T^*M.*

Proof. Using the injection $TM\subset T\Omega$ the Levi form becomes

$$L(v,w)=-i\partial\bar{\partial}\varphi(Jv,w). \tag{2.14}$$

On H, $\partial\bar{\partial}\varphi$ agrees with $\psi=\psi_{j\bar{k}}dz^j\wedge dz^{\bar{k}}$. The dual of the tensor $\Psi=-i\psi(J\cdot,\cdot)$ on $T\bar{\Omega}$ is just $\psi^{j\bar{k}}$ on $T^*\bar{\Omega}$. Over the boundary it follows from (2.7) and (2.8) that $h^{j\bar{k}}$ agrees with $\psi^{j\bar{k}}$ on the annihilator of ξ, and $h^{j\bar{k}}$ annihilates $i\partial\varphi$; this proves the lemma.

We define

$$T=-i(\xi-\bar{\xi}),\quad W=\xi+\bar{\xi}. \tag{2.15}$$

Then it is easy to check that

$$\begin{aligned} &T\lrcorner d\varphi=0,\quad T\lrcorner i\partial\varphi=1,\quad T\lrcorner -i\partial\bar{\partial}\varphi=-rd\varphi,\\ &W\lrcorner d\varphi=2,\quad W\lrcorner i\partial\varphi=i,\quad W\lrcorner -i\partial\bar{\partial}\varphi=ir(\partial\varphi-\bar{\partial}\varphi). \end{aligned} \tag{2.16}$$

In particular, W is proportional to the gradient of φ with respect to $g_{j\bar{k}}$, while T is tangent to $\partial\Omega$ and everywhere transversal to the maximal complex subspace H.

Given any boundary coordinates $y\in\mathbf{R}^{2n-1}$ for $M=\partial\Omega$, this provides a preferred set of coordinates (x,y) in Ω near $\partial\Omega$, where $x=-\varphi$ and the y^s are extended to be constant along the integral curves of W. We shall call such coordinates *normal coordinates*. In normal coordinates,

$$\partial_x=-\tfrac{1}{2}W. \tag{2.17}$$

Recall that on any manifold with boundary (see [4]) the ring of totally characteristic differential operators $\mathrm{Diff}_b(\bar{\Omega})$ consists of those operators which can be written as polynomials in C^∞ vector fields tangent to the boundary. In local coordinates (x,y) with $x\geq 0$, this condition requires that $P\in\mathrm{Diff}_b^m(\bar{\Omega})$ be of the form

$$P = \sum_{j+|\alpha|\le m} p_{\alpha,j}(x, y)\, D_y^\alpha (xD_x)^j. \tag{2.18}$$

PROPOSITION 2.19. *The Laplace-Beltrami operator Δ_g is totally characteristic on $\bar{\Omega}$ and in any local coordinates $x=-\varphi$, $y \in \mathbf{R}^{2n-1}$ near the boundary is of the form*

$$\Delta_g = I(xD_x) + xR_1 \tag{2.20}$$

where R_1 is also totally characteristic and the indicial polynomial is

$$I(\lambda) = \lambda^2 + in\lambda. \tag{2.21}$$

Proof. With $x=-\varphi$, $\mu_j^r = \partial y^r/\partial_{z^j}$, $r=1, \ldots, 2n-1$, and $\mu_{\bar{k}}^r = \overline{\mu_k^r}$, we have

$$\partial_{z^j} = -\varphi_j \partial_x + \mu_j^r \partial_{y^r}, \quad \partial_{z^{\bar{k}}} = -\varphi_{\bar{k}} \partial_x + \mu_{\bar{k}}^r \partial_{y^r}. \tag{2.22}$$

Substituting into $\Delta_g = g^{j\bar{k}}{}_8 D_j D_{\bar{k}}$ gives

$$\begin{aligned}\Delta_g = {} & g^{j\bar{k}}(-\varphi_j D_x + \mu_j^r D_{y^r})(-\varphi_{\bar{k}} D_x + \mu_{\bar{k}}^s D_{y^s}) \\ = {} & g^{j\bar{k}} \varphi_j \varphi_{\bar{k}} D_x^2 + g^{j\bar{k}} \varphi_j (D_x \varphi_{\bar{k}}) D_x - g^{j\bar{k}} \mu_j^r (D_{y^r} \varphi_{\bar{k}}) D_x - g^{j\bar{k}} \mu_j^r \varphi_{\bar{k}} D_{y^r} D_x - g^{j\bar{k}} \varphi_j \mu_{\bar{k}}^s D_x D_{y^s} \\ & - g^{j\bar{k}} \varphi_j (D_x \mu_{\bar{k}}^s) D_{y^s} + g^{j\bar{k}} (\mu_j^r D_{y^r})(\mu_{\bar{k}}^s D_{y^s}).\end{aligned} \tag{2.23}$$

From (2.9),

$$g^{j\bar{k}} \varphi_j \varphi_{\bar{k}} = \frac{x^2}{1+rx} \tag{2.24}$$

and from (2.22),

$$\begin{aligned}\varphi_{j\bar{k}} &= \partial_{z^j} \varphi_{\bar{k}} = -\varphi_j \partial_x \varphi_{\bar{k}} + \mu_j^r \partial_{y^r} \varphi_{\bar{k}} \\ &= -i\varphi_j D_x \varphi_{\bar{k}} + i\mu_j^r D_{y^r} \varphi_{\bar{k}}.\end{aligned} \tag{2.25}$$

Substituting these relations into the first three terms in (2.23) gives

$$\frac{x^2}{1+rx} D_x^2 + ig^{j\bar{k}} \varphi_{j\bar{k}} D_x = \frac{1}{1+rx} (xD_x)^2 + inxD_x. \tag{2.26}$$

Now the fourth and fifth terms in (2.23) are

$$-(g^{j\bar{k}} \varphi_{\bar{k}} \mu_j^r + g^{j\bar{k}} \varphi_j \mu_{\bar{k}}^r) D_x D_{y^r} = -\frac{x^2}{1+rx} (\xi^j \mu_j^r + \xi^{\bar{k}} \mu_{\bar{k}}^r) D_x D_{y^r}. \tag{2.27}$$

Thus Δ_g is certainly totally characteristic. Evaluating (2.26) at $x=0$ gives the totally characteristic ordinary differential operator

$$I(xD_x) = (xD_x)^2 + inxD_x. \tag{2.28}$$

This corresponds to (2.21), so it is only necessary to observe that the last three terms in (2.23) vanish at $x=0$ since the $g^{j\bar{k}}$ do so.

It is important to identify the first order terms in the Taylor series of Δ_g at $x=0$, as a totally characteristic operator.

THEOREM 2.29. *If (x, y) are normal coordinates near the boundary of Ω then*

$$\Delta_g = I(xD_x) + x(-r(xD_x)^2 + \Box_b + V) + x^2 R_2(x, y, xD_x, D_y) \tag{2.30}$$

where $\Box_b$ is the boundary Laplacian given by the Levi form on $\partial\Omega$ and the volume form $\theta \wedge d\theta^{n-1}$, V is a vector field tangent to $\partial\Omega$, and

$$I(xD_x) = (xD_x)^2 + inxD_x.$$

R_2 is elliptic where $\Box_b$ is characteristic.

Proof. Recall that $\Box_b$ is defined as $\bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*$, which reduces to $\bar{\partial}_b^* \bar{\partial}_b$ on functions. Here, $\bar{\partial}_b f$ is the coset of $\bar{\partial} f$ in $T^*\partial\Omega/\theta$ if f is extended to a neighborhood of $\partial\Omega$, and $\bar{\partial}_b^*$ is the formal adjoint. Thus, for any $u \in C_c^\infty(M)$,

$$\langle \Box_b f, u \rangle = \langle \bar{\partial}_b f, \bar{\partial}_b u \rangle = \int_M h^{j\bar{k}} f_{\bar{k}} u_j \psi \tag{2.31}$$

when f and u are extended to a neighborhood of $\partial\Omega$, and $\psi = \theta \wedge d\theta^{n-1}$. Using (2.22) and the fact that $h_{j\bar{k}}$ annihilates $\partial\varphi$, this becomes

$$\langle \Box_b f, u \rangle = \int_M h^{j\bar{k}} \mu_{\bar{k}}^r \partial_{y^r}(f) \mu_j^s \partial_{y^s}(u) \psi.$$

After the usual integration by parts, this shows that

$$\Box_b f = -(h^{j\bar{k}}(\mu_j^r \partial_{y^r})(\mu_{\bar{k}}^s \partial_{y^s}) f + Vf) \tag{2.32}$$

where V is some vector field tangent to $\partial\Omega$. Therefore, the last term in (2.23) can be replaced by $x(\Box_b + V)$.

Next observe that in normal (x, y) coordinates

$$0 = W(y^r) = \xi^j \mu_j^r + \xi^{\bar{k}} \mu_{\bar{k}}^r$$

so the fourth and fifth terms in (2.23) vanish, from (2.27). The first-order term of (2.26) at $x=0$ is clearly

$$-xr(0, y)(xD_x)^2.$$

Finally, the sixth term in (2.23) vanishes to order x^2, by virtue of (2.9). This shows that Δ_g is given by (2.30); to complete the proof we need only observe that the ellipticity of R_2 on $\mathbf{R}\theta$, where $\Box_b$ is characteristic, is a trivial consequence of (2.9).

Remark. By means of a rather laborious integration by parts, the vector field V in (2.30) can be shown to be equal to $\frac{1}{2}i(n-1)T$. Since we have no need for this result, we omit the proof.

3. Hölder spaces

In carrying out the analysis of Δ_g near $\partial\Omega$ we shall use spaces of functions satisfying certain degenerate Hölder estimates. On the half-space $Z = \bar{\mathbf{R}}^+ \times \mathbf{R}^{N-1}$, with natural coordinates (x, y) consider, for each $0<\alpha<1$, the subspace $B^\alpha(Z) \subset L^\infty(Z)$ of functions satisfying:

$$(x+x')^\alpha |f(x, y) - f(x', y')| \leq C(|x-x'|^\alpha + (x+x')^\alpha |y-y'|^\alpha) \tag{3.1}$$

for some constant C. If C is the smallest such constant set

$$\|f\|_\alpha = \|f\|_\infty + \mathrm{C}. \tag{3.2}$$

More generally if $K \in \mathbf{N}$ we write $B^{k,\alpha}(Z)$ for the space of functions $f \in L^\infty(Z)$ with

$$(xD_x)^p D_y^\beta f \in B^\alpha(Z), \quad \forall p + |\beta| \leq k, \tag{3.3}$$

where the derivatives are to be taken in the sense of distributions in Z. With the obvious norm $\|\cdot\|_{k,\alpha}$, $B^{k,\alpha}(Z)$ is a Banach space. In fact, under the diffeomrophism

$$\mathring{Z} \ni (x, y) \mapsto (\log(x), y) \in \mathbf{R}^N$$

these correspond to the usual Hölder spaces $C^{k,\alpha}(\mathbf{R}^N)$, of functions satisfying

$$|D^{\beta}_{(s,y)}g(s,y)-D^{\beta}_{(s,y)}g(s',y')|\leqslant C|(s,y)-(s',y')|,\quad \forall|\beta|\leqslant k, \tag{3.4}$$

where $s=\log(x)$.

The estimates (3.1), (3.3) are local in nature, in the sense that if $\varrho\in C^{\infty}_{c}(Z)$ and $f\in B^{k,\alpha}(Z)$ then $\varrho f\in B^{k,\alpha}(Z)$. In fact $B^{k,\alpha}(Z)$ is a ring under pointwise multiplication. We write $B^{k,\alpha}_{c}(Z), B^{k,\alpha}_{\mathrm{loc}}(Z)$, $B^{k,\alpha}(U)$ for the rings of compactly supported functions in Z, of functions locally in $B^{k,\alpha}(Z)$ and of functions defined analogously for any open subset $U\subset Z$, respectively.

Consider the space of almost regular distributions $\mathscr{A}(Z)\subset\mathscr{D}'(Z)$, defined in terms of the usual Sobolev spaces by

$$f\in\mathscr{A}(Z)\Leftrightarrow \exists s\in\mathbf{R}\text{ such that }(xD_x)^pD^{\beta}_{y}f\in H^{s}_{\mathrm{loc}}(Z),\ \forall p,\beta. \tag{3.5}$$

It is readily shown (see [4]) that $\mathscr{A}(Z)$ is just the space of extendible Lagrangian, or conormal, distributions on Z associated to the boundary, i.e. to the conormal bundle $N^*\partial Z$. Since $L^{\infty}(Z)\subset H^{0}_{\mathrm{loc}}(Z)$ it follows that

$$\bigcap_{k} B^{k,\alpha}(Z)=\mathscr{A}L^{\infty}(Z)\subset\mathscr{A}(Z), \tag{3.6}$$

with the intersection independent of α. This fact is very basic to the method used in this paper.

As in the standard case of Hölder (or Lipschitz) spaces there is a useful approximation criterion for a function f to be in $B^{k,\alpha}(Z)$. Indeed, suppose there is a decomposition of the form:

$$f=\sum_{k=0}^{\infty}f_k \tag{3.7}$$

where for some constant A,

$$\|f_k\|_{L^{\infty}}\leqslant A2^{-k\alpha} \tag{3.8}$$

$$\|xD_xf_k\|_{L^{\infty}},\ \|D_{y_s}f_k\|_{L^{\infty}}\leqslant A2^{k-k\alpha},\quad s=1,\dots,N-1. \tag{3.9}$$

LEMMA 3.10. *If f is given by* (3.7), *where* (3.8) *and* (3.9) *hold, then $f\in B^{\alpha}(Z)$.*

Proof. Using the diffeomorphism $s=\log(x)$ this reduces to the usual result for Hölder spaces (see for example [3]).

Next we introduce some even more degenerate Hölder spaces on the strictly pseudoconvex domain Ω. First choose a finite set of normal coordinate charts $(U(x,y))$

covering $\bar{\Omega}$, with range $\{0 \leq x < r\} \times \mathbf{R}^{N-1}$ for each chart. Given $p \in \partial\Omega$ choose a chart containing p. An affine linear transformation in the y variables, to $y' = A(p)y + B(p)$, depending smoothly on $p \in U$ can then be chosen so that at $p = (0, 0)$

$$\theta = d(y^{2n-1})', \tag{3.11}$$

$$d(y^1)', \ldots, d(y^{2n-1})' \text{ are orthogonal with respect to } h^{jk}. \tag{3.12}$$

Dropping the prime, we next define singular coordinates based at p,

$$v = x^{\frac{1}{2}}, \quad w^r = x^{-\frac{1}{2}} y^r, \quad t = x^{-1} y^{2n-1}, \quad r = 1, \ldots, 2n-2 = N-2. \tag{3.13}$$

These are defined in the preimage in Ω of the region

$$V_p = \{|y| \leq Rx^{\frac{1}{2}}, \ |y^{2n-1}| \leq Rx, \ x \leq r\} \tag{3.14}$$

under the normal coordinates (x, y') where R is independent of p. As p traverses the boundary it is clear that the V_p cover a neighborhood of $\partial\Omega$.

Definition 3.15. The space $\Lambda^{k,\alpha}(\Omega)$ consists of those functions $f \in L^\infty(\Omega)$ for which there exists a constant C such that $\|f\|_{k,\alpha} \leq C$ in each singular coordinate system (3.13), on the set V_p, corresponding to p and a finite covering of Ω by normal coordinate systems.

Clearly $\Lambda^{k,\alpha}(\Omega)$ is a Banach space with respect to the norm $\|f\|^*_{k,\alpha}$, given by the smallest constant C. It is necessary to show that the definition is independent of all choices made. In fact this follows immediately from the fact that a change of normal coordinates, or affine reduction to (3.11), (3.12) at p, induces a C^∞ diffeomorphism on each of the spaces V_p, depending smoothly on p. Since $B^{k,\alpha}(V)$ is coordinate invariant, $\Lambda^{k,\alpha}(\Omega)$ is well-defined. The main reason for introducing these spaces, and the singular coordinates (3.13), is that the geometry is bounded with respect to them, in the sense of Cheng and Yau [1].

LEMMA 3.16. *Let g_{ij}, g^{ij} be the entries of the metric tensor, derived from a strictly plurisubharmonic defining function for Ω, expressed in singular coordinates* $\log(v)$, w, t *as in* (3.13). *Then g^{ij}, g_{ij} are C^∞ in V_p with bounds on all derivatives independent of p, with respect to a suitable covering.*

Proof. We can use the calculations of Section 2. The metric g^{ij} occurs as the principal part of Δ_g in (2.30), (2.32). In terms of the singular coordinates, $s = \log(v)$, w, t,

$$2xD_x = D_s - w^r D_{w^r} - 2tD_t, \quad x^{\frac{1}{2}} D_{y^r} = D_{w^r}, \quad xD_{y^{2n-1}} = D_t, \quad r = 1, \dots, 2n-2. \tag{3.17}$$

Since the second order part of $\Box_b$, in (2.32) is composed of vector fields tangent to the maximal complex bundle H, and therefore of the form:

$$Z_j = \sum_{r=0}^{2n-2} M_j^r D_{y^r} + \sum_{r=0}^{2n-1} y^k M_{j,k}^r D_{y^r} + \sum_{r=0}^{2n-1} tN_j^r D_{y^r},$$

it is clear that $x\Box_b$ is a C^∞ differential operator in (s, w, t) in each V_p. Similarly, using (3.17) all the other terms in (2.30) are C^∞ in the singular coordinates. This shows that the g^{ij} are C^∞, clearly uniformly as p varies. Moreover, Δ_g is uniformly elliptic in V_p, so the inverse g_{ij} is also C^∞.

In [1], Cheng and Yau defined holomorphic coordinate charts covering Ω, such that each $z \in \Omega$ is contained in a chart in which the entries of the Hermitian matrices $g_{j\bar{k}}, g^{j\bar{k}}$ with their derivatives of any finite order are uniformly bounded, independently of z. They were then able to introduce spaces, $\tilde{C}^{k+\alpha}(\Omega)$, consisting of the functions with uniform Hölder estimates with respect to those coordinate charts respecting the bounded geometry. The fact that the coordinate charts are holomorphic is of no significance for such estimates, so as a corollary of Lemma 3.16 and Definition 3.15 we have

$$\Lambda^{k,\alpha}(\Omega) = \tilde{C}^{k+\alpha}(\Omega), \quad \forall k, \alpha. \tag{3.18}$$

4. Schauder estimates

In this section we show that if φ is any defining function such that $g = -\log(-\varphi)$ is strictly plurisubharmonic, the Laplacian Δ_g gives, for each $\varkappa > 0$, isomorphisms:

$$\Delta_g + \varkappa \colon \varphi^r \Lambda^{k+2,\alpha}(\Omega) \to \varphi^r \Lambda^{k,\alpha}(\Omega), \quad 0 < r < \tfrac{1}{2}\left(n + \sqrt{n^2 + 4\varkappa}\right) \tag{4.1}$$

of the degenerate Hölder spaces defined in Section 3 above. Here, we write $\varphi^r \Lambda^{k,\alpha}(\Omega)$ for the space of functions of the form $\varphi^r f, f \in \Lambda^{k,\alpha}(\Omega)$. Equipped with the obvious norm it is a Banach space whose topology is independent of the defining function φ. To prove that (4.1) is an isomorphism we use some facts concerning totally characteristic pseudodifferential operators. For the general theory of these operators the reader is referred to [4]. For convenience the definition and some elementary properties are recalled here.

A totally characteristic pseudodifferential operator on the half-space $Z=\bar{\mathbf{R}}_+\times\mathbf{R}^{N-1}$ is a continuous linear map $C_c^\infty(\mathring{Z})\to\mathscr{D}'(Z)$, where $\mathscr{D}'(Z)$ is the space of distributions on $\mathring{Z}$ which can be extended to a neighborhood of Z, of the form

$$Au(x,y)=(2\pi)^{-N}\int_Z\int_{\mathbf{R}^N}e^{i(1-t)\lambda+i(y-y')\cdot\eta}a(x,y,\lambda,\eta)\,u(xt,y')\,d\lambda\,d\eta\,dt\,dy'. \tag{4.2}$$

The amplitude $a\in S^m_{1,0}(Z\times\mathbf{R}^N)$ is required to satisfy the lacunary condition with respect to the λ-variable:

$$\int_{\mathbf{R}}e^{i(1-t)\lambda}a(x,y,\lambda,\eta)\,d\lambda=0\quad\text{for } t<0. \tag{4.3}$$

The space of such symbols of order m is denoted $S^m_{\mathrm{lac}}(Z\times\mathbf{R}^N)$ and the corresponding space of operators with kernels locally of the form (4.2) by $L^m_{\mathrm{b}}(Z)$. The residual space $L_{\mathrm{b}}^{-\infty}(Z)=\bigcap_m L^m_{\mathrm{b}}(Z)$ consists of operators mapping $\mathscr{E}'(Z)$ into $\mathscr{A}(Z)$.

Observe that a totally characteristic differential operator

$$P(x,y,xD_x,D_y)=\sum_{j+|\alpha|\le m}p_{j,\alpha}(x,y)\,x^jD_x^jD_y^\alpha$$

as in Section 2 is an element of $L^m_{\mathrm{b}}(Z)$ with symbol, in the sense of (4.2),

$$P(x,y,\lambda,\eta)=\sum_{j+|\alpha|\le m}p_{j,\alpha}(x,y)\,\lambda^j\eta^\alpha.$$

An operator $A\in L^m_{\mathrm{b}}(Z)$ is said to be elliptic if its symbol is elliptic in the usual sense. The invariance and symbolic properties of these operators are extensively discussed in [4].

We will show that any element of $L^m_{\mathrm{b}}(Z)$ is locally bounded on the appropriate singular Hölder spaces $B^{k,\alpha}(Z)$. The first step is to show that the error terms in the calculus are well-behaved.

LEMMA 4.4. *If $R\in L_{\mathrm{b}}^{-\infty}(Z)$, then $R\colon x^rL_c^\infty(Z)\to x^rB^{k,\alpha}_{\mathrm{loc}}(Z)$ for all $r,k\ge0$, all $0<\alpha<1$.*

Proof. By localizing it can be assumed that $r\in S^{-\infty}_{\mathrm{lac}}(Z\times\mathbf{R}^N)$ has support with compact projection onto the base Z. Then the corresponding element $R\in L_{\mathrm{b}}^{-\infty}(Z)$ can be written:

$$Ru(x,y)=\int_0^\infty\int_{\mathbf{R}^{N-1}} k_R(x,y,y',t)\,u(xt,y')\,dt\,dy' \tag{4.5}$$

where the kernel, in this sense, k_R, is C^∞ in all variables and, because of (4.3), rapidly decreasing with all derivatives as $t\to\infty$ and $t\searrow 0$. The integral above converges absolutely, and Ru is clearly bounded when u is bounded. In fact, for any integer j and multiindex α, $(xD_x)^j D_y^\alpha Ru$ is of the same form (4.5), and so R maps L_c^∞ into $B_{\mathrm{loc}}^{k,\alpha}$. Finally, since $R(x^r u)=x^r R' u$, where R' is again an element of $L_b^{-\infty}(Z)$, the lemma is proved.

THEOREM 4.6. *If* $A\in L_b^0(Z)$ *then* $A\colon B_c^{k,\alpha}(Z)\to B_{\mathrm{loc}}^{k,\alpha}(Z)$ *for each* $0<\alpha<1$, $k\in\mathbf{N}_0$.

Proof. Consider a useful partition of unity. Choose $\psi\in C_c^\infty(\mathbf{R}^N)$ with $\psi(\lambda,\eta)=1$ in $|(\lambda,\eta)|<\frac{1}{2}$, $\psi(\lambda,\eta)=0$ if $|(\lambda,\eta)|\geq 1$. Then set

$$\varphi_k(\lambda,\eta)=\psi(2^{-k}(\lambda,\eta))-\psi(2^{-k+1}(\lambda,\eta)),\quad k\geq 1. \tag{4.7}$$

These functions $\varphi_k\in C_c^\infty(\mathbf{R}^N)$ are uniformly bounded in $S_{1,0}^0(\mathbf{R}^N)$ and give a partition of unity as

$$\psi+\sum_{k\geq 1}\varphi_k=1.$$

They do not however satisfy the lacunary condition (4.3), so we modify them slightly to correct this. Set $\varrho_k=\varphi_k-T\varphi_k$ where

$$T\varphi(\lambda,\eta)=(2\pi)^{-1}\int e^{i(1-s)(\lambda-\lambda')}\varrho(s)\,\varphi(\lambda',\eta)\,d\lambda'\,ds \tag{4.8}$$

and $\varrho\in C^\infty(\mathbf{R})$ satisfies $\varrho(t)=1$ in $t<\frac{1}{2}$, $\varrho(t)=0$ in $t>\frac{3}{4}$. Since $T(1)=0$, the ϱ_k also give a partition of unity. Moreover (see [4]), T is bounded from S^∞ into $S^{-\infty}$, so the ϱ_k are bounded in $S_{\mathrm{lac}}^0(\mathbf{R}^N)$.

Now, suppose that $a\in S_{\mathrm{lac}}^0(Z\times\mathbf{R}^N)$ is such that the Fourier transform

$$\beta(x,y,s,\eta)=\int e^{i(1-s)\lambda}a(x,y,\lambda,\eta)\,d\lambda=0\quad\text{if } s<\tfrac{1}{2}, \tag{4.9}$$

strengthening (4.3) which demands that it be zero in $s<0$. The Fourier transform of the product

$$a_k(x,y,\lambda,\eta)=\varrho_k(\lambda,\eta)\,a(x,y,\lambda,\eta) \tag{4.10}$$

is the convolution of the two Fourier transforms, so has support in $s>0$, since the Fourier transform of ϱ, in the sense of (4.9), also has support in $s>\frac{1}{2}$. Thus, $a_k\in S^0_{\mathrm{lac}}$ is a bounded sequence. The corresponding kernels

$$\beta_k(x,y,s,w)=\int e^{i(1-s)\lambda+iw\cdot\eta}a_k(x,y,\lambda,\eta)\,d\lambda\,d\eta$$

satisfy, for each $k\geq 1$ and for each $p>0$ and compactum $K\Subset Z$:

$$|I_k|=\left|\int\beta_k(x,y,s,w)\,ds\,dw\right|\leq C_p 2^{-pk} \tag{4.11}$$

for some constant C_p. To see this note that from the definition of β_k

$$I_k=-(2\pi)^N a_k(x,y,0,0)=(2\pi)^N a(x,y,0,0)\,(T\varphi_k)\,(0,0)$$

since $\varphi_k(0,0)=0$. If the action of T is written as convolution then

$$T\varphi_k(0,0)=\int f(-\lambda,0)\,\varphi_k(\lambda,0)\,d\lambda$$

where f, coming from the Fourier transform of ϱ, is singular only at 0 and is rapidly decreasing as $|\lambda|\to\infty$. Recall the definition of the φ_k, which shown them to be supported in $2^{k-2}\leq|(\lambda,\eta)|\leq 2^k$. Now, changing variable to $2^{-k}\lambda$ immediately gives (4.11).

Let us consider the partition of unity φ_k, constructed above, more carefully.

LEMMA 4.12. *For each $p\in\mathbf{R}$, there exists a constant C_p such that*

$$|D^j_\lambda D^\beta_\eta T\varphi_k(2^k(\lambda,\eta))|\leq C_p 2^{-kp}(1+|(\lambda,\eta)|)^{-p},\quad |\beta|\leq p.$$

Proof. Inserting the change of variables $(\lambda,\eta)\mapsto 2^k(\lambda,\eta)$, $\lambda'\mapsto 2^k\lambda'$, $s\mapsto(1-2^{-k}r)$ in formula (4.8) for $T\varphi_k$ gives

$$T\varphi_k(2^k(\lambda,\eta))=(2\pi)^{-1}\int e^{-ir(\lambda-\lambda')}\varrho(1-2^{-k}r)\,(\lambda',\eta)\,d\lambda'\,dr$$

where we have used the fact that

$$\varphi_k(2^k(\lambda,\eta))=\varphi(\lambda,\eta)=\psi(\lambda,\eta)-\psi(\tfrac{1}{2}(\lambda,\eta)).$$

If we denote by $\hat{\varphi}(r,\eta)$ the Fourier transform of $\varphi(\lambda,\eta)$ with respect to λ then $\hat{\varphi}$ is rapidly decreasing and

$$|D_\lambda^j D_\eta^\beta T\varphi_k(2^k(\lambda,\eta))| = (2\pi)^{-1}\int r^j e^{-ir\lambda}\varrho(1-2^{-k}r)D_\eta^\beta(-r,\eta)\,dr$$

and since the integral is supported in $r\geq\frac{1}{2}2^k$ the estimates (4.13) follow by integration by parts, proving the lemma.

The estimates in (4.13) are clearly invariant under Fourier transformation, in particular they show that the correction terms $T\varphi_k$ make only a trivial change to the standard proof (see for example [3]) of the boundedness of pseudodifferential operators on Hölder spaces. Thus, to prove Theorem 4.4, suppose first that $u\in B_c^\alpha(Z)$. Modifying $A\in L_b^0(Z)$ by an element of $L_b^{-\infty}(Z)$ we can assume (4.9) holds. The error committed in doing this is negligible because of (4.4). Now,

$$Au\equiv\sum_{k\geq 1}A_k u\,(\mathrm{mod}\,\mathscr{A}L^\infty(Z))$$

where each $A_k\in L_b^{-\infty}(Z)$ has symbol a_k, as in (4.10). Using (4.13) we see that

$$\begin{aligned}|A_k u(x,y)| &\leq |u(0,0)\int\beta_k(x,y,s,w)\,ds\,dw|+C\int|\beta_k(x,y,s,w)\,(|s|+|w|)^\alpha|\,ds\,dw\\ &\leq C_\alpha 2^{-k\alpha}+2^{-Nk}C\int|\beta_k'(x,y,2^k(s,w))|(|s|+|w|)^\alpha ds\,dw\\ &\leq C2^{-k\alpha}\end{aligned}$$

where we have also used the fact that

$$\beta_k(x,y,s,w)=2^{-Nk}\beta_k'(x,y,2^k(s,w))$$

with β_k' a bounded sequence in the Schwartz space $\mathscr{S}$, from (4.13) and (4.7). Similarly (3.9) also holds for $A_k u$ so by applying Lemma 3.10, $Au\in B_{\mathrm{loc}}^\alpha(Z)$, and

$$A\colon B_c^\alpha(Z)\to B_{\mathrm{loc}}^\alpha(Z). \tag{4.14}$$

Similarly if $u\in B_c^{1,\alpha}(Z)$ then for $A\in L_b^0(Z)$ the commutators $[xD_x,A],[D_{y_j},A]$ are in $L_b^0(Z)$ (see [4]) so

$$xD_x u=A(xD_x u)+[xD_x,A]\,u\in B_{\mathrm{loc}}^\alpha(Z)$$

and similarly for $D_{y_r}u$. A simple inductive argument completes the proof of Theorem 4.6.

COROLLARY 4.15. *If* $A\in L_b^{-m}(Z)$, $m\in\mathbf{N}$, *then* $A\colon B_c^{k,\alpha}(Z)\to B_c^{k+m,\alpha}(Z)$.

Proof. $(xD_x)^jD_y^\alpha A\colon B_c^{k,\alpha}(Z)\to B_{\text{loc}}^{k,\alpha}(Z)$, $\forall|\alpha|+j\leq m$.

COROLLARY 4.16. *If* $A\in L_b^{-m}(Z)$ *then* $A\colon x^rB_c^{k,\alpha}(Z)\to x^rB_{\text{loc}}^{k+m,\alpha}(Z)$ *for all* $r>0$, $m\in\mathbf{N}$.

Proof. If $u\in B_c^{k,\alpha}(Z)$ we can use (4.2) to write

$$A(x^ru)=x^rBu, \tag{4.17}$$

where $B\in L_b^{-m}(Z)$ is of the form (4.2) with amplitude b satisfying

$$\int e^{i(1-t)}b(x,y,\lambda,\eta)\,d\lambda=t^r\int e^{i(1-t)\lambda}a(x,y,\lambda,\eta)\,d\lambda,$$

clearly in $S_{\text{lac}}^{-m}(Z\times\mathbf{R}^N)$. Thus Corollary 4.15 applies.

Returning now to the proof of the isomorphism (4.1), we note first that $\Delta_g+\varkappa$ is bounded:

$$\Delta_g+\varkappa\colon\varphi^r\Lambda^{k+2,\alpha}(\Omega)\to\varphi^r\Lambda^{k,\alpha}(\Omega),\quad\forall r>0,\ k\in\mathbf{N}. \tag{4.18}$$

Indeed, as noted in the proof of Lemma 3.16 above, in any singular coordinate chart (v,w,t) as discussed in Section 3, we have

$$xD_x=\tfrac{1}{2}vD_v-\tfrac{1}{2}w^rD_{w^r}-tD_t,\quad x^{\frac{1}{2}}D_{y^r}=D_{w^r},\quad xD_{y^{2n-1}}=D_t. \tag{4.19}$$

Making these substitutions into (2.30) it is clear that Δ_g is totally characteristic and uniformly bounded in such singular coordinates, which implies (4.18) directly from the definition of the $\Lambda^{k,\alpha}$.

In view of (3.18), we have the following result proved by Cheng and Yau [1]:

PROPOSITION 4.21. $\Delta_g+\varkappa$: $\Lambda^{k+2,\alpha}(\Omega)\to\Lambda^{k,\alpha}(\Omega)$ *is an isomorphism for all* $k>0$, $0<\alpha<1$.

Now, for any $s\in\mathbf{R}$, it follows from the decomposition (2.30) for Δ_g that

$$[\Delta_g,\varphi^s]=s\varphi^sQ_s, \tag{4.22}$$

where

$$Q_s\colon\Lambda^{k+2,\alpha}(\Omega)\to\Lambda^{k,\alpha}(\Omega) \tag{4.23}$$

is a differential operator of order 1 wich is C^∞ and totally characteristic in singular coordinates, and is bounded independently of s for s small. Thus, for s sufficiently small,

$$\varphi^{-s}(\Delta_g+\varkappa)\,\varphi^s=(\Delta_g+\varkappa)+sQ_s\colon\ \Lambda^{k+2,\alpha}(\Omega)\to\Lambda^{k,\alpha}(\Omega) \tag{4.24}$$

is also an isomorphism. In particular, if $(\Delta_g+\varkappa)\,u\in\varphi^r\Lambda^{k,\alpha}(\Omega)$ for $r>0$, then $u\to 0$ on $\partial\Omega$.

Now the isomorphism (4.1) follows easily from this result, together with the following proposition:

PROPOSITION 4.25. *If* $u\in C^0(\bar\Omega)$ *vanishes on* $\partial\Omega$, *and*

$$(\Delta_g+\varkappa)\,u=f\in\varphi^s\Lambda^{k,\alpha}(\Omega) \tag{4.26}$$

for $\varkappa>0$, $0<s<\frac{1}{2}(n+\sqrt{n^2+4\varkappa})$, *then* $u\in\varphi^s\Lambda^{k+2,\alpha}(\Omega)$.

Proof. First we use the standard maximum principle to show that $u\in\varphi^sL^\infty(\Omega)$. By direct computation, in some neighborhood N of $\partial\Omega$,

$$\Delta_g(-\varphi)^s=(-\varphi)^s\left(-s^2+sn-s^2\frac{r\varphi}{1-r\varphi}\right), \tag{4.27}$$

where r is as in (2.4). Thus, if we choose A large enough,

$$(\Delta_g+\varkappa)\,(u-A(-\varphi)^s)=f+A(s^2-sn-\varkappa)\,(-\varphi)^s-As^2\frac{r(-\varphi)^{s+1}}{1-r\varphi}<0$$

in N, provided $s^2-sn-\varkappa<0$. Then $u-A(-\varphi)^s\leqslant 0$, since otherwise the difference has a positive maximum in the interior, which is a contradiction. Similarly it follows that u is bounded below by a multiple of $(-\varphi)^s$.

As is clear from Lemma 3.16, Δ_g is elliptic as an element of L^2_{b} in the singular coordinates (v, w, t), uniformly in p. Thus it has a parametrix $B\in L^{-2}_{\mathrm{b}}$, with

$$B(\Delta_g+\varkappa)-\mathrm{Id}=R\in L^{-\infty}_{\mathrm{b}}. \tag{4.28}$$

So we can write u locally as

$$u=Bf-Ru.$$

Now Corollary 4.16 shows that $Bf\in\varphi^s\Lambda^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$, and Lemma 4.4 gives $Ru\in\varphi^s\Lambda^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$. By the uniformity of Δ_g with respect to p, these norms are globally bounded. This proves Proposition 4.25.

Remark 4.29. The isomorphism (4.1) can also be proved by applying the Riesz representation theorem to the continuous linear functional

$$v \mapsto \int f(x, y)\, \bar{v}(x, y)\, dg \tag{4.30}$$

on the Hilbert space H obtained by closing $C_c^\infty(\Omega)$ with respect to the first Sobolev norm of the metric g, i.e.

$$\|u\|_H^2 = \int |du|_g^2 + \varkappa |u|^2 dg.$$

The fact that the solution so obtained is actually in $\varphi^r \Lambda^{k+2,\alpha}(\Omega)$ can be proved by methods similar to those used in this section.

5. Commutation

In this section we will improve the regularity result of Section 4 by commuting operators in $\mathrm{Diff}_b(\bar{\Omega})$ through Δ_g. The non-degenerate Cauchy-Riemann structure on the boundary $\partial\Omega$ induces a filtration on $\mathrm{Diff}_b(\bar{\Omega})$, the properties of which are closely related to the nilpotency of the Heisenberg group, which is in turn related to the decomposition (2.30) of the Laplacian.

It was noted in Section 2 above that a choice of defining function φ fixes a contact form

$$\theta = \iota^*(i\partial\varphi), \ \iota : \partial\Omega \hookrightarrow \Omega.$$

The maximal complex subspace $H = \theta^\perp \subset T(\partial\Omega)$ carries a formally integrable complex structure J; if $H^{1,0} \subset H \otimes \mathbf{C}$ is the i-eigenspace of J the integrability conditon can be written

$$V, W \in C^\infty(H^{1,0}) \Rightarrow [V, W] \in C^\infty(H^{1,0}). \tag{5.1}$$

The choice of defining function also fixes a vector field $T \in C^\infty(T\partial\Omega)$, defined in (2.15). T can be characterized intrinsically in $M = \partial\Omega$ by

$$\theta(T) = 1, \quad T \lrcorner d\theta = 0. \tag{5.2}$$

If (x, y) are any normal coordinates near $\partial\Omega$, we can use them to extend vector fields unambiguously from the boundary to $\bar{\Omega}$; if $Z_1, \dots, Z_{n-1}$ is a frame for $H^{1,0}$ this gives a local basis $xD_x, T, Z_1, \dots, Z_{n-1}, \bar{Z}_1, \dots, \bar{Z}_{n-1}$ for vector fields on $\bar{\Omega}$ tangent to $\partial\Omega$.

Any totally characteristic differential operator $P \in \mathrm{Diff}_b^m(\bar{\Omega})$ can be written in the form

$$P = \sum_{k+j+|\beta| \leq m} p_{k,j,\beta}(x, y)\,(xD_x)^k Z^{\beta'} \bar{Z}^{\beta''} T^j \tag{5.3}$$

where $\beta = (\beta', \beta'')$ ranges over $(2n-2)$-multiindices. We now define a double filtration of Diff_b^m which reflects the number of factors, T, Z_j, and $\bar{Z}_k$ appearing in (5.3). If (m, w, s) is a triple of nonnegative integers we define $\mathrm{Diff}_b^{m,w,s}(\bar{\Omega}) \subset \mathrm{Diff}_b^m(\bar{\Omega})$ as the space of operators P which, in a covering of $\partial\Omega$ by normal coordinate systems, have the form (5.3) with each term satisfying

$$|\beta| + 2j - 2r \leq w, \quad |\beta| - 2r \leq s, \tag{5.4}$$

where r is the greatest integer such that $x^{-r} p_{k,j,\beta}$ is C^∞ up to $x=0$.

This definition depends on the choice of defining function φ, but once a defining function is given, it is independent of other choices; in particular, it is obviously independent of the choice of normal coordinates or frame Z. As a consequence of the following lemma, the definition is also independent of the order of factors appearing in (5.3), since rearranging the vector fields can only introduce additional terms of lower weight.

LEMMA 5.5. $[Z_j, Z_k]$, $[\bar{Z}_j, \bar{Z}_k]$, $[T, Z_j]$, $[T, \bar{Z}_k] \in \mathrm{Diff}_b^{1,1,1}(\bar{\Omega})$; $[Z_j, \bar{Z}_k] \in \mathrm{Diff}_b^{1,2,1}(\bar{\Omega})$.

Proof. Observe that Z_j and $\bar{Z}_k$ have weights $(1, 1)$, while T has weights $(2, 0)$. In view of the integrability condition (5.1), $[Z_j, Z_k]$ has weights $(1, 1)$, as does $[\bar{Z}_j, \bar{Z}_k]$. On the other hand, $[Z_j, \bar{Z}_k]$ may involve a term T, with weights $(2, 0)$, and terms in Z_j and $\bar{Z}_k$, with weights $(1, 1)$. Observe from (5.2) that

$$0 = d\theta(T, Z_j) = T\theta(Z_j) - Z_j\theta(T) - \theta([T, Z_j]) = -\theta([T, Z_j])$$

so $[T, Z_j]$ has weights $(1, 1)$, again satisfying the statement; the same applies to $[T, \bar{Z}_k]$.

We also remark in passing that the definition does not really depend on the fact that the vector fields $Z_j, \bar{Z}_k, T$ are extended normally from the boundary, since any other extension differs from this one by a vector field vanishing on $\partial\Omega$, which does not affect the inequalites (5.4).

The following proposition allows us to prove our regularity result by induction on the weights of operators in $\mathrm{Diff}_b(\bar{\Omega})$.

PROPOSITION 5.6. *If* $P_1 \in \mathrm{Diff}_b^{m_1, w_1, s_1}$, $P_2 \in \mathrm{Diff}_b^{m_2, w_2, s_2}$, *then*

$$[P_1, P_2] \in \mathrm{Diff}_b^{m_1+m_2-1, w_1+w_2, s_1+s_2-1} + \mathrm{Diff}_b^{m_1+m_2-1, w_1+w_2-2, s_1+s_2}.$$

Proof. From Lemma 5.5, the result is true for the vector fields $T, Z_j, \bar{Z}_k$. Now it suffices to consider the case $r=0$ in (5.4), since a factor of x^r effectively commutes through the expansion (5.3) without changing the weights. For any pair of operators P_1, P_2, writing them in the form (5.3) reduces the computation to the case of monomials

$$P_1 = Z^{\beta'} \bar{Z}^{\beta''} T^k, \quad P_2 = Z^{\alpha'} \bar{Z}^{\alpha''} T^j.$$

The commutator can be written as a sum of terms like $Z^{\beta'} \bar{Z}^{\beta''}[T^k, Z^{\alpha'}] \bar{Z}^{\alpha''} T^j$, each involving a commutator of two vector fields. By Lemma 5.5, each term has weight at most (w_1+w_2, s_1+s_2-1) or (w_1+w_2-2, s_1+s_2).

Now we can use these spaces to define subspaces of the Hölder spaces $\Lambda^{k,\alpha}(\Omega)$. Set

$$\Lambda^{k,\alpha;s}(\Omega) = \{u \in \Lambda^{k,\alpha}(\Omega) : Pu \in \Lambda^{0,\alpha} \text{ whenever } P \in \mathrm{Diff}_b^{k,s,r}\}. \tag{5.7}$$

This is a Banach space, and we have the obvious mapping properties:

$$P : x^t \Lambda^{k,\alpha;s} \to x^t \Lambda^{k-m,\alpha,s-w} \quad \text{if } P \in \mathrm{Diff}_b^{m,w,r} \text{ with } m \leq k,\ w \leq s. \tag{5.8}$$

With these preliminaries we can now improve the regularity result, Proposition 4.25, for the Laplacian Δ_g. Using the defining function φ to construct the spaces $\mathrm{Diff}_b^{m,w,s}(\bar{\Omega})$, we have from (2.27)

$$\Delta_g \in \mathrm{Diff}_b^{2,0,0}(\bar{\Omega}). \tag{5.9}$$

PROPOSITION 5.10. *Suppose* $u \in C^0(\bar{\Omega})$ *vanishes on* $\partial\Omega$, *and*

$$(\Delta_g + \varkappa)\, u = f \in \varphi^r \Lambda^{m,\alpha;s}(\Omega),$$

where $\varkappa > 0$, $1 < r < \frac{1}{2}(n + \sqrt{n^2 + 4\varkappa})$ *and* $s \leq 2m$. *Then* $u \in \varphi^r \Lambda^{m+2,\alpha;s}(\Omega)$.

Proof. For $s=0$, the conclusion reduces to Proposition 4.25. So we proceed by induction over s and for given m we suppose that $\varphi^r \Lambda^{m+2,\alpha;s-1}(\Omega)$. We want to show that $Qu \in \varphi^r \Lambda^{0,\alpha}(\Omega)$ for all $Q \in \mathrm{Diff}_b^{m+2,s,p}$; it is sufficient to show that

$$Qu \in \varphi^r \Lambda^{2,\alpha} \quad \text{for all } Q \in \mathrm{Diff}_b^{m,s,p} \tag{5.11}$$

since $Q \in \mathrm{Diff}_b^{m+2,s,p}$ can be written $\Sigma P_j Q_j$ with $P_j \in \mathrm{Diff}_b^{2,0,0}$ and $Q_j \in \mathrm{Diff}_b^{m,s,p}$, using the fact that $s \leq 2m$. We will show (5.11) by induction on p. Naturally, the method is to commute through Δ_g operators in $\mathrm{Diff}_b^{m,s,p}$. We start with the case $p=0$, and take $Q \in \mathrm{Diff}_b^{m,s,0}$. Then, by (5.9) and Proposition 5.6,

$$[Q, \Delta_g] \in \mathrm{Diff}_b^{m+1,s-1,0}(\bar{\Omega}),$$

so by the inductive hypothesis on s,

$$(\Delta_g + \varkappa)(Qu) = Qf + [\Delta_g, Q]\, u \in \varphi^r \Lambda^{0,\alpha}. \tag{5.12}$$

To apply Propostion 4.25, we need only show that Qu vanishes on $\partial\Omega$. Observe that Q can be written $Q = \Sigma\, V_j Q_j$ where V_j are vector fields tangent to $\partial\Omega$ and $Q_j \in \mathrm{Diff}_b^{m-1,s-1,0}$. By the inductive hypothesis, $Q_j u \in \varphi^r \Lambda^{2,\alpha}$. Now in any singular coordinates (v, w, t) it is easy to see that V_j maps $\varphi^r \Lambda^{1,\alpha}(\Omega)$, into $\varphi^{r-1} \Lambda^{0,\alpha}(\Omega)$. So Proposition 4.23 gives (5.11) for $p=0$.

Now suppose $p>0$, and $Q \in \mathrm{Diff}_b^{m,s,p}(\bar{\Omega})$. Again applying Proposition 5.6,

$$[Q, \Delta_g] \in \mathrm{Diff}_b^{m+1,s-1,p}(\bar{\Omega}) + \mathrm{Diff}_b^{m+1,s,p-1}(\bar{\Omega}).$$

We therefore deduce (5.12), by using the inductive hypothesis on s and that on p. As before, $Qu \in \varphi^{r-1} \Lambda^{0,\alpha}(\Omega)$, and so we have (5.11) for all p; this proves the proposition.

6. Complex Monge-Ampère operator

The results of the previous sections concerning the Laplace-Beltrami operator can be applied to the complex Monge-Ampère operator

$$M(u) = Y(u)\, e^{-(n+1)u}\, u \tag{6.1}$$

where

$$Y(u) = \det(g_{j\bar{k}} + u_{j\bar{k}}) \det(g_{j\bar{k}})^{-1} \tag{6.2}$$

to give corresponding regularity results for the solution to (1.4). The linearization of M about the funciton $u=0$ is just

$$M'(0) = -(\Delta_g + n + 1). \tag{6.3}$$

In [1] Cheng and Yau showed that there exists a unique solution $u \in \bigcap_k \Lambda^{k,\alpha}(\Omega)$ to

(1.4), (1.6), and that if F in (1.4) vanishes on $\partial\Omega$, then so does u. In this section we improve their result to show that if F vanishes to order r on $\partial\Omega$, $0<r<n+1$, then in fact

$$u \in \bigcap_{k,s} \varphi^r \Lambda^{k,\alpha;s}(\Omega).$$

We shall say that a nonlinear differential operator Y on $\bar{\Omega}$ is totally characteristic if it can be written in the form

$$Y(u) = F(x, y, u, P_1 u, \ldots, P_p u) \tag{6.4}$$

where the P_j are totally characteristic linear differential operators with C^∞ coefficients and F is a C^∞ function. Given a strictly plurisubharmonic defining function φ for Ω, the weights of the P_j are defined as in Section 5; we say Y has weights (w, s) if there is a representation (6.4) for Y in which all the $P_j \in \mathrm{Diff}_b^{m,w,s}(\bar{\Omega})$, provided the coefficients of the P_j are C^∞ when expressed in local coordinates $(v, y)=(x^{\frac{1}{2}}, y)$. The introduction of v is simply to make the vector field $x^{\frac{1}{2}} Z_j$, which is of weight zero, have C^∞ coefficients.

THEOREM 6.5. *If φ is any defining function for Ω such that $g=-\log(-\varphi)$ is strictly plurisubharmonic, then the operator $Y(u)$, given by* (6.2), *is totally characteristic with order* 2 *and weights* $(0,0)$.

Proof. The operator Y can be written in the form:

$$Y(u) = 1-\Delta_g u+G_2(x, y; u_{j\bar{k}})+\ldots+G_n(x, y; u_{j\bar{k}}) \tag{6.6}$$

where the G_k are C^∞ functions of x, y and homogeneous polynomials in the derivatives $u_{j\bar{k}}$ with respect to a given choice of complex coordinates in $\mathbf{C}^n$. The weights of the G_k depend on the number of factors Z_j, $\bar{Z}_k$, T, when $u_{j\bar{k}}$ is written in the form (5.3), and the order of vanishing of the coefficients at $x=0$. Thus in computing the weights of Y it suffices to freeze the coefficients G_k at some boundary point $y=\bar{y}$ and consider only the x-dependence.

If we choose complex coordinates $z^1, \ldots, z^n$ centered at $(0, \bar{y}) \in \partial\Omega$ such that

$$dx = 2\mathrm{Re}\,(dz^1), \quad \varphi_{j\bar{k}} = \delta_{j\bar{k}} \text{ at } (0, \bar{y}) \tag{6.7}$$

then (2.2) shows that, as a function of x, with $y=\bar{y}$ fixed,

$$g_{1\bar{1}} = x^{-2}+O(x^{-1}), \quad g_{j\bar{k}} = \delta_{j\bar{k}} x^{-1}+O(1) \quad \text{if } j\bar{k} \neq 1\bar{1}. \tag{6.8}$$

Now, recalling from (1.5) that

$$\det(g_{j\bar{k}})^{-1} = e^{F}x^{n+1} \tag{6.9}$$

we proceed to consider the various terms in (6.6). Of course the first two terms are taken care of by (5.9).

The expansion law for determinants, applied to the second factor in (6.2), shows that the coefficient of

$$u_{j_1\bar{k}_1}\cdot u_{j_2\bar{k}_2}\cdot u_{j_3\bar{k}_3}\cdot\ldots\cdot u_{j_m\bar{k}_m} \tag{6.10}$$

in G_m is a product of the form

$$C(x,y)\,x^{n+1}\cdot g_{j_{m+1}\bar{k}_{m+1}}\cdot\ldots\cdot g_{j_n\bar{k}_n} \tag{6.11}$$

where the indices $(j_1,\ldots,j_m)$, $(j_{m+1},\ldots,j_n)$ and $(\bar{k}_1,\ldots,\bar{k}_m)$, $(\bar{k}_{m+1},\ldots,\bar{k}_n)$ are partitions of $(1,\ldots,n)$. Now observe from (6.8) that the product (6.11) is of order x^{m+1} unless $1\notin(j_1,\ldots,j_m)$ and $\bar{1}\notin(\bar{k}_1,\ldots,\bar{k}_m)$, in which case it is of order x^m.

On the other hand, the definition (2.15) for T and W shows that, at the base point,

$$\partial_1 = -\tfrac{1}{2}(W+iT) = iD_x - \tfrac{1}{2}iT, \quad \partial_{\bar{1}} = -\tfrac{1}{2}(W-iT) = iD_x + \tfrac{1}{2}iT, \tag{6.12}$$

and since $\partial_j\in H$ at $(0,\bar{y})$ for $j>1$, we can take $Z_j=\partial_j$, $j=2,\ldots,n$ at $(0,\bar{y})$. Since any vector field which vanishes at $x=0$ has weights $(0,0)$, this shows that $x^2\partial_1\partial_{\bar{1}}$, $x^{\frac{3}{2}}\partial_1\partial_{\bar{k}}$, $x^{\frac{3}{2}}\partial_j\partial_{\bar{1}}$ and $x\partial_j\partial_{\bar{k}}$, where $j,\bar{k}\neq 1,\bar{1}$, are all totally characteristic with weights $(0,0)$. Combining this with the observation above on the vanishing order of (6.11), the theorem is proved.

In order to prove an analogue of Proposition 4.25 for the Monge-Ampère operator M, we will make use of the folowing lemma.

LEMMA 6.13. *Suppose $u\in\Lambda^{k,\alpha}(\Omega)$ vanishes on $\partial\Omega$. Then every singular coordinate chart V_p, all totally characteristic derivatives of u of order $\leq k$ vanish on $\partial\Omega$, uniformly in p.*

Proof. Recall that $u\in\Lambda^{k,\alpha}(\Omega)$ means that $u\in C^{k,\alpha}$ in logarithmic coordinates $(s,w,t)=(\log v,w,t)$. Now suppose $u\in\Lambda^{1,\alpha}(\Omega)$ and $u\to 0$ on $\partial\Omega$. If in logarithmic singular coordinates some derivative Du does not approach zero as $s\to-\infty$, the Hölder condition for Du implies the existence of $\delta,\varepsilon>0$, and a sequence of points p_n with $s(p_n)\to-\infty$ such that, say, $Du>\varepsilon$ in $B_\delta(p_n)$. Integrating, we obtain a contradiction to the fact that $u\to 0$ as $s\to-\infty$. Induction on k completes the proof.

PROPOSITION 6.14. *Suppose $F\in C^\infty(\bar\Omega)$ vanishes to order r on $\partial\Omega$, $0<r<n+1$, and $u\in\Lambda^{0,\alpha}(\Omega)$ is the unique solution to*

$$M(u)=Y(u)\,e^{-(n+1)u}=e^F. \tag{6.15}$$

Then $u\in\bigcap_k\varphi^r\Lambda^{k,\alpha}(\Omega)$.

Proof. Choose $\varrho\in C^\infty(\mathbf{R})$ with $\varrho(x)=1$ for $x\leq\frac12$, $\varrho(x)=0$ for $x\geq1$, and set $\varrho_R(x)=\varrho(x/R)$; then with $x=-\varphi$, ϱ_R is globally defined on $\bar\Omega$ with support in $\{-\varphi\leq R\}$. If we set $u^R=(1-\varrho_R)\,u$, $v^R=\varrho_R\,u$, and $g^R=g+u^R$,we can write (6.15) as

$$\det(g^R_{j\bar k}+v^R_{j\bar k})\det(g_{j\bar k})^{-1}e^{-(n+1)(u^R+v^R)}=e^F. \tag{6.16}$$

Linearizing this about $v=0$, we obtain:

$$-\det(g^R_{j\bar k})\det(g_{j\bar k})^{-1}e^{-(n+1)u^R}(\Delta_{g^R}+n+1)=-(\Delta_{g^R}+n+1)+Q^R \tag{6.17}$$

where $Q^R=0$ in $\{-\varphi\leq R\}$. Therefore, we can write (6.16) as

$$-(\Delta_{g^R}+n+1)\,v^R+G_2^R(v^R)+\ldots+G_n^R(v^R)=e^F-M(u^R)-Q^R(v^R) \tag{6.18}$$

where the right-hand side is in $\varphi^r\Lambda^{k,\alpha}(\Omega)$ for all $k\geq0$, and the G_j^R are nonlinear differential operators which are totally characteristic with weights $(0,0)$ and are at least quadratic in v^R. We can factor each $G_j^R(v^R)$ as $C_j^R(v^R)\,P_j^R(v^R)$, where the P_j^R are linear and the C_j^R are (possibly) nonlinear. Freezing the coefficients $C_j^R(v^R)$, we see that v^R satisfies the linear differential equation

$$(-\Delta_{g^R}-(n+1)+C_2^R(v^R)\,P_2^R+\ldots+C_n^R(v^R)\,P_n^R)\,v^R\in\varphi^r\Lambda^{k,\alpha}(\Omega). \tag{6.19}$$

Now observe that Lemma 6.13 implies that $v^R=\varrho_R u$ approaches zero in all $\Lambda^{k,\alpha}(\Omega)$ as $R\to0$, and that similarly $u^R=(1-\varrho_R)\,u\to u$; in particular u^R is uniformly bounded in all $\Lambda^{k,\alpha}(\Omega)$. This means first of all that the symbol of Δ_{g^R} and its inverse are uniformly bounded in all $\Lambda^{k,\alpha}(\Omega)$. The proofs of Theorem 4.6 and Corollaries 4.15 and 4.16 show that the norm of the parametrix B of (Δ_g+n+1) on the spaces $\varphi^r\Lambda^{k,\alpha}(\Omega)$ depends only on the estimates

$$|(xD_x)^kD_y^\alpha D^\beta_{(\lambda,\eta)}\,b(x,y,\lambda,\eta)|\leq C(1+|\lambda,\eta|)^{-2-|\beta|}$$

on its symbol; a similar remark applies to the residual term Ru. Combining these observations with (4.1), we see that

$$(\Delta_{g^R}+n+1)\colon \varphi^r\Lambda^{k+2,\alpha}(\Omega)\to\varphi^r\Lambda^{k,\alpha}(\Omega) \tag{6.20}$$

is an isomorphism with uniformly bounded inverse as $R\to 0$. Moreover, the coefficients of C_j^R and P_j^R in (6.19) are also uniformly bounded in R; thus since $v^R\to 0$ the operator norm of the perturbation terms in (6.19) can be made arbitrarily small by choosing R small. For some $R>0$, therefore, the operator (6.19) is invertible on $\varphi^r\Lambda^{k+2,\alpha}(\Omega)$. This proves the proposition.

THEOREM 6.21. *If $F\in C^\infty(\bar\Omega)$ vanishes to order r on $\partial\Omega$, $1<r<n+1$, and $u\in\Lambda^{0,\alpha}(\Omega)$ is the unique solution to* (6.15), *then $u\in\bigcap_{k,s}\varphi^r\Lambda^{k,\alpha;s}(\Omega)$.*

Proof. If we write

$$R(u)=M(u)-1+(\Delta_g+n+1)\,u \tag{6.22}$$

then from (6.6) it is clear that R is a nonlinear totally characteristic operator of weights $(0,0)$ with each term at least quadratic in u. Thus, for every k,r,α,s

$$R\colon \varphi^r\Lambda^{k+2,\alpha;s}(\Omega)\to\varphi^{2r}\Lambda^{k,\alpha;s}(\Omega). \tag{6.23}$$

For $r\geq 1$, $\varphi^r\Lambda^{k,\alpha;s}(\Omega)\subset\Lambda^{k,\alpha;s+1}(\Omega)$, so from (6.23),

$$R\colon \varphi^r\Lambda^{k+2,\alpha;s}(\Omega)\to\varphi^{r}\Lambda^{k,\alpha;s+1}(\Omega) \tag{6.24}$$

provided $r\geq 1$.

Now for any k, Proposition 6.14 shows that $u\in\varphi^r\Lambda^{k,\alpha}(\Omega)$. From (6.22) and the hypothesis,

$$-(\Delta_g+n+1)\,u=e^F-1-R(u)\in\varphi^r\Lambda^{k,\alpha;1}(\Omega). \tag{6.25}$$

Since Cheng and Yau showed that u vanishes on $\partial\Omega$, we can apply Proposition 5.10 to conclude that $u\in\varphi^r\Lambda^{k+2,\alpha;1}(\Omega)$. Then (6.24) allows us to complete the proof by induction on s.

7. Conormal distributions

If $Z=\bar{\mathbf{R}}_+\times\mathbf{R}^{N-1}$ is the standard half-space, we define the space of conormal, or almost regular, distributions $\mathscr{A}(Z)\subset\mathscr{D}'(Z)$ as in Section 3 by

$$u\in\mathscr{A}(Z)\text{ iff }\exists s\in\mathbf{R}\text{ such that }Pu\in H^s_{\mathrm{loc}}(Z),\text{ all }P\in\mathrm{Diff}_{\mathrm b}(Z), \tag{7.1}$$

where $H^s(Z)$ is the usual Sobolev space on Z. It is shown in [4] that $\mathscr{A}(Z)$ is just the space of extendible Lagrangian distributions associated to the conormal bundle $N^*\partial Z$.

In order to obtain an asymptotic expansion for the solution to (1.4) near the boundary of Ω, we shall construct a filtration of $\mathscr{A}(Z)$ that behaves well under multiplication. We define

$$\mathscr{A}L^\infty=\{u\in\mathscr{A}(Z): Pu\in L^\infty(Z)\quad\text{for all } P\in \mathrm{Diff}_b(Z)\}. \tag{7.2}$$

Then, for any $s\in\mathbf{R}$, define

$$\mathscr{A}^{(s)}(Z)=\bigcap_{r<s} x^r\mathscr{A}L^\infty(Z). \tag{7.3}$$

As usual, we set $\mathscr{A}^{(s)}_{\mathrm{loc}}(Z)=\{u\colon \varrho u\in\mathscr{A}^{(s)}(Z)$ whenever $\varrho\in C^\infty_c(Z)\}$.

We note first that all conormal distributions fall into some $\mathscr{A}^{(s)}_{\mathrm{loc}}$:

LEMMA 7.4. $\mathscr{A}(Z)=\bigcup_{s\in\mathbf{R}}\mathscr{A}^{(s)}_{\mathrm{loc}}(Z)$.

Proof. If $u\in\mathscr{A}(Z)$, the definition (7.1) shows that

$$D^\alpha_{(x,y)}(x^pu)\in H^s_{\mathrm{loc}}(Z)\quad\text{for all } |\alpha|\leq p.$$

This implies $x^pu\in H^{s+p}_{\mathrm{loc}}(Z)$. The Sobolev embedding theorem then shows that x^pu is locally bounded if we choose p large enough. A similar argument shows that all totally characteristic derivatives of x^pu are likewise locally bounded.

From definition (7.4), we have the characterization

$$\mathscr{A}^{(s)}(Z)=x^s\mathscr{A}^{(0)}(Z) \tag{7.5}$$

and the mapping properties

$$P\colon \mathscr{A}^{(s)}(Z)\to\mathscr{A}^{(s)}(Z),\quad\text{for any } P\in\mathrm{Diff}_b(Z) \tag{7.6}$$

$$X\colon \mathscr{A}^{(s)}(Z)\to\mathscr{A}^{(s-1)}(Z),\quad\text{for any } X\in C^\infty(TZ). \tag{7.6}$$

We also have the inclusions

$$\mathscr{A}^{(s)}(Z)\subset C^r(Z)\quad\text{for all } 0\leq r<s,$$

which follows from (7.7) and the fact that $\mathscr{A}^{(\varepsilon)}(Z)\subset C^0(Z)$ for $\varepsilon>0$.

Since elements of $\mathscr{A}(Z)$ are actually C^∞ functions on $\mathring{Z}$, they can be multiplied. The filtration $\{\mathscr{A}^{(s)}\}$ turns $\mathscr{A}(Z)$ into a filtered algebra, by virtue of the following lemma:

LEMMA 7.8. *If* $u \in \mathscr{A}^{(s)}(Z)$, $v \in \mathscr{A}^{(t)}(Z)$, *then* $uv \in \mathscr{A}^{(s+t)}(Z)$.

Proof. First assume $u, v \in \mathscr{A}L^{\infty}(Z)$. Then

$$(xD_x)^k D_y^{\alpha}(uv) = \sum_{\substack{p+q=k \\ \beta+\gamma=\alpha}} C_{p,\beta}((xD_x)^p D_y^{\beta} u)((xD_x)^q D_y^{\gamma} v).$$

Each term in this sum is bounded by hypothesis. Now in general, write $u = x^{s-\varepsilon}\eta$, $v = x^{t-\varepsilon}\xi$, with $\eta, \xi \in \mathscr{A}L^{\infty}$. Then $uv = x^{s+t-2\varepsilon}\eta\xi$,, which shows $uv \in \mathscr{A}^{(s+t)}(Z)$,

The most important examples of functions in $\mathscr{A}^{(s)}(Z)$ are $x^s\alpha$ and $x^s\alpha(\log x)^p$, where $\alpha \in C^{\infty}(Z)$. We shall say $u \in \mathscr{A}(Z)$ is *graded* if it admits an asymptotic expansion in functions of this form; more particularly, $u \in \mathscr{A}(Z)$ is graded if there exist real numbers $s_j \to \infty$ as $j \to \infty$, integers M_j, $j \geq 1$, and functions $\psi_{j,p} \in C^{\infty}(Z)$, such that

$$u - \sum_{j \leq N} \sum_{p=1}^{M_j} \psi_{j,p} x^{s_j} (\log x)^p \in \mathscr{A}^{(s_N)}(Z). \tag{7.9}$$

We shall be interested primarily in certain subspaces of the space of graded distributions. We define $\mathscr{A}_{p,q}(Z)$ as the space of graded conormal distributions having an expansion (7.9) in which the s_j are consecutive integers, and the M_j are dominated as follows

$$M_j \leq \min(1, s_j - p + 1) + \frac{s_j - p}{q}.$$

In other words, the first occurrence of $\log x$ is with x^p, and the power of $\log x$ increases by 1 only after q steps. We also set $\mathscr{A}^{(s)}_{p,q}(Z) = \mathscr{A}^{(s)}(Z) \cap \mathscr{A}_{p,q}(Z)$. It is easy to see from (7.6) and (7.8) that

$$P: \mathscr{A}^{(s)}_{p,q}(Z) \to \mathscr{A}^{(s)}_{p,q}(Z), \quad \text{for any } P \in \mathrm{Diff}_b(Z) \tag{7.10}$$

$$\mathscr{A}^{(0)}_{p,p}(Z) \quad \text{is closed under multiplication.} \tag{7.11}$$

On a strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$, we define $\mathscr{A}(\bar{\Omega})$ as the space of functions on Ω which restrict $\mathscr{A}_{\mathrm{loc}}$ in any local coordinates, and similarly for $\mathscr{A}^{(s)}(\bar{\Omega})$, $\mathscr{A}_{p,q}(\bar{\Omega})$. It is obvious from the definition of the spaces $\Lambda^{k,\alpha;s}(\Omega)$ that

$$\bigcap_{k,s} \varphi^r \Lambda^{k,\alpha;s}(\Omega) \subset \mathscr{A}^{(r)}(\bar{\Omega}). \tag{7.12}$$

Using the expressions (6.1), (6.6) for the complex Monge-Ampère operator M, it follows from (7.10) and (7.11) that

$$M\colon \mathscr{A}^{(0)}_{p,p}(\bar{\Omega}) \to \mathscr{A}^{(0)}_{p,p}(\bar{\Omega}) \quad \text{for all } p \geq 1. \tag{7.13}$$

8. Asymptotic expansions

In this section we show that the solution u to the complex Monge-Ampère equation is graded. We will construct the expansion (7.9) explicitly. The method relies on the fact that, due to the decomposition (2.30) for the Laplace-Beltrami operator, on $\mathscr{A}^{(s)}/\mathscr{A}^{(s+1)}$ the operator (Δ_g+n+1) reduces to

$$[(\Delta_g+n+1)u]_s = E[u]_s \tag{8.1}$$

where E is the totally characteristic ordinary differential operator

$$E = I(xD_x)+n+1 = (xD_x)^2+inxD_x+n+1. \tag{8.2}$$

As mentioned in the introduction, the symbol isomorphism (1.15) for conormal distributions defines E invariantly as an operator on $\mathscr{A}^{(s)}(N_+\,\partial\Omega)/\mathscr{A}^{(s+1)}(N_+\,\partial\Omega)$. Similarly, from (6.1) and (6.4) the Monge-Ampère operator M reduces to

$$[M(u)-1]_s = E[u]_s, \text{ provided } s \geq 1. \tag{8.3}$$

Observe that the kernel of E on $C^\infty(\mathbf{R}_+)$ is spanned by the functions x^{-1} and $x^{(n+1)}$. Since we will always be working in $\mathscr{A}^{(s)}$ for $s>0$, only the latter will appear. Thus if $\beta \in C^\infty(\bar{\Omega})$, a solution $\eta \in \mathscr{A}^{(s)}(\bar{\Omega})$ of

$$[E(\eta)]_s = [x^s\beta]_s$$

is given in normal coordinates by

$$\eta = \frac{-x^s\beta}{(s-n-1)(s+1)}, \quad \text{if } s \neq n+1, \tag{8.4}$$

while for $s=n+1$ we must take

$$\eta = \frac{-x^{n+1}\beta}{n+2}\log x + \gamma(y)\,x^{n+1} \tag{8.5}$$

where $\gamma \in C^\infty(\partial\Omega)$ is arbitrary. In general, if β is a finite sum of terms of the form

$\alpha(y)x^s(\log x)^p$, then $E(\eta)=\beta$ has a solution of the same form. Even more generally, we have,

LEMMA 8.6. *If* $\beta\in\mathscr{A}^{(s)}(Z)$, $s\geq 0$, *then*

$$E(\eta)=\beta \tag{8.7}$$

has a solution $\eta\in\mathscr{A}^{(s)}(Z)$.

Proof. We set

$$\eta(x,y)=\begin{cases}\dfrac{1}{n+2}\left(x^{-1}\displaystyle\int_0^x\beta(w,y)\,dw+x^{n+1}\int_x^1 w^{-n-2}\beta(w,y)\,dw\right) & \text{if } s\leq n+1\\[2ex] \dfrac{1}{n+2}\left(x^{-1}\displaystyle\int_0^x\beta(w,y)\,dw-x^{n+1}\int_0^x w^{-n-2}\beta(w,y)\,dw\right) & \text{if } s>n+1\end{cases} \tag{8.8}$$

Then η solves (8.7), as can be verified by direct calculation. Moreover, for any $t<s$, we can write $\beta(x,y)=x^t\alpha(x,y)$ for some $\alpha\in\mathscr{A}L^\infty(Z)$. Making this substitution in (8.8), it follows easily that for any q with $t<q<s$, $x^{-q}\eta$ is bounded with all its totally characteristic derivatives, and thus $\eta\in\mathscr{A}^{(s)}(Z)$.

COROLLARY 8.9. *Suppose* $\eta_1,\eta_2\in\mathscr{A}^{(s)}(\bar\Omega)$, $s\geq 0$, *and*

$$[E(\eta_1)]_s=[E(\eta_2)]_s. \tag{8.10}$$

Then

$$[\eta_1]_s=[\eta_2]_s \quad \textit{if } s\leq n \textit{ or } s>n+1; \tag{8.11}$$

$$[\eta_1]_s=[\eta_2+\gamma(-\varphi)^{n+1}]_s \quad \textit{for some function } \gamma\in C^\infty(\bar\Omega) \textit{ if } n<s\leq n+1. \tag{8.12}$$

Proof. Since $E(\eta_1-\eta_2)\in\mathscr{A}^{(s+1)}(\bar\Omega)$, Lemma (8.6) shows that there exists $\xi\in\mathscr{A}^{(s+1)}(\bar\Omega)$ with $E(\xi)=E(\eta_1-\eta_2)$. Then $\eta_1-\eta_2-\xi$ is in the kernel of E, which implies that there exists some $\gamma\in C^\infty(\bar\Omega)$ such that

$$\eta_1-\eta_2-\xi=\gamma(-\varphi)^{n+1}.$$

This immediately implies (8.11), (8.12).

In [2], Fefferman showed how to obtain a smooth defining function φ_0 for Ω such that $J(-\varphi_0)-1$ vanishes to order $n+1$ on $\partial\Omega$. For completeness, we give an alternate construction of φ_0.

THEOREM 8.13 (Fefferman, [2]). *There exists a smooth defining function φ_0 for Ω such that $g=-\log(-\varphi_0)$ is strictly plurisubharmonic, and*

$$\det(g_{j\bar{k}})\, e^{-(n+1)g} = 1+O(\varphi_0^{n+1}). \tag{8.14}$$

Proof. Let φ be any defining function such that $g=-\log(-\varphi)$ is strictly plurisubharmonic. First we find a function $\eta \in C^\infty(\bar{\Omega})$ such that

$$\det(g_{j\bar{k}}+\eta_{j\bar{k}})\, e^{-(n+1)(g+\eta)} \to 1 \text{ on } \partial\Omega. \tag{8.15}$$

Since $g_{j\bar{k}} \to \infty$ on $\partial\Omega$, for any such η,

$$\det(g_{j\bar{k}}+\eta_{j\bar{k}})\det(g_{j\bar{k}})^{-1} \to 1 \text{ on } \partial\Omega,$$

and so the choice $\eta=-F/(n+1)$, where F is as in (1.5), gives (8.15). Now $\varphi' = -e^{-(g+\eta)}$ is a smooth defining function for Ω; modifying φ' away from a neighborhood of $\partial\Omega$ we may assume that $-\log(-\varphi')$ is strictly plurisubharmonic.

Now suppose that by induction we have a smooth defining funciton φ, such that $g=-\log(-\varphi)$ is strictly plurisubharmonic and satisfies

$$\det(g_{j\bar{k}})\, e^{-(n+1)g} = 1+(-\varphi)^s\beta \tag{8.16}$$

with $\beta \in C^\infty(\bar{\Omega})$ and $1 \leq s \leq n$. If we define η as in (8.4), then η is globally defined and C^∞ on $\bar{\Omega}$, and (8.3) shows that

$$\det(g_{j\bar{k}}+\eta_{j\bar{k}})^{-1}\, e^{-(n+1)(g+\eta)} = 1+O(\varphi^{s+1}). \tag{8.17}$$

Again modifying $\varphi' = -e^{-(g+\eta)}$ away from $\partial\Omega$, we may assume $-\log(-\varphi')$ is strictly plurisubharmonic, thus completing the induction.

With φ_0 as in Theorem 8.13, Theorem 6.21 guarantees that the solution u to (1.4), (1.6) is in $\mathscr{A}^{(n+1)}(\bar{\Omega})$. We are now in a position to derive our principal result:

THEOREM 8.18. *Let φ_0 be as in Theorem 8.13. Then the solution $u \in \mathscr{A}^{(n+1)}(\bar{\Omega})$ to (1.4), (1.6) is in $\mathscr{A}^{(n+1)}_{n+1,n+1}(\bar{\Omega})$. Specifically, there exist functions $u_{j,p} \in C^\infty(\bar{\Omega})$ such that for all $N \in \mathbf{N}$,*

$$u-\sum_{j=1}^{N} \sum_{p \leq \left[\frac{n+j}{n+1}\right]} u_{j,p}\, \varphi_0^{n+j} (\log(-\varphi_0))^p \in \mathscr{A}^{(N+n+1)}(\bar{\Omega}). \tag{8.19}$$

Proof. We work in normal coordinates near the boundary; it is clear that the coefficient functions in (8.19) can be extended to all of $\bar{\Omega}$.

Directly from (8.3), u satisfies

$$[E(u)]_{n+1} = [e^F - 1]_{n+1}.$$

If we define

$$\eta = -\frac{e^F - 1}{n+2} \log x \in \mathscr{A}^{(n+1)}(\bar{\Omega}),$$

then $[E(\eta)]_{n+1} = [e^F - 1]_{n+1}$, and so Corollary 8.9 shows that there exists some $\gamma \in C^\infty(\bar{\Omega})$ such that

$$[u]_{n+1} = [\eta + \gamma x^{n+1}]_{n+1}.$$

Thus the result holds for $N=1$. Moreover, if we set $u_{(1)} = \eta + \gamma x^{n+1}$, then (7.13) shows that $M(u_{(1)}) - e^F \in \mathscr{A}^{(n+2)}_{n+1,n+1}(\bar{\Omega})$.

Now assume that by induction we have $u_{(j)} \in \mathscr{A}^{(n+j)}, j=1, \ldots, N-1$, with

$$u - \sum_{j \leq N-1} u_{(j)} \in \mathscr{A}^{(N+n)}(\bar{\Omega}), \tag{8.20}$$

$$M\left(\sum_{j \leq N-1} u_{(j)}\right) - e^F \in \mathscr{A}^{(N+n)}_{n+1,n+1}(\bar{\Omega}). \tag{8.21}$$

Setting $v = \Sigma u_{(j)}$, $\eta = u - v \in \mathscr{A}^{(N+n)}(\bar{\Omega})$, (6.6) gives

$$e^F = M(u) = M(v+\eta) = (1 - \Delta_g(v+\eta) + G_2(v+\eta) + \ldots + G_n(v+\eta))\, e^{-(n+1)(v+\eta)}. \tag{8.22}$$

Now observe that, since $v \in \mathscr{A}^{(n+1)}(\bar{\Omega})$, each of the nonlinear terms G_j satisfies $G_j(v+\eta) = G_j(v) + \mathscr{A}^{(N+n+1)}(\bar{\Omega})$. Therefore, we can write (8.22) as

$$e^F = M(v) - E(\eta) + \mathscr{A}^{(N+n+1)}(\bar{\Omega}). \tag{8.23}$$

If we take $u_{(N)} \in \mathscr{A}^{(N+n)}(\bar{\Omega})$ to be a graded solution to

$$[E(u_{(N)})]_{N+n} = [M(v) - e^F]_{N+n},$$

then we have (8.20) with $N-1$ replaced by N; and again (7.13) shows that (8.21) also holds for N. This completes the induction.

The expansion (8.19) can easily be converted to the form (1.7) given in the introduction by taking ψ_p to be a function on $\bar{\Omega}$ which has $u_{j,p}$, $j\geq 1$, as its Taylor coefficients.

We have a similar result for solutions to the linearized problem. The proof is similar but easier, and is omitted here.

THEOREM 8.24. *Suppose $\varkappa>0$, and let $m=\frac{1}{2}(n+\sqrt{n^2+4\varkappa})$. For $f\in\varphi^r C^\infty(\bar{\Omega})$, where r is an integer such that $0<r<m$, let $u\in\mathcal{A}^{(r)}(\bar{\Omega})$ be the unique solution to*

$$(\Delta_g+\varkappa)\,u=f.$$

If m is an integer, then $u\in\mathcal{A}^{(r)}_{n+1,\infty}(\bar{\Omega})$, and there exists a function $\psi\in C^\infty(\Omega)$ such that

$$u-\psi\log(-\varphi_0)\in C^\infty(\bar{\Omega}).$$

If m is not an integer, then there exists $\psi\in C^\infty(\bar{\Omega})$ such that

$$u-\psi(-\varphi_0)^m\in C^\infty(\bar{\Omega}).$$

Graham, in [6], has obtained some related results for the special case of the Bergman metric on the ball.

References

[1] CHENG, S.-Y. & YAU, S. T., On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33 (1980), 507–544.

[2] FEFFERMAN, C., Monge-Ampère equations, the Bergman kernel and geometry of pseudoconvex domains. *Ann. of Math.*, 103 (1976), 395–416.

[3] GREINER, P. C. & STEIN, E. M., *Estimates for the $\bar{\partial}$-Neumann Problem*. Mathematical Notes no. 19, Princeton University Press, Princeton N.J., 1977.

[4] MELROSE, R. B., Transformation of boundary problems. *Acta Math.*, 147:3–4 (1981), 149–236.

[5] YAU, S. T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. *Comm. Pure Appl. Math.*, 31 (1978), 339–411.

[6] GRAHAM, C. R., *The Dirichlet problem for the Bergman Laplacian*. Thesis, Princeton University, 1981.

Received April 9, 1981