BOUNDARY COHOMOLOGY OF SHIMURA VARIETIES, III: The nightmare continues

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Introduction

The present article continues the study of the boundary cohomology of Shimura varieties initiated in [HZ1, HZ2]. Let G be a reductive group over $\mathbb{Q}$, X the symmetric space associated to $G(\mathbb{R})$, and Γ an arithmetic subgroup (e.g., a congruence subgroup) of $G(\mathbb{Q})$. We consider the cohomology of $\Gamma\backslash X$ with coefficients in the local system $\widetilde{\mathbf{V}}$ constructed from a representation V *of* G, i.e., $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}}) \simeq H^\bullet(\Gamma, V)$. It is standard that this cohomology can be decomposed as the direct sum of "interior" cohomology, defined as the image of the cohomology with compact supports $H^\bullet_c(\Gamma\backslash X, \widetilde{\mathbf{V}})$, and a complementary "boundary cohomology" that restricts non-trivially to the boundary of the Borel-Serre (manifold-with-corners) compactification of $\Gamma\backslash X$. The designation of boundary cohomology is generally non-canonical, and much work has been devoted to constructing canonical decompositions using Eisenstein series.

By an elaboration on the de Rham theorem, one knows that the cohomology group $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ can be expressed as the relative Lie algebra cohomology of the space of V-valued C^∞ functions on $\Gamma\backslash G(\mathbb{R})$, or even the functions of moderate growth ([B2, §7]). Thanks to the work of Franke [Fr1], one can replace the functions of moderate growth by the subspace of automorphic forms, and this can provide the starting point for an approach to the boundary cohomology. However, in this series of articles we are concerned only tangentially with the relation between boundary cohomology and automorphic forms. We choose to work at a more intrinsic level, concentrating instead on the additional structures on $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ when X is a hermitian symmetric domain. In that case, $\Gamma\backslash X$ is an algebraic variety, and $\widetilde{\mathbf{V}}$ underlies a natural variation of Hodge structure. Morihiko Saito's theory of mixed Hodge modules [Sa3] then gives that $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ has a corresponding mixed Hodge

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structure (MHS). The nature of this MHS at the boundary—more accurately, the associated MHS on the deleted neighborhood cohomology of the boundary—was the subject of [HZ2].

The "adelic version" of $\Gamma\backslash X$ is the Shimura variety $Sh(G,X)$, whose connected components are of the form $\Gamma\backslash X$. This has a canonical model over a number field E. Via the Hodge to de Rham spectral sequence, $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ acquires an E-rational structure distinct from the topological rational structure coming from the coefficients $\widetilde{\mathbf{V}}$. In particular, $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ has a Hodge filtration whose graded pieces are given by the coherent cohomology with coefficients in certain automorphic vector bundles [H1, Mi2]; the latter have natural E-rational structure. (This E-rationality can be asserted for de Rham cohomology, without grading for the Hodge filtration, and that is conjecturally equivalent in this context.) Study of the boundary cohomology of such automorphic vector bundles was begun in [HZ1].

Both [HZ1] and [HZ2] made essential use of toroidal compactifications of Shimura varieties (which has its origin in [AMRT]), following [H3] and [H4]. The toroidal boundary of $Sh(G,X)$ (a divisor with normal crossings), like the Borel-Serre boundary, is stratified according to conjugacy classes of parabolic subgroups of G. The cohomology of the boundary, both in the topological setting (as above) and in the coherent setting (i.e., for canonically extended automorphic vector bundles), can be computed as the abutment of the spectral sequence for the closed covering given by this stratification; this is called the *nerve spectral sequence*. In [HZ1] we analyzed the contribution to the nerve spectral sequence from the strata associated to *maximal* parabolics in the coherent setting. The first task of the present article is to extend this analysis to general parabolics, thereby fulfilling our promise from [HZ1], and this is carried out in the first three Sections. This necessitated a generalization in Section 1 of much of the machinery of automorphic vector bundles to the toroidal compactifications of *mixed* Shimura varieties (constructed by Pink [P]).

Most of the calculations from [HZ1, §3] go over without change, but there are a few delicate points, notably the issue of basechange in (3.4). For the latter, we must recall the role of conditions of growth and decay. These entered in the coherent setting when we established the existence and degeneration of Leray spectral sequences for morphisms of toroidally compactified varieties given by canonical (or subcanonical) extension of the same for their interiors. In effect, it enabled us to circumvent the complications related to basechange at infinity. As suggested above, this last point recurs here. We are obliged to prove (in (2.3)) a generalization to mixed conditions of growth and decay, enabling us, in effect, to isolate a single boundary stratum.

Our main result in Section 3 is that the differentials in the E_1-term of the nerve spectral sequence for coherent cohomology decompose naturally into pieces that either are given in terms of restriction maps on pure Shimura varieties or are "purely topological" (see (3.5.4)). Via Franke's interpretation of cohomology in terms of automorphic forms, this implies (see (3.6)) that the constant term maps for cohomology, expressed as integration of an automorphic form along the unipotent radical of appropriate parabolic subgroups, are rational with respect to the de Rham rational structure; for maximal parabolics, this was already obtained in [HZ1, 4.8].

The nerve spectral sequence for the topological cohomology $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ was studied in detail in [HZ2]. Hodge-theoretic considerations require (algebraic) compactifications, and the toroidal compactifications were convenient to use for this purpose as well. It was a subtle matter to compare the deleted neighborhoods of the Borel-Serre and toroidal boundary strata associated to a given parabolic subgroup (see [HZ2, §2]). We constructed isomorphisms between them that are compatible with restriction maps, allowing for transport of structure from the latter to the former. From this, it follows that the differentials in the topological nerve spectral sequence are morphisms of mixed Hodge structures. In particular, they induce maps after grading for the Hodge filtration F. Since a morphism of mixed Hodge structures is determined by its gradation for F, it follows, for instance, that ghost classes exist in $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ if and only if they exist in $\mathrm{Gr}_F H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ (see (4.6.7)). (Recall that a ghost class in $H^\bullet(\Gamma\backslash X, \widetilde{\mathbf{V}})$ is a cohomology class whose restriction to the Borel-Serre boundary is non-zero, yet whose restriction to each face (stratum) thereof *is* zero.) In (4.1), we compare the graded differentials to the results obtained for the differentials in the case of the coherent cohomology. To that end, we derive a formula for the deleted neighborhood cohomology of a boundary stratum as de Rham cohomology on a suitable toroidal compactification of the associated (Baily-Borel) boundary component (see (4.1.9)).

Of course, the above can be repeated for the weight filtration. For an example of the use of weights to rule out ghost classes (cf. (4.6.14)), see [Z5, App. A].[3] We are still seeking a satisfactory way of dealing with the entire mixed Hodge structure. It is therefore strongly to be feared that the present article is not the last of the series .... The content of the first three sections of this article completes the verification of results announced in [HZ1, §5] and in [H5]. They can be summarized by saying that

[3]The correct outcome of the calculation presented in the latter is that there are no ghosts for $GSp(4)$ *when the representation V is generic*, i.e., where the highest weight for $Sp(4)$ has positive inner product with both simple roots. When V is trivial, on the other hand, the calculation does

the (topological) nerve spectral sequence is a spectral sequence of mixed Hodge-de Rham structures over the field of definition of the canonical model.

In Section 4, we continue to develop the Hodge theoretic material from [HZ2, §5]. In (4.2), we reformulate the results in (4.1) by using the "minimal model" of the holomorphic de Rham complex, viz., the dual Bernstein-Gelfand-Gelfand complex, and deduce the E-rational version of (4.1.9). A big surprise in this work was the discovery of another interesting filtration on the boundary complex, whose spectral sequence is, like the nerve spectral sequence, a spectral sequence of mixed Hodge structures. In a way, there is nothing new about this filtration, which we call the filtration by holomorphic rank; it is given by the pullback to the Borel-Serre boundary of the filtration of the Baily-Borel Satake boundary by (unions of) boundary strata of increasing dimension (see also (4.4.15)). In a sense that can be made precise, its E_1-term is closer to the abutment than that of the nerve spectral sequence, though further from the question of ghost classes. We treat the holomorphic rank filtration in (4.4), though the same considerations already show up in (3.5) in the coherent setting. Cases of the latter give the Hodge components for the E_1-term of the topological holomorphic rank spectral sequence, and this gets examined in (4.5).

Several fundamental questions remain open. The analysis of cohomology of Shimura varieties should be extended to the intersection cohomology of their minimal (Baily-Borel) compactifications. The Zucker conjecture, proved by Looijenga [L] and Saper-Stern [SS], asserts that this cohomology is isomorphic to the L_2-cohomology, or again to the Lie algebra cohomology of square-integrable C^∞ functions, or by [Fr1], of square-integrable automorphic forms. However, it is not known whether this isomorphism identifies Morihiko Saito's Hodge structure with the analytic Hodge structure on L_2-cohomology (the one given a priori by the L^2 harmonic forms). In Section 5 we obtain a partial result in this direction: we show that the map from L_2-cohomology of the open Shimura variety to ordinary cohomology is a morphism of (mixed) Hodge structures (this is a small improvement over what was asserted in [H5, 3.3.9]). We do not address the question of whether intersection cohomology carries a de Rham rational structure.

It is also true that not all questions are treated in maximum generality. For instance, we have not studied the cohomology of a general automorphic vector bundle or variation of mixed Hodge structure on a mixed Shimura variety, but have rather been content to work out the cases directly relevant to the cohomology of pure Shimura varieties. Experience suggests these omissions will return to haunt us (providing even more impetus for article IV?). Another thing absent is the exploration of relations between our constructions and the general polylogarithms constructed by Wildeshaus [W1, W2].

Much of this work was begun at the time of writing of [H5], where some of our results were announced. The actual writing of the present article did not get under way until the second-named author visited Université Paris 7 in May, 1997. We both wish to thank that institution for the hospitality extended on that occasion. Likewise, a large amount of the work and writing of this article was carried out while the second-named author was spending Academic Year 1998–99 on sabbatical at the Institute for Advanced Study in Princeton. We also wish to thank P. Polo for helpful discussions of the generalized Bernstein-Bernstein-Gelfand resolution, and Z. Mebkhout for help with the proof of Proposition (4.2.21).

Contents

Conventions and notation

(0.1) We let $\underline{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ be the torus whose real representations parametrize Hodge structures. To any real representation $h : \underline{S} \to GL(V)$ is associated a Hodge structure on V such that $V^{p,q}$ is the subspace of $V_{\mathbb{C}}$ on which $z \in \underline{S}(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^{\times}$ acts as $z^{-p}\overline{z}^{-q}$. Here we do not assume the Hodge structure to be pure. We let $\mu : \mathbb{G}_{m,\mathbb{C}} \to \underline{S}_{\mathbb{C}}$ be the cocharacter such that $h \circ \mu(z)$ acts as z^{-p} on $V^{p,q}$, for any h and V as above. We let $w : \mathbb{G}_{m,\mathbb{R}} \to \underline{S}$ be the natural map. Then with (h, V) as above, the associated Hodge structure is pure of weight $w(V)$ if and only if $h \circ w$ defines a character of $\mathbb{G}_{m,\mathbb{R}}$, and in that case $h \circ w(t) = t^{-w(V)}$.

(0.2) Let G be a connected reductive group over $\mathbb{C}$, $P \subset G$ a parabolic subgroup, $L \subset P$ a Levi subgroup, $H \subset L$ a maximal torus, Let $B \subset P$ be a Borel subgroup of G containing H. We use the lower case Gothic letters $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{l}$, $\mathfrak{b}$ to denote the corresponding Lie algebras, and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots, Φ^+ the positive roots corresponding to $\mathfrak{b}$, $\Phi_L = \Phi(\mathfrak{l}, \mathfrak{h})$ and Φ_L^+ respectively the roots and positive roots of $\mathfrak{h}$ in $\mathfrak{l}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group, $W_L = W(\mathfrak{l}, \mathfrak{h})$. The subset $W^P \subset W$ of *Kostant representatives* of $W_L \backslash W$ is the set

$$\{w \in W \mid w^{-1}(\alpha) > 0, \forall \alpha \in \Phi_L^+\}.$$

Then W^P is the set of representatives of shortest length for the right cosets $W_L \backslash W$.

The same terminology is used for the shortest representatives of W_L in W when G is quasi-split over a field of characteristic zero.

(0.3) Starting in §3 we will concentrate on the case of pure Shimura varieties. Notation is as in [HZ1] and [HZ2] (see, however, (0.5)). In particular, (G, X) is a Shimura datum, $K_p \subset G_{\mathbb{R}}$ is the stabilizer of a point $p \in X$, $H \subset K_p$ is a maximal torus, $\mathfrak{h}$ its Lie algebra. We assume a set of positive roots Φ^+ chosen as above, with decomposition $\Phi^+ = \Phi_c^+ \cup \Phi_n^+$ into compact and non-compact roots, as in [HZ1, 3.6] (where we used the letter R instead of Φ). All references to highest weights will be with respect to these choices. Let $\mathfrak{P}^+ \subset \mathfrak{g}$ (resp. $\mathfrak{P}^-$) be the maximal parabolic subalgebra with Levi component $\mathfrak{k}_p$ and unipotent radical $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$).

(0.4) We wish to recall the combinatorics of the intersection of boundary strata in the Borel-Serre and toroidal compactifications of the locally symmetric variety $\Gamma \backslash X$. This goes the same for both, so we will treat only the former.

The Borel-Serre compactification is $\Gamma \backslash \overline{X}$, where $\overline{X}$ is a certain manifold-with-corners with $G(\mathbb{Q})$-action, with X as interior. Its closed (or open) boundary faces are in natural one-to-one correspondence with the rational parabolic subgroups P of G, and are denoted $\overline{e(P)}$. Then $\overline{e(P)} \cap \overline{e(Q)} \neq \emptyset$ if and only if $P \cap Q$ is parabolic, and then the intersection is simply $\overline{e(P \cap Q)}$.

In $\Gamma\backslash\overline{X}$, one has boundary faces indexed by Γ-*conjugacy classes* of rational parabolic subgroups. It is customary to take representatives of these conjugacy classes, so one must understand the notational complication that ensues. The boundary face $\overline{e'(P)}$ of $\Gamma\backslash\overline{X}$ is an arithmetic quotient of $\overline{e(P)}$, so likewise for any Γ-conjugate of P. Thus, it is wrong to expect that $\overline{e'(P)} \cap \overline{e'(Q)} = \overline{e'(P \cap Q)}$ when $P \cap Q$ is parabolic. Indeed, $\overline{e'(P \cap Q)}$ is only one connected component of the intersection. Suppose that $P \cap Q$ is the parabolic subgroup R. Then $\overline{e'(P)} \cap \overline{e'(Q)}$ is a finite disjoint union of faces of the form $\overline{e'(gRg^{-1})}$, for g running over representatives of the fiber over the identity double coset in the right-hand side of

$$(\Gamma \cap P)\backslash P(\mathbb{Q})/R(\mathbb{Q}) \longrightarrow \Gamma\backslash G(\mathbb{Q})/Q(\mathbb{Q}).$$

This understanding underlies all treatment of intersections of boundary strata in this work. (For more on this, see the appendix to [HZ2, (3.5)].)

(0.5) We wish to remind the reader that it was necessary in [HZ2, (1.4)] to make a change of notation for the sequel, in particular this paper. In [HZ1], we used the symbol F to denote a boundary component, and associated objects such as its normalizing parabolic subgroup P_F, the Cayley transform c_F, the associated cone complex Σ_F, etc. carried the subscript F. Given that $P = P_F$ and F are equivalent data, we decided it was more convenient to switch to writing P, c_P, Σ_P, etc. As such, one must take this change into account when tracking down notions from [HZ1].

The notation $Z_\Sigma(R)$ (for the closed R-stratum of the toroidal boundary) was introduced in [HZ2, (1.5.2)]. It derives from [HZ2, (1.4.11)]; there, the F-stratum should have been denoted ${}^{<}Z_{F,\Sigma}$ (as in 1.5.2 of [HZ1]), not ${}^{0}Z_{F,\Sigma}$. Fortunately, this causes no trouble in the sequel.

(0.6) A single object (sheaf, group, etc.) can be regarded as a complex with [at most] one non-zero term *in degree* 0. A sheaf $\mathcal{S}$, or other such object, even if it "looks like it belongs in degree q," is placed in degree q only by specifying the standard shift to the right: $\mathcal{S}[-q]$. In general, $\mathcal{S}^\bullet[-q]$ has in degree i what $\mathcal{S}^\bullet$ has in degree $i-q$, whence the notation.

The cone $\mathcal{C}^\bullet$ of a morphism of complexes $\phi : \mathcal{S}^\bullet \to \mathcal{T}^\bullet$ has, by definition, as underlying sheaf $\mathcal{S}^\bullet[1] \oplus \mathcal{T}^\bullet$. Often it is more natural to work with $\mathcal{C}^\bullet[-1]$; in that case, this shift will be specified.

1. Automorphic vector bundles on mixed Shimura varieties

In §4 of [HZ1], although we did not draw attention to this fact, we tacitly developed the basic theory of automorphic vector bundles on the mixed Shimura varieties, in the sense of Pink's thesis [P], that arise as boundary strata of toroidal compactifications of pure Shimura varieties. The present discussion recapitulates the constructions of [HZ1] in more orderly fashion and in the full generality of [P]. The general setting includes the special case of Kuga's families of abelian varieties with additional structure, as well as their toroidal compactifications. These will not be discussed in the sequel but the results proved here appear to have interesting applications to the construction of mixed motives.

Most of the proofs are simple adaptations of the constructions in [H1, H2, H4], as well as [HZ1, HZ2, 5.1]. Where this is the case, we keep the details to a minimum, referring to the analogous proofs in the earlier papers. We follow the strategy of [HZ1] by reducing purely geometric theorems about (mixed) Shimura varieties to local assertions, which we then prove on connected models of the form $\Gamma\backslash D$.

(1.1) *Mixed Shimura varieties.* We refer to [P] for notation and precise definitions concerning mixed Shimura varieties. As in [HZ1, 1.6], a mixed Shimura variety is defined by a pair $(Q, \mathcal{X})$, where Q is a connected algebraic group over $\mathbb{Q}$ with a three step-filtration by normal subgroups

$$\{1\} \subset W_{-2}Q \subset W_{-1}Q = R_uQ \subset W_0Q = Q \tag{1.1.1}$$

and $\mathcal{X}$ is a homogeneous space for $Q(\mathbb{R}) \cdot W_{-2}Q(\mathbb{C})$ with a finite-to-one map

$$h : \mathcal{X} \rightarrow Hom(\underline{S}_{\mathbb{C}}, Q_{\mathbb{C}}). \tag{1.1.2}$$

We write h_x for $h(x)$. It is assumed that $W_{-2}Q$ is commutative, and for any arithmetic subgroup $\Gamma \subset Q$ the quotient $T_\Gamma(Q, \mathcal{X}) = \Gamma \cap W_{-2}Q(\mathbb{Q})\backslash W_{-2}Q(\mathbb{C})$ is viewed as the set of complex points of the split torus with character group $Hom(\Gamma \cap W_{-2}Q, \mathbb{Z})$. The additional conditions satisfied by the pair $(Q, \mathcal{X})$ are listed in [P, Definition 2.1] and will be recalled as needed. The mixed Shimura variety is denoted $Sh(Q, \mathcal{X})$.

We write $U_Q = W_{-2}Q$, $W_Q = W_{-1}Q$, $V_Q = W_Q/U_Q$, and drop the subscript Q when it is understood, as it will be for most of the remainder of this section. Then U and V are commutative unipotent algebraic groups. In particular the exponential maps $Lie(U) \rightarrow U$ and $Lie(V) \rightarrow V$ are isomorphisms of vector groups. We let $G = Q/W$ be the maximal reductive quotient of Q.

Let

$$\alpha_1 : Q \rightarrow Q/U,\ \ \alpha_2 : Q/U \rightarrow G, \alpha = \alpha_2 \circ \alpha_1 : Q \rightarrow G$$

be the natural maps. Conditions (i) and (ii) of [P, Definition 2.1] are that

(i) For any $x \in \mathcal{X}$, $\alpha_2 \circ h_x$ is defined over $\mathbb{R}$;

(ii) For any $x \in \mathcal{X}$, $\alpha \circ h_x \circ w$ is a cocharacter of the center of G.

Pink's remaining conditions imply that the pair $(G, \{\alpha \circ h_x | x \in \mathcal{X}\})$ is a datum defining a pure Shimura variety. Let $\rho : Q \to GL(L)$ be a $\mathbb{Q}$-rational representation. Then for any $x \in \mathcal{X}$, the map $\rho \circ h_x$ defines a rational mixed Hodge structure on L [P, Proposition 1.4], and, as x varies, the family of $\rho \circ h_x$ defines a variation of mixed Hodge structures over $\mathcal{X}$. In particular, the family satisfies Griffiths transversality. Taking ρ to be the adjoint representation, we obtain a family of mixed Hodge structures on $Lie(Q)$. For $x \in \mathcal{X}$, let $Q_x^0 \subset Q_{\mathbb{C}}$ denote the connected subgroup with Lie algebra $F_x^0(Lie(Q))$; then Q_x^0 is the subgroup stabilizing the Hodge filtration $F_x^{\bullet}(L)$ for any representation (ρ, L). Let $\check{\mathcal{X}}$ denote the quotient $Q_{\mathbb{C}}/Q_x^0$.

Let $\mathcal{H}_{\mathcal{X}} = im(h) \subset Hom(\underline{S}_{\mathbb{C}}, Q_{\mathbb{C}})$. The space $\mathcal{H}_{\mathcal{X}}$ has a $Q(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant complex structure determined by the following property: Suppose (ρ, L) is a faithful representation of Q. Let $\mathcal{F}(\rho)$ denote the variety of flags in L containing the Hodge filtration F_y attached to $\rho \circ y$ for any, and hence all, $y \in \mathcal{H}_{\mathcal{X}}$. Then the map

$$\beta' : \mathcal{H}_{\mathcal{X}} \to \mathcal{F}(\rho);\ y \mapsto F_y$$

is a complex analytic embedding [P, 1.18(a)] that naturally factors through the homogeneous space $\check{\mathcal{X}}(\mathbb{C})$. Since h is a local diffeomorphism, this also determines a unique $Q(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant complex structure on $\mathcal{H}$. The map $\beta = \beta' \circ h : \mathcal{X} \to \check{\mathcal{X}}$ is called the *Borel morphism*.

The *reflex field* $E(Q, \mathcal{X})$ of the pair $(Q, \mathcal{X})$ is the field of definition of the conjugacy class of $h_x \circ \mu : \mathbb{G}_m \to Q$, with μ as in (0.1), for any $x \in \mathcal{X}$ [P, Definition 11.1]. Just as in the case of pure Shimura varieties, the homogeneous algebraic variety $\check{\mathcal{X}}$ is canonically a quotient of this conjugacy class, and for the same reasons has a natural rational structure over $E(Q, \mathcal{X})$ (cf. [H1, §3]).

Let $K_f \subset Q(\mathbf{A}_f)$ be a compact open subgroup. Then the mixed Shimura variety ${}_{K_f}Sh(Q, \mathcal{X})$ is defined as a complex analytic variety, just as in the pure case, by

$$(1.1.3) \qquad {}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C}) = Q(\mathbb{Q}) \backslash (\mathcal{X} \times (Q(\mathbf{A}_f)/K_f)).$$

As K_f varies, the ${}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C})$ form a projective system, and one can define $Sh(Q, \mathcal{X})(\mathbb{C})$ to be the inverse limit of the ${}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C})$. Alternatively, we can view $Sh(Q, \mathcal{X})(\mathbb{C})$ as shorthand for the projective system. In either case, we obtain an action of the group $Q(\mathbf{A}_f)$ on $Sh(Q, \mathcal{X})(\mathbb{C})$ as "Hecke correspondences." However, we will mainly be concerned with the objects at finite level.

A morphism $(Q, \mathcal{X}) \to (Q', \mathcal{X}')$ of mixed Shimura data is a pair consisting of a group homomorphism $Q \to Q'$ defined over $\mathbb{Q}$, and a holomorphic map

$\mathcal{X} \to \mathcal{X}'$, compatible with the group actions on the two sides. To a morphism $\phi : (Q, \mathcal{X}) \to (Q', \mathcal{X}')$ of mixed Shimura data we can associate holomorphic maps of mixed Shimura varieties: if $K_f \subset Q(\mathbf{A}_f)$ and $K'_f \subset Q'(\mathbf{A}_f)$ are such that $\phi(K_f) \subset K'_f$, then there is a natural map

$$(1.1.4) \qquad [\phi] : {}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C}) \to {}_{K'_f}Sh(Q', \mathcal{X}')(\mathbb{C})$$

([P, 3.4]). As in the pure case, to a morphism of mixed Shimura data we obtain a map of reflex fields in the opposite direction: $E(Q', \mathcal{X}') \subset E(Q, \mathcal{X})$.

In particular, we can take $Q' = Q/U$ or $Q' = G = Q/W$ in the above construction, and $Q \to Q'$ to be the natural projection. If we define $\mathcal{X}'$ to be the quotient of $\mathcal{X}$ by $U(\mathbb{C})$, resp. by $U(\mathbb{C}) \cdot W(\mathbb{R})$, we thus obtain morphisms of Shimura data $(Q, \mathcal{X}) \to (Q', \mathcal{X}')$. Writing $U\backslash\mathcal{X}$ in place of $U(\mathbb{C})\backslash\mathcal{X}$ and $W\backslash\mathcal{X}$ in place of $U(\mathbb{C}) \cdot W(\mathbb{R})\backslash\mathcal{X}$, we thus obtain a short sequence of morphisms of mixed Shimura varieties:

$$(1.1.5) \qquad Sh(Q, \mathcal{X})(\mathbb{C}) \xrightarrow{\pi_2} Sh(Q/U, U\backslash\mathcal{X})(\mathbb{C}) \xrightarrow{\pi_1} Sh(G, W\backslash\mathcal{X})(\mathbb{C}).$$

We have already noted that $(G, W\backslash\mathcal{X})$ is a pure Shimura datum.

Among the main theorems of [P] are the existence of natural algebraic structures on the mixed Shimura varieties $Sh(Q, \mathcal{X})(\mathbb{C})$, the existence of smooth projective toroidal compactifications of the mixed Shimura varieties ${}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C})$ (provided the level subgroup K_f is *neat*, in the sense of [P, 0.6]), and the existence of canonical models of $Sh(Q, \mathcal{X})(\mathbb{C})$ and its toroidal compactifications over the reflex field $E(G, X)$ [P, Theorems 9.21, 11.18, and 12.4]. We mention for the sake of completeness that any K_f contains a neat subgroup of finite index, and that if K_f is neat then any subgroup of finite index in K_f is also neat. The algebraic structures are "natural" in that they are compatible with the morphisms introduced above, including the Hecke correspondences $Q(\mathbf{A}_f)$ and the morphisms $[\phi]$ of (1.1.4) associated to morphisms $\phi : (Q, \mathcal{X}) \to (Q', \mathcal{X}')$ of mixed Shimura data. The algebraization of the mixed Shimura varieties is actually accomplished by constructing ample line bundles on certain toroidal compactifications.

The canonical model, denoted $Sh(Q, \mathcal{X})$, is defined, as in the pure case, by functoriality with respect to morphisms of mixed Shimura data $(T, h) \to (Q, \mathcal{X})$. Here T is a torus, the "symmetric space" h is a single point, and the canonical model of $Sh(T, h)$ is defined by the analogue of the Shimura-Taniyama reciprocity law for complex multiplication. The points contained in the image of a morphism of the form $Sh(T, h) \to Sh(Q, \mathcal{X})$, or the corresponding points at finite level, are called *special points*. Thus the canonical models constructed by Pink, like those of Shimura, are characterized by the reciprocity law at the special points. Details can be found in [P, 11.5].

(1.1.6) The (partial) toroidal compactifications of ${}_{K_f}Sh(Q,\mathcal{X})(\mathbb{C})$ are associated to combinatorial data, just as in the pure case [cf. HZ1, 1.4, 1.7]. Following the conventions introduced in [HZ1], the set of combinatorial data will be denoted Σ and will be called a *K_f-admissible family of fans* for $(Q,\mathcal{X})$ (Pink denotes the set of data $\mathcal{S}$ and calls it a K_f-admissible partial cone decomposition; see [P, 6.4]). This aspect of Pink's theory was discussed in [HZ1, 1.6 and 4.1] in connection with the mixed Shimura varieties that arise as boundary strata of pure Shimura varieties, and the general case is identical. The (partial) toroidal compactification of ${}_{K_f}Sh(Q,\mathcal{X})(\mathbb{C})$ associated to the K_f-admissible family of fans Σ is denoted ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma(\mathbb{C})$. We assume Σ satisfies the hypotheses of [P, Theorem 12.4]. Then by [loc. cit.], ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma(\mathbb{C})$ exists as a complex algebraic variety, and descends to a scheme ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ over the reflex field $E(Q,\mathcal{X})$. Under supplementary hypotheses on Σ, we may assume ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ to be quasi-projective; in this case we call Σ *projective*. Just as in the pure case, the set of all K_f-admissible families of fans is an directed system under the relation of refinement. If Σ_1 is a refinement of Σ_2 then there is a natural morphism ${}_{K_f}Sh(Q,\mathcal{X})_{\Sigma_1} \rightarrow_{K_f} Sh(Q,\mathcal{X})_{\Sigma_2}$ [P, 6.7(b)]; moreover this morphism is projective [KKMS], and the set of projective Σ is cofinal in the set of all families of fans [P, Theorem 9.21].

If K_f is neat, then we may also assume ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ to be smooth and the complement $\partial_{K_f}Sh(Q,\mathcal{X})_\Sigma =_{K_f} Sh(Q,\mathcal{X})_\Sigma -_{K_f} Sh(Q,\mathcal{X})$ to be a divisor with normal crossings. In this case, ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ is said to be SNC.

(1.1.7) We will carry out certain calculations on connected components of the mixed Shimura varieties. Thus if $\Gamma \subset Q(\mathbb{Q})$ is a congruence subgroup, and if $\mathcal{X}^0 \subset \mathcal{X}$ is a connected component, we let $M_\Gamma = M_\Gamma(\mathcal{X}^0) = \Gamma\backslash\mathcal{X}^0$. The connected components of the mixed Shimura variety ${}_{K_f}Sh(Q,\mathcal{X})(\mathbb{C})$ are all of the form $M_\Gamma(\mathcal{X}^0)$ [P, 3.2]. The notation $(\mathcal{X}^0)$ will generally be dropped, since the choice of local component will be irrelevant.

Similarly, a toroidal compactification of M_Γ will be denoted $M_{\Gamma,\Sigma}$. Here it is understood that Σ is a Γ-admissible family of fans for the pair $Q,\mathcal{X}^0$, with the obvious definition. Alternatively, if M_Γ is realized as a connected component of some ${}_{K_f}Sh(Q,\mathcal{X})(\mathbb{C})$, then we may take Σ to be a K_f-admissible family of fans for $(Q,\mathcal{X})$, and define $M_{\Gamma,\Sigma}$ to be the closure of M_Γ in ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma(\mathbb{C})$. Again, this distinction will be irrelevant in calculations involving connected components.

(1.2) *Construction of automorphic vector bundles.* We keep the notation of §1.1. Henceforward, we make the following standard

Hypothesis (1.2.1). The pure Shimura datum $(G, W\backslash\mathcal{X})$ is *motivic* in the sense that, for any $h \in W\backslash\mathcal{X}$, the weight morphism $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$.

Hypothesis 1.2.1 guarantees that any algebraic representation of G gives rise to a rational local system, in the sense of [H1, 3.14]. The hypothesis also allows us to define the *standard principal bundle* $I(Q, \mathcal{X})(\mathbb{C})$ over $Sh(Q, \mathcal{X})(\mathbb{C})$ as the complex analytic variety

$$(1.2.2) \qquad I(Q, \mathcal{X})(\mathbb{C}) = \varprojlim Q(\mathbb{Q})\backslash(Q(\mathbb{C}) \times \mathcal{X} \times (Q(\mathbf{A}_f)/K_f)).$$

Here as in (1.1.3), the group $Q(\mathbb{Q})$ acts diagonally on the left on the product of the three factors. That this is indeed a principal bundle follows in the usual way from Hypothesis 1.2.1 (cf. [M2, Proposition 3.3]). Again, this can be viewed as an inverse system of principal $Q(\mathbb{C})$-bundles

$$_{K_f}I(Q, \mathcal{X})(\mathbb{C}) \to {}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C}).$$

The natural projection $I(Q, \mathcal{X})(\mathbb{C}) \to Sh(Q, \mathcal{X})(\mathbb{C})$ is denoted $q = q_{Q,\mathcal{X}}$. On the other hand, there is a natural map $p = p_{Q,\mathcal{X}} : I(Q, \mathcal{X})(\mathbb{C}) \to \check{\mathcal{X}}(\mathbb{C})$ defined in terms of (1.2.2) by

$$(1.2.3) \qquad p(q, x, q_f) = q^{-1} \cdot \beta(x), q \in Q(\mathbb{C}),\ x \in \mathcal{X},\ q_f \in (Q(\mathbf{A}_f)/K_f).$$

The right-hand side is obviously invariant under left-translation by $Q(\mathbb{Q})$ and defines a holomorphic map.

(1.2.4) Proposition. *The standard principal bundle $I(Q, \mathcal{X})(\mathbb{C})$ can be given a natural structure of algebraic principal bundle $I(Q, \mathcal{X})_{\mathbb{C}}$ over $Sh(Q, \mathcal{X})_{\mathbb{C}}$. Moreover, $I(Q, \mathcal{X})_{\mathbb{C}}$ has a canonical $Q(\mathbf{A}_f)$-equivariant model $I(Q, \mathcal{X})$ over the reflex field $E(Q, \mathcal{X})$, and the morphisms $q_{Q,\mathcal{X}}$ and $q_{Q,\mathcal{X}}$ descend to yield a canonical diagram*

$$\begin{array}{ccc} & I(Q, \mathcal{X}) & \\ {}^{p}\swarrow & & \searrow^{q} \\ \check{\mathcal{X}} & & Sh(Q, \mathcal{X}) \end{array}$$

over $E(Q, \mathcal{X})$.

(1.2.5) Explanation-Definition. The canonical model to which we refer in the proposition is defined by combining the definitions of Pink (for canonical models of mixed Shimura varieties) and Milne (for canonical models of standard principal bundles over pure Shimura varieties, see [M1]). Unfortunately, a complete explanation requires additional notation. We take as given the existence of a canonical model on $I(T,h)$ when T is a torus, defined as in [M1] in terms of the period torsor. If $(Q',\mathcal{X}') \to (Q,\mathcal{X})$ is an embedding of mixed Shimura data then $E(Q,\mathcal{X}) \subset E(Q',\mathcal{X}')$ [P, 11.2(b)]. Then a canonical model on $I(Q,\mathcal{X})$ is a $Q(\mathbf{A}_f)$-equivariant model such that, for any embedding of mixed Shimura data $(T,h) \to (Q,\mathcal{X})$, with T a torus, the natural morphism $I(T,h) \to I(Q,\mathcal{X})$ descends to $E(T,h)$. One verifies as for pure Shimura varieties that the canonical model is unique, if it exists.

Proof of Proposition 1.2.4. For any positive integer g, we let $(P_{2g},\mathcal{X}_{2g})$ be the mixed Shimura datum defined in [P, 2.25]. It is a mixed Shimura datum associated to the genus g (maximal) boundary component of the Siegel modular Shimura variety of genus $g+1$, denoted $Sh(GSp(g+1),\mathfrak{S}^{\pm}_{g+1})$. Here $GSp(g+1)$ is the group of similitudes of a $2(g+1)$-dimensional symplectic space and $\mathfrak{S}^{\pm}_{g+1}$ is the union of the Siegel upper and lower half-spaces of genus $g+1$. We can realize the parabolic subgroup $P_{2g} \subset GSp(g+1)$ as the stabilizer of a rational genus g boundary component of $\mathfrak{S}^{\pm}_{g+1}$. For a cofinal subset of the set of level subgroups $K_f \subset P_{2g}(\mathbf{A}_f)$, the the associated mixed Shimura variety ${}_{K_f}Sh(P_{2g},\mathcal{X}_{2g})$ can thus be realized as a union of boundary strata of a toroidal compactification of $Sh(GSp(g+1),\mathfrak{S}^{\pm}_{g+1})$ (at an appropriate finite level, depending on K_f). Since the center of the unipotent radical of P_{2g} is one-dimensional, the realization as a (union of) boundary strata does not depend on the choice of toroidal compactifications.

Now when $(Q,\mathcal{X})$ is a pure Shimura datum and the natural map $\mathcal{X} \to \mathcal{H}$ is the identity – call this a *standard* Shimura datum – then the proposition is proved in [H1] and [M1]. When Q is a torus, the proposition is tautologically true, whether or not $\mathcal{X}$ is connected. On the other hand, when ${}_{K_f}Sh(Q,\mathcal{X})$ is a (union of) boundary strata of a toroidal compactification attached to a rational boundary component of a pure Shimura variety, then the canonical model of $I(Q,\mathcal{X})_{\mathbb{C}}$ is constructed in [HZ1, Lemma 4.5.8]. The fact that this model is canonical is an immediate consequence of [HZ1, Lemma 4.6.9], since, when (T,h) is as in 1.2.5, the morphism $Sh(T,h) \to Sh(Q,\mathcal{X})$ factors through some section of the natural map $\pi_1 \circ \pi_2 : Sh(Q,\mathcal{X}) \to Sh(G, W\backslash\mathcal{X})$.

We remark that the $Q(\mathbf{A}_f)$-equivariance is not stated explicitly in [HZ1], and indeed the statement needs to be modified for boundary strata of toroidal compactifications to take account of the combinatorial data defining the compactification. We will only apply [HZ1, Lemma 4.5.8] to the case of $(P_{2g},\mathcal{X}_{2g})$. The fan in this case is uniquely determined. Therefore, the $P_{2g}(\mathbf{A}_f)$-equivariance follows immediately from the $GSp(g+1)(\mathbf{A}_f)$-equivariance of the canonical model of the standard principal bundle over $Sh(GSp(g+1), \mathfrak{S}^{\pm}_{g+1})$.

It follows in the usual way that if $(Q', \mathcal{X}') \hookrightarrow (Q, \mathcal{X})$ is an embedding of Shimura data, then the existence of a canonical model for $I(Q, \mathcal{X})$ implies the existence of a canonical model for $I(Q', \mathcal{X}')$. It thus follows from what has been established up to this point that, if $(Q, \mathcal{X})$ admits an embedding of the form

$$(1.2.6) \qquad (Q, \mathcal{X}) \to (G_1, X_1) \times (T, h) \times \prod_{i=1}^{n} (P_{2g}, \mathcal{X}_{2g})$$

where (G_1, X_1) is a standard Shimura datum, T is a torus, and n is a non-negative integer, then the proposition holds for $(Q, \mathcal{X})$.

Lemma 2.26 of [P] asserts that, we can replace $(Q, \mathcal{X})$ by a new pair $(Q', \mathcal{X}')$, where Q' is an extension of Q by $U_0 \simeq \mathbb{G}_a$ (contained in the center of the unipotent radical of Q') of a specific type, called a $(P_0, \mathcal{X}_0) \to (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$-torsor, such that $(Q', \mathcal{X}')$ admits an embedding of the form (1.2.6). Thus we have the Q'-torsor $I(Q', \mathcal{X}')$ over $Sh(Q', \mathcal{X}')$ with its canonical model over $E(Q', \mathcal{X}')$, which is easily checked to equal $E(Q, \mathcal{X})$. The quotient of $I(Q', \mathcal{X}')$ by U_0 is then a Q-torsor over $Sh(Q', \mathcal{X}')$, which is in turn a $\mathbb{G}_m$-torsor over $Sh(Q, \mathcal{X})$. Over $\mathbb{C}$, $U_0 \backslash I(Q', \mathcal{X}')$ is obviously isomorphic to the pullback of $I(Q, \mathcal{X})_{\mathbb{C}}$ from $Sh(Q, \mathcal{X})$. Thus the $\mathbb{G}_m$-action on $Sh(Q', \mathcal{X}')$ lifts canonically to an action on $U_0 \backslash I(Q', \mathcal{X}')$ and thus defines an $E(Q, \mathcal{X})$-rational model $I(Q, \mathcal{X})$ of $I(Q, \mathcal{X})_{\mathbb{C}}$. Since any morphism $(T, h) \to (Q, \mathcal{X})$, with T a torus, lifts to a morphism $(T, h) \to (Q', \mathcal{X}')$, the canonicity of $I(Q, \mathcal{X})$ follows from that of $I(Q', \mathcal{X}')$.

It remains to show that the morphism $p : I(Q, \mathcal{X}) \to \check{\mathcal{X}}$ can be descended to $E(Q, \mathcal{X})$. This follows again by reduction to the individual factors on the right-hand side of (1.2.6). For the first factor this is in [H1] and [M1], for the second factor it's trivial, and for the third factor it is contained in Lemma 4.5.8 of [HZ1]. This completes the proof.

Given the diagram in Proposition 1.2.4, automorphic vector bundles can be constructed on $Sh(Q, \mathcal{X})$ by the procedure described in [H1] and [M1], which we now recall. Let $\mathcal{W}$ be a Q-homogeneous vector bundle on $\check{\mathcal{X}}$. Then $p^*\mathcal{W}$ is a Q-equivariant vector bundle on $I(Q, \mathcal{X})$, hence descends to a $Q(\mathbf{A}_f)$-equivariant vector bundle $[\mathcal{W}]$ on $Sh(Q, \mathcal{X})$. The functor $\mathcal{W} \mapsto [\mathcal{W}]$ is rational over $E(Q, \mathcal{X})$, by construction, and defines canonical models, just as in [M1], on the complex points of $[\mathcal{W}]$, which just as in the classical case can be identified with a locally homogeneous vector bundle:

$$(1.2.7) \qquad [\mathcal{W}](\mathbb{C}) \xrightarrow{\sim} (Q(\mathbb{Q}) \backslash \beta^*(\mathcal{W}) \times (Q(\mathbf{A}_f)/K))$$

A Q-homogeneous vector bundle on $\check{\mathcal{X}}$ is called *locally-finite* if it is the direct limit of finite-rank homogeneous vector bundles on $\check{\mathcal{X}}$. The functor above obviously is defined on locally-finite homogeneous vector bundles and satisfies $[\varinjlim \mathcal{W}_\alpha] = \varinjlim [\mathcal{W}_\alpha]$. Defining $[\mathcal{W}](\mathbb{C})$ by (1.2.7), we resume this discussion as follows:

(1.2.8) Proposition. *The functor $\mathcal{W} \mapsto [\mathcal{W}](\mathbb{C})$, defined by (1.2.7), maps the tensor category of locally finite Q-homogeneous algebraic vector bundles on $\check{\mathcal{X}}$ to the tensor category of $Q(\mathbf{A}_f)$-equivariant algebraic vector bundles on $Sh(Q,\mathcal{X})$. This functor is rational over $E(Q,\mathcal{X})$ and is functorial with respect to morphisms $(Q',\mathcal{X}') \to (Q,\mathcal{X})$ of mixed Shimura data. When $(Q,\mathcal{X})$ is a pure Shimura datum, the functor coincides with the one defined in* [H1, M1]. *In particular, the bundles $[\mathcal{W}]$ have canonical models in the sense of* [M1].

The final assertion in (1.2.8) is an immediate consequence of the others, since the canonical models are defined with respect to embeddings of pure Shimura data of the form (T,h), with T a torus. Bundles of the form $[\mathcal{W}]$ are called *automorphic vector bundles.* The automorphic vector bundles form a tensor subcategory of the category of all $Q(\mathbf{A}_f)$-equivariant vector bundles on $Sh(Q,\mathcal{X})$.

Let (ρ, L) be an algebraic representation of Q. We say (ρ,L) is *locally finite* if it is the direct limit of finite-dimensional (algebraic) representations of Q. An automorphic vector bundle $[\mathcal{W}]$ on Sh is called *flat* if $\mathcal{W}$ is isomorphic to a Q-homogeneous vector bundle on $\check{\mathcal{X}}$ of the form $L \otimes \mathcal{O}_{\check{\mathcal{X}}}$, where (ρ,L) is a locally-finite representation of Q. We then write $[\mathcal{W}] = \tilde{L}$. Equivalently, on any connected component $M_\Gamma \subset Sh(\mathbb{C})$, $[\mathcal{W}]$ is the direct limit of vector bundles associated to finite-dimensional algebraic representations of the fundamental group Γ of M_Γ. A flat vector bundle always has a canonical flat connection $\nabla = \nabla_L : \tilde{L} \to \tilde{L} \otimes \Omega^1_{Sh}$, induced by functoriality from the constant connection $1 \otimes d : L \otimes \mathcal{O}_{\check{\mathcal{X}}} \to L \otimes \Omega^1_{\check{\mathcal{X}}}$.

The standard principal bundle $I(Q,\mathcal{X})$ has the following functorial description. Let (ρ,L) be a faithful $\mathbb{Q}$-rational representation with contragredient (ρ^*, L^*). Let $s_i \in L^{\otimes n_i} \otimes (L^*)^{\otimes m_i}$ $(i = 1, \ldots, r)$ be a finite set of $\mathbb{Q}$-rational tensors such that Q is the subgroup of $GL(L)$ stabilizing all s_i. Let $\mathbf{1}$ denote $\mathbb{Q}$, viewed as a trivial representation of Q. For each i, the morphism of Q-representations

$$\alpha_i : \mathbf{1} \to L^{\otimes n_i} \otimes (L^*)^{\otimes m_i}; \quad \alpha_i(1) = s_i$$

defines by functoriality an injective homomorphism

(1.2.9) $\tilde{\alpha}_i : \mathcal{O} \to \tilde{L}^{\otimes n_i} \otimes (\tilde{L}^*)^{\otimes m_i}$

of automorphic vector bundles. Let $\tilde{s}_i = \tilde{\alpha}_i(1)$, $i = 1, \ldots, r$. It is then tautological that $I(Q, \mathcal{X})$ represents the functor that, to any Sh-scheme T, associates

$$\{f : \tilde{L}_T \xrightarrow{\sim} L_T \mid (f^{\otimes n_i} \otimes (f^*)^{\otimes m_i})(\tilde{s}_i) = s_i \otimes 1\}. \tag{1.2.10}$$

Here L_T is the constant bundle $L \otimes \mathcal{O}_T$ and the remaining notation is obvious. In fact, the construction of the canonical model of $I(Q, \mathcal{X})$ in [H1,M1] in the pure case proceeds by constructing the flat filtered automorphic vector bundle $\tilde{L}$ and showing that it has an appropriate kind of canonical model; in particular, that the $[s_i]$ are rational over $E(Q, \mathcal{X})$. Then $I(Q, \mathcal{X})$ is defined by (1.2.10), and is shown to be independent of the choice of (ρ, L).

The morphism $\pi_2 : Sh(Q, \mathcal{X}) \to Sh(Q/U, U\backslash\mathcal{X})$ is a locally constant pro-torus fibration over $Sh(Q/U, U\backslash\mathcal{X})$ with fiber $\mathcal{T} = \varprojlim U(\mathbb{Q})\backslash U(\mathbb{C}) \times U(\mathbf{A}_f)/K_{U,f}$. Here $K_{U,f}$ runs over the set of open compact subgroups of $U(\mathbf{A}_f)$. At finite level K_f, π_2 is a locally constant torus fibration. It follows just as in [HZ1, (3.2.1)] that

$$[\mathcal{W}] = \pi_2^*(\pi_{2,*}([\mathcal{W}])^{\mathcal{T}}). \tag{1.2.11}$$

Here $\mathcal{T}$ acts naturally on the direct image $\pi_{2,*}([\mathcal{W}])$ (cf. [SGA 3]) and the superscript ${}^{\mathcal{T}}$ designates the subsheaf of $\mathcal{T}$-invariant sections.

This can be globalized, following [HZ1, §4.6]. The subgroup $W \subset Q$ acts freely on the principal bundle $I(Q, \mathcal{X})$. We let $I_2(Q, \mathcal{X})$ (resp. $I_1(Q, \mathcal{X})$) denote the quotient of $I(Q, \mathcal{X})$ by U (resp. by W). Then $I_2(Q, \mathcal{X})$ (resp. $I_1(Q, \mathcal{X})$) represents the functor defined by analogy with (1.2.10), in which the faithful representation (ρ, L) of Q is replaced by a representation of Q that factors through a faithful representation of Q/U (resp. Q/W). It follows immediately that

$$I_2(Q, \mathcal{X}) \xrightarrow{\sim} \pi_2^*(I(Q/U, U\backslash\mathcal{X})), \quad I_1(Q, \mathcal{X}) \xrightarrow{\sim} (\pi_1 \circ \pi_2)^*(I(Q/W, W\backslash\mathcal{X})), \tag{1.2.12}$$

the isomorphisms being canonical and, in particular, defined over $E(Q, \mathcal{X})$.

On the other hand, there are canonical isomorphisms

$$U\backslash\check{\mathcal{X}} \xrightarrow{\sim} (U\backslash\mathcal{X})^\vee, \quad W\backslash\check{\mathcal{X}} \xrightarrow{\sim} (W\backslash\mathcal{X})^\vee. \tag{1.2.13}$$

The isomorphisms (1.2.13) can be defined over $\mathbb{C}$ tautologically, in terms of the definition of $\check{\mathcal{X}}$ as a homogeneous space under $Q(\mathbb{C})$; it is then easy to see that the isomorphisms descend to $E(Q, \mathcal{X})$. Now any Q-equivariant vector bundle $\mathcal{W}$ on $\check{\mathcal{X}}$ is necessarily U-equivariant. Since U acts freely on $\check{\mathcal{X}}$, the natural map $\check{\pi}_2 : \check{\mathcal{X}} \to (U\backslash\mathcal{X})^\vee$ makes $\check{\mathcal{X}}$ a U-torsor over $(U\backslash\mathcal{X})^\vee$. It follows that

$$\mathcal{W} = \check{\pi}_2^*(\mathcal{W})$$

for some Q/U-homogeneous vector bundle $\mathcal{W}_U$ on $(U\backslash\mathcal{X})^\vee$. It is then easy to see that, in the notation of (1.2.11),

$$(\pi_{2,*}([\mathcal{W}])^{\mathcal{T}} \xrightarrow{\sim} [\mathcal{W}_U]. \tag{1.2.14}$$

In particular, $\pi_{2,*}([\mathcal{W}])^{\mathcal{T}}$ is an automorphic vector bundle over $Sh(Q/U, U\backslash\mathcal{X})$.

The analogue for π_1 is more complicated and will be discussed in §1.4.

(1.2.15) Remark Milne has proposed a generalization of Langlands' conjecture on conjugation of Shimura varieties to the setting of mixed Shimura varieties [M2, VI, Conjecture 2.1]. There is no doubt that this conjecture can be derived from the methods of [P], as in the proof of [P, Theorem 11.17], because $(P_{2g}, \mathcal{X}_2 g)$ has a canonical model over $\mathbb{Q}$. In other words, the mixed case admits a simple reduction to the pure case. It suffices to define the notation, and then the proof should be completely analogous to that of [P, Theorem 11.17].

We leave it to the reader to state the analogue of Milne's conjecture for the action of $Aut(\mathbb{C})$ on automorphic vector bundles over mixed Shimura varieties. We simply observe that, once again, the proof should be completely analogous to that of Proposition 1.2.4, above.

(1.2.16) Let $x \in \mathcal{X}$ and let $P_x \subset Q(\mathbb{R})\dot{U}(\mathbb{C})$ be its stabilizer. In the diagram (1.2.7), the homogeneous vector bundle $\beta^*(\mathcal{W})$ is determined by the isotropy representation of P_x on the fiber $\mathcal{W}_x$. We say $\mathcal{W}$ is *fully decomposed* if the isotropy representation at some point x is trivial on the unipotent radical of P_x. This condition is evidently independent of the choice of x, and we then say that the automorphic vector bundle $[\mathcal{W}]$ is fully decomposed.

(1.3) *Canonical extensions of automorphic vector bundles.* Let ${}_{K_f}Sh(Q, \mathcal{X}) \subset {}_{K_f}Sh(Q, \mathcal{X})_\Sigma$ be a (partial) toroidal compactification as an algebraic variety, as in (1.1.6). We retain Hypothesis (1.2.1) and assume for the remainder of this section that K_f is neat, in the sense of [P, 0.6], and that ${}_{K_f}Sh(Q, \mathcal{X})_\Sigma$ is an SNC compactification. We write $Sh = {}_{K_f}Sh(Q, \mathcal{X}), Sh_\Sigma = {}_{K_f}Sh(Q, \mathcal{X})_\Sigma, \partial Sh_\Sigma = \partial_{K_f}Sh(Q, \mathcal{X})_\Sigma$.

Our hypothesis that K_f is neat implies, as in the pure case (cf. the proof of [H2, Theorem 4.2]) that the connection ∇_L has unipotent monodromy. Deligne's existence theorem [De1] thus provides a unique extension of $\tilde{L}$ to a vector bundle $\tilde{L}_\Sigma$ over Sh_Σ such that ∇ extends to a connection ∇_Σ with regular singularities and nilpotent connection forms along ∂Sh_Σ.

(1.3.1) Definition. *Let $[\mathcal{W}]$ be an automorphic vector bundle on Sh. An admissible metric on $[\mathcal{W}]$ is a hermitian metric whose pullback, under the natural* *map* $\mathcal{X} \times Q(\mathbf{A}_f) \to Sh$*, is a* $Q(\mathbb{R}) \cdot U(\mathbb{C}) \times Q(\mathbf{A}_f)$*-invariant hermitian metric on* $\beta^*(\mathcal{W} \times \mathcal{X} \times Q(\mathbf{A}_f))$.

The stabilizer $Stab(x)$ in $Q(\mathbb{R}) \cdot U(\mathbb{C})$ of any point $x \in \mathcal{X}$ is a compact Lie group (Pink doesn't state this explicitly, but it is clear from Lemma 1.17 of [P]; cf. the proof of [P, Prop. 2.11].). Therefore, admissible metrics exist and their restrictions to indecomposable C^∞ summands of $[\mathcal{W}]$ are unique up to constant multiples. Note that the automorphic vector bundles that are indecomposable as C^∞ automorphic bundles are parametrized by irreducible representations of the compact group $Stab(x)$, for one (or any) $x \in \mathcal{X}$. The indecomposable *holomorphic* automorphic bundles are parametrized by locally-finite indecomposable representations of the algebraic subgroup $Q_x^0 \subset Q$, defined in (1.1).

(1.3.2) Lemma. *(i) Let* $[\mathcal{W}]$ *be an automorphic vector bundle on* Sh*. Then* $[\mathcal{W}]$ *has an admissible metric. (ii) Let* $h(\cdot,\cdot)$ *be an admissible metric on the flat vector bundle* $\tilde{L}$*. Let* $B \subset Sh_\Sigma$ *be an open polydisk with coordinates* $z_1, \ldots z_n$ *chosen so that* $B \cap \partial Sh_\Sigma$ *is the union of the coordinate hyperplanes defined by* $z_1, \ldots, z_r$ *for some* $r \leq n$*. Then*

(1.3.3) $\Gamma(B, \tilde{L}_\Sigma) =$

$\{s \in \Gamma(B \cap Sh, \tilde{L}) \mid |h(s,s)| = O(P(|log(z_1)|, \ldots, |log(z_r)|))$ *for some polynomial* $P\}$.

Proof. Part (i) has already been explained. To prove Part (ii), we reduce to the case of an equivariant torus embedding. Indeed, let $Q' \subset Q$ be a rational boundary subgroup and let $\mathcal{X}'$ be the corresponding rational boundary component of $\mathcal{X}$, in the sense of [P, §4]. We let $U' \subset W' = R_u Q' \subset Q'$ be the canonical filtration of Q', and let $\pi'_2 : Sh(Q', \mathcal{X}') \to Sh(Q'/U', U' \backslash \mathcal{X}')$ be the canonical map, representing $Sh(Q', \mathcal{X}')$ as the locally constant pro-torus fibration over $Sh(Q'/U', U' \backslash \mathcal{X}')$ with structure group $\mathcal{T}'$, as in the discussion preceding (1.2.11). There is a natural embedding $\check{\mathcal{X}}' \subset \check{\mathcal{X}}$ (cf. [HZ1, Lemma 4.5.8]), extending the inclusion of $\beta(\mathcal{X}')$ in $\check{\mathcal{X}}$ as a Siegel domain of the third kind. There is also a neighborhood $Sh(Q', \mathcal{X}')^+_{\Sigma_{Q'}}$ of the boundary $\partial Sh(Q', \mathcal{X}')_{\Sigma_{Q'}}$ admitting a natural map

$$f_{Q'} : Sh(Q', \mathcal{X}')^+_{\Sigma_{Q'}} \to Sh_\Sigma. \tag{1.3.4}$$

The map $f_{Q'}$ is a local isomorphism in a neighborhood of the part of the boundary of Sh_Σ corresponding to Q'.

We assume Q' chosen so that the polydisk B lies in the image of $f_{Q'}$. The flat vector bundle $\tilde{L}$ is defined by the representation ρ, say of Q. The restriction of ρ to Q' defines a flat vector bundle $(\tilde{L})'$ on $Sh(Q',\mathcal{X}')$ whose restriction to $Sh(Q',\mathcal{X}')^+_{\Sigma_{Q'}} \cap Sh(Q',\mathcal{X}')$ is equivalent to $f^*_{Q'}(\tilde{L})$. Then it is clear by uniqueness that the admissible metric $h(\cdot,\cdot)$ on $\tilde{L}$ pulls back to the restriction to $Sh(Q',\mathcal{X}')^+_{\Sigma_{Q'}}$ of an admissible metric on $(\tilde{L})'$. Thus we may assume $Q' = Q$ and that B is not contained in the image of $f_{Q''}$ for any proper boundary subgroup Q'' of Q. In other words, possibly shrinking B, the map π_2 can be written as a product

$$\pi_2 : B \xrightarrow{\sim} B_0 \times \mathcal{T}_{\Sigma(\mathcal{T})}$$

where B_0 is open in $Sh(Q/U, U\backslash\mathcal{X})$ and $\mathcal{T} \hookrightarrow \mathcal{T}_{\Sigma(\mathcal{T})}$ is an equivariant torus embedding, with $\mathcal{T}$ a torus of the form $U(\mathbb{Q})\backslash U(\mathbb{C}) \times U(\mathbf{A}_f)/K_{U,f}$ as in the discussion at the end of §1.2. Near B the local system is thus defined by a representation of the fundamental group of $\mathcal{T}$. In this way, we may assume $Q = U$, Sh is a torus of the form $U(\mathbb{C})/\Lambda$, for some lattice Λ in $U(\mathbb{R})$, $\tilde{L}$ is the local system defined by a representation (ρ, L) of Λ, and $h(\cdot,\cdot)$ is a $U(\mathbb{C})$-invariant metric. The representation ρ is the restriction to Λ of a unipotent representation of the algebraic group U. Moreover, we may assume $n = r$ and $Sh \hookrightarrow Sh_\Sigma$ is isomorphic to the obvious embedding $\mathbb{C}^{\times,n} \to \mathbb{C}^n$, with coordinates $z_1,\ldots,z_n$. In particular, Sh_Σ is affine, hence $\tilde{L}_\Sigma$ generated by its global sections.

Let $j : U(\mathbb{C}) \times U(\mathbf{A}_f) \to Sh$ denote the natural map. We fix a basepoint, say o, in $U(\mathbb{C})$, and a basis $\{e_1,\ldots,e_N\}$ of L, such that $h(e_i,e_j)_o$ is the identity matrix. It follows from the $U(\mathbb{C})$-invariance of $h(\cdot,\cdot)$ that, with respect to the trivialization $j^*(\tilde{L}) \xrightarrow{\sim} U(\mathbb{C}) \times L$, the matrix of $h(e_i,e_j)_u$ is polynomial in the variable $u \in U(\mathbb{C})$, hence have logarithmic growth in terms of the coordinates $z_1,\ldots,z_n$. Now let $\{v_1,\ldots,v_N\}$ denote a basis of global sections of $\tilde{L}_\Sigma$, and express the v_i in terms of the horizontal sections e_j over the universal cover $U(\mathbb{C})$ of Sh:

$$v_i = \sum_j f_{ij} e_j,$$

where f_{ij} are functions on $U(\mathbb{C})$. Since the monodromy of $\tilde{L}$ is unipotent, the f_{ij} have logarithmic growth in terms of $z_1,\ldots,z_n$. The inclusion of the left-hand side of (1.3.3) in the right-hand side is now clear. The opposite inclusion then follows because any section satisfying an inequality of the indicated form has removable singularities with respect to the basis $\{v_1,\ldots,v_N\}$.

A local section of $\tilde{L}$ satisfying an inequality of type (1.3.3) is said to be *slowly increasing*.

In (ii) of the following Definition-Proposition, we let $(Q',\mathcal{X}')$, U', and so on be as in the proof of Lemma 1.3.2.

(1.3.5) Definition-Proposition. *Let $[\mathcal{W}]$ be an automorphic vector bundle on Sh. A canonical extension $[\mathcal{W}]_\Sigma$ of $[\mathcal{W}]$ over Sh_Σ is a vector bundle over Sh_Σ satisfying one of the following equivalent conditions:*

(i) If $h(\cdot,\cdot)$ be an admissible metric on $[\mathcal{W}]$ and if B is as in Lemma (1.3.2,ii), then

$$\Gamma(B,[\mathcal{W}]_\Sigma) = \{s \in \Gamma(B\cap Sh,[\mathcal{W}]) : |h(s,s)| = O(P(|log(z_1)|,\dots|log(z_r)|))$$

for some polynomial P, which may depend on s.

(ii) Let $(Q',\mathcal{X}')$ be a rational boundary pair for $(Q,\mathcal{X})$, as above. Write $\Sigma = \bigcup \Sigma_P$, the union taken over the rational boundary subgroups P of Q. If $\sigma \in \Sigma_{Q'}$, let $Sh(Q',\mathcal{X}')_\sigma \subset Sh(Q',\mathcal{X}')_{\Sigma_{Q'}}$ denote the corresponding partial compactification, and let

$$\pi'_{2,\sigma} : Sh(Q',\mathcal{X}')_\sigma \to Sh(Q'/U',U'\backslash\mathcal{X}')$$

denote the natural extension of π'_2 (cf. [H2, 4.1] for the analogue in the pure case). Let $[\mathcal{W}]'$ denote the automorphic vector bundle on $Sh(Q',\mathcal{X}')$ associated to the Q'-homogeneous vector bundle $\mathcal{W}$, restricted to $\check{\mathcal{X}}' \subset \check{\mathcal{X}}$. Then via the natural map (1.3.4) we have canonical isomorphisms

$$\phi_\sigma : f^*_{Q'}([\mathcal{W}]_\sigma) \xrightarrow{\sim} (\pi'_{2,\sigma})^*(\pi'_{2,*}([\mathcal{W}]')^{\mathcal{T}'}),$$

and these isomorphisms are compatible with respect to inclusions $\sigma \subset \tau$.

(iii) There is a flat automorphic vector bundle $\tilde{L}$ such that $[\mathcal{W}]$ can be realized as an automorphic subquotient of $\tilde{L}$, and $[\mathcal{W}]_\Sigma$ is a locally free subquotient of the Deligne extension $\tilde{L}_\Sigma$, compatibly with the given realization over Sh.

Proof. The existence of $[\mathcal{W}]_\Sigma$ satisfying (iii) is proved in exactly the same way as the corresponding assertion for pure Shimura varieties [H2, Theorem 4.2]. The equivalence of (iii) with (ii), similarly, is identical to the proof of (4.2.2) of [H2].

Let B be as in (i). It follows from Lemma (1.3.2,ii) that, if $[\mathcal{W}]_\Sigma$ satisfies (iii), then there is an inclusion

(1.3.6) $$\Gamma(B,[\mathcal{W}]_\Sigma) \subset \{s \in \Gamma(B\cap Sh,[\mathcal{W}]) | s \text{ is slowly increasing}\}.$$

But then every $s \in \Gamma(B\cap Sh,[\mathcal{W}])$ in the right-hand side of (1.3.6) has a removable singularity along ∂Sh_Σ and therefore extends to an element of $\Gamma(B,[\mathcal{W}]_\Sigma)$. This shows the equivalence of (iii) with (i).

Henceforward, unless an explicit statement is made to the contrary, we assume our partial compactifications Sh_Σ are actually compact, as well as SNC. Then ∂Sh_Σ is a complete divisor with normal crossings on Sh_Σ, defined by the invertible sheaf of ideals $\mathcal{I}_{\partial Sh_\Sigma}$. If $[\mathcal{W}]$ is an automorphic vector bundle on Sh with canonical extension $[\mathcal{W}]_\Sigma$, we define the *subcanonical extension* of $[\mathcal{W}]$ to be

$$[\mathcal{W}]^{sub} = [\mathcal{W}]^{sub}_\Sigma = [\mathcal{W}]_\Sigma \otimes \mathcal{I}_{\partial Sh_\Sigma},$$

the subscript $_\Sigma$ will be omitted whenever possible. Similarly, we will write $[\mathcal{W}]^{can}$ in place of $[\mathcal{W}]_\Sigma$ when Σ is understood.

The theory of coherent cohomology developed in [H3, H4] extends without change to the case of mixed Shimura varieties. We state the main results in the present context.

(1.3.7) Proposition. *Let $(Q, \mathcal{X})$ be a mixed Shimura datum, and let $[\mathcal{W}]$ be an automorphic vector bundle on $Sh = {}_{K_f}Sh(Q, \mathcal{X})$, for some open compact subgroup K_f of $Q(\mathbf{A}_f)$. Let Sh_Σ be a toroidal compactification of Sh. (i) Let Σ_1 be a refinement of Σ, and let $f_{\Sigma_1,\Sigma} : Sh_{\Sigma_1} \to Sh_\Sigma$ be the corresponding morphism of compactifications. Then there are isomorphisms*

$$f^*_{\Sigma_1,\Sigma}([\mathcal{W}]_\Sigma) \xrightarrow{\sim} [\mathcal{W}]_{\Sigma_1}, \quad f^*_{\Sigma_1,\Sigma}([\mathcal{W}]^{sub}_\Sigma) \xrightarrow{\sim} [\mathcal{W}]^{sub}_{\Sigma_1},$$

functorial in $\mathcal{W}$, and the natural morphisms

$$H^q(Sh_\Sigma, [\mathcal{W}]_\Sigma) \to H^q(Sh_{\Sigma_1}, [\mathcal{W}]_{\Sigma_1}),$$

$$H^q(Sh_\Sigma, [\mathcal{W}]^{sub}_\Sigma) \to H^q(Sh_{\Sigma_1}, [\mathcal{W}]^{sub}_{\Sigma_1})$$

are isomorphisms for all q.

(ii) For each q, we define

$$\tilde{H}^q({}_{K_f}Sh(Q, \mathcal{X}), [\mathcal{W}]^{can}) = H^q({}_{K_f}Sh(Q, \mathcal{X})_\Sigma, [\mathcal{W}]_\Sigma)$$

for any Σ; we can also write

$$\tilde{H}^q({}_{K_f}Sh(Q, \mathcal{X}), [\mathcal{W}]^{can}) = \varinjlim_{\Sigma} H^q({}_{K_f}Sh(Q, \mathcal{X})_\Sigma, [\mathcal{W}]_\Sigma),$$

the limit taken over the system of all Σ's, all maps being isomorphisms. Similarly, we define

$$\tilde{H}^q({}_{K_f}Sh(Q, \mathcal{X}), [\mathcal{W}]^{sub}) = \varinjlim H^q({}_{K_f}Sh(Q, \mathcal{X})_\Sigma, [\mathcal{W}]^{sub}_\Sigma),$$

Let

$$\tilde{H}^q([\mathcal{W}]^{can}) = \varinjlim_{K_f} \tilde{H}^q({}_{K_f}Sh(Q,\mathcal{X}), [\mathcal{W}]^{can}),$$

$$\tilde{H}^q([\mathcal{W}]^{sub}) = \varinjlim_{K_f} \tilde{H}^q({}_{K_f}Sh(Q,\mathcal{X}), [\mathcal{W}]^{sub}).$$

Then $\tilde{H}^q([\mathcal{W}]^{can})$ *and* $\tilde{H}^q([\mathcal{W}]^{sub})$ *are naturally admissible* $Q(\mathbf{A}_f)$*-modules.*

The proof of Proposition 1.3.7 follows word for word the proofs of Propositions 2.4 and 2.6 of [H4]. The action of $Q(\mathbf{A}_f)$ arising in (ii) is defined as in [H4, (2.5.2)]: to any $h \in Q(\mathbf{A}_f)$ and any fixed toroidal compactification ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ at level K_f, there is an analogous compactification ${}_{K_f^h}Sh(Q,\mathcal{X})_{\Sigma^h}$ of ${}_{K_f^h}Sh(Q,\mathcal{X})$, where $K_f^h = h^{-1}K_f h$. Right-multiplication by h defines an isomorphism ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma \xrightarrow{\sim} {}_{K_f^h}Sh(Q,\mathcal{X})_{\Sigma^h}$ that respects automorphic vector bundles and canonical extensions. By functoriality, we thus obtain an operator t_h in the limit on $\tilde{H}^q([\mathcal{W}]^{can})$ and $\tilde{H}^q([\mathcal{W}]^{sub})$.

For any automorphic vector bundle $[\mathcal{W}]$ on Sh we consider the Dolbeault complex $\mathcal{A}^{0,\bullet}([\mathcal{W}])$ on Sh. For each q, $\mathcal{A}^{0,q}([\mathcal{W}])$ will denote the sheaf of C^∞-sections of the bundle $\Omega^{0,q} \otimes [\mathcal{W}]$ of $(0,q)$-forms with values in $[\mathcal{W}]$. Letting j_Σ denote the embedding $Sh \hookrightarrow Sh_\Sigma$, we have, for each q, the direct image sheaf $j_{\Sigma,*}(\mathcal{A}^{0,q}([\mathcal{W}]))$ on Sh_Σ. We let $\mathcal{A}^{0,q}([\mathcal{W}]))_{si} \subset j_{\Sigma,*}(\mathcal{A}^{0,q}([\mathcal{W}]))$ denote the subsheaf of sections s such that s and $\overline{\partial}s$ are both slowly increasing, in the sense of (1.3.3), with respect to admissible metrics on the bundles $\Omega^{0,q} \otimes [\mathcal{W}]$ and $\Omega^{0,q+1} \otimes [\mathcal{W}]$. Similarly, we define the *rapidly decreasing* sections, and the sheaf $\mathcal{A}^{0,q}([\mathcal{W}]))_{rd}$, as in [HZ1, 3.8.2]. We let $C^{0,q}([\mathcal{W}])_* = \varinjlim_{K_f,\Sigma} \Gamma(Sh_\Sigma, \mathcal{A}^{0,q}([\mathcal{W}])$, where $*$ denotes either si or rd.

(1.3.8) Theorem. *There is a natural commutative diagram of* $Q(\mathbf{A}_f)$*-modules, where the horizontal maps are isomorphisms:*

$$\begin{array}{ccc} \tilde{H}^\bullet([\mathcal{W}]^{sub}) & \xrightarrow{\sim} & H^\bullet(C^{0,\bullet}([\mathcal{W}])_{rd}) \\ \downarrow & & \downarrow \\ \tilde{H}^\bullet([\mathcal{W}]^{can}) & \xrightarrow{\sim} & H^\bullet(C^{0,\bullet}([\mathcal{W}])_{si}) \end{array}$$

The proof is identical to that of [H4, Corollary 3.4].

A purely algebraic construction of the canonical extension of an automorphic vector bundle was given in [HZ1], in the case of pure Shimura varieties. The analogous construction works here as well.

(1.3.9) Proposition. *We retain the notation of Proposition 1.3.7.*

(i) The standard principal bundle $I(Q,\mathcal{X})$ *extends to an* $E(Q,\mathcal{X})$*-rational* $Q(\mathbf{A}_f)$*-equivariant principal* Q*-bundle* $I(Q,\mathcal{X})$ *over* $Sh(Q,\mathcal{X})$

(ii) The morphism $p : I(Q, \mathcal{X}) \to \check{\mathcal{X}}$ extends to a Q-equivariant morphism p_Σ : $I(Q, \mathcal{X})_\Sigma \to \check{\mathcal{X}}$, rational over $E(Q, \mathcal{X})$.

(iii) For any automorphic vector bundle $[\mathcal{W}]$ on $Sh(Q, \mathcal{X})$, there is a canonical isomorphism of vector bundles over $Sh(Q, \mathcal{X})_\Sigma$:

$$[\mathcal{W}]_\Sigma \xrightarrow{\sim} p_\Sigma^*(\mathcal{W})/Q.$$

This isomorphism is rational over the field of definition of $\mathcal{W}$ as a Q-equivariant vector bundle over $\check{\mathcal{X}}$. Over $Sh(Q, \mathcal{X})$ this isomorphism restricts to the construction of $[\mathcal{W}]$ given in §1.2.

Proof. The proof is identical to that of Lemma 4.4.2 of [HZ1].

(1.3.10) Remark. Just as in Remark (1.2.15), we can formulate a conjecture for the action of $Aut(\mathbb{C})$ on the canonical extensions of automorphic vector bundles. It should again be a simple matter to reduce this conjecture to the case of pure Shimura varieties, where it is known [BHR, Proposition 1.4.3].

(1.4) *Functorial properties of canonical extensions.* We retain the notation of the previous sections, and let $\pi_2 : Sh(Q, \mathcal{X}) \to Sh(Q/U, U\backslash\mathcal{X})$ and $\pi_1 : Sh(Q/U, U\backslash\mathcal{X}) \to Sh(G, W\backslash\mathcal{X})$ be the morphisms of (1.1.5). Let $K_f \subset Q(\mathbf{A}_f)$ be any neat open compact subgroup and let $K_f^2 = K_f/(U(\mathbf{A}_f) \cap K_f) \subset (Q/U)(\mathbf{A}_f)$, $K_f^1 = K_f/(W(\mathbf{A}_f) \cap K_f) \subset G(\mathbf{A}_f)$. Then there are natural morphisms at level K_f, also denoted π_2 and π_1:

$$(1.4.1) \qquad {}_{K_f}Sh(Q, \mathcal{X}) \xrightarrow{\pi_2} {}_{K_f^2}Sh(Q/U, U\backslash\mathcal{X}) \xrightarrow{\pi_1} {}_{K_f^1}Sh(G, W\backslash\mathcal{X}).$$

Now we can find families of fans Σ, Σ_2, Σ_1, admissible relative to K_f, K_f^2, and K_f^1, respectively, such that ${}_{K_f}Sh(Q, \mathcal{X})_\Sigma$ is compact and such that the morphisms in (1.4.1) extend to the corresponding toroidal compactifications [P, 6.7(b)], which we assume to be SNC and rational over $E(Q, \mathcal{X})$:

$$(1.4.2) \qquad {}_{K_f}Sh(Q, \mathcal{X})_\Sigma \xrightarrow{\pi_{2,\Sigma}} {}_{K_f^2}Sh(Q/U, U\backslash\mathcal{X})_{\Sigma_2} \xrightarrow{\pi_{1,\Sigma}} {}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}.$$

In what follows, we let $\mathcal{W}$ be a Q-homogeneous vector bundle on $\check{\mathcal{X}}$ and let $[\mathcal{W}]$ be the corresponding automorphic vector bundle on ${}_{K_f}Sh(Q, \mathcal{X})$.

(1.4.3) Proposition. *With the above notation, the higher direct images $R^q\pi_{2,\Sigma,*}[\mathcal{W}]^{can} = 0$, $R^q\pi_{2,\Sigma,*}[\mathcal{W}]^{sub} = 0$ for $q > 0$, and we have canonical isomorphisms*

(1.4.3.1)

$$\pi_{2,\Sigma,*}[\mathcal{W}]^{sub} \xrightarrow{\sim} (\pi_{2,\Sigma,*}[\mathcal{W}])^{sub} \xrightarrow{\sim} [\mathcal{W}_U]^{sub}, \tag{1.4.3.2}$$

in the notation of (1.2.14). Moreover, there is a canonical isomorphism

$$[\mathcal{W}]^{can} \xrightarrow{\sim} \pi^*_{2,\Sigma}(\pi_{2,\Sigma,*}[\mathcal{W}])^{can}. \tag{1.4.3.3}$$

Proof. The isomorphism (1.4.3.3) is analogous to (i) of [HZ1, Proposition 3.12.2] and is proved in the same way. The vanishing of the higher direct images of $[\mathcal{W}]^{can}$, together with the first isomorphism in (1.4.3.1), is a mild generalization of (ii) of [HZ1, Proposition 3.12.2]. The proof in the present situation is identical, the basic local ingredient being a calculation [HZ1, Lemma 1.6.8 (iii)] in the setting of proper morphisms of torus embeddings. (A sharper statement could have been made in [HZ1]: there it is asserted that there is a quasi-isomorphism $(\pi_{2,\Sigma,*}[\mathcal{W}])^{can} \approx \mathbf{R}\pi_{2,\Sigma,*}[\mathcal{W}]^{can}$, but the proof actually verifies the precise vanishing and isomorphism indicated here.) The second isomorphism in (1.4.3.1) is clear from (1.2.14).

The assertions for subcanonical extensions follow as in [HZ1, (3.14.1)] from the toric version proved in [HZ1, Lemma 1.6.8 (iii)].

The following proposition is more substantial. In what follows, we let $\mathfrak{v} = Lie(V)$. Any $x \in \mathcal{X}$, determines a subalgebra $F^0_x(Lie(Q)) \subset Lie(Q)$, as in (1.1), and we let $\mathfrak{v}^-_x = F^0_x(Lie(Q)) \cap \mathfrak{v}$, as in [HZ1, 1.8]. The (abelian) Lie algebra $\mathfrak{v}^-_x$ depends only on the image $\check{\pi}_1 \circ \check{\pi}_2$ of x in $W\backslash\mathcal{X}$, which we denote $\overline{x}$ in the following proposition.

(1.4.4) Proposition. *Let $\mathcal{V}$ be a Q/U-homogeneous vector bundle on $U\backslash\check{\mathcal{X}}$ and let $[\mathcal{V}]$ be the corresponding automorphic vector bundle on ${}_{K^2_f}Sh(Q/U, U\backslash\mathcal{X})$. For any integer q, let $\mathcal{V}^{(q)}$ be the G-homogeneous vector bundle on $W\backslash\check{\mathcal{X}} \simeq (W\backslash\mathcal{X})^\vee$ whose fiber at the point $\overline{x}$ is the representation of $F^0_{\overline{x}}(Lie(G))$ on $H^q(\mathfrak{v}^-_x, \mathcal{V}_x)$, for any $x \in \check{\mathcal{X}}$. Then for all q, there are canonical isomorphisms*

$$R^q\pi_{1,\Sigma,*}[\mathcal{V}]^{can} \xrightarrow{\sim} (R^q\pi_{1,\Sigma,*}[\mathcal{V}])^{can} \xrightarrow{\sim} [\mathcal{V}^{(q)}]^{can} \tag{1.4.4.1}$$

$$R^q\pi_{1,\Sigma,*}[\mathcal{V}]^{sub} \xrightarrow{\sim} (R^q\pi_{1,\Sigma,*}[\mathcal{V}])^{sub} \xrightarrow{\sim} [\mathcal{V}^{(q)}]^{sub} \tag{1.4.4.2}$$

of vector bundles on ${}_{K^1_f}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}$. The family of such isomorphisms, as $\mathcal{V}$ varies, is rational over $E(Q,\mathcal{X})$; in other words, the isomorphisms respect canonical models.

Proof. Using the characterization of the canonical extensions by growth conditions in (1.3.5(i)), the first isomorphism in (1.4.4.1) is a slight generalization of Proposition 3.13.4(i) of [HZ1], and is proved in the same way. The second isomorphism which incorporates the rationality assertion, is proved by the argument used to prove Lemma 4.7.14 of [HZ1].

Similarly, the subcanonical extension $[\mathcal{V}]^{sub}$ can be defined as the subsheaf of $[\mathcal{V}]^{can}$ of sections which are rapidly decreasing at the boundary, cf. 1.3.8. The proof of Proposition 3.13.4(i) of [HZ1] works just as well for rapidly decreasing sections, yielding the first isomorphism of (1.4.4.2). The second isomorphism then follows from the second isomorphism of (1.4.4.1).

(1.4.5) Corollary. *Write* $\pi_\Sigma = \pi_{1,\Sigma} \circ \pi_{2,\Sigma}$. *There are spectral sequences:*

(1.4.5.1)
$$\begin{aligned} E_2^{p,q} = H^p({}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}, (R^q\pi_{\Sigma,*}[\mathcal{W}])^{can}) \\ \Rightarrow\ H^{p+q}({}_{K_f^2}Sh(Q/U, U\backslash\mathcal{X})_{\Sigma_2}, [\mathcal{W}_U]^{can}) \\ \xrightarrow{\sim}\ H^{p+q}({}_{K_f}Sh(Q, \mathcal{X})_\Sigma, [\mathcal{W}]^{can}); \end{aligned}$$

(1.4.5.2)
$$\begin{aligned} E_2^{p,q} = H^p({}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}, (R^q\pi_{\Sigma,*}[\mathcal{W}])^{sub}) \\ \Rightarrow\ H^{p+q}({}_{K_f^2}Sh(Q/U, U\backslash\mathcal{X})_{\Sigma_2}, [\mathcal{W}_U]^{sub}) \\ \xrightarrow{\sim}\ H^{p+q}({}_{K_f}Sh(Q, \mathcal{X})_\Sigma, [\mathcal{W}]^{sub}). \end{aligned}$$

Suppose $[\mathcal{W}]$ *is fully decomposed (see (1.2.16)). Then these spectral sequences degenerate at* E_2 *and the filtration induced by the* E_2 *term splits canonically.*

Proof. Given Propositions 1.4.3 and 1.4.4, this is just the Leray spectral sequence. Degeneration and splitting at E_2 in the fully decomposed case is proved as in [HZ1, Cor. 3.13.5, Prop. 3.14.2].

(1.4.6) At no point have we seriously used the fact that W is the full unipotent radical of Q. Thus suppose $W' \subset W'' \subset R_uQ$ is a sequence of unipotent subgroups, and let $U' = W_{-2}Q \cap W'$, $Q' = Q/W'$, and define U'' and Q'' analogously. There are morphisms of mixed Shimura data

$$(Q, \mathcal{X}) \to (Q', \mathcal{X}') \to (Q'', \mathcal{X}''),$$

and corresponding morphisms of mixed Shimura varieties

(1.4.6.1)
$$_\bullet Sh(Q, \mathcal{X}) \xrightarrow{\pi'} {}_\bullet Sh(Q', \mathcal{X}') \xrightarrow{\pi''} {}_\bullet Sh(Q'', \mathcal{X}'');$$

here $\mathcal{X}' = W'\backslash\mathcal{X}$, $\mathcal{X}'' = W''\backslash\mathcal{X}$, and we write $\bullet$ to avoid having to keep track of level subgroups.

We can factor $\pi' = \pi_1' \circ \pi_2'$, $\pi'' = \pi_1'' \circ \pi_2''$ as above, with $\pi_2' : {}_\bullet Sh(Q, \mathcal{X}) \to {}_\bullet Sh(Q/U', U' \backslash \mathcal{X})$, etc. We choose families of fans Σ, Σ_2', Σ', Σ_2'', and Σ'' that define compatible toroidal compactifications

$$
\begin{aligned}
{}_\bullet Sh(Q,\mathcal{X})_\Sigma \overset{\pi'_{2,\Sigma}}{\to} {}_\bullet Sh(Q/U', U'\backslash\mathcal{X})_{\Sigma_2'} \overset{\pi'_{1,\Sigma}}{\to} {}_\bullet Sh(Q',\mathcal{X}')_{\Sigma'} \\
\overset{\pi''_{1,\Sigma}}{\to} {}_\bullet Sh(Q'/U'', U''\backslash\mathcal{X}')_{\Sigma_2''} \overset{\pi''_{1,\Sigma}}{\to} {}_\bullet Sh(Q'',\mathcal{X}'')_{\Sigma''}
\end{aligned}
$$

in the obvious notation.

(1.4.7) Proposition. *Let $[\mathcal{W}]$ be a completely decomposed automorphic vector bundle on $Sh(Q, \mathcal{X})$. There are spectral sequences*

$$
\begin{aligned}
E_2^{p,q} = H^p({}_\bullet Sh(Q^?, \mathcal{X}^?)_{\Sigma^?}, (R^q \pi^?_{\Sigma,*}[\mathcal{W}])^{can}) \\
\Rightarrow \; H^{p+q}({}_\bullet Sh(Q/U^?, U^?\backslash\mathcal{X})_{\Sigma_2^?}, [\mathcal{W}_{U^?}]^{can}) \\
\overset{\sim}{\longrightarrow} \; H^{p+q}({}_\bullet Sh(Q,\mathcal{X})_\Sigma, [\mathcal{W}]^{can});
\end{aligned} \tag{1.4.7.1}
$$

and for $j \geq 0$ there are canonical isomorphisms

$$
[\mathcal{W}_{U^?}^{(j),?}]^{can} \overset{\sim}{\longrightarrow} R^j \pi^?_{\Sigma,*}[\mathcal{W}])^{can}, \tag{1.4.7.2}
$$

with $? = ', ''$, where $\mathcal{W}_{U^?}$ $\mathcal{W}_{U^?}^{(j),?}$ are defined as in Propositions 1.4.3 and 1.4.4, respectively. These spectral sequences degenerate and split canonically at E_2.

Likewise, for each index j, there is a spectral sequence

(1.4.7.3)

$$
\begin{aligned}
E_2^{p,q} = H^p({}_\bullet Sh(Q'', \mathcal{X}'')_{\Sigma''}, (R^q \pi''_{\Sigma,*}[\mathcal{W}^{(j)}_{U''}])^{can}) \\
\Rightarrow \; H^{p+q}({}_\bullet Sh(Q'/U'', U''\backslash\mathcal{X}')_{\Sigma_2''}, [\mathcal{W}^{(j)}_{U''}]^{can}) \\
\overset{\sim}{\longrightarrow} \; H^{p+q}({}_\bullet Sh(Q',\mathcal{X}')_{\Sigma'}, [\mathcal{W}^{(j)}_{U'}]^{can})
\end{aligned}
$$

canonically degenerate and split at E_2. Finally, for $p, q, j \geq 0$, the composition of the splittings

$$
\begin{aligned}
H^p({}_\bullet Sh(Q'', \mathcal{X}'')_{\Sigma''}, (R^q \pi''_{\Sigma,*}([\mathcal{W}^{(j)}_{U''}])^{can})) \to H^{p+q}({}_\bullet Sh(Q',\mathcal{X}')_{\Sigma'}, [\mathcal{W}^{(j)}_{U'}]^{can}) \\
\to H^{p+q+j}({}_\bullet Sh(Q,\mathcal{X})_\Sigma, [\mathcal{W}]^{can}),
\end{aligned}
$$

where the first arrow is as in (1.4.7.3) and the second is as in (1.4.7.1) and we identify

$$
[\mathcal{W}^{(j)}_{U'}]^{can} \overset{\sim}{\longrightarrow} R^j \pi'_{\Sigma,*}[\mathcal{W}]^{can}
$$

as in (1.4.7.2), is the canonical splitting

$$
H^p({}_\bullet Sh(Q'', \mathcal{X}'')_{\Sigma''}, (R^{q+j}(\pi''_\Sigma \circ \pi'_\Sigma)_*[\mathcal{W}])^{can})) \to H^{p+q+j}({}_\bullet Sh(Q,\mathcal{X})_\Sigma, [\mathcal{W}]^{can})
$$

of (1.4.7.1).

All of these assertions remain true if ^{can} *is replaced by* ^{sub}.

Proof. The assertions regarding (1.4.7.1), resp. (1.4.7.2), are proved as in Corollary (1.4.5), resp. Proposition (1.4.4). The assertion regarding the composition of the splittings is an easy consequence of the spectral sequence of a composite functor.

(1.5) *Periods and rationality.* The articles [H1] and [M1] define canonical rational structures on automorphic vector bundles in terms of CM motives. This construction is adapted in section 4.3 of [HZ1] to the mixed Shimura varieties treated there. We briefly show how this construction extends to general mixed Shimura varieties. Then, following sections 4.7 and 4.8 of [HZ1], we show that the isomorphisms of Propositions (1.4.3) and (1.4.4) are compatible with these canonical rational structures.

Let $\mathcal{W}$ be a Q-homogeneous vector bundle on $\check{\mathcal{X}}$, and let $[\mathcal{W}]$ be the corresponding automorphic vector bundle. If $(x, q_f) \in \mathcal{X} \times Q(\mathbf{A}_f)$, we let $[x, q_f]$ denote the corresponding point in $Sh(Q, \mathcal{X})(\mathbb{C})$. As in the pure case, there is a canonical "periods" isomorphism

$$\text{(1.5.1)} \qquad \mathrm{Per}_{(x,q_f)} \mathcal{W}_{\beta(x)} \xrightarrow{\sim} [\mathcal{W}]_{[x,q_f]}$$

(cf. [HZ1, (4.3.5)]). Suppose $(Q, \mathcal{X}) = (T, h)$, with T a torus, or more generally that $[x, q_f]$ is a special point corresponding to a morphism $(T, h) \to (Q, \mathcal{X})$. Then the homogeneous vector bundle $\mathcal{W}$ defines a representation $\sigma = \sigma_{\mathcal{W}}$ of T and a period element

$$p(h, \sigma_{\mathcal{W}}) \in Aut([\mathcal{W}]|_{Sh(T,h)}(\mathbb{C})),$$

well-defined modulo $\sigma_{\mathcal{W}}(T(\mathbb{Q}))$, that compares the de Rham and Betti rational structures of $[\mathcal{W}]|_{Sh(T,h)}(\mathbb{C})$. See [HZ1, (4.3.6)] for the precise definition.

Suppose $\mathcal{W}$ is rational over the extension $E(\mathcal{W})$ of $E(Q, \mathcal{X})$, as Q-homogeneous vector bundle. By Proposition 1.2.8, $[\mathcal{W}]$ has a canonical model over $E(\mathcal{W})$. This model has the following description in terms of the period elements: For any map $(T, h) \hookrightarrow (Q, \mathcal{X})$, with T a torus, let $E(T, h; \mathcal{W}) = E(T, h) \cdot E(\mathcal{W})$. Then for any field $L \supset E(T, h; \mathcal{W})$ and any $t_f \in T(\mathbf{A}_f)$, we have

$$\text{(1.5.2)} \qquad [\mathcal{W}](L)|_{[h,t_f]} = p(h, \sigma_{\mathcal{W}}) \cdot \mathrm{Per}_{(h,t_f)}(\mathcal{W}_{\beta(h)}(L)) \subset [\mathcal{W}]|_{[h,t_f]}(\mathbb{C}).$$

This summarizes and generalizes the discussion following [HZ1, (4.3.6)].

Now let $W' \subset R_u Q$ and $U' = W' \cap W_{-2} R_u Q$ as in (1.4.6), and let $\pi : \mathcal{X} \to W' \backslash \mathcal{X}$ be the natural map. For $j \geq 0$ we define the homogeneous vector bundle $\mathcal{W}'^{(j)}$ on

$W'\backslash\mathcal{X}$ as in the previous section; its fiber at the point $\pi(x) \in W'\backslash\mathcal{X}$, for $x \in \mathcal{X}$, is given by

$$\mathcal{W}_x^{(j)} \stackrel{def}{=} H^j((\mathfrak{v}'_x)^-, \mathcal{W}_x), \tag{1.5.3}$$

where $(\mathfrak{v}'_x)^- = \mathfrak{v}_x \cap Lie(W')$. Let $\sigma^j_{x,\mathcal{W}}$ denote the representation of $Stab_{W'\backslash Q}(\pi(x))$ on $\mathcal{W}_x^{(j)}$. The following lemma is the analogue of Lemma 4.7.14 of [HZ1], and is proved in the same way:

(1.5.4) Lemma. *Let $(T,h) \hookrightarrow (Q,\mathcal{X})$ be a CM pair, as above. Let $L \supset E(T,h;\mathcal{W})$ and $t_f \in T(\mathbf{A}_f)$. Then we have*

$$p(h, \sigma^j_{x,\mathcal{W}}) \cdot \mathrm{Per}_{(h,t_f)} \mathcal{W}_x^{(j)}(L) = R^j\pi_*[\mathcal{W}](L)_{[h,t_f]} \subset R^j\pi_*[\mathcal{W}]|_{[h,t_f]}(\mathbb{C}).$$

(1.6) *Boundary strata* For future reference, we introduce notation for the stratification of the toroidal boundary. For this we need to recall the classification of rational boundary components of $Sh(Q,\mathcal{X})$. Let $G^{ad} = G_1 \times \cdots \times G_r$ be the decomposition into $\mathbb{Q}$-simple factors. Let $P_i \subset G_i$ be a $\mathbb{Q}$-parabolic subgroup for every i, and let $P \subset Q$ be the inverse image of $P_1 \times \cdots \times P_r$. Then [P, Definition 4.5] P is an *admissible* $\mathbb{Q}$-parabolic subgroup of Q if every P_i is either equal to G_i or to a maximal proper $\mathbb{Q}$-parabolic subgroup of G_i.

In particular, every admissible $\mathbb{Q}$-parabolic subgroup of Q is the pullback of an admissible $\mathbb{Q}$-parabolic subgroup of G. Let $\overline{P}$ be an admissible $\mathbb{Q}$-parabolic subgroup of G. We define the $\overline{P}$-stratum ${}_{K_f^1}Sh(G, W\backslash\mathcal{X})^{\overline{P}}_{\Sigma_1}$ of ${}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}$ as in [HZ1, 1.7], and define the P-stratum of ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ to be

$$ {}_{K_f}Sh(Q,\mathcal{X})^P_\Sigma = (\pi_{2,\Sigma} \circ \pi_{1,\Sigma})^{-1}({}_{K_f^1}Sh(G, W\backslash\mathcal{X})^{\overline{P}}_{\Sigma_1}). \tag{1.6.1}$$

We note that these strata are in general not Zariski closed.

There is a notational inconsistency in [HZ1] (cf. (0.5)). Along with the $\overline{P}$-stratum we defined the F-stratum of ${}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}$ in [HZ1, 1.7], where F denotes the rational boundary component fixed by $\overline{P}$, also denoted P_F. However, in §5.3 of [HZ1] the F-stratum was denoted the $\overline{P}$-stratum. In the latter setting we had also defined R-strata for general parabolic subgroups R of G (i.e., not only for maximal parabolics) by the formula

$$ {}_{K_f^1}Sh(G, W\backslash\mathcal{X})^R_{\Sigma_1} = \bigcap_{\substack{\overline{P}\supset R \\ \overline{P} \text{ maximal}}} {}_{K_f^1}Sh(G, W\backslash\mathcal{X})^{\overline{P}}_{\Sigma_1}. \tag{1.6.2}$$

Here the final $\overline{P}$ should have been an F. However, we will retain the notation (1.6.2) and rename the F-stratum the $\overline{P}(*)$-stratum, denoted ${}_{K_f^1}Sh(G, W\backslash\mathcal{X})^{\overline{P}(*)}$

We recall that this is the union of the $G(\mathbf{A}_f)$-translates of the P_α-strata for all maximal parabolics P_α conjugate to $\overline{P}$. Similarly, if R is any standard parabolic subgroup of G, we let

$$ {}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}^{R(*)} = \bigcap_{\substack{\overline{P}\supset R \\ \overline{P}\text{ maximal}}} {}_{K_f^1}Sh(G, W\backslash\mathcal{X})_{\Sigma_1}^{\overline{P}(*)}. \tag{1.6.3}$$

If now P is any $\mathbb{Q}$-parabolic subgroup of Q, we define the P-stratum of ${}_{K_f}Sh(Q, \mathcal{X})_\Sigma$ by analogy with (1.6.1); the $P(*)$-stratum ${}_{K_f}Sh(Q, \mathcal{X})_\Sigma^{P(*)}$ of ${}_{K_f}Sh(Q, \mathcal{X})_\Sigma$ is defined in the same way, with $\overline{P}$ replaced by $\overline{P}(*)$.

In what follows, we will also use the less cumbersome notation of [HZ2, (1.5.2)] for *closed* boundary strata. Thus the Zariski closure of the R-stratum will be denoted $Z_\Sigma(R)$ and the Zariski closure of the $P(*)$-stratum will be denoted $Z_\Sigma(P(*))$, where appropriate.

2. Mixed growth conditions and coherent cohomology

In this Section, we treat the conditions of moderate growth and rapid decrease on differential forms—and also combinations of the two—along divisors with normal crossings and discuss their implications on Dolbeault cohomology. In the case of the boundary divisor of a toroidal compactification of the mixed Shimura variety associated to the pair $(Q, \mathcal{X})$, we convert these conditions to growth and decay conditions on the corresponding vector-valued functions on the group Q on various Siegel sets for its admissible parabolic subgroups. This enables us to deduce the useful isomorphism (2.7.1) on cohomology that seems inaccessible by geometric methods.

(2.1) $\overline{\partial}$ *with logarithmic growth in one variable* (cf. [HP],[H4, §3], [HZ1, 3.8]). Let f be a C^∞ function on the punctured disc $\Delta_a^* = \{z \in \mathbb{C} : 0 < |z| \leq a\}$, with $a < 1$. Given an integer N, we say that f has *logarithmic growth* of degree $\leq N$ when

$$ |f(z)|\,|\log|z||^{-N} \tag{2.1.1}$$

is bounded from above on Δ_a^*. The set of all such functions is denoted A_N^0. It is obvious that $A_N^0 \subset A_{N'}^0$ for all $N' \geq N$.

One says that f is *slowly increasing* if it has logarithmic growth of degree $\leq N$ for some N, and *rapidly decreasing* if it has logarithmic growth of degree $\leq N$ for all N.[4] It is wise to restrict one's attention to functions f such that the derivatives

[4] Clearly, one can assume without loss of generality that N is taken from any set of numbers

of f of all orders are slowly increasing (resp. rapidly decreasing). The space of all such functions is denoted A^0_{sia} (resp. A^0_{rda}).

If ω is a differential form of bidegree (0,1), one can write $\omega = g(z)d\overline{z}/\overline{z}$. We say that ω has logarithmic growth of degree $\leq N$ whenever g does; in effect, we declare that $d\overline{z}/\overline{z}$ is slowly increasing. We can then define the notion of (0,1)-forms that are slowly increasing or rapidly decreasing to all orders, and denote them by $A^{(0,1)}_{sia}$ and $A^{(0,1)}_{rda}$ respectively. We obtain Dolbeault complexes $A^\bullet_{sia}$:

$$A^0_{sia} \xrightarrow{\overline{\partial}} A^{(0,1)}_{sia},$$

and the analogously defined $A^\bullet_{rda}$, which is a subcomplex of $A^\bullet_{sia}$.

Proposition (2.1.2). *(Dolbeault lemma with logarithmic growth). i) $H^0(A^\bullet_{sia})$ is the space of holomorphic functions on the disc Δ_a of radius a.*

ii) $H^0(A^\bullet_{rda})$ is the space of holomorphic functions on Δ_a that vanish at the origin.

iii) $H^1(A^\bullet_{rda}) = H^1(A^\bullet_{sia}) = 0$.

As the first two assertions of the Proposition are obvious, we concentrate on (iii). This is seen to come down to verifying:

Proposition (2.1.3) [HP]. *Let N be an integer different from -1. If $g \in A^0_N$, there exists $f \in A^0_{N+1}$ such that $\overline{\partial} f(z) = g(z)d\overline{z}/\overline{z}$.*

For completeness, we give:

Outline of the proof of (2.1.3). There is a standard solution of the $\overline{\partial}$-equation in a disc Δ, which gets written here as $\overline{\partial} f/\partial\overline{z} = g(z)/\overline{z}$, viz. $f = I(g)$ where

(2.1.4) $$I(g)(z) = (2\pi i)^{-1} \int_{w\in\Delta} \frac{g(w)dw \wedge d\overline{w}/\overline{w}}{w - z}.$$

This is an absolutely convergent integral if g has logarithmic growth; note that $I(g)$ is independent of N. One then shows that logarithmic growth for g implies the corresponding growth for f. By decomposing (2.1.4) into pieces, as in [HP], one reduces the verification to the following elementary calculations:

Lemma (2.1.5). *Let N be an integer different from -1. Then there is a constant $C_N > 0$ such that whenever $r \leq a$:*

(i) $$r^{-1} \int_0^r |log\, \rho|^N \, d\rho \leq C_N \, |log\, r|^N,$$

(ii+) $$\int^r \rho^{-1} \, |log\, \rho|^N \, d\rho \leq C_N \, |log\, r|^{N+1}, \quad \textit{if } N \geq 0$$

(ii–) $$\int_r^a \rho^{-1} \left| log\, \rho \right|^N \, d\rho \le C_N \left| log\, r \right|^{N+1} \quad \textit{if } N < 0.$$

Moreover, when $g \in A^0_{sia}$ *(resp.* $g \in A^0_{rda}$*), the formula*

$$\frac{\partial I(g)}{\partial z} = I(\frac{\partial g}{\partial z})$$

implies that $I(g) \in A^0_{sia}$ *(resp.* $I(g) \in A^0_{rda}$*) likewise.*

(2.2) *Logarithmic growth in several variables.* In the our setting, the natural notions of logarithmic growth and decay are the following. Given an integer N, a function f of $(\mathbf{z}, \mathbf{w}) \in (\Delta^*_a)^m \times (\Delta_a)^n$ is said to have has *logarithmic growth* of degree $\le N$ when

(2.2.1) $$|f(\mathbf{z}, \mathbf{w})| \left(\sum_{j=1}^{m} |\log |z_j|| \right)^{-N}$$

is bounded from above. The set of all such functions is again denoted A^0_N. We can then define A^0_{sia} and A^0_{rda} as in (2.1). Also as in (2.1), we regard $d\overline{z}_j/\overline{z}_j$ as logarithmic (after all, it has logarithmic growth in the sense of (1.3.5)). It is not hard to see that these growth conditions allow one to define, on any complex manifold M, *sheaves* of forms having slow increase or rapid decrease along the divisor with normal crossings D, for the notion of logarithmic growth is independent of local coordinates, and likewise for all derivatives. We denote these by $\mathcal{A}^\bullet_{sia}(M, D)$ and $\mathcal{A}^\bullet_{rda}(M, D)$.

An alternate simpler notion of logarithmic growth of degree N is provided by the boundedness of

(2.2.2) $$|f(\mathbf{z}, \mathbf{w})| \prod_{j=1}^{m} |\log |z_j||^{-N} ,$$

which is asymptotically different from (2.2.1). However, (2.2.2) yields the same notions of slow increase or rapid decrease in view of the following:

Lemma (2.2.3). *If* $\lambda_j \ge 1$ *for* $1 \le j \le m$*, then*

$$\left(\prod_{j=1}^{m} \lambda_j \right)^{\frac{1}{m}} \le \tfrac{1}{m} \sum_{j=1}^{m} \lambda_j \le \prod_{j=1}^{m} \lambda_j .$$

This allow us to replace the growth condition of (2.2.1) by that of (2.2.2), for which we have, in effect, separation of variables (cf. [CH, p. 25]). One obtains from (2.2.3):

Proposition (2.2.4). *Let D be a divisor with normal crossings on a complex manifold M. Then $\mathcal{A}^{\bullet}_{sia}(M, D)$ is a fine resolution of $\mathcal{O}_M$, and $\mathcal{A}^{\bullet}_{rda}(M, D)$ is a fine resolution of $\mathcal{O}_M(-D)$.*

Corollary (2.2.5). *Let $\mathcal{F}$ be a locally-free sheaf of $\mathcal{O}_M$-modules. Then $\mathcal{A}^{\bullet}_{sia}(M, D) \otimes_{\mathcal{O}_M} \mathcal{F}$ is a fine resolution of $\mathcal{F}$, and $\mathcal{A}^{\bullet}_{rda}(M, D) \otimes_{\mathcal{O}_M} \mathcal{F}$ is a fine resolution of $\mathcal{F}(-D)$.*

(2.2.6) *Remark.* The issues involved in the proof of (2.2.4) were misrepresented in [HP], as was pointed out to us by J. I. Burgos, who also provided a correct argument along the lines outlined here.

(2.3) *Forms with mixed growth conditions.* We continue to let D denote a divisor with normal crossings on the complex manifold M. Consider any 2-partition of the set of irreducible components of D, and write D correspondingly as the union $D_1 \bigcup D_2$. We define the middle term in

$$\mathcal{A}^{\bullet}_{rda}(M, D) \subseteq \mathcal{A}^{\bullet}_{mxa}(M; D, D_2) \subseteq \mathcal{A}^{\bullet}_{sia}(M, D) \tag{2.3.1}$$

by the condition that a differential form should be slowly increasing along D_1, and rapidly decreasing along D_2, and likewise for all derivatives. Our purpose is to obtain the following variants of (2.2.4) and (2.2.5):

Proposition (2.3.2). *$\mathcal{A}^{\bullet}_{mxa}(M; D, D_2)$ is a fine resolution of $\mathcal{O}_M(-D_2)$.*

Corollary (2.3.3). *For any locally-free sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules, $\mathcal{A}^{\bullet}_{mxa}(M; D, D_2) \otimes_{\mathcal{O}_M} \mathcal{F}$ is a fine resolution of $\mathcal{F}(-D_2)$.*

One is led to define:

(2.3.4) *Definition.* A function f of $(\mathbf{z}, \mathbf{u}, \mathbf{w}) \in (\Delta_a^*)^{m_1} \times (\Delta_a^*)^{m_2} \times (\Delta_a)^n$ is said to have *logarithmic growth* of bidegree (N_1, N_2) when

$$|f(\mathbf{z}, \mathbf{u}, \mathbf{w})| \left(\sum_{j=1}^{m_1} |\log |z_j|| \right)^{-N_1} \left(\sum_{j=1}^{m_2} |\log |u_j|| \right)^{-N_2}.$$

is bounded from above. Differential $(0, k)$-forms of logarithmic growth of bidegree (N_1, N_2) are defined just as before, and the space of such is denoted $A^k_{(N_1,N_2)}$. Then Proposition (2.3.2) and indeed Proposition (2.2.4) are deduced from the following multivariate version of (2.1.3), which is proved by iteration on the latter.

Proposition (2.3.5). *Let N_1 and N_2 be integers different from -1. If $k > 0$ and $\omega \in A^k_{(N_1,N_2)}$, there exists $\phi \in A^{k-1}_{(N_1+1,N_2+1)}$ such that $\overline{\partial}\phi = \omega$.*

(2.4) *Siegel sets in mixed Shimura varieties.* We will retain the notation from (1.1) as far as possible. Thus, let $(Q, \mathcal{X})$ be a mixed Shimura datum. By convention (as before) we also allow $\mathcal{X}$ to denote a connected component of this homogeneous space. Let $(G, W\backslash\mathcal{X})$ be the corresponding pure Shimura datum for $G = Q/W$ and $\pi : Q \to G$ the natural map. For any connected component M_Γ of ${}_{K_f}Sh(Q, \mathcal{X})(\mathbb{C})$ and corresponding component $\widehat{M}_\Gamma$ of ${}_{\widehat{K}_f}Sh(G, W\backslash\mathcal{X})(\mathbb{C})$, where $\widehat{K}_f = \pi(K_f)$, we have the commutative diagram of complex manifolds

$$\begin{array}{cccc} & \mathcal{X} & \longrightarrow & M_\Gamma \\ & \downarrow & & \downarrow \\ \widehat{\mathcal{X}} = & W\backslash\mathcal{X} & \longrightarrow & \widehat{M}_\Gamma \end{array} \tag{2.4.1}$$

in which the spaces $\mathcal{X}$ and $\widehat{\mathcal{X}}$ are contractible.

The notion of going to infinity in M_Γ or $\widehat{M}_\Gamma$ is expressible in terms of Siegel sets, equivalently in the group or on the homogeneous space, associated to parabolic subgroups. For the present purposes, it is the admissible parabolic subgroups that enter. Recall that the mapping $\mathcal{P}(G) \to \mathcal{P}(Q)$ that assigns $\widetilde{P} = \pi^{-1}(P)$ to $P \in \mathcal{P}(G)$ is bijective. Since the notion of admissibility is determined on G, the admissible parabolic subgroups of G and Q are thus in canonical one-to-one correspondence.

We work with the pair $(G, \widehat{\mathcal{X}})$, though the story for $(Q, \mathcal{X})$ is parallel. Let $P \in \mathcal{P}(G)$; then $P(\mathbb{R})$ acts transitively on $\widehat{\mathcal{X}}$. Select a basepoint $\widehat{x}_0 \in \widehat{\mathcal{X}}$. Let A_P denote the associated lift to $P(\mathbb{R})$ of the connected component of the center of P/W_P. Then $P^0 \simeq ({}^0P)^0 \times A_P$, where ${}^0P$ denotes, as in [BS, 1.1], the intersection of the kernels of the squares of the rational characters of P; we make this identification routinely. A *Siegel set* for P in $G(\mathbb{R})$ is a subset of $P(\mathbb{R})$ of the form $\mathfrak{S} = \mathfrak{S}_{\kappa,t} = \kappa \times A_{P,t}$, where κ is a compact subset of ${}^0P(\mathbb{R})$, and

$$A_{P,t} = \{a \in A_P : a^\beta \geq t \text{ for all simple roots } \beta \text{ occurring in } W_P\}. \tag{2.4.2}$$

It is also standard usage to call $\mathfrak{S} \cdot \widehat{x}_0$ a Siegel set for P in $\widehat{\mathcal{X}}$.

The associated Siegel domain coordinates are deduced from embedding $\widehat{\mathcal{X}}$ as an open subset of the total space $\widehat{\mathcal{X}}_P$ of a mixed Shimura variety,

$$\widehat{\mathcal{X}} \subset \widehat{\mathcal{X}}_P \to U_P(\mathbb{C})\backslash\widehat{\mathcal{X}}_P \to \widehat{F}_P, \tag{2.4.3}$$

where $\widehat{F}_P$ is the boundary component of $\widehat{\mathcal{X}}$ corresponding to P, and trivializing the fibration analytically:

$$\widehat{\mathcal{X}}_P \simeq \widehat{F}_P \times V_P \times U_P(\mathbb{C}). \tag{2.4.4}$$

This yields

$$\widehat{\mathcal{X}} \simeq \{(z, v, u) \in \widehat{F}_P \times V_P \times U_P(\mathbb{C}) : \operatorname{Im} u - \Psi(z, v) \in C_P\} \tag{2.4.5}$$

where C_P is a homogeneous cone in U_P, in which the action of A_P on $\widehat{\mathcal{X}}$ is given by the dilations of C_P. In the above, $\Psi : \widehat{F}_P \times V_P \to U_P$ has well-known properties; aside from continuity, these will not concern us here.

(2.4.6) *Remark.* When G is $\mathbb{Q}$-simple and P is maximal, one places a point at infinity in $V_P \times U_P(\mathbb{C})$ in (2.4.2) to attach (topologically) the Baily-Borel boundary component corresponding to P. It is reached by letting $a^\beta \to \infty$ (see (2.4.2)), while keeping the simple roots of $G_{h,P}$ bounded.

Next, let $\mathcal{T}$ be any compact cone in C_P; by "compact", we mean that $\widehat{\mathcal{T}} = (\mathcal{T} - 0)/\mathbb{R}^+$ is compact in $\widehat{C}_P$. A *complex Siegel set* for P in $\widehat{\mathcal{X}}$, built from $\mathcal{T}$, will be a set S of the form

$$\{(y, v, u) \in \widehat{F}_P \times V_P \times U_P(\mathbb{C}) : y \in \kappa_h,\ v \in \kappa_V,\ \mathrm{Re}\ u \in \kappa_U,\ \mathrm{Im}\ u \in (t_0 + \mathcal{T})\}, \tag{2.4.7}$$

for κ_h, κ_V, κ_U compact and $t_0 \in \mathcal{T}$. It is easy to verify that the two notions of Siegel set are commensurate: every Siegel set for P is contained in a complex Siegel set for P, and conversely. (For the same reason, one sees that in defining a Siegel set, one could take the κ to be a compact subset of $P(\mathbb{R})$ or even of $G(\mathbb{R})$.)

We turn now briefly to the pair $(Q, \mathcal{X})$. As our current concern is over mixed Shimura varieties for which π_2 in (1.1.5) is an isomorphism (that is to say, abelian fibrations), *we assume that* $W_{-2}Q = \{1\}$. Let $\widetilde{P} \in \mathcal{P}(Q)$ correspond to $P \in \mathcal{P}(G)$. There are objects for $\widetilde{P}$ analogous to those for P given above, as defined in [P, §4]. This yields a diagram:

$$\begin{array}{ccccccc} \mathcal{X} & \subset & \mathcal{X}_{\widetilde{P}} & \longrightarrow & U_{\widetilde{P}}(\mathbb{C})\backslash\mathcal{X}_{\widetilde{P}} & \longrightarrow & F_{\widetilde{P}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathcal{X}} & \subset & \widehat{\mathcal{X}}_P & \longrightarrow & U_P(\mathbb{C})\backslash\widehat{\mathcal{X}}_P & \longrightarrow & \widehat{F}_P \end{array} \tag{2.4.8}$$

in which the vertical arrows are surjections, and a surjection of cones $C_{\widetilde{P}} \twoheadrightarrow C_P$. (Note, however, that the homomorphism $U_{\widetilde{P}} \to U_P$ need not be injective.) Moreover, if we select a basepoint $x_0 \in \mathcal{X}$ with $\pi(x_0) = \widehat{x}_0$, the projection of $A_{\widetilde{P}}$ in $Q(\mathbb{R})$ to $A_P \subset G(\mathbb{R})$ is an isomorphism. Thus Siegel sets for $\widetilde{P}$ in $\mathcal{X}$, defined in the same way as those for P, map onto Siegel sets for P in $\widehat{\mathcal{X}}$. In actuality, though, it will suffice for our purposes to consider the Siegel sets for P and $\widetilde{P}$ independently, i.e., on the same footing.

The following is well-known (see [B3]):

Proposition (2.4.9). *Let κ be a compact subset of ${}^0P(\mathbb{R})$, $\mathfrak{S}_{\kappa,t}$ the Siegel set $\kappa \times A_{P,t}$ and $(\mathfrak{S}_{\kappa,t})_\Gamma$ the image of $\mathfrak{S}_{\kappa,t}$ in $\widehat{M}_\Gamma \subset {}_{\widehat{K}_f}Sh(G, \widehat{\mathcal{X}})(\mathbb{C})$. Then for t sufficiently large,* *where* κ_Γ *is the image of* κ *in* $\widehat{M}_\Gamma$*, such that the projection* $\mathfrak{S}_{\kappa,t} \to (\mathfrak{S}_{\kappa,t})_\Gamma$ *decomposes as* $(\kappa \to \kappa_\Gamma) \times \mathbf{1}_{A_{P,t}}$.

Corollary (2.4.10). *Let* $(\mathfrak{S}_{\kappa,t})_\Gamma$ *be as in (2.4.9). Then for* t *sufficiently large, there is a canonical projection* $a : (\mathfrak{S}_{\kappa,t})_\Gamma \to A_P$ *(*A_P*-coordinate).*

(2.5) *Toroidal compactifications and Siegel sets.* For simplifying notation only, we assume henceforth that G is $\mathbb{Q}$-simple.

We start by taking a slight variation of the basic diagram (induced by (2.4.3)) in [HZ1, 1.2.5]:

$$\text{(2.5.1)} \qquad \begin{array}{ccccc} \widetilde{M}'_P = & \Gamma_{U_P}\backslash\mathcal{X}_P & \longrightarrow & M'_P \\ & \downarrow & & \downarrow{\scriptstyle \pi_2} \\ & U_P(\mathbb{C})\backslash\mathcal{X}_P & \longrightarrow & A_P \\ & \downarrow & & \downarrow{\scriptstyle \pi_1} \\ & \widehat{\mathcal{X}}_P & \longrightarrow & M_P \end{array}$$

We have made the upper square cartesian: $\widetilde{M}'_P$ is the pullback of M'_P to the universal cover $U_P(\mathbb{C})\backslash\mathcal{X}_P$ of A_P, replacing $\mathcal{X}_P$. The coordinates in (2.4.4) induce an analytic isomorphism

$$\text{(2.5.2)} \qquad \widetilde{M}'_P \simeq \widehat{\mathcal{X}}_P \times V \times T_P.$$

We next review some pertinent features the structure of the boundary of toroidal compactifications $M_{\Gamma,\Sigma}$ of connected components of $Sh(G,\mathcal{X})(\mathbb{C})$, following [HZ1, (1.5)]. Let $P \in \mathcal{P}(G)$. The data in Σ_P are used to construct $\pi : (M'_P)_{\Sigma_P} \to M_P$, with boundary divisor written as

$$\text{(2.5.3)} \qquad {}^<\widetilde{Z}_\Sigma(\succeq P) \overset{\text{def.}}{=} \bigcup_{P' \succeq P} {}^<\widetilde{Z}_\Sigma(P').$$

The quotient of (2.5.3) by $\Gamma_{\ell,P}$ will be denoted ${}^<Z_\Sigma(\succeq P)$; in the notation of [HZ1] it is the union of all ${}^<Z_\Sigma(P')$ for $P' \succeq P$. Recall that a neighborhood of the latter is a building block in the construction of $M_{\Gamma,\Sigma}$. It will be useful to keep in mind that ${}^<Z_\Sigma(\succeq P)$ is more than the P-stratum of the boundary unless P is maximal with respect to $\succ$, yet contains all of $Z_\Sigma(P)$ (the closure of the P-stratum) only when P is minimal with respect to $\succ$. We also observe that T_P acts on (2.5.3).

The toroidal constructions with T_P lift to ones on $\widetilde{M}'_P$. In particular, we have

$$\text{(2.5.4)} \qquad (\widetilde{M}'_P) \simeq \widehat{\mathcal{X}}_P \times V \times (T_P)$$

Let ${}^{<}\widetilde{\widetilde{Z}}_{\Sigma}(P)$ and ${}^{<}\widetilde{\widetilde{Z}}_{\Sigma}(\succeq P)$ denote the respective pullbacks of ${}^{<}\widetilde{Z}_{\Sigma}(P)$ and ${}^{<}\widetilde{Z}_{\Sigma}(\succeq P)$; we can write, for instance,

$$ {}^{<}\widetilde{\widetilde{Z}}_{\Sigma}(\succeq P) \simeq \widehat{\mathcal{X}}_P \times V \times \partial(T_P)_{\Sigma_P}. \tag{2.5.5}$$

In [HZ2, 1.4(d)], we introduced a topological compactification of M_Γ, the *real boundary quotient* $M_{\Gamma,\Sigma,\partial_{\mathbb{R}}}$ of $M_{\Gamma,\Sigma}$, which was designed for the present discussion (also, cf. [HZ1, 2.3]), with structure map

$$p : M_{\Gamma,\Sigma} \longrightarrow M_{\Gamma,\Sigma,\partial_{\mathbb{R}}}.$$

By construction, $p({}^{<}Z_{\Sigma}(\succeq P))$ is given naturally as $\Gamma_{\ell,P}\backslash\left({}^{<}\widetilde{Z}_{\Sigma}(\succeq P)/T_P^c\right)$, where T_P^c is the maximal compact real torus in T_P. Likewise, we can define the real boundary quotient

$$(\widetilde{M}'_P)_{\Sigma_P,\partial_{\mathbb{R}}} \simeq \widehat{\mathcal{X}}_P \times V \times \partial(T_P)_{\Sigma_P,\partial_{\mathbb{R}}}, \tag{2.5.6}$$

for which $\partial(\widetilde{M}'_P)_{\Sigma_P,\partial_{\mathbb{R}}} \simeq \widehat{\mathcal{X}}_P \times V \times (T_P)_{\Sigma_P,\partial_{\mathbb{R}}}$. In terms of the product structures in (2.5.4) and (2.5.6), the quotient map $(\widetilde{M}'_P)_{\Sigma_P} \to (\widetilde{M}'_P)_{\Sigma_P,\partial_{\mathbb{R}}}$ is induced by the natural projection $(T_P)_{\Sigma_P} \to (T_P)_{\Sigma_P,\partial_{\mathbb{R}}}$.

For any character χ of T_P, there is a morphism of commutative diagrams,[5] which we also denote by χ:

$$\left\{\begin{array}{ccc} U_P(\mathbb{C}) & \xrightarrow{\mathrm{Im}} & U_P(\mathbb{C})/U_P(\mathbb{R}) \simeq U_P(\mathbb{R}) \\ {\scriptstyle\exp}\downarrow & & {\scriptstyle\exp}\downarrow\simeq \\ U_P(\mathbb{C})/U_P(\mathbb{Z}) \simeq T_P & \xrightarrow[\mathrm{Mod}]{} & T_P/T_P^c \end{array}\right\} \xrightarrow{\chi} \left\{\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathrm{Im}} & \mathbb{C}/\mathbb{R} \simeq \mathbb{R} \\ {\scriptstyle\exp}\downarrow & & {\scriptstyle\exp}\downarrow\simeq \\ \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^* & \xrightarrow[\mathrm{Mod}]{} & \mathbb{C}^*/S^1 \simeq \mathbb{R}^+ \end{array}\right\} \tag{2.5.7}$$

In other words, up to a constant multiple, the logarithm of the modulus of a toroidal variable is induced by the imaginary part of a linear function on $U_P(\mathbb{C})$. In this way, approaching the boundary of $\widetilde{X}_{\Sigma,\partial_{\mathbb{R}}}$ corresponds to going to infinity in the radial direction of the cone C_P. Specifically, $\partial(T_P)_{\Sigma_P,\partial_{\mathbb{R}}}$ can be viewed as a copy of $\widehat{\Sigma}_P$ attached at infinity to the cone, with $\widehat{\Sigma}_P^{(1),c}$ giving $p({}^{<}Z_{\Sigma}(P))$ (see the illustration [HZ1, (2.3.10)]), while T_P^c-orbits are getting collapsed to a point.

We make the following:

Observations (2.5.8). *Let P be a maximal $\mathbb{Q}$-parabolic subgroup of G, $\mathcal{T}$ a compact cone in C_P, and S a complex Siegel set for P built from $\mathcal{T}$, as in (2.4.7). Then:*

[5]In the literature (see [HZ1, (2.3.2)]), the [illegible] T_P [illegible] $U_P(\mathbb{R})$ is called “real”

i) the closure of the image of S *in* $(\widetilde{M}'_P)_{\Sigma_P,\partial_{\mathbb{R}}}$ *meets the boundary in* $\kappa_h \times \kappa_V \times \mathcal{T}$,

ii) Take $\mathcal{T}$ *to be any union of finitely many top-dimensional simplicial cones in* $\Sigma_P^{(1),c}$ *that maps onto* $\Gamma_{\ell,P}\backslash\Sigma_P^{(1),c}$*, and assume that* κ_V *and* κ_U *are sufficiently large. Then the closure of the image of* S *in* $M_{\Gamma,\Sigma}$ *meets the boundary in* $\pi^{-1}(\widehat{\kappa}_h) \cap {}^{<}Z_P$*, where* $\widehat{\kappa}_h$ *denotes the image of* κ_h *in* M_P*.*

iii) For any $P' \succ P$*, there is a dense open subset (determined by* $\mathcal{Y}_P(P')$ *below) of* $Z_\Sigma(P')$ *of which each point is in the closure of some Siegel set for* P*.*

In practice, the above asserts that every point of $Z_\Sigma(P)$ (the union of components of the toroidal divisor generated by Σ_P^c) is a limit point of some Siegel set; however, one must consider Siegel sets for all parabolic subgroups dominated by (i.e., $\prec$) P, as well as those for P itself.

To be more specific, if we let $V(L)$ stand for the set of vertices of a simplicial complex L, then

$$V(\widehat{\Sigma}_P) = \coprod\{V(\widehat{\Sigma}^c_{P'}) : P' \succeq P\} \tag{2.5.9}$$

is a $\Gamma_{\ell,P}$-equivariant partition (cf. [HZ2, (2.4.2)]). By taking the real projection of the toroidal divisors corresponding to each $\widehat{\Sigma}^c_{P'}$, one induces a decomposition of $\widehat{C}_P$ (which is an open subset of $|\widehat{\Sigma}_P|$) into sets $\widehat{\mathcal{Y}}_P(P')$ that intersect only at their boundaries, and which are the closure of their interiors; of these, only $\widehat{\mathcal{Y}}_P(P)$ is compact modulo $\Gamma_{\ell,P}$. In [HZ2, (2.2)], $\widehat{\mathcal{Y}}_P(P')$ gets called the *open thick* P'*-stratum* of $\widehat{\Sigma}_P^{(1)}$, and the decomposition (2.5.9) plays an important role. When $\mathcal{T} \subset C_P$ as before, we will say that $\mathcal{T}$ *abuts on* $\widetilde{Z}_\Sigma(P')$ whenever $\widehat{\mathcal{T}} \subset \widehat{\mathcal{Y}}_P(P')$. Letting $\mathcal{Y}_P(P')$ denote the corresponding cone in C_P, we conclude:

Proposition (2.5.10). *Let* P *be a maximal* $\mathbb{Q}$*-parabolic subgroup of* G*. Then:*

i) The set of all complex Siegel sets for admissible parabolics that abut on $\widetilde{Z}_\Sigma(P)$ *consists precisely of those complex Siegel sets, for some* $Q \preceq P$*, built from compact cones contained in the subcone* $\mathcal{Y}_Q(P)$ *of* C_Q*.*

ii) Let $\widetilde{Z}^Q_\Sigma(P)$ *denote the union of the limit points of such Siegel sets for a fixed* Q*. Then this set is open and dense in* $\widetilde{Z}_\Sigma(P)$*.*

iii) Let $Z^Q_\Sigma(P) \subseteq Z_\Sigma(P)$ *denote the image of* $\widetilde{Z}^Q_\Sigma(P)$ *in* $M_{\Gamma,\Sigma}$*. Then*

$$Z_\Sigma(P) = \bigcup\{Z^Q_\Sigma(P) : Q \in \mathcal{M}(\Gamma)\},$$

where $\mathcal{M}(\Gamma)$ *is a set a* Γ*-conjugacy classes of admissible parabolic subgroups* Q *with* $Q \preceq P$*.*

iv) $\bigcap\{Z^Q_\Sigma(P) : Q \in \mathcal{M}(\Gamma)\}$ *is an open and dense subset of* $Z_\Sigma(P)$*. It is open in the Zariski topology only when* P *is minimal with respect to* $\preceq$*, i.e., when* $\mathcal{M}(\Gamma)$ *has only one element (representable by* $\{P\}$*).*

(2.6) *Growth conditions at the toroidal boundary.* Given $P \in \mathcal{P}(G)$, let β denote the positive simple root occurring in W_P. We now prove:

Proposition (2.6.1). *Let* $(\mathbf{z}, \mathbf{w}) \in (\Delta^*_r)^n \times \Delta^m_r$*, with* $r < 1$*, give local coordinates on a deleted neighborhood of* ${}^{<}Z_\Sigma(\succeq P)$ *associated to an* n*-fold intersection of components of this divisor. Then on the intersection of this coordinate neighborhood with* $(\mathfrak{S}_{\kappa,t})_\Gamma$*, the functions* a^β *(determined by (2.4.10)) and* $\sum_{j=1}^n \log|z_j|$ *grow at the same rate along* ${}^{<}Z_\Sigma(\succeq P)$*.*

An argument proving this is given for pure Shimura varieties in [Mu, p. 264], though it seems a bit unfocused. The proof is geometric, and goes the same in the mixed case; we present it here:

Proof. We appeal to (2.5.7). Write λ_j for $\log|z_j|$; these are linear functions on the cone that are positive in the region under consideration. We may assume without loss of generality that the Siegel set $(\mathfrak{S}_{\kappa,t})_\Gamma$ involves, as a factor, a compact subset of the "standard" cross-section of $C_P - \{0\} \to \widehat{C}_P$ determined by $G_{\ell,P}(\mathbb{R})^{\mathrm{der}}$. Then, the value of a^β at any point is given by the amount of dilation in C_P required to get to the point from this cross-section. By compactness, this differs from the amount of dilation required to get from any other cross-section over the same compact set by an amount that is bounded from above and away from zero. In other words, the rate of growth of the A_P-coordinate is independent of cross-section. Then, just take the cross-section to be the one defined by $\sum_j \lambda_j = 1$.

Next, let $\widehat{x}_0$ be the basepoint of $\widehat{\mathcal{X}}$, which was used in definition of Siegel sets in (2.4). Let $\widehat{K}_0$ denote the stabilizer in $G(\mathbb{R})^{\mathrm{der}}$ of $\widehat{x}_0$. It is well-known (see [B1]) that there are norms $\|\cdot\|$ on $G(\mathbb{R})^{\mathrm{der}}$ that satisfy:

i) $\|gk\| = \|g\|$ whenever $k \in \widehat{K}_0$;

ii) $\|gg'\| \le \|g\|\|g'\|$.

The following is easy to verify:

Proposition (2.6.2). *Let* P *be a maximal parabolic subgroup of* G*. Then:*

i) On A_P*,* $\|a\|$ *grows like* $(a^\beta)^\ell$ *for some* $\ell > 0$*;*

ii) For g *in the Siegel set* $\mathfrak{S}$*,* $\|g\|$ *and* $\|a(g)\|$ *have the same rate of growth*

We recall that a C^∞ $(0,q)$-form ω on M_Γ with values in $\mathcal{W}$ is determined by a unique smooth Γ-invariant function f_ω on $G(\mathbb{R})$ with values in $\bigwedge^q(\mathfrak{p}^-)^*\otimes W$. When we combine the preceding with (2.5.10), we get:

Proposition (2.6.3).

i) An element ω of the Dolbeault complex for $\mathcal{W}$ on (an open subset of) M_Γ has moderate growth along $Z_\Sigma(P)$ in the sense of (2.2.4) if and only if f_ω has moderate growth on all complex Siegel sets for Q that are built from subcones of $\mathcal{Y}_Q(P)\subset C_Q$, for all $Q\preceq P$.

ii) ω lies in $\mathcal{A}^\bullet_{\mathrm{mxa}}(M_{\Gamma,\Sigma};Z_\Sigma,Z_\Sigma(P))\otimes\mathcal{W}^{\mathrm{can}}$ (of (2.3)) if and only if f_ω has moderate growth and has rapid decrease on all complex Siegel sets for Q that are built from subcones of $\mathcal{Y}_Q(P)\subset C_Q$, for all $Q\preceq P$.

(2.7) *Mixed growth conditions and basechange.* We now take in (2.4.1) the lower part of the tower for the mixed Shimura variety for the admissible parabolic subgroup P of G (this is denoted $\pi_1: A_P\to M_P$ in [HZ1]). According to [P], there exist compatible toroidal compactifications for π_1. Let $\widetilde{\pi}_1:\widetilde{A}_{P,\Xi}\to\widetilde{M}_{P,\Xi}$ (as in [HZ1, 1.6]) be one of them. We let $Z^A_\Xi(P')$ and $Z_\Xi(P')$ resp. denote the P'-strata of the respective toroidal boundaries. The main goal of this section is to deduce the following variant and extension of [HZ1, 3.14]:

Proposition (2.7.1). *In the above situation, there is a commutative diagram of quasi-isomorphisms of sheaves on $\widetilde{M}_{P,\Xi}$:*

$$\begin{array}{ccc} R\widetilde{\pi}_{1,*}\{[\mathcal{V}^A]^{\mathrm{can}}(-\widetilde{\pi}_1^*Z_\Xi(P'))\} & \xrightarrow{\approx} & R\widetilde{\pi}_{1,*}\{[\mathcal{V}^A]^{\mathrm{can}}\}(-Z_\Xi(P')) \\ \downarrow & & \downarrow\approx \\ R\widetilde{\pi}_{1,*}\{[\mathcal{V}^A]^{\mathrm{can}}(-Z^A_\Xi(P'))\} & \overset{\approx}{\leftrightarrow} & \mathcal{H}^\bullet(\mathfrak{s}_P,V)^{\mathrm{can}}(-Z_\Xi(P')) = \bigoplus_i\mathcal{H}^i(\mathfrak{s}_P,V)^{\mathrm{can}}(-Z_\Xi(P'))[-i] \end{array}$$

This implies at once:

Corollary (2.7.2). *In the above situation,*

$$H^\bullet(\widetilde{A}_{P,\Xi},[\mathcal{V}^A]^{\mathrm{can}}(-Z^A_\Xi(P')))\simeq H^\bullet(\widetilde{M}_{P,\Xi},\mathcal{H}^\bullet(\mathfrak{s}_P,V)^{\mathrm{can}}(-Z_\Xi(P'))).$$

(2.7.3) *Remark.* There would be nothing to prove in (2.7.1) if one knew that $Z^A_\Xi(P')$ were equal to $\widetilde{\pi}_1^*Z_\Xi(P')$ in the sense of divisors, i.e., that the fibers of $\widetilde{\pi}_1$ over $Z_{P'}$ are reduced. One might hope that this can be arranged by choosing Ξ suitably, but that is not apparent from [P].

Proof of (2.7.1). We show first that in the derived category of $\widetilde{M}_{P,\Xi}$, both sides of the isomorphism in the bottom row of (2.7.1), are computable by means of complexes of C^∞ differential forms, on M_P and A_P respectively, with moderate growth and that decrease rapidly in the direction of the P'-strata.

We use the mechanism from [HZ1, 3.5.12]. Consider the Dolbeault complexes:

$$\mathcal{A}^{0,\bullet}_{M_P}(\mathcal{H}^\bullet(\mathfrak{s}_P, V)) \overset{\iota_1}{\hookrightarrow} \mathcal{A}^{0,\bullet}_{M_P}(\bigwedge \mathfrak{s}_P^* \otimes \mathcal{V}) \overset{\iota_2}{\hookrightarrow} \pi_{1*}\mathcal{A}^{0,\bullet}_{A_P}(\mathcal{V}^A); \tag{2.7.4}$$

up to quasi-isomorphism, this is

$$\mathcal{H}^\bullet(\mathfrak{s}_P, V) \to \bigwedge \mathfrak{s}_P^* \otimes \mathcal{V} \to R\pi_{1,*}\mathcal{V}^A. \tag{2.7.5}$$

The complexes in (2.7.4) can be written as sheaves of relative Lie algebra cochains:

$$\mathcal{C}_1^\bullet \hookrightarrow \mathcal{C}_2^\bullet \hookrightarrow \mathcal{C}_3^\bullet, \tag{2.7.6}$$

where, for O open in M_P,

$$\mathcal{C}_1^\bullet(O) = \{C^\infty(O_{h,P}) \otimes \bigwedge(\mathfrak{p}_h^-)^* \otimes H^\bullet(\mathfrak{s}_P, V)\}^{K_{h,P}}, \tag{2.7.6.1}$$

$$\mathcal{C}_2^\bullet(O) = \{C^\infty(O_{h,P}) \otimes \bigwedge(\mathfrak{p}_h^-)^* \otimes \bigwedge(\mathfrak{s}_P)^* \otimes V\}^{K_{h,P}}, \tag{2.7.6.2}$$

where $O_{h,P}$ is the inverse image of O under $\Gamma_{h,P}\backslash G_{h,P} \to M_P$; also for O' the inverse image of O in A_P

$$\mathcal{C}_3^\bullet(O) = \{C^\infty(O_1') \otimes \bigwedge(\mathfrak{p}_h^- \oplus \mathfrak{s}_P)^* \otimes V\}^{K_{h,P}}, \tag{2.7.6.3}$$

with O_1' the inverse image of $O_{h,P}$ in $\Gamma_{A_P}\backslash(P'/U_P)(\mathbb{R})$. For (2.7.6.3), we are using the fact that $\mathcal{V}^A$ is a homogeneous vector bundle (recall (1.2.14)), determined by the representation of $K_{h,P}$ on $H^0(T_P, \mathcal{V}_P')^{T_P}$, which is isomorphic (via evaluation at the basepoint) to V as a $K_{h,P}$-module.

There are homotopy inverses to the maps in (2.7.6) via projections

$$\mathcal{C}_1^\bullet \overset{\psi_1}{\longleftarrow} \mathcal{C}_2^\bullet \overset{\psi_2}{\longleftarrow} \mathcal{C}_3^\bullet, \tag{2.7.7}$$

with ψ_1 given by the semi-simplicity of representations of a compact group—note that we are invoking here the assumption that $\mathfrak{p}^-$ acts trivially on V—and ψ_2 given by taking the constant term along the fibers of A_P.

Here and again, one makes use of the following homological lemma:

Lemma 2.7.8. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be cochain complexes [of sheaves], and assume given morphisms $i : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi : \mathcal{C}_2 \to \mathcal{C}_1$ such that*

i) $\psi \circ i = 1$,

ii) $i \circ \psi = 1 + dB + Bd$.

for some mapping $B : \mathcal{C}_2 \to \mathcal{C}_1$ of degree -1. Let $\mathcal{C}_2' = \ker \psi$. If $i' : \mathcal{C}_2' \to \mathcal{C}_2$ denotes the inclusion, the morphism

$$\mathcal{C}_1 \oplus \mathcal{C}_2' \xrightarrow{i \oplus i'} \mathcal{C}_2.$$

is an isomorphism. Moreover, $\mathcal{C}_2' \approx 0$, and i and ψ are quasi-isomorphisms inducing mutually inverse isomorphisms on cohomology [sheaves]. Furthermore, the above is functorial with respect to the triple (i, ψ, B).

This yields that the maps in (2.7.4), and therefore also (2.7.5), are quasi-isomorphisms. This was extended to the toroidal boundary for the canonical extension in [HZ1, 3.13] and the subcanonical extension in [HZ1, 3.14]. These were obtained by observing that ψ_1 and ψ_2 respect the growth conditions on (2.7.7) given as in (2.6.1), as do their homotopy operators, and then appealing to (2.7.8). Said another way, imposing the growth conditions preserved the homological situation, and we could again invoke (2.7.8).

We apply (2.3.3) to both sides of bottom row in (2.7.1), writing them as Dolbeault complexes with mixed growth conditions on $\widetilde{A}_{P,\Xi}$ and $\widetilde{M}_{P,\Xi}$ resp., taking D_2 to be the respective P'-stratum of the toroidal boundary, for some $\mathbb{Q}$-parabolic subgroup Q of P. This reduces us to verifying that

(2.7.9)

$$\mathcal{A}^\bullet_{mxa}(\widetilde{M}_{P,\Xi}; Z_\Xi, Z_\Xi(P')) \otimes \mathcal{H}^\bullet(\mathfrak{s}_P, V)^{\mathrm{can}} \to \widetilde{\pi}_{1,*}\{\mathcal{A}^\bullet_{mxa}(\widetilde{A}_{P,\Xi}; Z^A_\Xi, Z^A_\Xi(P')) \otimes [\mathcal{V}^A]^{\mathrm{can}}\}.$$

is a quasi-isomorphism, extending that of (2.7.4) to the toroidal boundary. For use in conjunction with the Lie algebra cochains of (2.7.6), we appeal to (2.6.3, ii): the mixed growth conditions are equivalent to moderate growth, together with decay conditions on a specific class of Siegel sets. That these are preserved by ψ_1 and ψ_2 in (2.7.7) follows as before. This gives that (2.7.9) is a quasi-isomorphism, and (2.7.1) follows.

To complete the proof of (2.7.1), we must show that the square commutes. First, we make the following observation, which follows from (2.2.5) and (2.3.3):

Lemma 2.7.10. *The inclusions*

(2.7.10.1)

are compatible fine resolutions of

$$[\mathcal{V}^A]^{\mathrm{can}} \otimes \widetilde{\pi}^* \mathcal{I}_{Z_\Xi(P')} \hookrightarrow [\mathcal{V}^A]^{\mathrm{can}} \otimes \mathcal{I}_{Z^A_\Xi(P')} \hookrightarrow [\mathcal{V}^A]^{\mathrm{can}}. \tag{2.7.10.2}$$

Applying the functor $R\widetilde{\pi}_*$ to (2.7.10.1) gives

$$\begin{aligned}\widetilde{\pi}_*\{\mathcal{A}^\bullet_{sia}(\widetilde{A}_{P,\Xi}, Z^A_\Xi) \otimes [\mathcal{V}^A]^{\mathrm{can}}\} \otimes \mathcal{I}_{Z_\Xi(P')} &\hookrightarrow \{\widetilde{\pi}_*\mathcal{A}^\bullet_{mxa}(\widetilde{A}_{P,\Xi}; Z^A_\Xi, Z^A_\Xi(P')) \otimes [\mathcal{V}^A]^{\mathrm{can}}\} \\ &\hookrightarrow \{\widetilde{\pi}_*\mathcal{A}^\bullet_{sia}(\widetilde{A}_{P,\Xi}, Z^A_\Xi) \otimes [\mathcal{V}^A]^{\mathrm{can}}\}.\end{aligned}$$

The above are taken as $\mathcal{C}_2$'s in (2.7.8); for $\mathcal{C}_1$'s, we take

$$\begin{aligned}\mathcal{A}^\bullet_{sia}(\widetilde{M}_{P,\Xi}, Z_\Xi) \otimes \mathcal{H}^\bullet(\mathfrak{s}_P, V)^{\mathrm{can}} \otimes \mathcal{I}_{Z_\Xi(P')} &\hookrightarrow \mathcal{A}^\bullet_{mxa}(\widetilde{M}_{P,\Xi}, Z_\Xi) \otimes \mathcal{H}^\bullet(\mathfrak{s}_P, V)^{\mathrm{can}} \\ &\hookrightarrow \mathcal{A}^\bullet_{sia}(\widetilde{M}_{P,\Xi}, Z_\Xi) \otimes \mathcal{H}^\bullet(\mathfrak{s}_P, V)^{\mathrm{can}}.\end{aligned}$$

¿From here, it is obvious that the square in (2.7.1) commutes.

The above implies a useful fact:

Proposition (2.7.11). *The morphism*

$$H^\bullet(\widetilde{A}_{P,\Xi}, [\mathcal{V}^A]^{\mathrm{can}}) \to H^\bullet(\widetilde{A}_{P,\Xi}, [\mathcal{V}^A]^{\mathrm{can}} \otimes \mathcal{O}_{Z^A_\Xi(P')})$$

is given by the direct sum of mappings

$$H^\bullet(\widetilde{M}_{P,\Xi}, \mathcal{H}^i(\mathfrak{s}_P, V)^{\mathrm{can}}) \to H^\bullet(\widetilde{M}_{P,\Xi}, \mathcal{H}^i(\mathfrak{s}_P, V)^{\mathrm{can}} \otimes \mathcal{O}_{Z_\Xi(P')}).$$

Proof. The sheaf $[\mathcal{V}^A]^{\mathrm{can}} \otimes \mathcal{O}_{Z^A_\Xi(P')}$ is quasi-isomorphic to the complex

$$\{[\mathcal{V}^A]^{\mathrm{can}} \otimes \mathcal{I}_{Z^A_\Xi(P')} \to [\mathcal{V}^A]^{\mathrm{can}}\}[1] \tag{2.7.11.1}$$

Then

$$R\widetilde{\pi}_*\{[\mathcal{V}^A]^{\mathrm{can}} \otimes \mathcal{I}_{Z^A_\Xi(P')} \to [\mathcal{V}^A]^{\mathrm{can}}\}[1]$$

is quasi-isomorphic to

$$\bigoplus_i \{\mathcal{H}^i(\mathfrak{s}_P, V)^{\mathrm{can}} \otimes \mathcal{I}_{Z_\Xi(P')} \to \mathcal{H}^i(\mathfrak{s}_P, V)^{\mathrm{can}}\}[1], \tag{2.7.11.2}$$

which is in turn quasi-isomorphic to $\mathcal{H}^\bullet(\mathfrak{s}_P, V)^{\mathrm{can}} \otimes \mathcal{O}_{Z_\Xi(P')}$. Restriction to $Z^A_\Xi(P')$ and $Z_\Xi(P')$ is induced by inclusion into (2.7.11.1) and (2.7.11.2). Our assertion follows.

3. The nerve spectral sequence for coherent cohomology

For the remainder of the article, we fix a pure Shimura datum (G, X). Thus G is a reductive group over $\mathbb{Q}$; for simplicity we assume G^{ad} to be $\mathbb{Q}$-simple. All mixed Shimura varieties will be realized as boundary strata of toroidal compactifications of ${}_{K_f}Sh(G, X)_\Sigma$. Here and in what follows the notation K_f and Σ are used as in §1, relative to (G, X), and will be used without comment. Moreover, all toroidal data Σ will be assumed to be *full*, in the sense of [HZ1, 2.2.6].

(3.1) *Cohomology on boundary strata of toroidal compactifications.* As in [HZ2] we let $\mathcal{P}(G)$ denote the set of rational $\mathbb{Q}$-parabolic subgroups of G, $\mathcal{P}_{\max}(G) \subset \mathcal{P}(G)$ the subset of maximal proper parabolics; the analogous notation is used for other reductive groups. To any $P \in \mathcal{P}_{\max}(G)$ we associate its Levi quotient L_P and subgroups $G_{h,P}$ and $G_{\ell,P}$ of L_P, as in [HZ1, §1.2], where the "P" was sometimes dropped; see also (0.5). As in [HZ1], we choose liftings of L_P to a subgroup of P, also denoted L_P. Later we will restrict attention to standard parabolic subgroups, relative to a choice of minimal parabolic P_0 and then the Levi subgroups L_P will all be assumed to contain L_{P_0}.

As in [HZ2, (2.2)], for $P \in \mathcal{P}_{\max}(G)$ we define an injection $\varepsilon_P : \mathcal{P}(G_{\ell,P}) \to \mathcal{P}(G)$, with image denoted $\mathcal{P}_P(G)$, such that

$$\mathcal{P}(G) - \{G\} = \coprod_{P \in \mathcal{P}_{\max}(G)} \mathcal{P}_P(G).$$

If $R \in \mathcal{P}_P(G)$, then R is said to be subordinate to P, and we write $\Pi(R) = P$; this is the case if $G_{h,P}$ is maximal among the $G_{h,Q}$ contained in the Levi quotient of R, as Q runs through $\mathcal{P}_{\max}(G)$ (this was stated differently in [HZ2]). For brevity, we write $G_{h,R} = G_{h,P}$ if R is subordinate to P. Let $R_{\ell,P} = R \cap G_{\ell,P}$ (in [HZ2,(2.2.3)], this was called $\iota_P(R)$). It is a $\mathbb{Q}$-parabolic subgroup of $G_{\ell,P}$, and every $\mathbb{Q}$-parabolic of $G_{\ell,P}$ is of the form $R_{\ell,P}$ for some R subordinate to P. We also let $G_{\ell,R}$ denote the Levi subgroup $L_R \cap G_{\ell,P}$ of $R_{\ell,P}$.

For $R \in \mathcal{P}(G)$, R subordinate to P, we define the R-stratum $Z_\Sigma(R)$ of ${}_{K_f}Sh(G, X)_\Sigma$ as in (1.5.2) and [HZ2, (1.5.2)]. We revert momentarily to the classical (non-adelic) language of [HZ1, §§1–3]. When $R = P$ is maximal, $Z_\Sigma(R)$ is itself the union of irreducible divisors with normal crossings corresponding to 1-dimensional cones in Σ_P^c modulo conjugation by $\Gamma_{\ell,P}$ (notation as in [HZ1], esp. §1.5). The nerve $\mathfrak{N}_\Sigma(R)$ of the closed covering of $Z_\Sigma(R)$ by its irreducible components (denoted $\mathfrak{N}(\mathcal{Z}_{\Sigma_P})$ in [HZ1]) is a simplicial complex canonically isomorphic to $\Gamma_{\ell,P} \backslash \hat{\Sigma}_P^c$ [HZ1, Lemma 3.7.2]. More generally, if r is the parabolic rank of R, then $Z_\Sigma(R)$ has a closed covering by irreducible components of codimension r in ${}_{K_f}Sh(G, X)_\Sigma$. However, the nerve $\mathfrak{N}_\Sigma(R)$ of this closed covering does not have a transparent description in terms of Σ_F. This is because, for instance, if $P \prec P_1$, $R = P \cap P_1$, and σ (resp. σ') $\in \Sigma_P$ is a two-dimensional (resp. three-dimensional) cone with one (resp. two) edge(s) in Σ_{P_1} and one in Σ_P^c, then the corresponding closed toroidal stratum $\overline{Z}_\sigma$ (resp. $\overline{Z}_{\sigma'}$) is of codimension 0 (resp. 1) in $Z_\Sigma(R)$ but the facets of σ and σ' in Σ_P have the same dimension. The constructions in [HZ2, §2] were designed to deal with this problem.

Let $i_R : Z_\Sigma(R) \hookrightarrow {}_{K_f}Sh(G,X)_\Sigma$ denote the canonical closed immersion. For $R = P$ maximal, the geometric description of $\mathfrak{N}_\Sigma(P)$ in [HZ1, §3] yields a simple expression for the coherent cohomology $H^\bullet(Z_\Sigma(P), i_P^*[\mathcal{W}]^{can})$, when $[\mathcal{W}]$ is an automorphic vector bundle on $Sh(G,X)$ [HZ1, 3.7.8, 3.13.6]. The main step is the calculation [HZ1, 3.6.4] in terms of the pure Shimura variety attached to $G_{h,P}$ of the cohomology $H^\bullet(\overline{Z}_\sigma, i_\sigma^*[\mathcal{W}]^{can})$ where $\overline{Z}_\sigma$ is an individual closed stratum as above and $i_\sigma : \overline{Z}_\sigma \hookrightarrow {}_{K_f}Sh(G,X)_\Sigma$ is the inclusion. More precisely, the argument in [HZ1, 3.6.4] only concerns the open stratum Z_σ, and expresses its cohomology in terms of that of a family of automorphic vector bundles $\mathcal{V}_{\lambda(h,w)}$ on the base M_P. For $\overline{Z}_\sigma$ we obtain the analogous expression in terms of the canonical extensions $[\mathcal{V}_{\lambda(h,w)}]^{can}$ on the toroidal compactification $M_{P,\Sigma(P)}$, using Proposition 1.4.4 and Corollary 1.4.5; see (3.1.2), below. In particular, the calculation works in exactly the same way when $\mathfrak{N}_\Sigma(P)$ is replaced by $\mathfrak{N}_\Sigma(R)$ for general R.

Remark. Careful readers of the proofs in [HZ1] may note an apparent dependence on analytic considerations not discussed here. Specifically, the reduction of the cohomology of a boundary stratum to cohomology of reductive groups is carried out in Corollary 3.7.8 of [HZ1]. An intermediate step in the proof of this corollary is provided by Lemma 3.7.5 of [op. cit.], which (in the present setting) calculates the coherent cohomology of $Z_\Sigma(P)$ in terms of the $\Gamma_{\ell,P}$-equivariant coherent cohomology of an abelian scheme, the analogue of $Sh(Q/U, U\backslash\mathcal{X})_{\Sigma_2}$. In turn, the proof of Lemma 3.7.5 makes forward reference to [HZ, 3.9.4], whose proof is analytic and homotopy-theoretic. However, Lemma 3.7.5 is actually a simple consequence of (3.7.3), (3.7.4), and (3.7.6).

We first state the result (for general R) in the classical setting. Thus D is a connected component of X and $\Gamma \subset G$ is a neat arithmetic subgroup of $G(\mathbb{R})$ that fixes X. We retain the notation $\Gamma_{\ell,R} = \Gamma \cap G_{\ell,P}(\mathbb{Q}) \cap R(\mathbb{Q})$, when R is subordinate to P, from [HZ2]. We fix a point $p \in D$ as in [HZ1, §3.5], let $K_p \subset G(\mathbb{R})$ denote its stabilizer. We can and do always choose p so that $K_p \cap P(\mathbb{R})$ contains a maximal connected compact subgroup of $P(\mathbb{R})$ for every standard parabolic P. As in [HZ1], we restrict attention to fully decomposed automorphic vector bundles; i.e., those associated to representations of K_p ([HZ1, Definition 3.1.2]).

We let M_P denote the (connected) pure Shimura variety associated to P. Similarly, we let $W^{P,p}$ denote the subset of the Weyl group of K_p from [HZ1, §3.6] (see (0.5)). Recall that this is the set of Kostant representatives for the maximal parabolic denoted $Q^{P,p} \subset K_p$ with Levi subgroup $K_p^{(2)}$ and unipotent radical S_p. The Lie algebras are denoted by lower case Gothic characters, as usual. Recall further [HZ1, (1.8.3), (1.8.6)] that the Cayley transform induces isomorphisms

$$K_p^{P,(2)} \xrightarrow{\sim} K_{h,P} \times G_{\ell,P};\ \mathfrak{s}_{p,P} \xrightarrow{\sim} \mathfrak{v}_p^- . \tag{3.1.0}$$

Here $K_{h,P} \subset G_{h,P}$ is the maximal compact subgroup associated to p. More generally, if R is subordinate to the maximal parabolic P, we let $Q^{R,p} \subset Q^{P,p}$ be the parabolic with Levi factor isomorphic via Cayley transform to $K_{h,P} \times G_{\ell,R}$ and unipotent radical $S_{p,R} \supset S_{p,P}$ and let $W^{R,p}$ be the corresponding set of Kostant representatives in the Weyl group of K_p.

Proposition 3.1.1. *Let $[\mathcal{W}]$ be an irreducible automorphic vector bundle on $Sh(G,X)$ and let R be a proper $\mathbb{Q}$-parabolic subgroup of G, subordinate to the maximal parabolic P. Then*

(i) For each $b \geq 0$, there is a locally constant sheaf $\mathbf{L}^b(\bullet, \mathcal{W})$ on the simplicial complex $\mathfrak{N}_\Sigma(R)$ and there is a spectral sequence

$$E_2^{a,b} = H^a(\mathfrak{N}_\Sigma(R), \mathbf{L}^b(\bullet, \mathcal{W})) \Rightarrow H^{a+b}(Z_\Sigma(R), i_R^*[\mathcal{W}]^{can}).$$

(ii) Over the universal cover of $\mathfrak{N}_\Sigma(R)$, $\mathbf{L}^b(\bullet, \mathcal{W})$ is isomorphic to

$$\bigoplus_{w \in W^{P,p}} H^{b-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes V_{\lambda(\ell,w)}. \tag{3.1.2}$$

Here $M_{P,\Sigma(P)}$ is any admissible toroidal compactification of M_P, $\ell(w)$ is the length of w and $\mathcal{V}_{\lambda(h,w)}$ and $V_{\lambda(\ell,w)}$ are defined (relative to $[\mathcal{W}]$) as in [HZ1, (3.6.1)].

(iii) The spectral sequence in (i) degenerates at E_2, and is naturally split (cf. [HZ1, 3.7]).

Remark. The hypothesis that $[\mathcal{W}]$ be irreducible is made in order to obtain the simple decomposition (3.1.2). This hypothesis was inadvertently omitted in the discussion in [HZ1, §3].

Proof. For $R = P$ maximal the above is a strengthening of (3.7.3), (3.7.4) and (3.7.7) of [HZ1], which pertain to the R-stratum. Here, we are working with the closed R-stratum $Z_\Sigma(R)$, for arbitrary R. Correspondingly the formula in (ii) involves the canonical extension over some toroidal compactification $M_{\ldots}$. To obtain it, we first replace Corollary 3.6.3 of [loc. cit.] by Proposition 1.4.4 here: a fixed closed stratum $\overline{Z}_\sigma \subset Z_\Sigma(R)$, whose interior is constructed from a cone in Σ_P, can be realized as a toroidal compactification of a (connected) mixed Shimura variety of the form ${}_{K_f}Sh(Q,\mathcal{X})_\Sigma$ considered in (1.4) [P, Corollary 7.17]. For appropriately chosen data we have a morphism

$$\pi_\sigma = \pi_{2,\sigma} \circ \pi_{1,\sigma} : \overline{Z}_\sigma \to M_{P,\Sigma(P)}$$

as in (1.4.2). Corollary 1.4.5 (which replaces the Leray spectral sequences (3.5.10.2) and (3.5.10.4) of [HZ1]) yields as a special case:

$$E_2^{p,q} = H^p(M_{P,\Sigma(P)}, (R^q\Phi_\sigma i_\sigma^*[\mathcal{W}])^{can}) \Rightarrow H^{p+q}(\overline{Z}_\sigma, i_\sigma^*[\mathcal{W}]^{can}).$$

The proof of degeneration of this Leray spectral sequence at E_2 is contained in the proof of [HZ1, Proposition 3.13.4] (cf. [HZ1, (3.13.5.1)]).

This argument proves (i) and (ii). Then (iii) follows as in the proof of Proposition 3.7.7 of [HZ1]. For R maximal this is also a less precise form of Corollary 3.13.6, in that the geometric description of $\mathfrak{N}_\Sigma(R)$ is not used.

In §2 of [HZ2] the nerves $\mathfrak{N}_\Sigma(R)$ are identified for general R, up to homotopy type. Let $X(\Gamma_{\ell,P})$ be the locally symmetric space associated to the arithmetic subgroup $\Gamma_{\ell,P}$ (denoted $X(\Gamma_\ell)$ in [HZ1, 2.2.10]). Let $X(\Gamma_{\ell,P}) \hookrightarrow \overline{X}(\Gamma_{\ell,P})$ denote the Borel-Serre compactification. The result is the following:

Proposition 3.1.3. *Let R be a proper $\mathbb{Q}$-parabolic subgroup of G, subordinate to the maximal parabolic P. As above, let $R_{\ell,P}$ denote the corresponding $\mathbb{Q}$-parabolic subgroup of $G_{\ell,P}$. Then*

(i) The nerve $\mathfrak{N}_\Sigma(R)$ is homotopy equivalent to the $R_{\ell,P}$-stratum $\overline{e'(R_{\ell,P})} \subset \overline{X}(\Gamma_{\ell,P})$.

(ii) Let $R \subset R'$ be two $\mathbb{Q}$-parabolic subgroups of G subordinate to P. Under the homotopy equivalences in (i), the natural inclusion $\mathfrak{N}_\Sigma(R) \subset \mathfrak{N}_\Sigma(R')$ corresponds to the natural inclusion $\overline{e'(R_{\ell,P})} \subset \overline{e'(R'_{\ell,P})}$.

This is essentially contained in Proposition 2.6.4 of [HZ2]; the translation into the present language is provided by Corollaries 2.5.9 and 2.5.10 of [loc. cit.].

The fundamental group of $\overline{e'(R_{\ell,P})}$ is just $\Gamma_{\ell,R}$. Let $\widetilde{\mathbf{V}}_{\lambda(\ell,w)}$ denote the local system on $\overline{X}(\Gamma_{\ell,P})$ associated to the representation $\lambda(\ell,w)$, as in the statement of Corollary 3.7.8 of [HZ1]. Just as in the case of maximal strata (cf. [HZ1, Corollary 3.7.8]), Proposition 3.1.2 and (3.1.3) allow us to identify

$$\begin{aligned} H^a(\mathfrak{N}_\Sigma(R), \mathbf{L}^b(\bullet, \mathcal{W})) &\simeq \oplus_{w \in W^{P,p}} H^a(\Gamma_{\ell,R}, H^{b-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes V_{\lambda(\ell,w)}) \\ &\simeq \oplus_{w \in W^{P,p}} H^{b-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H^a(\Gamma_{\ell,R}, V_{\lambda(\ell,w)}) \\ &\simeq \oplus_{w \in W^{P,p}} H^{b-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H^a(\overline{e'(R_{\ell,P})}, i^*_{R_{\ell,P}} \widetilde{\mathbf{V}}_{\lambda(\ell,w)}) \end{aligned} \tag{3.1.4}$$

where the second isomorphism expresses the fact that $\Gamma_{\ell,R}$ acts trivially on the first factor, and in the third isomorphism $i_{R_{\ell,P}} : \overline{e'(R_{\ell,P})} \hookrightarrow \overline{X}(\Gamma_{\ell,P})$ is the natural inclusion.

On the other hand, the calculation of $H^\bullet(\overline{e'(R_{\ell,P})}, i^*_{R_{\ell,P}} \widetilde{\mathbf{V}}_{\lambda(\ell,w)})$ is standard. Let $W_{\ell,P}$ denote the relative Weyl group (over $\mathbb{Q}$) of $G_{\ell,P}$, and let $W_\ell^R \subset W_{\ell,P}$ denote the subset of Kostant representatives relative to $R_{\ell,P}$ (cf. Notation). Identify $G_{\ell,R}$ with a Levi *quotient* of $R_{\ell,P}$ and let $\Gamma^{red}_{\ell,R}$ denote the image of $\Gamma_{\ell,R}$ in $G_{\ell,R}(\mathbb{Q})$; let $X(\Gamma^{red}_{\ell,R})$ be the corresponding locally symmetric space, the quotient by $\Gamma^{red}_{\ell,R}$ of the symmetric space attached to $G_{\ell,R}$. Let $\mu(\ell, w)$ denote the highest weight of $\lambda(\ell, w)$ relative to the choice of positive roots for $G_{\ell,P}$ made in [HZ1, 3.6]. Then [Ha]

$$H^a(\overline{e'(R_{\ell,P})}, i^*_{R_{\ell,P}} \widetilde{\mathbf{V}}_{\lambda(\ell,w)}) \xrightarrow{\sim} \bigoplus_{\omega \in W_\ell^R} H^{a-\ell(\omega)}(X(\Gamma^{red}_{\ell,R}), \widetilde{\mathbf{V}}_{\lambda(\ell,w;\omega)}) \tag{3.1.5}$$

Here $\widetilde{\mathbf{V}}_{\lambda(\ell,w;\omega)}$ is the local system on $X(\Gamma^{red}_{\ell,R})$ associated to the representation $\lambda(\ell, w; \omega)$ of $G_{\ell,R}$, and this in turn is the representation with highest weight $\omega(\mu(\ell, w) + \rho_\ell) - \rho_\ell$, where ρ_ℓ is the half-sum of positive roots of $G_{\ell,P}$.

The following lemma will be used repeatedly in what follows.

Lemma 3.1.6. *Let H be a reductive group over the field k and let $P_1 \supset P_2$ be a pair of k-parabolic subgroups of G, with Levi decompositions $P_1 = L_1 \cdot U_1$, $P_2 = L_2 \cdot U_2$, $L_1 \supset L_2$. Let $Q = P_2 \cap L_1$. Let $T \subset L_2$ be a maximal k-split torus and let $W = W(H, T)$ and $W_1 = W(L_1, T)$ be the relative Weyl groups. Choose a minimal parabolic subgroup $P_0 \subset P_2$. Let $W^i \subset W$ be the sets of Kostant representatives for P_i, $i = 1, 2$; let $W^Q \subset W_1$ be the set of Kostant representatives for Q. (In both cases the Kostant representatives are defined relative to the ordering determined by P_0.) Under the natural identification of W_1 with a subgroup of W, we then have*

$$W^2 = \{\omega \cdot w \,|\, \omega \in W^Q, w \in W^1\}.$$

Moreover, with the above notation, $\ell_W(\omega \cdot w) = \ell_W(w) + \ell_{W_1}(\omega)$, where ℓ_W and ℓ_{W_1} are the length functions on W and W_1, respectively.

Proof.

Applying Lemma 3.1.6, with $P_1 = Q^{P,p}$, $P_2 = Q^{R,p}$, and thus $Q \simeq R_{\ell,P}$, to (3.1.4) and (3.1.5), we thus find for (3.1.1,(i))

$$E_2^{a,b} \simeq \bigoplus_{w=w_1\cdot w_2\in W^{R,p}} H^{b-\ell(w_2)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can})\otimes H^{a-\ell(w_1)}(X(\Gamma^{red}_{\ell,R}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)}).$$

Here $w \in W^{R,p}$ is factored as in Lemma 3.1.6 as $w_1 \cdot w_2$ with $w_1 \in W_\ell^R$ and $w_2 \in W^{P,p}$; $\ell(w_i)$ is given by the appropriate length function; and $\lambda(\ell, w) = \lambda(\ell, w_2; w_1)$ in the previous notation. Moreover, we have used Kostant's theorem to write

$$\text{(3.1.7)} \qquad H^i(\mathfrak{s}_{p,R}, V_\lambda) = \bigoplus_{w\in W^{R,p};\ell(w)=i} V_{\lambda(h,w)} \otimes V_{\lambda(\ell,w)}$$

where $V_{\lambda(h,w)} = V_{\lambda(h,w_2)}$ is a representation of $K_{h,P}$ and $V_{\lambda(\ell,w)}$ is a representation of $G_{\ell,R}$.

Combining this with (3.1.1,(iii)), we obtain the following formula:

Proposition (3.1.8). *Let $[\mathcal{W}]$ be an irreducible automorphic vector bundle on $Sh(G, X)$. Then*

$$H^k(Z_\Sigma(R), i_R^*[\mathcal{W}]^{can}) \simeq \bigoplus_{a+b=k}\bigoplus_{w\in W^{R,p}} H^{b-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can})\otimes H^a(X(\Gamma^{red}_{\ell,R}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)}).$$

(3.1.9) *Remark.* (i) In the setting of Lemma 3.1.6, let $\mathfrak{u}_i$ denote the Lie algebra of U_i for $i = 1, 2$ and $\mathfrak{u}_Q$ the Lie algebra of the unipotent radical of Q. Then $\mathfrak{u}_2 = \mathfrak{u}_1 \oplus \mathfrak{u}_Q$. If follows from Lemma (3.1.6) that the spectral sequence

$$E_2^{p,q} = H^q(\mathfrak{u}_Q, H^p(\mathfrak{u}_1, V)) \Rightarrow H^{p+q}(\mathfrak{u}_2, V).$$

degenerates at E_2.

(ii) Here and in what follows, the passage from the sum over $W^{P,p}$ to a sum over $W^{R,p}$ should be regarded as a purely topological operation. All the arithmetic information is carried by the $G_{h,P}$-factor, and is independent of the parabolic subgroup R subordinate to P. In particular:

(a) The isomorphism (3.1.5) is rational over the field of coefficients of the automorphic vector bundle $[\mathcal{W}]$. This is because the spectral sequence in (i) above, with $Q = R_{\ell,P}$, is purely algebraic and $\mathfrak{u}_Q$ is a $\mathbb{Q}$-rational nilpotent Lie subalgebra of $\mathfrak{g}$. In particular, the adelic reformulation (3.2.9) of the calculations of this section is compatible with the canonical models of $Sh(G, X)$ and $Sh(G_{h,P}, X(P))$.

(b) Similarly, for the Hodge-theoretic applications in §4, it makes no difference whether the cohomology of the R-stratum is written as a sum indexed by $W^{P,p}$ or by $W^{R,p}$. (Compare (4.2.27) and (4.2.29)). Again, this is because the contribution of the $G_{\ell,R}$-factor is purely topological.

(3.2) *The nerve spectral sequence for boundary cohomology.* The spectral sequence in question is the one associated to the closed covering $\{Z_\Sigma(P) : P \text{ maximal}\}$ of Z_Σ. In classical language, the E_1-term is

$$(3.2.1) \qquad E_1^{r,s} = \bigoplus_{r(R)=r+1} E_1^{r,s}(R) \Rightarrow H^{r+s}(Z_\Sigma, i^*[\mathcal{W}]^{can}), \quad \text{where}$$

$$(3.2.2) \qquad E_1^{r,s}(R) = H^s(Z_\Sigma(R), i_R^*[\mathcal{W}]^{can}).$$

Using the description of the right-hand side of (3.2.2) given in (3.1.8), we can write the above as

(3.2.3)
$$E_1^{r,s}(R) = \bigoplus_{a+b=s} \bigoplus_{w \in W^{R,p}} H^{b-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H^a(X(\Gamma_{\ell,R}^{red}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)}).$$

We set up the adelic version of the above. We drop the level subgroup K_f from the notation and write Sh_Σ for $Sh(G,X)_\Sigma$, ∂Sh_Σ for its (toroidal) boundary, $Sh_\Sigma^{R(*)}$ for its $R(*)$-stratum, and $\overline{Sh}_\Sigma^{R(*)}$ for the Zariski closure of $Sh_\Sigma^{R(*)}$ in ∂Sh_Σ. As in [HZ1, 5.3], the $\overline{Sh}_\Sigma^{P(*)}$, as P runs through the maximal admissible parabolics, form a closed cover of ∂Sh_Σ. Taking the nerve of this closed cover thus yields a spectral sequence for the cohomology of ∂Sh_Σ with coefficients in a canonically extended automorphic vector bundle. For simplicity we describe this spectral sequence when G^{ad} is assumed $\mathbb{Q}$-simple. Let $i_R : \overline{Sh}_\Sigma^{R(*)} \hookrightarrow Sh_\Sigma$ denote the corresponding closed embedding, and let $r(R)$ denote the parabolic rank of R. Taking the limit over neat open compact subgroups and families of fans, we obtain the nerve spectral sequence in the following form:

$$(3.2.4) \qquad E_1^{r,s} = \bigoplus_{r(R)=r+1} \varinjlim_{K_f,\Sigma} H^s(\overline{Sh}_\Sigma^{R(*)}, i_R^*([\mathcal{W}]^{can})) \Rightarrow H^{r+s}([\mathcal{W}](\infty)).$$

Here $H^\bullet([\mathcal{W}](\infty))$ denotes $\varinjlim_{K_f,\Sigma} H^\bullet(\partial Sh_\Sigma, [\mathcal{W}]^{can} \otimes \mathcal{O}_{\partial Sh_\Sigma})$, as in [HZ1].

The individual summands on the left-hand side of (3.2.4) can be written as induced representations, just as in [HZ1]. It is most convenient to express each term as a tensor product of the coherent cohomology of a (pure) Shimura variety by the cohomology of a locally symmetric space. Let R be any rational parabolic, subordinate to the maximal rational parabolic P, and define $G_{\ell,R}$ as before. We define the adelic locally symmetric space

$$(3.2.5) \qquad X(G_{\ell,R}) = \lim G_{\ell,R}(\mathbb{Q})\backslash G_{\ell,R}(\mathbf{A})/(K_{\ell,R} \cdot Z_{\ell,R})(\mathbb{R}) \cdot K_f$$

where $Z_{\ell,R}$ is the center of $G_{\ell,R}$, $K_{\ell,R} = K_p \cap G_{\ell,R}(\mathbb{R})$, and $K_{\ell,R,f}$ runs through compact open subgroups of $G_{\ell,R}(\mathbf{A}_f)$. Let L_R be a (standard) Levi subgroup of R. We also let $Sh(G_{h,P}, X(P))$ denote the Shimura variety attached to $G_{h,P}$. For brevity we write $Sh(R) = Sh(G_{h,P}, X(P))$ when R is subordinate to P. For each $w \in W^{R,p}$ we have the automorphic vector bundle $[\mathcal{V}_{\lambda(h,w)}]$ over $Sh(R)$, as above, and therefore we can define the cohomology of its canonical extensions in the adelic limit

$$\widetilde{H}^{\bullet}([\mathcal{V}_{\lambda(h,w)}])$$

(notation as in [H4] and [HZ1, §4]). We define

$$\mathcal{H}^{\bullet}(w) = \widetilde{H}^{\bullet}([\mathcal{V}_{\lambda(h,w)}]) \otimes H^{\bullet}(X(G_{\ell,R}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)})[-\ell(w)]. \tag{3.2.6}$$

by analogy with [HZ1, (4.1.10)]

Parametrization of connected components introduces annoying complications, as in [HZ1, 4.1.12]. We let

$$\Delta_{1,R} = Ker: \ G_h(\mathbf{A}_f) \times G_{\ell,R}(\mathbf{A}_f) \to G_h(\mathbf{A}_f) \cdot G_{\ell,R}(\mathbf{A}_f)$$

(the product viewed as a subgroup of $R(\mathbf{A}_f)$) and let

$$\Delta_{0,R} = L_R(\mathbb{Q})^+ / (G_h(\mathbb{Q})^+ \cdot G_{\ell,R}(\mathbb{Q})^+); \ \Delta_R = \Delta_{0,R} \times \Delta_{1,R}. \tag{3.2.7}$$

(For the relation between (3.2.7) and connected components of the R stratum when R is maximal, see [HZ1, (4.1.10,4.1.11)]. The analogous relation holds for general R.) Then $G_h(\mathbf{A}_f) \times G_{\ell,R}(\mathbf{A}_f)$ acts on $\mathcal{H}^{\bullet}(w)$, and therefore so does its subgroup $\Delta_{1,R}$. Moreover $\Delta_{0,R}$ acts on $Sh(R) \times X(G_{\ell,R})$ by the analogue of [HZ1, (4.1.11)]. We define

$$I^R\{\mathcal{H}^{\bullet}(w)\} = Ind_{G_h(\mathbf{A}_f)\cdot G_{\ell,R}(\mathbf{A}_f) \rtimes \Delta_{0,R}}^{L_R(\mathbf{A}_f)} \{\mathcal{H}^{\bullet}(w)^{\Delta_{1,R}}\}, \tag{3.2.8}$$

just as in [HZ1, (4.1.13)]. Then we have

Corollary 3.2.9. *i) In the spectral sequence (3.2.4), there is a natural decomposition* $E_1^{r,s} = \bigoplus_{r(R)=r+1} E_1^{r,s}(R)$, *where*

$$E_1^{r,s}(R) = Ind_{R(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \bigoplus_i \bigoplus_{w \in W^{R,p}} I^R\{\widetilde{H}^{s-i-\ell(w)}([\mathcal{V}_{\lambda(h,w)}]) \otimes H^i(X(G_{\ell,R}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)})\}.$$

ii) Moreover, suppose $R' \subset R$ *are two* $\mathbb{Q}$*-parabolic subgroups of* G, *both subordinate to* P, *with* $r(R') = r+2$, $r(R) = r+1$. *Let* $Q' = R'_{\ell,P}$, $Q(R') = Q' \cap G_{\ell,R}$.
Let d ... *denote the restriction* $E_1^{r,s}(R)$ *of the nerve spectral sequence differential*

$d_1 : E_1^{r,s} \to E_1^{r+1,s}$, *followed by projection to* $E_1^{r+1,s}(R')$. *Then, with respect to the above formula,* $d_{R',R}$ *is induced by the identity on the holomorphic component tensored with the boundary map induced by the inclusion of the boundary stratum* $\overline{e'(Q(R'))}$ *in the Borel-Serre compactification of* $X(G_{\ell,R})$.

Proof. The first statement generalizes Corollary 4.1.14 of [HZ1], and is proved in the same way. The second statement follows immediately from (ii) of Proposition 3.1.3 (as such, it deduced from the same for (3.2.2)).

(3.3) *Differentials in the nerve spectral sequence.* Corollary 3.2.9 completes the verification of [HZ1, (5.3.12.4)], and computes half of the differentials d_1 in the nerve spectral sequence in terms of the cohomology of locally symmetric spaces attached to reductive groups. The present section will begin the computation of the remaining half: the $d_{R',R}$ for $r(R') = r(R) + 1$ with R and R' subordinate to distinct maximal parabolics, say P and P', respectively. Let $P(P') = P' \cap G_{h,P}$; then $P(P')$ is a maximal proper parabolic subgroup of $G_{h,P}$, hence corresponds to a boundary stratum in the toroidal compactification of $Sh(G_{h,P}, X(P))$. In terms of the expression for $E_1^{r,s}$ in Corollary 3.2.9, $d_{R',R}$ will be expressed in terms of the restriction map from the first (holomorphic) tensor factor to the boundary stratum corresponding to $P(P')$. In other words, $d_{R',R}$, which is given *a priori* by a restriction map on the coherent cohomology of mixed Shimura varieties – more precisely, the cohomology of simplicial schemes whose components are all toroidal compactifications of mixed Shimura varieties – can be expressed as a restriction map whose source is the coherent cohomology of a toroidally compactified pure Shimura variety. This is the opposite of the story in (ii) in Corollary 3.2.9, and will be proved in (3.4) below.

As usual, we will verify this at the level of connected Shimura varieties at finite level, the extension to the adelic setting being routine. Thus we have isomorphisms

(3.3.1)

$$E_1^{r,s}(R) \xrightarrow{\sim} \oplus_{w \in W^{R,p}} \oplus_i H^{s-i-\ell(w)}(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H^i(X(\Gamma_{\ell,R}^{red}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)})$$

and

(3.3.2)

$$E_1^{r+1,s}(R') \xrightarrow{\sim} \bigoplus_{w' \in W^{R',p}} \oplus_j H^{s-j-\ell(w')}(M_{P',\Sigma(P')}, [\mathcal{V}_{\lambda(h,w')}]^{can}) \otimes H^j(X(\Gamma_{\ell,R'}^{red}), \widetilde{\mathbf{V}}_{\lambda(\ell,w')})$$

On the other hand, letting $[\mathcal{V}]$ denote any automorphic vector bundle on $M_{\dots}$

we have by (3.1.8) applied to $G_{h,P}$

(3.3.3) $$H^t(\overline{Sh_{\Sigma}^{P(P')(*)}}, i^*_{P(P')}[\mathcal{B}]^{can}) \xrightarrow{\sim} \bigoplus_{\omega \in W^{P(P'),p'}} H^{t-k-\ell(\omega)}(M_{P',\Sigma(P')}, [\mathcal{B}_{\lambda(h,\omega)}]^{can}) \otimes H^k(X(\Gamma_{\ell,P(P')}), \widetilde{\mathbf{B}}_{\lambda(\ell,\omega)})$$

Here $[\mathcal{B}_{\lambda(h,\omega)}]$ and $\widetilde{\mathbf{B}}_{\lambda(\ell,\omega)}$ bear the same relation to $[\mathcal{B}]$ (for the boundary component $M_{P',\Sigma(P')}$ of $M_{P,\Sigma(P)}$) as $[\mathcal{V}_{\lambda(h,\omega)}]$ and $V_{\lambda(\ell,\omega)}$ bear to $[\mathcal{W}]$ (for the boundary component $M_{P,\Sigma(P)}$ of $Sh(G,X)$). Finally, p' is the fixed point of the subgroup $K_p \cap G_{h,P}$. Bear in mind that

(3.3.4) $$G_{\ell,R'} = G_{\ell,R} \cdot G_{\ell,P(P')},$$

where $G_{\ell,P(P')} \subset G_{h,P}$ is the G_ℓ-factor of the maximal parabolic $P(P') \subset G_{h,P}$. Then for sufficiently small Γ, we have

$$\Gamma_{\ell,R'} = \Gamma_{\ell,R} \times \Gamma_{\ell,P(P')}; \quad \Gamma^{red}_{\ell,R'} = \Gamma^{red}_{\ell,R} \times \Gamma_{\ell,P(P')}.$$

Moreover, Lemma 3.1.6 allows us to write

$$W^{R',p} = W^{P(P'),p'} \cdot W^{R,p}.$$

Say $w' \in W^{R',p}$ is the product $\omega \cdot w$, with $\omega \in W^{P(P'),p'}$ and $w \in W^{R,p}$. We then have canonical isomorphisms

(3.3.5) $$V_{\lambda(\ell,w')} \xrightarrow{\sim} V_{\lambda(h,w)(\ell,\omega)} \otimes V_{\lambda(\ell,w)};$$

(3.3.6)
$$H^j(X(\Gamma^{red}_{\ell,R'}), \widetilde{\mathbf{V}}_{\lambda(\ell,w')}) \xrightarrow{\sim} \oplus_{i+k=j} H^k(X(\Gamma_{\ell,P(P')}), \widetilde{\mathbf{V}}_{\lambda(h,w)(\ell,\omega)}) \otimes H^i(X(\Gamma^{red}_{\ell,R}, \widetilde{\mathbf{V}}_{\lambda(\ell,w)}).$$

Here $V_{\lambda(h,w)(\ell,\omega)}$ is the representation of $G_{\ell,P(P')}$ associated to the highest weight $\lambda(h,w)$ of $K_{h,P}$ and the Kostant representative ω.

We consider the toroidally compactified (connected) Shimura variety $M_{P,\Sigma(P)}$; let

$$i_{P(P')} : Z_{\Sigma(P)}(P(P')) \hookrightarrow M_{P,\Sigma(P)}$$

denote the inclusion of the closed $P(P')$ stratum of its boundary. If in (3.3.3) we now let $[\mathcal{B}]$ vary over the $[\mathcal{V}_{\lambda(h,w)}]$ from (3.3.1), we conclude that the right-hand side of (3.3.2) is the sum over w of the right-hand sides of (3.3.3), tensored with the sum over i of $H^i(\Gamma_{\ell,R}, V_{\lambda(\ell,w)})$:

(3.3.7)
$$E_1^{r+1,s}(R') \xrightarrow{\sim} \bigoplus \bigoplus E_1^{0,s-i-\ell(w)}(P(P'); [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H^i(X(\Gamma^{red}_{\ell,R}, \widetilde{\mathbf{V}}_{\lambda(\ell,w)}).$$

Here we have written $E_1^{0,s}(P(P'); [\mathcal{B}])$ for the $P(P')$ piece of the $E_1^{0,s}$-term of the nerve spectral sequence calculating the boundary cohomology of $[\mathcal{B}]^{can}$ on $M_{P,\Sigma(P)}$; it is the cohomology of the pullback of $[\mathcal{V}_{\lambda(h,w)}]$ to $Z_{\Sigma(P)}(P(P'))$.

For use in the next section, we note that the above factorization of $W^{R',p}$ can be continued. As in (3.1), we have Levi subgroups $K_p^{P',(2)} \subset K_p$, $K_{p'}^{P(P'),(2)}$, etc., and the corresponding sets of Kostant representatives. Now we have the factorizations:

$$(3.3.8)\qquad \begin{aligned} W^{R',p} &= W^{P(P'),p'} \cdot W^{R,p} = W^{P(P'),p'} \cdot W^{P,p} \cdot W_\ell^R = W^{P\cap P',p} \cdot W_\ell^R \\ &= W^{Q_\ell} \cdot W^{P',p} \cdot W_\ell^R. \end{aligned}$$

Here W_ℓ^R is a set of Kostant representatives for $R \cap G_{\ell,P}$ in $G_{\ell,P}$ and the factorization $W^{R,p} = W^{P,p} \cdot W_\ell^R$ is the application of Lemma 3.1.6, via Cayley transform, to the Levi subgroup $K_{h,P} \times G_{\ell,R} \subset K_{h,P} \cdot G_{\ell,P}$. Similarly, $W^{P\cap P',p}$ is a set of Kostant representatives in K_p obtained by identifying $K_p^{P,(2)} \xrightarrow{\sim} K_{h,P} \times G_{\ell,P}$ and then taking the parabolic subgroup associated to $P(P')$ in $K_{h,P}$; the last equality on the first line is the application of Lemma 3.1.6 to the pair of inclusions

$$K_{p'}^{P(P'),(2)} \times G_\ell \subset K_{h,P} \times G_\ell$$

Finally, we can identify $K_{p'}^{P(P'),(2)} \xrightarrow{\sim} K_{h,P(P')} \times G_{\ell,P(P')}$ via Cayley transform on $G_{h,P}$, and $K_{h,P(P')}$ is just $K_{h,P'}$. Applying Lemma 3.1.6 to the standard parabolic R'_ℓ of $G_{\ell,P'}$ with Levi factor $G_{\ell,R'} = G_{\ell,R} \times G_{\ell,P(P')}$, we obtain the factorization $W^{P\cap P',p} = W^{R'_\ell} \cdot W^{P',p}$, whence the second line.

(3.4) *Differentials in the nerve spectral sequence, concluded.* For any automorphic vector bundle $[\mathcal{B}]$ on M_P, let

$$d_{P(P')} : H^\bullet(M_{P,\Sigma(P)}, [\mathcal{B}]^{can}) \rightarrow H^\bullet(Z_{\Sigma(P)}(P(P')), i^*_{P(P')}[\mathcal{B}]^{can})$$

denote the natural restriction. We use the same notation to designate the homomorphism

$$H^\bullet(M_{P,\Sigma(P)}, [\mathcal{B}]^{can}) \rightarrow E_1^{0,\bullet}(P(P'), \mathcal{B})$$

obtained by composing the restriction map with the decomposition (3.2.3), and analogously in the adelic setting.

Proposition (3.4.1). *Let $[\mathcal{W}]$ be an automorphic vector bundle on $Sh(G,X)$, as above, and let R and R' be a pair of proper parabolic subgroups of G, with $R' \subset R$ $r(R') = r(R)+1$. Suppose that R and R' are subordinate to the maximal parabolics*

P and P' respectively ($P' \prec P$). In terms of the (adelic versions of the) isomorphisms (3.3.1) and (3.3.7), the map $d_{R,R'}$ defined as in Corollary 3.2.9 is given by

$$\bigoplus_{w \in W^{R,p}} \bigoplus_i d_{P(P')} \otimes \mathbf{1}_{i,\lambda(\ell,w)},$$

where $\mathbf{1}_{i,\lambda(\ell,w)}$ is the identity map of $H^i(X(G_{\ell,R}), \widetilde{\mathbf{V}}_{\lambda(\ell,w)})$.

Proof. The proposition is stated in the adelic framework, but for the proof it suffices to work with connected Shimura varieties. Let $\overline{Z}_\sigma \subset Z_\Sigma(R)$ be an irreducible stratum and let $\overline{Z}_{\sigma(R')} = \overline{Z}_\sigma \cap Z_\Sigma(R') = \overline{Z}_\sigma \cap Z_\Sigma(P')$. Let i_σ and $i_{\sigma(R')}$ be the inclusions of $\overline{Z}_\sigma$ and $\overline{Z}_{\sigma(R')}$, respectively, in ${}_{K_f}Sh(G,X)_\Sigma$.

We will show that the following diagram commutes:

(3.4.2)
$$\begin{array}{ccc} H^\bullet(\overline{Z}_\sigma, i_\sigma^*([\mathcal{W}]^{can})) & \longrightarrow & H^\bullet(\overline{Z}_{\sigma(R')}, i^*_{\sigma(R')}([\mathcal{W}]^{can})) \\ {\scriptstyle f}\uparrow & & {\scriptstyle g'}\uparrow \\ \bigoplus_{w \in W^P} H^\bullet(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) & \xrightarrow{d_{P(P')}} & \bigoplus_{w \in W^P} H^\bullet(Z_{\Sigma(P)}(P(P')), i^*_{P(P')}[\mathcal{V}_{\lambda(h,w)}]^{can}) \end{array}$$

Here the top arrow is the natural restriction map, the left-hand vertical arrow is given by Corollary 1.4.5, and the right-hand vertical arrow is to be constructed. There is difficulty caused by the fact that $\overline{Z}_{\sigma(R')}$, by which we mean the reduced divisor on $\overline{Z}_\sigma$, need not be the pullback of $Z_{\Sigma(P)}(P(P'))$.

We abstract the situation and study the restriction map $d_{R',R}$ in the context of base change. Consider the following diagram of morphisms of schemes of finite type over $\mathbb{C}$:

(3.4.3)
$$\begin{array}{ccccc} Z'' = Z'_{red} & \xrightarrow{i_0} & Z' & \xrightarrow{i'} & A \\ & & {\scriptstyle g}\downarrow & & {\scriptstyle \pi}\downarrow \\ & & Z & \xrightarrow{i} & M \end{array}$$

We assume that the square is cartesian. The notation Z'_{red} refers to the underlying reduced scheme of Z'. The schemes A and M are assumed smooth, and Z is a (reduced) divisor on M. The morphism π is proper. More generally, we will be later considering situations in which

(3.4.4) i) A is a finite union of irreducible components, each of which is a smooth scheme, as are the intersections of any finite set of irreducible components;

ii) any intersection of irreducible components of A is smooth and the restriction of π to it is proper.

In effect, we will take A to be a simplicial scheme, each of whose components is as in (3.4.3).

Let $\mathcal{F}$ be a locally-free sheaf on A. Consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}\otimes\mathcal{I}_{Z'} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}\otimes\mathcal{O}_{Z'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}\otimes\mathcal{I}_{Z''} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}\otimes\mathcal{O}_{Z''} & \longrightarrow & 0
\end{array} \tag{3.4.5}
$$

The rows are exact because $\mathcal{F}$ is locally-free. Applying the (exact) functor $R\pi_*$ to the first row, we get

$$0 \to R\pi_*(\mathcal{F}\otimes\mathcal{I}_{Z'}) \to R\pi_*\mathcal{F} \to R\pi_*(i'^*\mathcal{F}) \to 0. \tag{3.4.6}$$

Since $\mathcal{I}_{Z'} = \pi^*\mathcal{I}_Z$ and $\mathcal{I}_Z$ is locally-free, we can rewrite (3.4.6) by “adjunction” as

$$0 \to (R\pi_*\mathcal{F})\otimes\mathcal{I}_Z \to R\pi_*\mathcal{F} \to R\pi_*(i'^*\mathcal{F}) \to 0. \tag{3.4.7}$$

This yields the isomorphism

$$R\pi_*(i'^*\mathcal{F}) \approx i^*R\pi_*\mathcal{F}. \tag{3.4.8}$$

We get from (3.4.5):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}\otimes\mathcal{I}_Z & \longrightarrow & \mathcal{S} & \longrightarrow & R\pi_*(i'^*\mathcal{F}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & R\pi_*(\mathcal{F}\otimes\mathcal{I}_{Z''}) & \longrightarrow & \mathcal{S} & \longrightarrow & R\pi_*(i''^*\mathcal{F}) & \longrightarrow & 0
\end{array} \tag{3.4.9}
$$

where we have written $\mathcal{S}$ for $R\pi_*\mathcal{F}$, and $i'' = i'\circ i_0$.

We consider the diagram (3.4.3), where $\pi : A \to M$ is a morphism of toroidally compactified mixed Shimura varieties ${}_\bullet Sh(Q,\mathcal{X})_\Sigma \longrightarrow {}_\bullet Sh(Q',\mathcal{X}')_{\Sigma_1}$, with $Q' = Q/R_uQ$ (so M is pure), and Z is the (reduced) boundary divisor associated to a maximal parabolic subgroup of Q'. Take $\mathcal{F}$ to be a canonically extended fully decomposed automorphic vector bundle $[\mathcal{V}^A]^{can}$ on A, and $\mathcal{S} = R\pi_*\mathcal{F}$ again. We want to compare the restriction map $H^\bullet(A,\mathcal{F}) \longrightarrow H^\bullet(Z'', i'',^*\mathcal{F})$ to the map $H^\bullet(M,\mathcal{S}) \longrightarrow H^\bullet(Z, i^*\mathcal{S})$.

We are actually interested in the case in (3.4.2): $A = \overline{Z}_\sigma$, $M = M_{P,\Sigma(P)}$, $Z = Z_{\Sigma(P)}(P(P'))$, and $\mathcal{F} = i_\sigma^*([\mathcal{W}]^{can})$. However, we see by Proposition 1.4.3 that in general we may replace Q by Q/W_2Q, thereby reducing the issue to the case where $W_2Q = (0)$. Then $\pi : A \to M$ above is generically an abelian scheme, and that puts us into the situation of Proposition 2.7.1 (with P' replaced here by $P(P')$ and $Z'' = Z^A$.)

In the present situation, it follows from Proposition 2.7.1 that the left vertical arrow in (3.4.9) is a quasi-isomorphism. This provides the top half of the following commutative diagram, where the vertical arrows are just the Leray spectral sequence:

(3.4.10)
$$\begin{array}{ccccc}
H^\bullet(A,\mathcal{F}) & \to & H^\bullet(A,i^{\prime,*}\mathcal{F}) & \xrightarrow{\simeq} & H^\bullet(A,i^{\prime\prime,*}\mathcal{F}) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
H^\bullet(M,R\pi_*\mathcal{F}) & \to & H^\bullet(M,R\pi_*(i^{\prime,*}\mathcal{F})) & \xrightarrow{\simeq} & H^\bullet(M,R\pi_*(i^{\prime\prime,*}\mathcal{F})) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
H^\bullet(M,\oplus\,\mathcal{S}_w) & \to & H^\bullet(M,\oplus\, i^*\mathcal{S}_w) & = & H^\bullet(M,\oplus\, i^*\mathcal{S}_w)
\end{array}$$

As for the bottom half of (3.4.10), the pairs of opposing vertical maps (all isomorphisms) are given by the splitting in the derived category provided by (2.7.6) and (2.7.7) for all three instances. More precisely, from Lemma (2.7.10) and the subsequent discussion we obtain the left half of the following commutative diagram:

(3.4.11)
$$\begin{array}{ccccc}
H^\bullet(A,\mathcal{F}\otimes\mathcal{I}_{Z'}) & \to & H^\bullet(A,\mathcal{F}) & \to & H^\bullet(A,i^{\prime,*}\mathcal{F}) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
H^\bullet(M,R\pi_*\mathcal{F}\otimes\mathcal{I}_Z)) & \to & H^\bullet(M,R\pi_*\mathcal{F}) & \to & H^\bullet(M,R\pi_*(i^{\prime,*}\mathcal{F})) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
H^\bullet(M,\oplus\,\mathcal{S}_w\otimes\mathcal{I}_Z) & \to & H^\bullet(M,\oplus\,\mathcal{S}_w) & \to & H^\bullet(M,\oplus\, i^*\mathcal{S}_w)
\end{array}$$

The top of (3.4.11) is obtained from (3.4.7). Using (3.4.8), we can replace $R\pi_*(i^{\prime,*}\mathcal{F})$ by $i^*R\pi_*\mathcal{F}$ in the second row of (3.4.11). That the last pair of opposing vertical maps are isomorphisms then follows by the five lemma. It follows that restriction in (3.4.10) respects the splittings. This completes the verification of (3.4.2).

Now the natural projection $Z_{\Sigma(P)}(P(P'))\to M_{P',\Sigma(P')}$ —we are assuming our toroidal compactifications have been chosen compatibly—defines isomorphisms

$$g'' : H^\bullet(Z_{\Sigma(P)}(P(P')), i^*_{P(P')}[\mathcal{V}_{\lambda(h,w)}]^{can}) \xrightarrow{\sim} E_1^{0,\bullet}(P(P'),[\mathcal{V}_{\lambda(h,w)}]).$$

We claim that the composite

$$g''\circ g' : H^\bullet(\overline{Z}_{\sigma(R')}, i^*_{\sigma(R')}([\mathcal{W}]^{can})) \xrightarrow{\sim} \oplus_{w\in W^P}\, E_1^{0,\bullet}(P(P'),[\mathcal{V}_{\lambda(h,w)}])$$

equals the isomorphism given by applying Proposition 1.4.5 simplicially to the individual closed strata of $\overline{Z}_{\sigma(R')}$.

More precisely, if $\overline{Z}_{\sigma'}$ is an irreducible stratum of $\overline{Z}_{\sigma(R')}$, then (1.4.5), combined with (3.1.2), asserts an isomorphism between $H^\bullet(\overline{Z}_{\sigma'}, i^*_{\sigma'}([\mathcal{W}]^{can}))$ and a sum over terms indexed by $W^{P',p}$. Similarly, (3.3.3) writes each $E_1^{0,\bullet}(P(P'),[\mathcal{V}_{\lambda(h,w)}])$ as a sum of terms indexed by $W^{P(P'),p'}$. As in (3.3.8), we have

$$W^{P(P'),p'} \quad W^{P,p} \quad W^{P\cap P',p} \quad W^{R'_\ell} \quad W^{P',p}$$

Thus this sum can be regrouped as a sum over terms indexed by $W^{P',p}$. The claim, from which the proposition will follow, is that the composite $g'' \circ g'$ can be identified with the isomorphism of (1.4.5) after regrouping in this way. But this follows formally by applying Proposition 1.4.7 simplicially to the individual closed strata.

We conclude as in [HZ1, §4.8]:

Theorem (3.4.12). *Let $[\mathcal{W}]$ be an automorphic vector bundle on $Sh(G,X)$, with canonical model over the field $E(\mathcal{W})]$. Suppose $R' \subset R$ are two $\mathbb{Q}$-parabolics of G with $r(R') = r+2 = r(R)+1$. Write $E_1^{r,s}(R)$ and $E_1^{r+1,s}(R')$ as in Corollary 3.2.9. Then the differential $d_{R,R'} : E_1^{r,s}(R) \to E_1^{r+1,s}(R')$ is rational with respect to the $E(\mathcal{W})]$-structures on the two sides, for all s.*

Proof. If R' and R are subordinate to two distinct maximal parabolics, this follows from Theorem 4.8.1 of [HZ1] and Proposition 3.4.1 above. If R' and R are subordinate to the same maximal parabolic, the theorem is a simple consequence of Corollary 3.2.9.

(3.5) *Towards the determination of E_2.* It is a good idea at this point to recall how the nerve spectral sequence (3.2.1) arises as the spectral sequence of a filtered complex. Writing $C^\bullet(\cdot)$ for any functorial cochain complex representing $R\Gamma(\cdot)$, one has that $C^\bullet(i^*[\mathcal{W}]^{can})$ is quasi-isomorphic to the double complex $K^\bullet$, with terms

(3.5.1) $$K^{r,s} = \bigoplus_{r(R)=r+1} C^s(i_R^*[\mathcal{W}]^{can}),$$

in which the differential d is the sum of the differential of $C^\bullet$ (which increases s by one), and restriction (which increases r by one). We denote the latter by d_{par}. The filtration $\mathcal{R}$ according to the value of r is a decreasing filtration of $K^\bullet$, for which

(3.5.2) $$\mathrm{Gr}^t_{\mathcal{R}} K^\bullet = \bigoplus_{r(R)=t+1} C^\bullet(i_R^*[\mathcal{W}]^{can})[-t],$$

in which d_{par} vanishes. This is the source of the nerve spectral sequence, with E_1-term $E_1(K^\bullet, \mathcal{R})$, and d_1 induced by d_{par}.

For comparison later, it is convenient to replace parabolic rank by actual rank. Let ρ denote the rank of G (over $\mathbb{Q}$). A $\mathbb{Q}$-parabolic subgroup R of G has $\mathbb{Q}$-rank ρ_R, with $\rho_R + r(R) = \rho$. Thus, $r(R) = t+1$ if and only if $\rho_R = \rho - 1 - t$. The increasing filtration by $\mathbb{Q}$-rank determines the same spectral sequence (up to a translation) that $\mathcal{R}$ does; *we allow $\mathcal{R}$ to refer to that from now on.*

Our calculations in this Section suggest the following notion. Let R be a parabolic subgroup of G that is subordinate to the maximal parabolic P. Then R has a Levi subgroup G G

Definition (3.5.3). *The holomorphic rank of R is the $\mathbb{Q}$-rank of $G_{h,P}$, and is denoted $\rho_h(R)$.*

Thus, all parabolic subgroups subordinate to a given P have the same holomorphic rank. We use this notion now to combine (3.2.9) and (3.4.1) into a single statement:

Theorem (3.5.4). *The differential d_{par} of $K^\bullet$ decomposes as $d_{par} = d^\ell + d^h$, where d^ℓ preserves holomorphic rank and d^h lowers holomorphic rank. This induces the same for the differential d_1 of the E_1-term $E_1(K^\bullet, \mathcal{R})$ of the nerve (rank) spectral sequence: $d_1 = d_1^\ell + d_1^h$. The determination of d_1^ℓ is given in (3.2.9); that of d_1^h is given in (3.4.1).*

It seems right at this point to shift gears and look instead at the spectral sequence for the filtration $\mathcal{R}^h$ of $K^\bullet$ by *holomorphic* rank: $\mathcal{R}_j^h K^\bullet$ is the direct sum of those terms in (3.5.1) with R of holomorphic rank $\leq j$. This is clearly closed under d. We have

$$\text{(3.5.5)} \qquad \mathrm{Gr}_t^{\mathcal{R}^h} K^\bullet = \bigoplus_{\rho_h(R) = t} C^\bullet(i_R^*[\mathcal{W}]^{can})[1 - r(R)],$$

on which d^h vanishes (compare (3.5.2)).

Though we were preoccupied with cohomology calculations in (3.1), we point out that the discussion in (3.1.1) and (3.1.4) actually give assertions in the derived category, viz.,

Proposition (3.5.6). *Let P be a maximal parabolic subgroup of G, and R a parabolic subordinate to P. Let $\overline{\pi}_P : Z_\Sigma(R) \to M_{P,\Sigma(P)}$ be the boundary projection. Then there is a canonical isomorphism in the derived category of $M_{P,\Sigma(P)}$:*

$$\begin{aligned} R(\overline{\pi}_P)_*(i_R^*[\mathcal{W}]^{can}) &\approx \bigoplus_{w \in W^{P,p}} [\mathcal{V}_{\lambda(h,w)}]^{can} \otimes C^\bullet(\Gamma_{\ell,R}, V_{\lambda(\ell,w)})[-\ell(w)] \\ &\approx \bigoplus_{w \in W^{R,p}} [\mathcal{V}_{\lambda(h,w)}]^{can} \otimes C^\bullet(\Gamma_{\ell,R}^{red}, V_{\lambda(\ell,w)})[-\ell(w)] \end{aligned}.$$

Note that the first factor of the tensor product above is independent of R (for R subordinate to P). If we combine (3.5.6) with (3.5.5), we obtain something "familiar":

Corollary (3.5.7). *There is an isomorphism in the derived category of $M_{P,\Sigma(P)}$:*

$$Gr_t^{\mathcal{R}^h} R(\overline{\pi}_P)_* K^\bullet \approx \bigoplus [\mathcal{V}_{\lambda(h,w)}]^{can} \otimes K_c^\bullet(X(\Gamma_{\ell,P}), V_{\lambda(\ell,w)})[-\ell(w)],$$

where

$$K_c^\bullet(X(\Gamma_{\ell,P}), V_{\lambda(\ell,w)}) = \bigoplus_{\Pi(R)=P} C^\bullet(\Gamma_{\ell,R}, V_{\lambda(\ell,w)})$$

is a cochain complex for the cohomology of the pair $(\overline{X}(\Gamma_{\ell,P}), \partial\overline{X}(\Gamma_{\ell,P}))$.

As suggested by the notation, one can identify $H^\bullet(\overline{X}(\Gamma_{\ell,P}), \partial\overline{X}(\Gamma_{\ell,P}); V_{\lambda(\ell,w)})) \simeq H_c^\bullet(X(\Gamma_{\ell,P}), V_{\lambda(\ell,w)}))$. Numbering the standard maximal parabolics by omitted simple root, so that $P_i \succ P_{i+1}$, we can now assert:

Corollary (3.5.8). *In the spectral sequence for* $\mathcal{R}^h$, *the* E_1*-term is*

$$\begin{aligned} E_1^{p,q}(K^\bullet, \mathcal{R}^h) &= H^{p+q}(Gr_{-p}^{\mathcal{R}^h} K^\bullet) \\ &= \bigoplus_{a,w;P\sim P_{\rho+p}} H^a(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H_c^{p+q-a-l(w)}(X(\Gamma_{\ell,P}), \widetilde{V}_{\lambda(\ell,w)}). \end{aligned}$$

NB—The non-trivial terms in the above are for $p \le 0$.

Next, we note that the differential d_1 in (3.5.8) is induced by d^h. We can deduce the effect of d_1 by observing that $\mathcal{R}$ is the composite, or *convolution*, of $\mathcal{R}^h$ and an evident analogously defined filtration $\mathcal{R}^\ell$, written

$$\mathcal{R} = \mathcal{R}^h * \mathcal{R}^\ell. \tag{3.5.9}$$

This implies that

$$\mathrm{Gr}_j^{\mathcal{R}} K^\bullet \simeq \bigoplus_{t+u=j} \mathrm{Gr}_t^{\mathcal{R}^h} \mathrm{Gr}_u^{\mathcal{R}^\ell} K^\bullet \tag{3.5.10}$$

(see [Z3, §1(6)]). We can see, from the calculations that produced (3.5.4), that d_1 does "nothing" on the second factor of the tensor product in (3.5.7). In actuality, that means we are looking at the inclusions of $\overline{X}(\Gamma_{\ell,P})$ in $\partial\overline{X}(\Gamma_{\ell,P'})$, with $\rho_h(P') = \rho_h(P) - 1$. Also, note that there is a cohomology equivalence by excision

$$\coprod_P (\overline{X}(\Gamma_{\ell,P}), \partial\overline{X}(\Gamma_{\ell,P})) \approx (\partial\overline{X}(\Gamma_{\ell,P'}), \partial\overline{X}(\Gamma_{\ell,P'}) - \coprod_P X(\Gamma_{\ell,P})).$$

Thus:

Proposition (3.5.11). *In the spectral sequence for* $\mathcal{R}^h$, *the differential*

$$d_1 : E_1^{p,q}(K^\bullet, \mathcal{R}^h) \to E_1^{p+1,q}(K^\bullet, \mathcal{R}^h)$$

is given by

$$\bigoplus_{a,w;P\sim P_{\rho+p}} \{H^a(M_{P,\Sigma(P)}, [\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H_c^{p+q-a-l(w)}(X(\Gamma_{\ell,P}), \widetilde{V}_{\lambda(\ell,w)})\} \xrightarrow{d_1^h \otimes \delta}$$

$$\bigoplus \{H^a(M_{P',\Sigma(P')}, i_{P(P')}^*[\mathcal{V}_{\lambda(h,w)}]^{can}) \otimes H_c^{p+q+1-a-l(w)}(X(\Gamma_{\ell,P'}), \widetilde{V}_{\lambda(\ell,w)})\};$$

in the above, d_1^h is induced by d^h and acts between consecutive boundary components ($P \succ P'$), and δ is the connecting homomorphism in the exact sequence of the triple

$$(\overline{X}(\Gamma_{\ell,P'}), \partial\overline{X}(\Gamma_{\ell,P'}), \partial\overline{X}(\Gamma_{\ell,P'}) - \bigsqcup_P X(\Gamma_{\ell,P})),$$

where P runs over a finite set of conjugates of $P_{\rho+p}$ (see [HZ2, App. to (3.5)]).

The E_2-term of the holomorphic rank spectral sequence, $E_2(K^\bullet, \mathcal{R}^h)$, is, of course, isomorphic to the cohomology of $(E_1(K^\bullet, \mathcal{R}^h), d_1)$. This can be determined as in the appendix below.

Appendix: *Zipper products.*

Let $C^\bullet$ be a cochain complex with differential d, and $V^\bullet$ a chain of vector spaces:

$$V^0 \xrightarrow{T_0} V^1 \xrightarrow{T_1} V^2 \xrightarrow{T_2} \cdots.$$

For the next definition, one really need assume only that $C^\bullet$ is a chain of vector spaces.

Definition (3.A.1). *The zipper product $Zip(C^\bullet, V^\bullet)$ of $C^\bullet$ and $V^\bullet$ is the chain of vector spaces*

$$V^0 \otimes C^0 \xrightarrow{T_0 \otimes d} V^1 \otimes C^1 \xrightarrow{T_1 \otimes d} V^2 \otimes C^2 \xrightarrow{T_2 \otimes d} \cdots.$$

The following is easy to verify:

Proposition (3.A.2). *(i) If $C^\bullet$ is a complex, then $Zip(C^\bullet, V^\bullet)$ is actually a complex.*

(ii) $$H^i(Zip(C^\bullet, V^\bullet)) \simeq \{(V^i \otimes \ker d_i) + (\ker T_i) \otimes C^i\}/(im\ T_{i-1}) \otimes (im\ d_{i-1}).$$

Our goal is to express the above in terms of the cohomology of $C^\bullet$. We point out that (3.5.8) is the direct sum of zipper products. The calculation is based on the unremarkable:

Lemma (3.A.3). *Let A and B be subspaces of a vectorspace, and let E be a subspace of both. Then there is a short exact sequence*

$$0 \longrightarrow A/E \longrightarrow (A+B)/E \longrightarrow (A+B)/A \simeq B/(A \cap B) \longrightarrow 0.$$

In our case (3.A.2, (ii)), we compute

$$A/E = (V^i \otimes \ker d_i)/(\text{im } T_{i-1}) \otimes (\text{im } d_{i-1}) \simeq (\text{im } T_{i-1}) \otimes H^i(C^\bullet) \oplus (\widetilde{V}^i \otimes \ker d_i),$$

where $\widetilde{V}^i$ is a complement to im T_{i-1} in V^i. On the other hand,

$$B/(A \cap B) = \{(\ker T_i) \otimes C^i\}/\{[(\ker T_i) \otimes C^i] \cap [V^i \otimes (\ker d_i)]\} = (\ker T_i) \otimes (C^i/\ker d_i).$$

Putting these together, we obtain:

Proposition (3.A.4).

$$H^i(Zip(C^\bullet, V^\bullet)) \simeq \{(im\ T_{i-1}) \otimes H^i(C^\bullet)\} \oplus \{(coker\ T_{i-1}) \otimes (\ker d_i)\} \oplus \{(\ker T_i) \otimes (C^i/\ker d_i)\}.$$

(3.6) *Differentials and automorphic forms.* It remains to express the E_1 terms and the differentials d_1 in terms of automorphic forms. The terms $E_1^{1,s}$ have already simply state the result. However, Franke's theorem [Fr1] justifies replacing slowly increasing smooth functions by automorphic forms in the Lie algebra cohomology groups, and we will do so. We warn the reader that [Fr1] only treats (topological) cohomology with local coefficients. The extension to coherent cohomology is contained in an unpublished early version of [Fr1].

Suppose H is a reductive algebraic group over $\mathbb{Q}$, $P \subset H$ a parabolic subgroup, $N \subset P$ its unipotent radical, $L = P/N$ the Levi quotient. The *constant term map* $c_{H,P} : \mathcal{A}(H) \rightarrow \mathcal{A}(L)$ is defined as usual:

$$c_{H,P}(f)(g) = \int_{N(\mathbb{Q})\backslash N(\mathbf{A})} f(ng)dn, \tag{3.6.1}$$

where dn is the right-invariant measure on $N(\mathbb{Q})\backslash N(\mathbf{A})$ with total mass 1. If now $R' \subset R$ is a pair of parabolic subgroups of G, with Levi quotients $L_{R'}$ and L_R, respectively, we write $c_{R,R'}$ for $c_{L_R,L_{R'}}$.

As in [HZ1, §4], we write $I_R^G = Ind_{R(\mathbf{A}_f)}^{G(\mathbf{A}_f)}$, when R is a parabolic subgroup of G. Given R subordinate to the maximal parabolic P, we write $\mathfrak{p}_{h,R}^- = \mathfrak{p}_{h,P}^-$. Let

$$\mathfrak{g}_{\ell,R} = \mathfrak{k}_{\ell,R} \oplus \tilde{\mathfrak{p}}_{\ell,R} \tag{3.6.2}$$

be the (complex) Cartan decomposition. As in [HZ1, §4.2], we write

$$\tilde{\mathfrak{p}}_{\ell,R} = \mathfrak{a}_R \oplus \mathfrak{p}_{\ell,R},$$

where $\mathfrak{a}_R$ is the Lie algebra of a split component A_R of G_ℓ containing the center Z_G of G. Finally, write $\mathfrak{s}_R$ instead of $\mathfrak{s}_{p,R}$, and let $\mathfrak{u}_P(R)$ denote the Lie algebra of the unipotent radical of $R_{\ell,P}$. Then we can decompose

$$\mathfrak{s}_R \xrightarrow{\sim} \mathfrak{s}_P \oplus \mathfrak{u}_P(R) \tag{3.6.3}$$

Recall from [HZ1, (3.10.3.2),(4.2.6)] that we can use the Cayley transform to identify

$$\mathfrak{p}^- \xrightarrow{\sim} \mathfrak{p}_{h,P}^- \oplus \mathfrak{u}_{P,\mathbb{C}} \oplus \mathfrak{v}_P^-.$$

It is better for our purposes to use the further identification

$$\mathfrak{p}^- \xrightarrow{\sim} \mathfrak{p}_{h,P}^- \oplus \tilde{\mathfrak{p}}_{\ell,P} \oplus \mathfrak{s}_P \tag{3.6.4}$$

derived from [HZ1, (1.8.6)]. Now projection of $Lie(R_{\ell,P}) = \mathfrak{g}_{\ell,R} \oplus \mathfrak{u}_P(R)$ onto $\tilde{\mathfrak{p}}_{\ell,P}$ via the Cartan decomposition for $\mathfrak{g}_{\ell,P}$ defines an isomorphism of $K_{\ell,R}$-modules

$$\tilde{\mathfrak{p}}_{\ell,R} \oplus \mathfrak{u}_P(R) \xrightarrow{\sim} \tilde{\mathfrak{p}}_{\ell,P}. \tag{3.6.5}$$

Combining (3.6.3–3.6.5), we obtain an isomorphism of $K_{h,R} \cdot K_{\ell,R} \cdot A_R(\mathbb{R})$-modules

$$\mathfrak{p}^- \xrightarrow{\sim} \mathfrak{p}_{h,P}^- \oplus \tilde{\mathfrak{p}}_{\ell,R} \oplus \mathfrak{s}_R. \tag{3.6.6}$$

Now

Theorem (3.6.7). *(i) For each r, s, there is an isomorphism*

$$E_1^{r,s} \xrightarrow{\sim} \bigoplus_{r(R)=r+1} I_R^G \left\{ \bigoplus_{w \in W^{R,p}} [\wedge^\bullet(\mathfrak{p}_{h,R}^- \oplus \tilde{\mathfrak{p}}_{\ell,R})^* \otimes \mathcal{A}(L_R) \otimes H^\bullet(\mathfrak{s}_R, V_\lambda)]^{K_h \cdot K_{\ell,R} \cdot A_R(\mathbb{R})}] \right\}^s$$

(on the right-hand side: terms of total degree s).

(ii) For each r, s, there is a commutative diagram

$$\begin{array}{ccc}
E_1^{r,s} & \xrightarrow{\sim} & \displaystyle\bigoplus_{r(R)=r+1} I_R^G \left\{ \bigoplus_{w \in W^{R,p}} [\wedge^\bullet(\mathfrak{p}_{h,R}^- \oplus \tilde{\mathfrak{p}}_{\ell,R})^* \otimes \mathcal{A}(L_R) \otimes H^\bullet(\mathfrak{s}_R, V_\lambda)]^{K_h \cdot K_{\ell,R} \cdot A_R(\mathbb{R})}] \right\}^s \\
{\scriptstyle d}\downarrow & & \downarrow \displaystyle\bigoplus_{R' \subsetneq R, r(R')=r(R)+1=r+2} c_{R,R'} \\
E_1^{r+1,s} & \xrightarrow{\sim} & \displaystyle\bigoplus_{r(R')=r+2} I_{R'}^G \left\{ \bigoplus_{w \in W^{R',p}} [\wedge^\bullet(\mathfrak{p}_{h,R'}^- \oplus \tilde{\mathfrak{p}}_{\ell,R'})^* \otimes \mathcal{A}(L_{R'}) \otimes H^\bullet(\mathfrak{s}_{R'}, V_\lambda)]^{K_h \cdot K_{\ell,R'} \cdot A_{R'}(\mathbb{R})}] \right\}^s .
\end{array}$$

Here the constant term maps are applied to functions, and the map on coefficients:

$$[\wedge^\bullet(\mathfrak{p}_{h,R}^- \oplus \tilde{\mathfrak{p}}_{\ell,R})^* \otimes H^\bullet(\mathfrak{s}_R, V_\lambda)] \to [\wedge^\bullet(\mathfrak{p}_{h,R'}^- \oplus \tilde{\mathfrak{p}}_{\ell,R'})^* \otimes H^\bullet(\mathfrak{s}_{R'}, V_\lambda)]$$

is obtained by applying (3.6.6) to identify

$$\wedge^\bullet(\mathfrak{p}_{h,R}^- \oplus \tilde{\mathfrak{p}}_{\ell,R} \oplus \mathfrak{s}_R)^* \xrightarrow{\sim} \wedge^\bullet (\mathfrak{p}^-)^*,$$

and likewise for R'.

Proof. That this holds for the $d_{R,R'}$ with R and R' subordinate to the same maximal parabolic follows from Corollary 3.2.9 and [Sch,1.10]. For R and R' subordinate to distinct maximal parabolics, this follows from Proposition 3.4.1 and diagram (4.2.4) of [HZ1].

We have been working implicitly with a maximal compact subgroup K_p, which we assume to be the stabilizer of a CM point $p \in X$, corresponding to an inclusion $(T, p) \subset (G, X)$ of Shimura data. Let $E(T, p)$ be the corresponding reflex field; it necessarily contains $E(G, X)$. As in 4.9 of [HZ1], we may define canonical trivializations of the terms $E_1^{r,s}$ by choosing $E(T, p)$-rational coordinates of V_λ and the vector spaces appearing in (3.6.3-3.6.6).

Corollary (3.6.8). *With respect to a canonical trivialization, the constant term maps $c_{R,R'}$ above are rational over the reflex field $E(T, p)$.*

Proof. This follows immediately from Theorem 3.4.12.

4. Hodge theoretic applications

In this Section, we will tie together the results and methods of [HZ1] (as generalized here in §3) with the Hodge theoretic considerations from [HZ2].

(4.1) *The Hodge filtration for boundary cohomology.* Let (G, X) be a pair defining a (pure) Shimura variety, E a finite-dimensional vector space over $\mathbb{Q}$, and $\rho : G \to GL(E)$ a homomorphism of algebraic groups. This determines a compatible family of local systems $\widetilde{\mathbf{E}}$ on the associated complex varieties $Sh(G, X)(\mathbb{C})$. Moreover, the space X, by its nature (see (1.1)), defines a family of variations of Hodge structure on $Sh(G, X)(\mathbb{C})$, with underlying local system $\widetilde{\mathbf{E}}$.

Let $M = M_\Gamma$ denote a connected component of $Sh(G, X)(\mathbb{C})$. Let $j : M \to M_\Sigma$ be the inclusion of M in a smooth toroidal compactification, and let Z_Σ denote the corresponding boundary divisor ∂M_Σ, a divisor with normal crossings in M_Σ. This implies that the holomorphic logarithmic complex,

$$\Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes \widetilde{\mathcal{E}}^{can} \tag{4.1.1}$$

is quasi-isomorphic to $Rj_*\widetilde{\mathbf{E}}$. In the above, $\widetilde{\mathcal{E}}^{can}$ denotes the canonical extension of the flat vector bundle $\mathcal{E}$ determined by E, as in [HZ1, 4.4]. The complex (4.1.1) is equipped with the decreasing *Hodge filtration* F, and the increasing *weight filtration* W induced from one on $Rj_*\widetilde{\mathbf{E}}$. These complete the data of a cohomological mixed Hodge complex that induces, upon taking hypercohomology, the mixed Hodge structure on $H^\bullet(M, \widetilde{\mathbf{E}})$.[6]

Though the description of W on (4.1.1) is rather complicated, it is easy to specify F, and we will study the latter. Let $\{F^p\Omega^\bullet_M(\log Z_\Sigma)\}$ be the usual Hodge filtration, given simply by truncation from below, and let $\{\mathcal{F}^s\}$ be the filtration of $\mathcal{E}$ that gives the variation of Hodge structure. Then the filtration F of (4.1.1) is given by the tensor product of these two:

$$F^p(\Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes \widetilde{\mathcal{E}}^{can}) = \sum_{r+s=p} F^r\Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes (\mathcal{F}^s)^{can}. \tag{4.1.2}$$

For this, there is a general definition of $(\mathcal{F}^s)^{can}$ for a variation of Hodge structure, given by $j_*\mathcal{F}^s \bigcap \widetilde{\mathcal{E}}^{can}$. In the present case, each $\mathcal{F}^s$ is an automorphic vector bundle and $(\mathcal{F}^s)^{can}$ coincides with the canonical extension in the sense of automorphic vector bundles (see [HZ1, 3.2]). Because $(\Omega^p_M)^{can}$ is $\Omega^p_{M_\Sigma}(\log Z_\Sigma)$, we can rewrite (4.1.2) as:

$$F^p(\Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes \widetilde{\mathcal{E}}^{can}) = \sum_{r+s=p} (F^r\Omega^\bullet_M \otimes \mathcal{F}^s)^{can}. \tag{4.1.3}$$

[6] It may be useful to keep in mind that this remains true for admissible variations of *mixed* Hodge structure [S, 3].

By Hodge theory [De2, (8.1.9)], the spectral sequence for F degenerates at E_1.

There are parallel assertions for deleted neighborhood cohomology. For each admissible parabolic subgroup R of G, let $i_R : Z_\Sigma(R) \to M_\Sigma$ denote the inclusion of the R-stratum of ∂M_Σ. The complex

$$\mathcal{C}^\bullet_{\mathrm{dn}}(Z_\Sigma(R), \widetilde{\mathbf{E}}) = i_R^* R j_* \widetilde{\mathbf{E}} \tag{4.1.4}$$

is called the *deleted neighborhood complex* for $Z_\Sigma(R)$ in M_Σ, for its hypercohomology is naturally isomorphic to

$$H^\bullet_{\mathrm{dn}}(Z_\Sigma(R), \widetilde{\mathbf{E}}) = H^\bullet(N(R) - Z_\Sigma(R), \widetilde{\mathbf{E}}), \tag{4.1.5}$$

where $N(R)$ is a regular neighborhood of $Z_\Sigma(R)$ in M_Σ. For R' parabolic in R, let

$$i_{R',R} : Z_\Sigma(R') \to Z_\Sigma(R)$$

denote the inclusion; note that the evident relation $i_{R'} = i_{R',R} \circ i_R$ yields:

$$\mathcal{C}^\bullet_{\mathrm{dn}}(Z_\Sigma(R'), \widetilde{\mathbf{E}}) \approx i_{R',R}^* \mathcal{C}^\bullet_{\mathrm{dn}}(Z_\Sigma(R), \widetilde{\mathbf{E}}). \tag{4.1.6}$$

It is not hard to see that the following complex is quasi-isomorphic to $\mathcal{C}^\bullet_{\mathrm{dn}}(Z_\Sigma(R), \widetilde{\mathbf{E}})$:

$$i_R^*(\Omega^\bullet_{M_\Sigma}(\log Z) \otimes \widetilde{\mathcal{E}}^{can}), \tag{4.1.7}$$

the complex of locally-free sheaves on $Z_\Sigma(R)$ given by restricting (4.1.1). This complex inherits a filtration F from (4.1.2). Again, for general reasons, the filtration F induces the Hodge filtration of the natural mixed Hodge structure on $H^\bullet_{\mathrm{dn}}(Z_\Sigma(R), \widetilde{\mathbf{E}})$ [Sa3]; and the spectral sequence for F degenerates at E_1.

We now introduce the notation

$$\mathcal{DR}(M, \widetilde{\mathbf{E}}) = \Omega^\bullet_M \otimes \widetilde{\mathbf{E}} \simeq \mathcal{C}^\bullet(\mathfrak{p}^+, E), \tag{4.1.8}$$

so that, for instance, (4.1.1) becomes $\mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}$ (cf. (4.1.3)). This is a complex of automorphic vector bundles that are fully decomposed only when the representation ρ is trivial. We need to determine the direct images of deleted neighborhood complexes, first in the unfiltered category (the filtered version will be deduced later, in (4.3)):

Proposition (4.1.9). *Let P be a maximal $\mathbb{Q}$-parabolic subgroup of G, and R a parabolic subordinate to P. Let $\overline{\pi}_P : Z_\Sigma(R) \to (M_P)_\Xi$ be the boundary projection. Then there is a canonical isomorphism in the derived category of $(M_P)_\Xi$:*

$$R(\overline{\pi}_P)_*(i_R^*)\{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can} \simeq \{\mathcal{DR}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma, H^\bullet(\mathfrak{n}, E)))\}^{can}$$

Proof. Using the composite $\overline{\pi}_P = \overline{\pi}_1 \circ \overline{\pi}_2$, as per (1.4.2), we calculate that:

$$\begin{aligned} R(\overline{\pi}_P)_*(i_R^*\{\mathcal{DR}(M,\widetilde{\mathbf{E}})\}^{can}) &\approx R\overline{\pi}_{1,*}(\overline{\pi}_{2,*}i_R^*\mathcal{C}^\bullet(\mathfrak{p}^+,E)^{can}) && \text{(by (4.1.8))} \\ &\approx \mathcal{C}^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{s}'_P,(C^\bullet(\mathfrak{p}^+,E))))^{can} && \text{(by (2.7.5) and (3.5.6))} \\ &\approx \mathcal{C}^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{s}'_P \oplus \mathfrak{p}^+,E))^{can} && \text{(by Remark 3.1.9),} \end{aligned}$$

where $\mathfrak{s}'_P$ denotes the complex conjugate of $\mathfrak{s}_P$. The Cayley transform switches us from G to $G_{h,P}$ as follows:

$$\begin{aligned} R(\overline{\pi}_P)_*(i_R^*)\{\mathcal{DR}(M,\widetilde{\mathbf{E}})\}^{can}) &\approx \mathcal{C}^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{s}'_P \oplus \mathfrak{p}^+,E))^{can} && \text{(repeated from above)} \\ &\approx \mathcal{C}^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{p}^+_{h,P} \oplus \mathfrak{w}_P,E))^{can} && \text{(induced by } \overline{c}_P) \\ &\approx \mathcal{C}^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{p}^+_{h,P}, C^\bullet(\mathfrak{w}_P,E)))^{can} && \text{((3.1.9) again)} \\ &\approx \mathcal{C}^\bullet(\mathfrak{p}^+_{h,P}, C^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{w}_P,E)))^{can} && \text{(as } G_{h,P} \text{ and } G_{\ell,R} \text{ commute)} \\ &\approx \{\mathcal{DR}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R}, C^\bullet(\mathfrak{w}_P,E)))\}^{can} && \text{(equivalent formula)} \\ &\approx \{\mathcal{DR}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P,E)))\}^{can} && \text{(follows by Kostant; cf. (2.7.5)).} \end{aligned}$$

(4.1.10) *Remarks.* i) Recall (cf. [HZ1, 1.8.3]) that conjugation by c_P^{-1} takes a subgroup K_t of K_p onto G_ℓ, and transforms the adjoint action of K_t on $\mathfrak{s}_P \oplus \mathfrak{p}^-$ to the action of G_ℓ on $\mathfrak{p}^-_{h,P} \oplus \mathfrak{w}_P$. Thus the action of $\Gamma_{\ell,R}$ on $C^\bullet(\mathfrak{s}_P \oplus \mathfrak{p}^+, E)$ in the first line above comes from the identification of $\Gamma_{\ell,R}$, *via* $\overline{c}_P = c_P^{-1}$, with a subgroup of K_p, or equivalently, the identification

$$C^\bullet(\mathfrak{s}'_P \oplus \mathfrak{p}^+, E) \simeq C^\bullet(\mathfrak{p}^+_{h,P} \oplus \mathfrak{w}_P, E).$$

The above is the basis of the calculations in [HZ1] that underlie this isomorphism (see [HZ1, Lemma 3.5.11]). The action on the second line of the above series of quasi-isomorphisms, by contrast, is via the usual (adjoint) action of $G_{\ell,R}$ on $\mathfrak{w}_P$; $G_{\ell,R}$ and $\overline{c}_P$ both act as the identity on $\mathfrak{p}^+_{h,P}$.

Moreover, for $P' \prec P$, one has $c_{P'} = c_P \circ c_{P',P}$, where $c_{P',P}$ denotes the Cayley transform for $\widehat{\mathcal{X}}_{P'}$ as a boundary component of $\widehat{\mathcal{X}}_P$.

ii) Note that we have elected not to make use of the Kostant decomposition of $H^\bullet(\mathfrak{s}_P, V)$ (thus also its complex conjugate), utilized in [HZI, 3.6] and Section 3 here. The conclusion of the above result can be viewed as the extension of (3.5.6) to special complexes that are not fully decomposed. Also, the final answer is, as one expects after [HZ1], independent of the choice of toroidal compactification. Introducing the Kostant decomposition,

(4.1.10.1) $$H^\bullet(\mathfrak{w}_P, E) = \bigoplus E_{\mu(h,w)} \otimes E_{\mu(\ell,w)}[-l(w)],$$

we can rewrite the formula of (4.1.9) as:

$$
\begin{aligned}
R(\overline{\pi}_P)_* i_R^* R j_* \widetilde{\mathbf{E}} &\approx \bigoplus_{w \in W^P} R(j_P)_* \widetilde{\mathbf{E}}_{\mu(h,w)} \otimes H^\bullet(\Gamma_{\ell,R}, E_{\mu(\ell,w)})[-l(w)], \\
&\approx \{\mathcal{DR}(M_P, \widetilde{\mathbf{L}}_R)\}^{can}
\end{aligned}
\tag{4.1.10.2}
$$

where $j_P : M_P \hookrightarrow (M_P)_\Xi$ is the inclusion, and $L_R = H^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E)))$.

iii) Throughout, we can express the $\Gamma_{\ell,R}$-cohomology as $\Gamma_{\ell,R}^{red}$-cohomology, using the maneuver that produced (3.1.8). We would then write

$$L_R = H^\bullet(\Gamma_{\ell,R}^{red}, H^\bullet(\mathfrak{w}_R, E)).$$

Next, let $R' = R \cap P'$, with P' maximal. We consider the restriction mappings from Z_R to $Z_{R'}$ in conjunction with Proposition (4.1.9). If $P' \succ P$, then R' is also subordinate to P. In that case,

$$R(\overline{\pi}_P)_*((i_R^*)\{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can}) \longrightarrow R(\overline{\pi}_P)_*(i_{R'}^*\{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can}) \tag{4.1.11.1}$$

is given tautologically:

$$\{\mathcal{DR}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E)))\}^{can} \longrightarrow \{\mathcal{DR}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R'}, H^\bullet(\mathfrak{w}_P, E)))\}^{can}, \tag{4.1.11.2}$$

which is induced by the morphism of local systems coming by restriction on the Γ_ℓ's. After grading for F, the above is essentially equivalent to what we discussed in (3.2). On the other hand, if $P' \prec P$, then R' is subordinate to P'. In that case, one reverts to the determination in (3.4).

The treatment of d^h has the same problem and the same remedy as before. Consider the diagram:

$$
\begin{array}{ccc}
Z_\Sigma(R) & \xleftarrow{i_{R',R}} & Z_\Sigma(R') \\
\Big\downarrow{\scriptstyle \overline{\pi}_P} & & \Big\downarrow{\scriptstyle \tau} \\
M_{P,\Sigma(P)} & \xleftarrow{i_{P(P')}} & Z_{\Sigma(P)}(P(P')) \\
& & \Big\downarrow{\scriptstyle \overline{\pi}_{P,P'}} \\
& & M_{P',\Sigma(P')}
\end{array}
\tag{4.1.12}
$$

Let $L_R^\bullet = C^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E))$. Then the calculation:

$$
\begin{aligned}
R(\overline{\pi}_{P'})_*(i_{R'}^*\{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can}) &= R(\overline{\pi}_{P,P'})_* R\tau_*(i_{R',R}^* i_R^*)\{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can} \\
&\approx R(\overline{\pi}_{P,P'})_* i_{P(P')}^* R(\overline{\pi}_P)_* i_R^* \{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can} \quad \text{(from (2.7.11))} \\
&\approx R(\overline{\pi}_{P,P'})_* i_{P(P')}^* \{\mathcal{DR}(M_P, \widetilde{\mathbf{L}}_R)\}^{can} \quad \text{(by (4.1.9))} \\
&\approx \{\mathcal{DR}(M_{P'}, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,P(P')}, H^\bullet(\mathfrak{u}_{P(P')}, L_R)))\}^{can} \\
&= \{\mathcal{DR}(M_{P'}, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,P(P')}, H^\bullet(\mathfrak{u}_{P(P')}, H^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E)))\}^{can} \\
&\approx \{\mathcal{DR}(M_{P'}, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R'}, H^\bullet(\mathfrak{w}_{P'}, E)))\}^{can} \quad \text{(as in (3.1.6))}
\end{aligned}
\tag{4.1.13}
$$

recovers the direct computation of $R(\overline{\pi}_{P'})_* i^*_{R'}\{\mathcal{DR}(M, \widetilde{\mathbf{E}})\}^{can}$, because of (4.1.10) and the way in which the Cayley transforms entered. We now see that d^h is induced by *restriction* on the M_P's.

When one grades (4.1.2) for F, the differentials become linearized, and each $\mathrm{Gr}^p_F(\Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes \widetilde{\mathcal{E}}^{can})$ becomes a complex of canonically-extended, fully decomposed automorphic vector bundles, placing us in the context of (3.2). For (4.1.7),

$$i^*_R(\mathrm{Gr}^p_F(\Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes \widetilde{\mathcal{E}}^{can})) \tag{4.1.14}$$

becomes the restriction of the same to $Z_\Sigma(R)$.

We briefly recall the topological nerve spectral sequence that is of fundamental interest, which we treated in [HZ2, (3.5.4)]; it is the one associated to the local system $\widetilde{\mathbf{E}}$ and the covering of the Borel-Serre boundary by its closed faces, $\{\overline{e'(P)} : P \text{ maximal}\}$:

$$E_1^{r,s} = \bigoplus_{r(R)=r+1} H^s(\overline{e'(R)}, \widetilde{\mathbf{E}}) \Rightarrow H^{r+s}(\partial \overline{M}, \widetilde{\mathbf{E}}). \tag{4.1.15}$$

By [HZ2, (3.5.5)], the above coincides with the corresponding spectral sequence for deleted neighborhood cohomology on M_Σ, viz.,

$$E_1^{r,s} = \bigoplus_{r(R)=r+1} H^s_{\mathrm{dn}}(Z_\Sigma(R), \widetilde{\mathbf{E}}) \Rightarrow H^{r+s}_{\mathrm{dn}}(Z_\Sigma, \widetilde{\mathbf{E}}). \tag{4.1.16}$$

We can use (4.1.7) and (4.1.8) to rewrite (4.1.16) in the form

(4.1.17)
$$E_1^{r,s} = \bigoplus_{r(R)=r+1} \mathbb{H}^s(Z_\Sigma(R), i^*_R \mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}) \Rightarrow \mathbb{H}^{r+s}(Z_\Sigma, i^*\mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}),$$

where $\mathbb{H}$ indicates hypercohomology. This is a spectral sequence of mixed Hodge structures (see [HZ2, (5.5.2)]; cf. (4.5.2) here). A morphism of mixed Hodge structures is completely determined by its gradation for the Hodge filtration F, so we can grade for F without losing information about kernels and images; this reduces such questions about the spectral sequence (4.1.12) to analogous considerations for spectral sequences of the form (3.2.1). Thus,

$$\begin{aligned} \mathrm{Gr}^\bullet_F E_1^{r,s} = \bigoplus_{r(R)=r+1} & \mathrm{Gr}^\bullet_F \mathbb{H}^s(Z_\Sigma(R), i^*_R \mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}) \\ & \Rightarrow \mathrm{Gr}^\bullet_F \mathbb{H}^{r+s}(Z_\Sigma, i^*\mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}). \end{aligned} \tag{4.1.18}$$

By basic Hodge theory, we can write (4.1.18) as

$$\begin{aligned} \mathrm{Gr}^\bullet_F E_1^{r,s} = \bigoplus_{r(R)=r+1} & \mathbb{H}^s(Z_\Sigma(R), \mathrm{Gr}^\bullet_F\{i^*_R \mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}\}) \\ & \Rightarrow \mathbb{H}^{r+s}(Z_\Sigma, \mathrm{Gr}^\bullet_F\{i^*\mathcal{DR}(M, \widetilde{\mathbf{E}})^{can}\}) \end{aligned}$$

Using (4.1.14) and (4.1.16), we obtain:

$$(4.1.19)\quad \mathrm{Gr}^{\bullet}_F E_1^{r,s} = \bigoplus_{r(R)=r+1} H^s(Z_\Sigma(R), i_R^*\{\mathrm{Gr}^{\bullet}_F\mathcal{H}^{\bullet}(\mathfrak{p}^+, E)\}^{can})$$
$$\Rightarrow H^{r+s}(Z_\Sigma, i^*\{\mathrm{Gr}^{\bullet}_F\mathcal{H}^{\bullet}(\mathfrak{p}^+, E)\}^{can}).$$

(4.2) *The dual Bernstein-Gelfand-Gelfand complex.* Following Faltings [F], we can replace the de Rham complex $\mathcal{DR}(M, \widetilde{\mathbf{E}})$ by a quasi-isomorphic filtered subcomplex, the *dual BGG complex* $\mathcal{BGG}(M, \widetilde{\mathbf{E}})$, which is minimal in the sense that each degree is given by a sum of irreducible automorphic vector bundles. After taking cohomology, the quasi-isomorphism between $\mathcal{BGG}(M, \widetilde{\mathbf{E}})$ and $\mathcal{DR}(M, \widetilde{\mathbf{E}})$ recovers the results of [Z1]. Because its terms are irreducible automorphic vector bundles, the dual BGG complex has particularly nice rationality and integrality properties. Moreover, its cohomology is directly expressible in terms of automorphic forms

We use the formalism introduced in [H5, §2], but with slightly different notation. We assume first that E is an absolutely irreducible representation of G, with highest weight $\Lambda \in \mathfrak{h}^*$ (see (0.3)); the irreducibility hypothesis will be removed at the end of the section. Let W^1 denote the set of Kostant representatives for the parabolic subalgebra $\mathfrak{P}^+$ (see (0.3)). For any $t \in W^1$, we let $\Lambda(t) = t(\Lambda + \rho) - \rho$, where ρ is the half-sum of positive roots. Then $\Lambda(t)$ is the highest weight of an irreducible representation of K_p, hence defines an irreducible automorphic vector bundle which we denote $\mathcal{BGG}^t(M, \widetilde{\mathbf{E}})$. For $0 \leq i \leq dim_{\mathbb{C}}(M)$, let

$$\mathcal{BGG}^i(M, \widetilde{\mathbf{E}}) = \bigoplus_{\ell(t)=i} \mathcal{BGG}^t(M, \widetilde{\mathbf{E}}).$$

The $\mathcal{BGG}^i(M, \widetilde{\mathbf{E}})$ form a complex $\mathcal{BGG}(M, \widetilde{\mathbf{E}})$ in which the differentials are algebraic differential operators, and there is a canonical inclusion [F]

$$(4.2.1)\qquad \mathcal{BGG}(M, \widetilde{\mathbf{E}}) \hookrightarrow \mathcal{DR}(M, \widetilde{\mathbf{E}}).$$

In (4.2.1), one places the Hodge filtration F on the left-hand side in the usual way for homogeneous vector bundles (see [H5,(2.2.3)]), and we have likewise the Hodge filtration from (4.1) on the right-hand side.

Theorem (4.2.2) [F]. *The inclusion (4.2.1) is a filtered quasi-isomorphism that extends to a filtered quasi-isomorphism of complexes of canonical extensions.*

Remark. It is implicit in the second part of the above theorem that the complex $CalBGG(M, \widetilde{\mathbf{E}})^{can}$ is functorial with respect to change of toroidal compactification. This follows easily from Proposition 1.3.7, or alternatively from the functoriality of the canonical extension of the de Rham complex.

Theorem (4.2.2) yields immediately:

Corollary (4.2.3). *There is a spectral sequence*

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_F^p \mathcal{BGG}(M, \widetilde{\mathbf{E}})^{can}) \Rightarrow H^{p+q}(M, \widetilde{\mathbf{E}})$$

that degenerates at E_1. The filtration induced by F on the abutment is the Hodge filtration in its mixed Hodge structure.

Using (4.2.2) on both sides of (4.1.9), we obtain the following reformulation of the latter:

Corollary (4.2.4). *In the situation of (4.1.9), there is a canonical isomorphism in the derived category of $(M_P)_\Xi$:*

$$R(\overline{\pi}_P)_*(i_R^*)\{\mathcal{BGG}(M, \widetilde{\mathbf{E}})\}^{can} \approx \{\mathcal{BGG}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E)))\}^{can}.$$

We present below a more concrete way to see this quasi-isomorphism.

We begin with some general considerations involving Weyl group combinatorics. Let G be a reductive group as in (0.2), with chosen set Φ^+ of positive roots, and let $P, Q \subset G$ denote two standard parabolics; let $R = P \cap Q$. Let $\mathfrak{w}_P$, $\mathfrak{w}_Q$, $\mathfrak{w}_R$ denote the Lie algebras of the unipotent radicals of P, Q, and R, respectively; Similarly, let $L_?$, for $? = P, Q, R$, be a compatible set of Levi factors. Then we can decompose

$$\mathfrak{w}_R = \mathfrak{w}_{PQ} \oplus \mathfrak{w}_P^Q \oplus \mathfrak{w}_Q^P \tag{4.2.5}$$

in such a way that

$$\mathfrak{w}_P = \mathfrak{w}_{PQ} \oplus \mathfrak{w}_P^Q, \quad \mathfrak{w}_Q = \mathfrak{w}_{PQ} \oplus \mathfrak{w}_Q^P$$

(so $\mathfrak{w}_{PQ} = \mathfrak{w}_P \cap \mathfrak{w}_Q$) and $\Phi_R = \Phi_{PQ} \sqcup \Phi_P^Q \sqcup \Phi_Q^P$; here Φ_R, Φ_{PQ}, Φ_P^Q, Φ_Q^P denote the sets of positive roots in $\mathfrak{w}_R$, etc. Let $W^? \subset W$, for $? = P, Q, R$, denote the corresponding sets of Kostant representatives, and let $W_? \subset W$ be the Weyl group of $L_?$. Finally, $L_P \cap R = L_P \cap Q$ is parabolic in L_P and we let $W_P^R = W_P^Q$ denote the Kostant representatives in $W_{L_P} = W_P$ relative to $W_{L_P \cap R}$; define $W_Q^R = W_Q^P$ similarly. As in Lemma 3.1.6, we have canonical decompositions

$$W^R = W_P^R \cdot W^P = W_Q^R \cdot W^Q. \tag{4.2.6}$$

We denote the elements of W^R by ω, and those of W_P^Q, W^P, W_Q^P, W^Q by a, b, s, and t, respectively. For any $\omega \in W^R$ we write

$$\Phi_{PQ}(\omega) = \omega(\Phi^+) \cap -\Phi_{PQ};\ \Phi_P^Q(\omega) = \omega(\Phi^+) \cap -\Phi_P^Q;\ \Phi_Q^P(\omega) = \omega(\Phi^+) \cap -\Phi_Q^P.$$

Then

$$\ell(\omega) = |\Phi_{PQ}(\omega)| + |\Phi_P^Q(\omega)| + |\Phi_Q^P(\omega)|. \tag{4.2.7}$$

If $\omega = ab = st$ by the factorizations of (4.2.6), we then have

$$\ell(\omega) = \ell(a) + \ell(b) = \ell(s) + \ell(t), \tag{4.2.8}$$

$$\ell(a) = |\Phi_P^Q(\omega)|,\ \ell(b) = |\Phi_{PQ}(\omega)| + |\Phi_Q^P(\omega)|;\ \ell(s) = |\Phi_Q^P(\omega)|,\ \ell(t) = |\Phi_{PQ}(\omega)| + |\Phi_P^Q(\omega)|. \tag{4.2.9}$$

Finally, if $\omega, \omega' \in W$, we write $\omega \rightarrow \omega'$ if there exists $\gamma \in \Phi^+$ such that $\omega = r_\gamma \omega'$ and $\ell(\omega) = \ell(\omega') + 1$; here r_γ is the reflection attached to γ.

Lemma 4.2.10. . *Let $\omega, \omega' \in W^R$, with factorizations $\omega = ab = st$, $\omega' = a'b' = s't'$ as above. Suppose $\omega \rightarrow \omega'$. Suppose further that $b = b'$. Then $s = s'$ and $t \rightarrow t'$.*

Proof. It follows from (4.2.8) and (4.2.9) that $\ell(t) = \ell(t') + 1$. Indeed, $\Phi_{PQ}(\omega) = \Phi_{PQ}(b) = \Phi_{PQ}(\omega')$. Suppose we know that $s = s'$. Writing $st = r_\gamma st'$, we see that $t = r_{s^{-1}\gamma} t'$, hence $t \rightarrow t'$. So it remains to prove $s = s'$. For this it suffices to show that $\Phi_P^Q(s) = \Phi_P^Q(s')$, since both s and s' preserve $\mathfrak{w}_Q$. But

$$\Phi_P^Q(s) = \Phi_P^Q(\omega) = \Phi_P^Q(b),\ \Phi_P^Q(s') = \Phi_P^Q(\omega') = \Phi_P^Q(b'),$$

where the first equalities follow from the definition of W^Q and the second from the fact that W_P^Q preserves $\mathfrak{w}_P$. The assertion is now clear.

Let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ denote the Lie algebras of G, P, Q, and R respectively. Letting E and Λ be as above, we consider the generalized Bernstein-Gelfand-Gelfand resolutions of E with respect to the parabolic subalgebras we have introduced. Thus for $\omega \in W^R$ let

$$\mathbb{M}_R(\Lambda(\omega)) = U(\mathfrak{g}) \otimes_{U(\mathfrak{r})} E(\Lambda(\omega)), \tag{4.2.11}$$

with $\Lambda(\omega)$ defined as above and $E(\Lambda(\omega))$ the finite-dimensional L_R module with highest weight $\Lambda(\omega)$, extended as usual to an irreducible $U(\mathfrak{r})$ module. The *generalized BGG complex $BGG^R(E)$ of E, relative to* $\mathfrak{r}$, is the complex

(4.2.12)

$$\ldots \rightarrow \bigoplus_{\omega \in W^R, \ell(w) = i+1} \mathbb{M}_R(\Lambda(\omega)) \overset{d_i^R}{\rightarrow} \bigoplus_{\omega \in W^R, \ell(\omega) = i} \mathbb{M}_R(\Lambda(w)) \rightarrow \ldots \rightarrow \mathbb{M}_R(\Lambda) \rightarrow 0.$$

Here d_i^R is the differential constructed in [Le, §4]. It can be written as a sum

$$d_i^R = \sum_{\omega, \omega' \in W^R, \omega \rightarrow \omega', \ell(\omega') = i} d^R_{\omega, \omega'}, \tag{4.2.13}$$

where $d^R_{\omega,\omega'} : \mathbb{M}_R(\Lambda(\omega)) \rightarrow \mathbb{M}_R(\Lambda(\omega'))$. These differentials can be defined uniquely up to sign ([RC, Lemma 10.5]), and the natural surjection $\mathbb{M}_R(\Lambda) \rightarrow E$ (defined up to scalar multiples) defines a quasi-isomorphism $BGG^R(E) \approx E$ in the category of $U(\mathfrak{g})$-modules. It follows from the results of [RC, §§10-11] that changing the sign of one differential changes all signs simultaneously, and that the different choices of sign yield isomorphic complexes.

Let $\overline{P} = L_P \overline{N}_P$ be the parabolic opposite to P, with Lie algebra $\overline{\mathfrak{p}} = \mathfrak{l} \oplus \overline{\mathfrak{w}}_P$. As $U(\overline{\mathfrak{p}})$-module we have $\mathbb{M}_R(\Lambda(\omega)) = U(\overline{\mathfrak{p}}) \otimes_{U(\mathfrak{l}_P \cap \mathfrak{r})} E(\Lambda(\omega))$. For any irreducible finite-dimensional L_P-module V, we let $BGG^{L_P \cap R}(V)$ denote the generalized BGG resolution of V relative to $L_P \cap R$, and define the maps $d^{L_P \cap R}_{a,a'}$ by analogy with (4.2.13).

Now, $BGG^R(E)$ is $\bar{\mathfrak{n}}_P$-acyclic, hence the complex $H_0(\overline{\mathfrak{w}_P}, BGG^R(E))$ computes $H_\bullet(\mathfrak{w}_P, E)$, via the quasi-isomorphism $BGG^R(E) \approx E$. For any ω we have natural isomorphisms

$$(4.2.14) \qquad H_0(\overline{\mathfrak{w}}_P, \mathbb{M}_R(\Lambda(\omega))) \xrightarrow{\sim} U(\mathfrak{l}_P) \otimes_{U(\mathfrak{l}_P \cap \mathfrak{r})} E(\Lambda(\omega)).$$

We let $\mathbb{M}(\Lambda(\omega); L_P)$ denote the right-hand side of (4.2.14). Then

Proposition 4.2.15. *(i) There is a canonical isomorphism of complexes*

$$H_0(\bar{\mathfrak{n}}_P, BGG^R(E)) \xrightarrow{\sim} \bigoplus_{b \in W^P} BGG^{R(P)}(E(\Lambda(b))[\ell(b)].$$

Here $E(\Lambda(b))$ is the irreducible representation of L_P with highest weight $\Lambda(b)$.

(ii) More precisely, for any pair $a, a' \in W^Q_P = W^{R(P)}$, with $a \rightarrow a'$, and for any $b \in W^P$, there is a pair $t, t' \in W^Q$ and $s \in W^P_Q$ such that the differential $d^{R(P)}_{a,a'}$ in $BGG^{R(P)}(E(\Lambda(b))[\ell(b)]$ is obtained by applying $H_0(\bar{\mathfrak{n}}_P, \bullet)$ to $d^R_{\omega,\omega'}$, with $\omega = ab = st$, $\omega' = a'b = st'$.

(iii) For any pair $\omega, \omega' \in W^R$ not obtained as in (ii), the morphism $H_0(\bar{\mathfrak{n}}_P, d^R_{\omega,\omega'})$ vanishes.

Proof. We first verify (ii) and (iii). Assertion (ii) is a consequence of Lemma 4.2.10. Suppose $\omega \rightarrow \omega'$ in W^R, and write $\omega = ab$, $\omega' = a'b'$. Now it follows from the theory of Verma modules for L_P that the infinitesimal character separates $\mathbb{M}(\Lambda(\omega); L_P)$ from $\mathbb{M}(\Lambda(\omega'); L_P)$ if $b \neq b'$. Thus $H_0(\overline{\mathfrak{n}_P}, d^R_{\omega,\omega'}) = 0$ unless $b = b'$, and (iii) follows from (ii).

The assertion (i) follows easily from (ii) and (iii) by checking degrees. Indeed, by writing $\omega = ab$ as above, we see that the assertion is true in each degree separately, and it remains only to check that the differentials correspond (up to sign). But this follows from the explicit construction of the generalized BGG resolutions in [Le,RC].

We fix a toroidal compactification $Sh(G, X)_\Sigma$ (omitting the left subscript for the level subgroup), with Σ assumed to be sufficiently fine for the following arguments. Fix a maximal parabolic subgroup $P \subset G$. Let $P' \subset P$ denote $G_{h,P} \cdot W_P$, as in [HZ1, 1.2.3]. Thus there is a mixed Shimura datum $(P', \mathcal{X})$ and a partially compactified mixed Shimura variety $Sh(P', \mathcal{X})_{\Sigma'}$ corresponding to the P-stratum of the boundary of $Sh(G, X)_\Sigma$. We refer to (1.4) for the morphisms $\pi_{1,\Sigma'}$, $\pi_{2,\Sigma'}$, and let $\pi_{\Sigma'} = \pi_{2,\Sigma'} \circ \pi_{1,\Sigma'}$ denote the composite morphism from $Sh(P', \mathcal{X})_{\Sigma'}$ to M

where $M_{P,\Sigma(P)}$ is a toroidal compactification of the pure Shimura variety associated to $(G_{h,P}, W\backslash\mathcal{X})$. We let i_P^o denote the inclusion of the *open* P-stratum in the boundary, and let $\pi : Sh(P', \mathcal{X}) \rightarrow M_P$ be the corresponding map of uncompactified (mixed) Shimura varieties. The differentials in the complex $i_P^{o,*}\mathcal{BGG}(M, \widetilde{\mathbf{E}})$ (with M a connected component of $Sh(G, X)$) are differential operators of positive degree, hence not $\mathcal{O}$-linear. But they are morphisms of abelian sheaves, hence give rise to functorial morphisms under $R^i\pi_*$, for each i, and these are also differential operators. Our goal is to calculate these directly at the level of $\mathcal{BGG}(M_P, \widetilde{\mathbf{C}}^\bullet(\Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E)))$, using Proposition 4.2.4.

We apologize that the standard maximal parabolic P here will play the role of $\overline{P}$ in the applications of Proposition 4.2.15 in what follows; this is because we are ultimately interested in homology and cohomology of $\mathfrak{w}_P$, not $\overline{\mathfrak{w}}_P$. Thus, our notation is correspondingly contorted. First, let Q_1 be the complex parabolic with Lie algebra $\mathfrak{k} \oplus \mathfrak{P}^-$, and $\overline{Q} = Ad(c_P^{-1})(Q_1)$. Then, let $\overline{Q} \subset G$ denote the standard complex parabolic conjugate to $Q_1 = \mathfrak{k} \oplus \mathfrak{P}^-$; and Q the parabolic opposite to $\overline{Q}$, viz., $Ad(c_P)(\mathfrak{k} \oplus \mathfrak{P}^-)$. Let $R = \overline{P} \cap Q$, and $R_1 = Ad(c_P)R \subset Q_1$. We use the letters a, b, s, t as above to denote the corresponding Kostant representatives.

Via $Ad(c_P)$ we may identify $W^Q = W^{Q_1}$, $W^R = W^{R_1}$. We use the same letters a, b, s, t as above to denote corresponding Kostant representatives. In particular, we denote the components of the BGG complex by $\mathcal{BGG}^t(\widetilde{\mathbf{E}})$, dropping the "$M$" in (4.2.1). Similarly, for each q and t, we have

$$R^q\pi_*\mathcal{BGG}^t(\widetilde{\mathbf{E}}) \xrightarrow{\sim} \oplus_{s \in W_Q^{\overline{P}}, \ell(s)=q} [\mathcal{BGG}^t(\widetilde{\mathbf{E}})]^s. \tag{4.2.16}$$

Here if $\mathcal{BGG}^t(\widetilde{\mathbf{E}})$ is the automorphic vector bundle associated to the representation $E(t)$ of K_p, $[\mathcal{BGG}^t(\widetilde{\mathbf{E}})]^s$ is the automorphic vector bundle on M_P associated to $\mathbf{C}^\bullet(\Gamma_{\ell,P}, H^s(\mathfrak{s}'_P, E(t)))$, where H^s denotes the Kostant constituent of the cohomology associated to $s \in W_Q^{\overline{P}}$. We are abusing notation (in a harmless way) by writing in the discrete group $\Gamma_{\ell,P}$ in this adelic situation. If now $\omega = st = ab$ as in (4.2.6), then we can identify

$$[\mathcal{BGG}^t(\widetilde{\mathbf{E}})]^s = \mathbf{C}^\bullet(\Gamma_{\ell,P}, \mathcal{BGG}^a(\widetilde{\mathbf{H}}^b(\mathfrak{w}_P, E))). \tag{4.2.17}$$

It then follows from (4.2.6) that

Lemma 4.2.18. *The identifications (4.2.16) and (4.2.17) define an isomorphism between the constituents of the two quasi-isomorphic complexes in (4.2.4).*

It remains to show that these isomorphisms are compatible with the differentials on the two sides of (4.2.4). Recall that, if $\mathcal{S}$ and $\mathcal{T}$ are vector bundles on a smooth complex variety Z, a $\mathbb{C}$-linear map $\Delta : \mathcal{E} \to \mathcal{F}$ is a *differential operator of order* n if it can be factored as

$$\mathcal{E} \xrightarrow{j^n} jet^n(\mathcal{E}) \to \mathcal{F},$$

where $jet^n(\mathcal{E})$ is the bundle of n-jets of sections of $\mathcal{E}$, j^n is the canonical differential operator, and the right-hand arrow is $\mathcal{O}_Z$-linear. (Here the notation for jets is as in [H1].) Classification of differential operators $\Delta : \mathcal{E} \to \mathcal{F}$ thus comes down to classification of $\mathcal{O}_Z$-linear maps $jet^\infty(\mathcal{E}) \to jet^\infty(\mathcal{F})$, where jet^∞ denotes the inverse limit over the jet bundles of finite order.

For our purposes, it is more useful to look at the dual picture. Let $\mathcal{D}_Z$ denote the sheaf of differential operators on Z; we write $\mathcal{D}$ instead of $\mathcal{D}_Z$ when the base Z is understood. For $\mathcal{G}$ a vector bundle on Z, let $\mathcal{D}(\mathcal{G}) = \mathcal{D}_Z(\mathcal{G})$ denote the locally free $\mathcal{D}$-module $\mathcal{D} \otimes_{\mathcal{O}_Z} \mathcal{G}^\vee$. Then to Δ as above we associate a homomorphism of $\mathcal{D}$-modules

(4.2.19) $$\Delta^\vee : \mathcal{D}(\mathcal{F}^\vee) \to \mathcal{D}(\mathcal{E}^\vee).$$

More precisely, $jet^\infty(\mathcal{E})$ is naturally isomorphic to the $\mathcal{O}_Z$-linear dual of $\mathcal{D}_Z(\mathcal{E}^\vee)$ and the dual of $\Delta^\vee$ is the map of infinite jet bundles mentioned above. We call $\Delta^\vee$ the *linearization* of Δ.

We fix a point $p \in X$ compatible with our system of standard rational parabolic subgroups. If $[W]$ is the automorphic vector bundle corresponding to the irreducible representation (ρ, W_ρ) of the stabilizer K_p of p, then $\mathcal{D}([W])$ is the automorphic vector bundle corresponding to the natural representation of $\mathfrak{P}_p^-$ on the generalized Verma module

$$\mathbb{D}(W_\rho^*) = U(\mathfrak{g}) \otimes_{U(\mathfrak{P}_p^-)} W_\rho^*.$$

We point out that $\mathcal{D}([W]^{can}) \neq \mathcal{D}([W])^{can}$. However, we will only be concerned with the latter, to which *logarithmic* differential operators naturally extend.

We now fix a maximal standard rational parabolic P and the subgroup $P' \subset P$ to which the corresponding boundary mixed Shimura datum $(P', \mathcal{X})$ is attached. The constructions in (1.2) associate automorphic vector bundles on the associated mixed Shimura varieties to representations of the stabilizer in P' of the fixed point $p \in \mathcal{X}$. This stabilizer will be denoted F^0P', since its Lie algebra is $F^0Lie(P')$ for the Hodge filtration associated to p; in particular, we drop the base point p from the notation. In what follows, we will identify automorphic vector bundles with the locally homogeneous vector bundles attached to representations of

$$F^0 Lie(P') = \mathfrak{k} \oplus \mathfrak{p}^- \oplus \mathfrak{n}^-$$

and we will also drop the subscript P from the above.

Without loss of generality we may assume that, for fixed x in the P-stratum, there is an étale analytic morphism from $Sh(P', \mathcal{X})_{\Sigma'}$ to a neighborhood of x. The pullback of $\mathcal{D}([W])$ to $Sh(P', \mathcal{X})$ via this morphism is the $\mathcal{D}(\bullet)$ of the pullback of $[W]$. This can be determined just as above, and we find that it is the automorphic vector bundle corresponding to the natural representation of $F^0Lie(P')$ on the tensor product module

(4.2.20)

$$\mathbb{D}_P(W_\rho^*) = U(Lie(P')) \otimes_{U(F^0Lie(P'))} W_\rho^* \xrightarrow{\sim} U(Lie(P)) \otimes_{U(F^0Lie(P') \oplus \mathfrak{g}_\ell)} W_\rho^*,$$

where the second isomorphism just serves to indicate that arithmetic subgroups of $G_\ell = G_{\ell,P}$ act on the coefficients and commute with all homogeneous differential operators.

Now we concentrate on the map, again denoted $\pi : Z = Sh(P', \mathcal{X})_{\Sigma'} \to M_P$, where the subscript Σ' corresponds to a compactification in the vertical direction. In particular the map π is smooth and proper. Henceforward, we let

$$\mathcal{E} = \mathcal{BGG}^{t'}(\widetilde{\mathbf{E}}), \ \mathcal{F} = \mathcal{BGG}^{t}(\widetilde{\mathbf{E}}),$$

with $t \to t'$ in W^Q as above. Let Λ be the highest weight of the contragredient of E. Then the fibers at p of $\mathcal{F}^\vee$ and $\mathcal{E}^\vee$ are $\mathbb{D}_P(H^t(\mathfrak{p}^+, E)^\vee)$ and $\mathbb{D}_P(H^{t'}(\mathfrak{p}^+, E)^\vee)$, respectively. In turn, these can be identified respectively with $\mathbb{M}_{Q_1}(\Lambda(t))$ and $\mathbb{M}_{Q_1}(\Lambda(t'))$. The differential $d_{t,t'}^{Q_1} : \mathbb{M}_{Q_1}(\Lambda(t)) \to \mathbb{M}_{Q_1}(\Lambda(t'))$ in the generalized BGG complex gives rise to the differential operator $\Delta_{t,t'} : \mathcal{E} \to \mathcal{F}$, corresponding to a component of the differential in $\mathcal{BGG}(\widetilde{\mathbf{E}})$.

To determine the image of $\Delta_{t,t'}$ under $R^i\pi_*$, we pass to linearizations. The map $\Delta_{t,t'}^\vee$ of (4.2.19) is a morphism of $\mathcal{D}_Z$-modules, hence it is natural to consider its image under the direct image in the category of $\mathcal{D}_Z$-modules. Following [B4], we denote this derived functor $R\pi_+$.

Proposition 4.2.21. *(i) Let $\mathcal{G} = \mathcal{E}$ or $\mathcal{F}$, and $d = \dim Z - \dim M$. For each $q \geq 0$, there is a natural isomorphism*

$$R^\bullet\pi_+\mathcal{D}_Z(\mathcal{G}^\vee) \xrightarrow{\sim} \mathcal{D}_{M_P}(R^{d-\bullet}\pi_*(\mathcal{G}^\vee)).$$

(ii) Under this isomorphism, the image of $\Delta_{t,t'}^\vee$ is the linearization of the image of $\Delta_{t,t'}$ via $R^{d-\bullet}\pi_*$.

Proof. The first assertion is a simple consequence of the definitions. Recall that π is smooth and proper. By definition, for any $\mathcal{D}_Z$-module $\mathcal{N}$,

$$R\pi_+\mathcal{N} = R\pi_*(\mathcal{D}_{\ldots} \overset{\mathbb{L}}{\otimes} \mathcal{N})$$

Here $\mathcal{D}\{M_P \leftarrow Z\} = \pi^*(\mathcal{D}_{M_P}) \otimes \omega_{Z/M_P}$, where π^* is inverse image in the category of $\mathcal{O}$-modules and $\omega_{Z/M_P} = \Omega^d_{Z/M_P}$ is the relative dualizing sheaf. In our case, $N = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{G}^\vee$ is a free $\mathcal{D}_Z$-module, hence we can replace $\overset{\mathbb{L}}{\otimes}$ by $\otimes$ above, and we have

$$
\begin{aligned}
R^\bullet\pi_+\mathcal{D}_Z(\mathcal{G}^\vee) \ &= \ R^\bullet\pi_*[\pi^*(\mathcal{D}_{M_P}) \otimes \omega_{Z/M_P} \otimes \mathcal{G}^\vee] \\
&\xrightarrow{\sim} \mathcal{D}_{M_P} \otimes R^\bullet\pi_*(\omega_{Z/M_P} \otimes \mathcal{G}^\vee) \\
&\xrightarrow{\sim} \mathcal{D}_{M_P} \otimes (R^\bullet\pi_*\mathcal{G})^\vee \ = \ \mathcal{D}_{M_P}(R^{d-\bullet}\pi_*(\mathcal{G}^\vee))
\end{aligned}
\tag{4.2.22}
$$

where the first isomorphism is the projection formula and the second is Serre duality. The second assertion is well-known (see [Sa1, Lemme 2.3.6]).

We rewrite the formulas (4.2.22) in terms of representation theory. Recall that

$$
\mathfrak{w}_{P,\mathbb{C}} = \mathfrak{u}_\mathbb{C} \oplus \mathfrak{v}^+ \oplus \mathfrak{v}^- \tag{4.2.23}
$$

as modules over $\mathfrak{k}_h$; here $\mathfrak{u} = Lie(U)$ with U the center of W_P, as usual. We write $\mathfrak{w}^+ = \mathfrak{u}_\mathbb{C} \oplus \mathfrak{v}^+_p$. Then the bundle ω_{Z/M_P} is the automorphic vector bundle on M_P associated to the adjoint representation of $\mathfrak{k}_h$ on the highest exterior power of the holomorphic cotangent space at the image of p in M_P, i.e. on $\wedge^d(\mathfrak{w}^+)^*$. Next, $\pi^*(\mathcal{D}_{M_P})$ is the bundle associated to the adjoint representation of $F_0Lie(P')$ (or $F_0Lie(P)$, if we want to keep track of the G_ℓ-action) on $H_0(\mathfrak{w}^+, U(Lie(P')))$ (or $H_0(\mathfrak{w}^+, U(Lie(P)))$, but the addition of $\mathfrak{g}_\ell$ at this point changes nothing in the final calculation). Finally, the functor $R\pi_*$ corresponds to the standard bar complex $C^\bullet(\mathfrak{v}^-, \bullet)$ for Lie algebra cohomology. Thus the first line of (4.2.22) expresses $R\pi_+\mathcal{D}_Z(\mathcal{G}^\vee)$ as the automorphic vector bundle attached to

$$
C^\bullet(\mathfrak{v}^-, H_0(\mathfrak{w}^+, U(Lie(P)) \otimes_{U(F^0Lie(P))} \wedge^d(\mathfrak{w}^+)^* \otimes W)),
$$

where $\mathcal{G}^\vee$ is the automorphic vector bundle attached to the representation of $\mathfrak{k}$ on W. Since $\mathfrak{v}^- \subset F^0Lie(P')$ the action passes across the tensor product to yield

$$
H_0(\mathfrak{w}^+, U(Lie(P)) \otimes_{U(F^0Lie(P))} C^\bullet(\mathfrak{v}^-, \wedge^d(\mathfrak{w}^+)^* \otimes W))
$$

Koszul duality [Kn, Theorem 6.10 (6.30)] provides a canonical identification

$$
C^\bullet(\mathfrak{v}^-, ?) \xrightarrow{\sim} C_{d_- - \bullet}(\mathfrak{v}^-, \wedge^{\dim \mathfrak{v}^-}(\mathfrak{v}^-)^* \otimes ?),
$$

where $C_\bullet(\mathfrak{v}^-, ?)$ is the standard complex for Lie algebra homology and $d_- = \dim \mathfrak{v}^-$. Thus we can replace the last complex by

$$
H_0(\mathfrak{w}^+, U(Lie(P' \cdot W_2P'_\mathbb{C}) \otimes_{U(F^0Lie(P'))} C_{d_- - \bullet}(\mathfrak{v}^-, \wedge^{d_P}(\mathfrak{w}_P)^* \otimes W)) \tag{4.2.24}
$$

Here $d_P = \dim \mathfrak{w}_P$ and we have used (4.2.23) and the fact that $\mathfrak{v}^-$, being unipotent, has trivial determinant on $\mathfrak{w}_P$.

Remember that $\mathfrak{v}^-$ acts on W via an identification with $\mathfrak{s} \subset \mathfrak{k}$, the unipotent radical of a parabolic subalgebra we will denote $\bar{\mathfrak{r}}(Q)$. Similarly, we identify $\mathfrak{g}_\ell$ with $Ad(c_P)(\mathfrak{g}_\ell) \subset \mathfrak{k}$. Then there is a Levi decomposition

$$\bar{\mathfrak{r}}(Q) = \mathfrak{k}_h \oplus \mathfrak{g}_\ell \oplus \mathfrak{s}$$

and we let $\mathfrak{r}(Q) \subset \mathfrak{k}$ denote the opposite parabolic, $R(Q)$ the corresponding parabolic subgroup. We can replace W as $U(\mathfrak{k})$-module by the quasi-isomorphic complex $BGG^{R(Q)}(W)$. Then there is a canonical quasi-isomorphism

$$\begin{aligned} C_\bullet(\mathfrak{s}, W) &\cong H_0(\mathfrak{s}, BGG_\bullet^{R(Q)}(W)) \\ &\xrightarrow{\sim} H_0(\mathfrak{s}, \bigoplus_{\ell(s)=\bullet} U(\mathfrak{k}) \otimes_{U(\mathfrak{r}(Q))} E(\Lambda(W)(s))). \end{aligned} \tag{4.2.25}$$

Here s runs through the set of Kostant representatives relative to $R(Q)$, as in the first part of this section, and we use $\Lambda(W)$ to designate a highest weight of W.

We now return to the notation $\mathcal{E}, \mathcal{F}$, associated to the pair $t \rightarrow t' \in W^Q$; i.e., associated respectively to the representations $E(t')$ and $E(t)$ of $\mathfrak{k}$. We extend $U(\mathfrak{k}) \otimes_{U(\mathfrak{r}(Q))} E(t)^*(s))$ trivially to a module over $U(\mathfrak{k} \oplus \mathfrak{p}^-)$ and then restrict to $U(\mathfrak{k}_h \oplus \mathfrak{p}_h^-) \subset U(F^0 Lie(P'))$. We extend from $F^0 Lie(P')$ to $F^0 Lie(P)$ by letting G_ℓ act via the Levi decomposition above. Then as $U(F^0 Lie(P))$-module the right-hand side of the last formula in (4.2.25) is

$$U(F^0 Lie(P)) \otimes U(\mathfrak{r}(Q)) E(t)^*(s)).$$

So substituting (4.2.25) into (4.2.24), we obtain

$$H_0(\mathfrak{w}^+, U(Lie(P)) \otimes_{U(F^0 Lie(P))} \wedge^{d_P} (\mathfrak{n}_P)^* \otimes H_0(\mathfrak{s}, \bigoplus_{\ell(s)=\bullet} U(F^0 Lie(P)) \otimes_{U(\mathfrak{r}(Q))} E(t)^*(s))$$

which simplifies as

$$H_0(\mathfrak{w}_P, U(Lie(P)) \otimes_{U(\mathfrak{r}(Q))} E(t)^*(s)) = H_0(\mathfrak{w}_P, \mathbb{M}(\Lambda(t)(s))). \tag{4.2.26}$$

The same calculation, with t replaced by t', shows that $R^\bullet \pi_+ \mathcal{D}_Z(\mathcal{F}^\vee)$ is the automorphic vector bundle associated to

$$\bigoplus H_0(\mathfrak{w}_P, \mathbb{M}(\Lambda(t')(s)))$$

We recognize (4.2.26) as the left-hand side of (4.2.14), where (unfortunately) P and $\overline{P}$ have been interchanged. From here we apply Prop. 4.2.15 (b) to obtain that the correspondence in Lemma 4.2.18 is compatible with the differentials on the two sides of (4.2.4), i.e. that for any $s \in W_Q^{\overline{P}}$ the linearization of the direct image

$$R^q\pi_*^{(s)}\Delta(t,t') : R^q\pi_*(\mathcal{E})^s \rightarrow R^q\pi_*(\mathcal{F})^s$$

is the homomorphism

$$\mathcal{D}_M(R^{d_P-q}\pi_*(\mathcal{F})^\vee)^s \rightarrow \mathcal{D}_M(R^{d_P-q}\pi_*(\mathcal{E}^\vee))^s$$

associated to the homomorphism $\mathbb{M}(\Lambda(t)(s)) \rightarrow \mathbb{M}(\Lambda(t')(s))$ of generalized Verma modules. We conclude:

Theorem (4.2.27). *In the situation of (4.1.9), there is a canonical isomorphism of complexes:*

$$R(\overline{\pi}_P)_*(i_R^*)\{\mathcal{BGG}(M,\widetilde{\mathbf{E}})\}^{can} \xrightarrow{\sim} \bigoplus_{w\in W^P}\{\mathcal{BGG}(M_P,\widetilde{\mathbf{E}}_{\mu(h,w)})^{can}\otimes\mathbf{C}^\bullet(\Gamma_{\ell,R},H^\bullet(E_{\mu(\ell,w)})[-\ell(w)]\}.$$

Here the notation is as in (4.1.10.1).

We adelize the above theorem using the notation Sh_Σ, $Sh(R)$, etc., of (3.2). The least awkward expression is in terms of the inverse limit over all toroidal compactifications

$$\widetilde{Sh} = \widetilde{Sh}(G,X) = \varprojlim_{K_f,\Sigma} Sh_\Sigma$$

(recall that the level subgroup K_f is implicit in the notation). Let $i_R : \widetilde{Sh}^{R(*)} \hookrightarrow \widetilde{Sh}$ denote the inclusion of the $R(*)$-stratum (inverse limit over all $R(*)$-strata at finite level). Let $\Delta_{1,R,h}$ denote the projection of the group $\Delta_{1,R}$ of (3.2) on $G_{h,P}(\mathbf{A}_f)$, and let $Sh(R)^+$ denote the quotient $Sh(G_{h,P},X(P))/\Delta_{1,R,h}$. Then $R(\mathbf{A}_f)$ acts on $Sh(R)^+$, and the adelic version of $\overline{\pi}_P$ is a natural morphism

$$\overline{\pi}_R : \widetilde{Sh}^{R(*)} \rightarrow Ind_{R(\mathbf{A}_f)}^{G(\mathbf{A}_f)}\widetilde{Sh}(R)^+, \tag{4.2.28}$$

where $\widetilde{Sh}(R)^+$ is again the projective limit over all toroidal compactifications. For $w \in W^R$, let

$$\mathcal{BGG}(Sh(R),w,E) = \mathcal{BGG}(Sh(G_{h,P},X(P)),\widetilde{\mathbf{E}}_{\mu(h,w)})^{can}\otimes C^\bullet(X(G_{\ell,R}),E_{\mu(\ell,w)})[-\ell(w)]$$

. Then the formula (3.2.8), with the space $\mathcal{H}^\bullet(w)$ replaced by the complex $\mathcal{BGG}(Sh(R),w,E)$, defines an $R(\mathbf{A}_f)$-equivariant vector bundle $I^R(\mathcal{BGG}(Sh(R),w,E))$ on $\widetilde{Sh}(R)^+$. The adelic version of Theorem (4.2.27) is

Theorem (4.2.29). *(a) In the above situation, there is a canonical isomorphism of complexes:*

$$R(\overline{\pi}_P)_*(i_R^*)\{\mathcal{BGG}(\widetilde{Sh}(G,X),\widetilde{\mathbf{E}})\}^{can}\xrightarrow{\sim} Ind_{R(\mathbf{A}_f)}^{G(\mathbf{A}_f)}\bigoplus_{w\in W^R}\{I^R(\mathcal{BGG}(Sh(R)),w,E)).$$

(b) This isomorphism is rational over the reflex field $E(G,X)$.

Proof. Part (a) is just the adelic version of (4.2.27), or rather of its obvious variant for the R-stratum. (Here we are using the fact that the Kostant decomposition for the parabolic subgroup $R_{\ell,P}$ of $G_{\ell,P}$ defines a direct sum decomposition in the derived category for the cohomology complex of the $R_{\ell,P}$-stratum of the boundary of $X(G_{\ell,P})$.) Then (b) is immediate because the identification in Lemma 4.2.18 is rational over $E(G,X)$, as are all subsequent constructions.

(4.2.30) *Remark.* Upon taking fixed vectors for a level subgroup K_f, the above theorem can be given a reasonable expression in terms of toroidal compactifications at finite level.

(4.3) *Hodge theory at the boundary, revisited.* Let M'_P denote a connected component of the mixed Shimura variety associated to a maximal parabolic subgroup P of G, and

$$M'_P \xrightarrow{\pi_2} A_P \xrightarrow{\pi_1} M_P \tag{4.3.1}$$

the associated tower of algebraic fibrations (see (1.4.1) or [HZ1, (1.2.5)]). Let, as usual, π (or π_P) denote $\pi_1\circ\pi_2$. For $\rho: P\to GL(E)$ a rational representation, there are isomorphisms of local systems on M_P:

$$R^k\pi_*\widetilde{\mathbf{E}}\simeq\widetilde{\mathbf{H}}^k(\mathfrak{w}_P,E). \tag{4.3.2}$$

This follows from the fact that the fibers of π have the homotopy type of the compact nilmanifold $\Gamma_{W_P}\backslash W_P(\mathbb{R})$. In fact, (4.3.2) is a consequence of the more basic fact at the cochain level:

$$R\pi_*\widetilde{\mathbf{E}}\simeq\pi_*\{\widetilde{\mathbf{C}}^\bullet(\mathfrak{w}_P,E)|_{S_L}\}, \tag{4.3.3}$$

where S_L is the cross-section of π given by a choice of rational Levi subgroup $L\subset {}^0P$ and basepoint $c_P(x_0)$ (the latter is fixed by $G_{\ell,P}$).

In [HZ2, §5], we saw (among other things) that the above assertions remain true in the Hodge theoretic sense. As J. Wildeshaus has pointed out to us, the arguments in [HZ2] are really assertions about the associated *mixed Hodge modules*—much of it is already there—and we wish to recast them here.

Let $M'_{P,\Sigma}$ denote the toroidal partial compactification of M'_P determined by Σ; it is *not* an algebraic variety, as its boundary has infinitely many irreducible components. It contains the boundary stratum that is called ${}^{<}\widetilde{Z}_{P,\Sigma}$ in [HZ1]. Taking the quotient of the latter by $\Gamma_{\ell,P}$ gives rise to the P-stratum ${}^{<}Z_{P,\Sigma}$ in M_Σ. And $Z_\Sigma(P)$ is the just the closure of ${}^{<}Z_{P,\Sigma}$ in M_Σ. Of course, $Z_\Sigma(P)$ contains $Z_\Sigma(R)$ whenever $R \subseteq P$, but $Z_\Sigma(R)$ maps surjectively to $(M_P)_\Xi$ under the projection $\overline{\pi} : Z_\Sigma(P) \to (M_P)_\Xi$ (as in (4.1.10.2)) if and only if R is subordinate to P.

We recall some constructions from [HZ2, (5.2)]. We denote by j_Σ the inclusion of M'_P in $M'_{P,\Sigma}$. Let $\mathcal{M}_\Sigma$ (called $\mathcal{M}$ in [HZ2]) be the unique mixed Hodge module on $M'_{P,\Sigma}$ with underlying perverse sheaf $Rj_*\widetilde{\mathbf{E}}$, such that its restriction to M'_P is the variation of mixed Hodge structure defined by the representation of P on E. Then $(\pi_*\mathcal{M}_\Sigma, R\pi_*\widetilde{\mathbf{E}})$ is a mixed Hodge module on M_P *with action of* $\Gamma_{\ell,P}$.

Proposition (4.3.4). [HZ2, (5.2.9),(5.4.19)] *In the derived category of mixed Hodge modules,* $(\pi_*\mathcal{M}_\Sigma, R\pi_*\widetilde{\mathbf{E}})$ *is* $\Gamma_{\ell,P}$*-equivariantly isomorphic to the complex of variations of mixed Hodge structure* $\pi_*\{\widetilde{\mathbf{C}}^\bullet(\mathfrak{w}_P, E)|_{S_L}\}$ *with filtrations induced by the mixed Shimura data for* P*. In particular, the latter is admissible.*

Though unusual in general, we expected in this case that the mixed sheaf $R\pi_*\widetilde{\mathbf{E}}$ would split as the direct sum of pure sheaves (of different weights). Indeed, the following is an immediate consequence of (4.3.4):

Theorem (4.3.5). *In the derived category of mixed Hodge modules on* M_P*,*

$$R\pi_*\widetilde{\mathbf{E}} \approx \widetilde{\mathbf{H}}^\bullet(\mathfrak{w}, E) \simeq \bigoplus_{w \in W^P} \widetilde{\mathbf{E}}_{\mu(h,w)} \otimes E_{\mu(\ell,w)}[-l(w)]$$

underlies a decomposition of mixed Hodge modules. Each $\widetilde{\mathbf{E}}_{\mu(h,w)}$*, being determined by an irreducible representation of* $G_{h,P}$*, is pure.*

Corollary (4.3.6). [HZ2, (5.4.20)] *The mixed Hodge structure of* $H^i(M'_P, \widetilde{\mathbf{E}})$ *decomposes as the direct sum of mixed Hodge structures:*

$$\bigoplus_{w \in W^P} H^{i-l(w)}(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)}) \otimes E_{\mu(\ell,w)}.$$

It is time to move on to the boundary cohomology of $Z_\Sigma = \partial M_\Sigma$. The final result in [HZ2, §5] is the following, which should be compared to (4.3.5) and (4.3.6). It is the Hodge theoretic version of (4.1.9), and, by iteration, it provides the Hodge theoretic version of (4.1.10).

Theorem (4.3.7). [HZ2, (5.6.10)] *Let E be a representation of G, so that $\widetilde{\mathbf{E}}$ underlies a variation of Hodge structure on M. Let $j_P : M_P \to M_{P,\Sigma(P)}$ be a suitable toroidal compactification of M_P, as in* (3.1.2). *Suppose that* $\Pi(R) = P$. *Then*

i) there is a decomposition in the derived category of mixed Hodge modules on $M_{P,\Sigma(P)}$

$$\begin{aligned} R\overline{\pi}_* i_R^* \widetilde{\mathbf{E}} &\approx \bigoplus_{w \in W^P} R(j_P)_* \widetilde{\mathbf{E}}_{\mu(h,w)} \otimes C^\bullet(\Gamma_{\ell,R}, E_{\mu(\ell,w)})[-l(w)] \\ &\approx \bigoplus_{w \in W^R} R(j_P)_* \widetilde{\mathbf{E}}_{\mu(h,w)} \otimes C^\bullet(\Gamma_{\ell,R}^{red}, E_{\mu(\ell,w)})[-l(w)] \\ &\approx R(j_P)_* C^\bullet(\Gamma_{\ell,R}, \widetilde{\mathbf{H}}^\bullet(\mathfrak{w}, E)); \end{aligned}$$

ii) there is a decomposition of mixed Hodge structures

$$H^q_{dn}(Z_\Sigma(R), \widetilde{\mathbf{E}}) \simeq \bigoplus_{a;\, w \in W^P} H^a(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)}) \otimes H^{q-a-l(w)}(\Gamma_{\ell,R}, E_{\mu(\ell,w)}).$$

Remark. The passage from the sum over W^P to the sum over W^R is an isomorphism of mixed Hodge modules; cf. Remark (3.1.9.ii)(b).

There are four main ingredients in the proof of the above assertions: the simplicial structure of the boundary (which produces the $\Gamma_{\ell,R}$-cohomology in (ii) above), the calculation on M'_P for a single simplex, the irrelevance of the boundary of M_P, and the inability of the boundary cohomology here to distinguish the variation of mixed Hodge structure from M'_P from the variation of pure Hodge structure on M.

For the second item, one reverts back to $M'_{P,\Sigma}$, as the structure of both at the P-stratum is locally the same. Let τ be any cone in Σ_P, and $i_\tau : Z_\tau \to M'_{P,\Sigma}$ the inclusion. Then,

Proposition (4.3.8). *There is an isomorphism in the derived category of mixed Hodge modules on M_P, given by restriction of that in (4.3.4):*

$$R\pi_* i_\tau^* R j_* \widetilde{\mathbf{E}} \approx \pi_* \{\widetilde{\mathbf{C}}^\bullet(\mathfrak{w}_P, E)|_{S_L}\}.$$

The next proposition, tacit in [HZ2, §5], asserts that a divisor with normal crossings behaves like the boundary of a manifold-with-corners with regard to deleted neighborhood cohomology. (This is consistent with the comparison of nerves in [HZ2, (2.7.8)].)

Proposition (4.3.9). *Let D be a divisor with normal crossings, whose irreducible components will be denoted D, on the complex manifold X, and let $j : X - D \hookrightarrow X$*

be the inclusion. Let $D_A = \bigcap_{i \in A} D_i$ *be an intersection of components of* D*, and denote by* D_A° *the set of points of* D_A *that lie on no additional components of* D*, and* $\nu_A : D_A^\circ \hookrightarrow D_A$*. Finally, let* i_A *and* i_A° *denote the respective inclusions of* D_A *and* D_A° *in* Y*. Then for any local system* $\widetilde{\mathbf{L}}$ *on* $Y - D$*,*

$$i_A{}^* Rj_* \widetilde{\mathbf{L}} \xrightarrow{\sim} R\nu_{A,*}(i_A^\circ)^* Rj_* \widetilde{\mathbf{L}}.$$

Proof. The assertion is local on D_A. Also, there is nothing to prove at points of D_A°. Thus, suppose that $y \notin D_A^\circ$, and let D_B denote the (non-empty) intersection of the additional divisors passing through y. Then there are local coordinates centered at y such that near y, $j : Y - D \hookrightarrow Y$ is $(\Delta^*)^a \times (\Delta^*)^b \times \Delta^{n-b-a} \subset \Delta^n$, where $a = \#A$ and $b = \#B$. In these terms, $D_A^\circ = \{0\} \times (\Delta^*)^b \times \Delta^{n-b-a}$, and $D_A = \{0\} \times \Delta^b \times \Delta^{n-b-a}$. Thus, we may as well assume that $n = b + a$. Then the stalk of $i_A{}^* Rj_* \widetilde{\mathbf{L}}$ is $C^\bullet((\Delta^*)^a \times (\Delta^*)^b, \widetilde{\mathbf{L}})$, while that of $R\nu_{A,*}(i_A^\circ)^* Rj_* \widetilde{\mathbf{L}}$ is $C^\bullet((\Delta^*)^b, C^\bullet(\Delta^*)^a, \widetilde{\mathbf{L}})$. In both cases, one has a complex for computing $H^\bullet((\Delta^*)^a \times (\Delta^*)^b, \widetilde{\mathbf{L}})$, and our assertion follows.

In short, one does not have to worry about the compactification of a stratum of the boundary in treating the deleted neighborhood cohomology. This addresses the third item in the outline of the proof of (4.3.7) above.

The fourth and final one is worded imprecisely, and we elaborate now. As mentioned before, $\Gamma_{\ell,P} \backslash {}^< \widetilde{Z}_{P,\Sigma}$ is analytically isomorphic to ${}^< Z_{P,\Sigma}$; indeed, there is a $\Gamma_{\ell,P}$-invariant deleted neighborhood $\widetilde{O}$ of ${}^< \widetilde{Z}_{P,\Sigma}$ in $M'_{P,\Sigma}$ such that $\Gamma_{\ell,P} \backslash \widetilde{O}$ is analytically equivalent to an deleted neighborhood O of ${}^< Z_{P,\Sigma}$ in M_Σ.

Of course, the local system $\widetilde{\mathbf{E}}|_O$ underlies a variation of (pure) Hodge structure, viz., the restriction of the one on M. On the other hand, the variation of mixed Hodge structure determined by E as a representation of P is $\Gamma_{\ell,P}$-equivariant, so its restriction to $\widetilde{O}$ descends to a variation of *mixed* Hodge structure on O, with the same underlying local system; it is an admissible variation. To distinguish them as mixed Hodge modules, we write $\widetilde{\mathbf{E}}_{(G)}|_O$ and $\widetilde{\mathbf{E}}_{(P)}|_O$ resp. The following is an elaboration on [HZ2, (5.6.12)]:

Proposition (4.3.10). *Suppose that* $\Pi(R) = P$*. Then the mixed Hodge modules* $i_R^* \widetilde{\mathbf{E}}_{(G)}|_O$ *and* $i_R^* \widetilde{\mathbf{E}}_{(P)}|_O$ *are isomorphic.*

Proof. We begin by recalling that $\widetilde{\mathbf{E}}_{(G)}|_O$ and $\widetilde{\mathbf{E}}_{(P)}|_O$ have the same Hodge filtration, but they differ in their weight filtrations. The latter are determined respectively by the weight homomorphisms w_G and w_P (as in [HZ1, 1.2.2], where the first one is written $h \circ w$). As such, they have the same asymptotic Hodge filtrations, and we want to show that the associated weight filtrations along $Z_\Sigma(R)$ coincide. Let $Z_\Sigma^\circ(R)$ be the locus of smooth points of $Z_\Sigma(R)$. By (4.3.9) and the uniqueness property [Sa3, 2.11], it is enough to verify that the weights coincide along $Z_\Sigma^\circ(R)$.

The local monodromy transformations of $\widetilde{\mathbf{E}}$ are unipotent. Until we say otherwise, we continue in the setting of an arbitrary admissible variation of mixed Hodge structure $(\widetilde{\mathbf{V}}, \widetilde{\mathbf{W}}, \mathcal{F})$ with unipotent local monodromy, defined in a neighborhood of $Z_\Sigma^\circ(R)$. Then $i_R^*(\widetilde{\mathbf{V}}, \widetilde{\mathbf{W}}, \mathcal{F})$, can be determined as a mixed Hodge module by iteration on the following construction (iterated deleted neighborhood cohomology, discussed in [HZ2, (3.1.7)]). Let $y \in Z_\Sigma^\circ(R)$; because $\Pi(R) = P$, there is a component D of $Z_\Sigma(P)$ passing through y. Also, let $i_D \hookrightarrow M_\Sigma$ be the inclusion. Then $i_D^*(\widetilde{\mathbf{V}}, \widetilde{\mathbf{W}}, \mathcal{F})$ is a one-variable degeneration with D as "parameter."[7]

The issues that occur are at the level of a filtered local system on the punctured disc Δ^* (the situation transverse to D) with unipotent monodromy. Let N denote the (nilpotent) monodromy logarithm, acting on any reference fiber V of $\widetilde{\mathbf{V}}$. The weight filtration of $i_D^*(\widetilde{\mathbf{V}}, \widetilde{\mathbf{W}}, \mathcal{F})$ is given by the *weight filtration of N relative to W*, which is denoted $M(N;W)$. If W is trivial, then $M(N;W)$ is the absolute weight filtration $M(N)$ of N, equal to the convolution of the kernel and image filtrations of N (see [StZ, (2.3)]), which is characterized by the statement

$$\text{(4.3.11)}\quad NM_k(N) \subseteq M_{k-2}(N), \text{ and } \quad N^k : \mathrm{Gr}_k^{M(N)} V \to \mathrm{Gr}_{-k}^{M(N)} V \text{ is an isomorphism.}$$

In general, $M(N;W)$ is characterized (if it exists) by

$$\text{(4.3.12)}\quad NM_k(N;W) \subseteq M_{k-2}(N;W), \text{ and } \quad N^k : \mathrm{Gr}_{k+i}^{M(N;W)} \mathrm{Gr}_i^W V \to \mathrm{Gr}_{-k+i}^{M(N;W)} Gr_i^W V \text{ is an isomorphism};$$

in other words, $M(N;W)$ induces $M(Gr_i^W N)[i]$ on $Gr_i^W V$. In total generality, there is no reason that a filtration satisfying (4.3.12) should exist. However, it is one of the conditions defining admissibility that the relative weight filtration exist.

At this point, one appeals to the following fact that we used in [HZ2, (5.6)]:

Proposition (4.3.13). [StZ, (2.14)] *Let V be a vector space with increasing filtration W, and N a nilpotent endomorphism of V such that $NW_i \subseteq W_{i-1}$. Then $M(N;W)$ exists if and only if $NW_i \subseteq W_{i-2}$, in which case $M(N;W) = W$.*

We check that this is satisfied in the Shimura variety setting. Let σ be the one-dimensional cone in Σ_P^c that defines D. The monodromy logarithm N around D

[7] The situation in one variable was a basic, pre-existing element in the foundation of Saito's

is given by some non-zero integral element of σ, and this is interior to C_P. Thus, its weight filtration is just the one defined by w_P. This lowers weights by two, so by (4.3.13), $M(N;W) = W$, which is also $M(N)$. Thus, there is an isomorphism of mixed Hodge modules:

$$i_D^*\widetilde{\mathbf{E}}_{(P)}|_O \simeq i_D^*\widetilde{\mathbf{E}}_{(G)}|_O.$$

Using the fact that $i_R^\circ = i_D \circ (i_R^D)^\circ$, where $(i_R^D)^\circ$ denotes the inclusion of $Z_\Sigma^\circ(R)$ in D, we get that

$$i_R^*\widetilde{\mathbf{E}}_{(P)}|_O \simeq i_R^*\widetilde{\mathbf{E}}_{(G)}|_O,$$

on $Z_\Sigma^\circ(R)$, hence on $Z_\Sigma(R)$, which is what we needed to show.

We conclude by formulating the nerve spectral sequence in the language of this section:

(4.3.14) Theorem. *(see [HZ2, (5.5.2)])*

i) The mixed Hodge module $i^*Rj_*\widetilde{\mathbf{E}}$ is canonically isomorphic in the derived category of mixed Hodge modules to the chain complex $S_\bullet$, where

$$S_r = \bigoplus_{r(R)=r+1} i_R^*\, Rj_*\widetilde{\mathbf{E}}$$

and the differentials are given by restriction. On hypercohomology, the spectral sequence associated to filtration by degree in the latter is the nerve spectral sequence, which is thereby a spectral sequence of mixed Hodge structures.

ii) Whenever $\Pi(R) = P$, one has, as in (4.3.7, i),

$$\begin{aligned} R(\overline{\pi}_P)_* i_R^* Rj_*\widetilde{\mathbf{E}} &\approx R(j_P)_*\, C^\bullet(\Gamma_{\ell,R}, \widetilde{\mathbf{H}}^\bullet(\mathfrak{n}_P, E)) \\ &\approx \bigoplus_{w\in W^P} R(j_P)_*\widetilde{\mathbf{E}}_{\mu(h,w)} \otimes C^\bullet(\Gamma_{\ell,R}, E_{\mu(\ell,w)})[-l(w)]. \end{aligned}$$

iii) Grading for the Hodge filtration F, one has that $\mathrm{Gr}_F^p(i^*Rj_*\widetilde{\mathbf{E}})$ is canonically quasi-isomorphic to $\mathrm{Gr}_F^p S_\bullet$. The corresponding "nerve" spectral sequence is:

$$\begin{aligned} E_1^{r,s} &= \bigoplus_{r(R)=r+1} H^s(Z_R, i_R^*\mathrm{Gr}_F^p(Rj_*\widetilde{\mathbf{E}})) \\ &\simeq \bigoplus_{r(R)=r+1} H^s(Z_R, i_R^*\mathrm{Gr}_F^p\mathcal{BBG}(M,\widetilde{\mathbf{E}})^{can}) \Rightarrow H^{r+s}(Z, i^*\mathrm{Gr}_F^p\mathcal{BBG}(M,\widetilde{\mathbf{E}})^{can}). \end{aligned}$$

iv) Whenever $\Pi(R) = P$, one has:

$$R(\overline{\pi}_P)_* i_R^*\mathrm{Gr}_F^p\mathcal{BBG}(M,\widetilde{\mathbf{E}})^{can} \simeq C^\bullet(\Gamma_{\ell,R}, \{\mathcal{H}^\bullet(\mathfrak{n}_P, \mathrm{Gr}_F^p\mathcal{BBG}(M,\widetilde{\mathbf{E}}))\}^{can})$$

(4.3.15) *Remarks.* (i) Theorem (4.3.14) has an adelic version, along the lines of Theorem (4.2.29), whose formulation we leave to the reader, noting that it is most easily effected upon replacing the sum over W^P by a sum over W^R; see the remark following (4.3.7) and the horizontal arrows in (ii), below. We also note that (4.2.29)(b) implies that (4.3.14.i) defines a canonical isomorphism in the derived category of mixed Hodge modules with $E(G, X)$-rational structure.

(ii) Proposition (3.4.1) also has a version in terms of mixed Hodge modules: if $R \supset R'$ are two proper parabolics of G, with $r(R') = r(R) + 1$, then the following diagram commutes:

$$\begin{array}{ccc} R(\overline{\pi}_R)_* i_R^* Rj_*(\widetilde{\mathbf{E}}) & \xrightarrow{\approx} & Ind_{R(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \bigoplus_{w \in W^R} \{ I^R(R(j_R)_* \, C^\bullet(X(G_{\ell,R}), E_{\mu(\ell,w)})[-\ell(w)]) \} \\ \downarrow & & \downarrow \\ R(\overline{\pi}_{R'})_* i_{R'}^* Rj_*(\widetilde{\mathbf{E}}) & \xrightarrow{\approx} & Ind_{R'(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \bigoplus_{w \in W^{R'}} \{ I^{R'}(R(j_{R'})_* \, C^\bullet(X(G_{\ell,R'}), E_{\mu(\ell,w)})[-\ell(w)]) \} \end{array}$$

Here the horizontal arrows are the quasi-isomorphisms of (4.3.14.ii), and the left-hand vertical arrow is defined by the natural restriction. When $\Pi(R) = \Pi(R')$ (resp. $\Pi(R) \neq \Pi(R')$) the right-hand vertical arrow is determined as in (3.2.9)(ii) (resp. as in (3.4.1)). We leave the precise formulation to the reader, noting only that in the $\mathcal{BGG}$ realization, the quasi-isomorphisms can be replaced by isomorphisms of complexes.

(4.4) *The topological cohomology.* As usual, let $R \in \mathcal{P}(G)$. In the "standard" way of doing things, one writes $R = L_R \cdot W_R$; then by [Ha],

$$(4.4.1) \quad H^\bullet(\overline{e'(R)}, \widetilde{\mathbf{E}}) \simeq H^\bullet(e'(R), \widetilde{\mathbf{E}}) \simeq H^\bullet(\Gamma_R, E) \simeq H^\bullet(X(\Gamma_{L_R}), \widetilde{\mathbf{H}}^\bullet(\mathfrak{w}_R, E)).$$

In actuality, the above holds at the cochain level, which can be viewed as an instance of (2.7.8):

$$(4.4.2) \qquad C^\bullet(\overline{e'(R)}, \widetilde{\mathbf{E}}) \approx C^\bullet(X(\Gamma_{L_R}), \widetilde{\mathbf{H}}^\bullet(\mathfrak{w}_R, E)) \approx C^\bullet(\Gamma_{L_R}, H^\bullet(\mathfrak{w}_R, E)).$$

If we have that $\Pi(R) = P$, then $G_{h,P}$ is a direct factor of L_R; indeed, we have, as we noted before, that $R = G_{h,P} \cdot Q \cdot W_P$, for Q a parabolic subgroup of $G_{\ell,P}$. The group $\Gamma \cap Q$ has been denoted $\Gamma_{\ell,R}$ (see (3.1.4)). When this is taken into account, we continue as in (3.3) to see that (4.4.2) can be written as:

$$(4.4.3) \qquad C^\bullet(e'(R), \widetilde{\mathbf{E}}) \approx C^\bullet(\Gamma_{h,P} \cdot \Gamma_{\ell,R}, H^\bullet(\mathfrak{w}_P, E)).$$

We observe that the only thing on the right-hand side of (4.4.3) that varies when R is restricted to be subordinate to P, is the "$\Gamma_{\ell,R}$."

Analogous to (3.5.1), a complex for computing the topological cohomology of the Borel-Serre boundary $\partial\overline{X}$ is the double complex

$$\overline{K}^{r,s} = \bigoplus_{r(R)=r+1} C^s(e'(R), \widetilde{\mathbf{E}}), \tag{4.4.4}$$

in which the differential $\widetilde{d}$ is the sum of the differential of $C^\bullet$ (which increases s by one), and restriction (which increases r by one). We denote the latter again by d_{par}. The nerve spectral sequence in this setting (see [HZ2, (3.5)]) is associated to the filtration $\mathcal{R}$ by rank, starting from the analogue of (3.5.2):

$$\mathrm{Gr}^t_{\mathcal{R}}\overline{K}^\bullet = \bigoplus_{r(R)=t+1} C^\bullet(e'(R), \widetilde{\mathbf{E}})[-t], \tag{4.4.5}$$

in which d_{par} vanishes. This produces for the E_1-term, as in (4.1.15):

$$E_1^{p,q}(\overline{K}^\bullet, \mathcal{R}) = \bigoplus_{r(R)=p+1} E_1^{p,q}(R), \quad \text{where}$$

$$\begin{aligned} E_1^{p,q}(R) &= H^q(e'(R), \widetilde{\mathbf{E}}) \\ &\simeq \bigoplus_{a+j+k=q} H^a(M_P, \widetilde{\mathbf{H}}^k(\Gamma_{\ell,R}, H^j(\mathfrak{n}_P, E))) \quad \text{by (4.4.3).} \end{aligned} \tag{4.4.6}$$

When we invoke Kostant's theorem, this time for the rational parabolic subgroup P of G, with Levi subgroup $G_{h,P} \times G_{\ell,P}$:

$$H^\bullet(\mathfrak{n}_P, E) \simeq \bigoplus_{w\in W^P} \left(E_{\mu(h,w)} \otimes E_{\mu(\ell,w)}\right)[-l(w)], \tag{4.4.7}$$

we obtain

$$\begin{aligned} E_1^{p,q}(R) &\simeq \bigoplus_{a;w\in W^P} H^a(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)}) \otimes H^{q-a-l(w)}(\Gamma_{\ell,R}, E_{\mu(\ell,w)}) \\ &\simeq \bigoplus_{a;w\in W^P} H^a(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)}) \otimes H^{q-a-l(w)}(X(\Gamma_{\ell,R}), \widetilde{\mathbf{E}}_{\mu(\ell,w)}). \end{aligned} \tag{4.4.8}$$

There is some reason to work instead with the filtration $\mathcal{R}^h$ by *holomorphic* rank, which is the filtration determined by the decomposition of $\overline{M}$ induced by the natural mapping of $\overline{X}$ onto the Baily-Borel compactification of M (see [Z6, (1.6)(11)]). For this, we have instead

$$\begin{aligned} &i) \quad \mathcal{R}^h_t\overline{K}^\bullet = \bigoplus_{\rho_h(R)\le t} C^\bullet(e'(R), \widetilde{\mathbf{E}})[1-r(R)] \\ &ii) \quad \mathrm{Gr}^{\mathcal{R}^h}_t\overline{K}^\bullet = \bigoplus_{\rho_h(R)=t} C^\bullet(e'(R), \widetilde{\mathbf{E}})[1-r(R)] \\ &\qquad = \bigoplus C^\bullet(\Gamma_{h,P}, K^\bullet_c(X(\Gamma_{\ell,P}), H^\bullet(\mathfrak{n}_P, E))), \end{aligned} \tag{4.4.9}$$

with $K_c^\bullet$ as in (3.5.7), and the differential induced by $d + d^\ell$.

Remark. We note that the differential d^h, viewed as a summand of d_{par}, decomposes canonically as

$$d^h = \sum_r d^{h,r},$$

where, $d^{h,r}$ decreases holomorphic rank by r, and increases $\mathcal{R}^\ell$ by $r-1$.

¿From (4.4.9, ii), we can now write down a convenient expression for the E_1-term of the holomorphic rank spectral sequence (cf. (3.5.8)):

Proposition (4.4.10). *In the spectral sequence for $\mathcal{R}^h$, the E_1-term is*

$$E_1^{p,q}(\overline{K}^\bullet, \mathcal{R}^h) = \bigoplus_{P \sim P_{\rho+p}} \widetilde{E}_1^{p,q}(P), \qquad \textit{where}$$

$$\begin{aligned}
\widetilde{E}_1^{p,q}(P) &= \bigoplus_{a+j+k=p+q} H^a(M_P, \widetilde{\mathbf{H}}_c^k(X(\Gamma_{\ell,P}), \widetilde{\mathbf{H}}^j(\mathfrak{w}_P, E))) \\
&= \bigoplus_{a+j+k=p+q} H^a(\Gamma_{h,P}, H_c^k(X(\Gamma_{\ell,P}), \widetilde{\mathbf{H}}^j(\mathfrak{w}_P, E))) \\
&= \bigoplus_{a+j+k=p+q} H_c^k(X(\Gamma_{\ell,P}), \widetilde{\mathbf{H}}^a(M_P, \widetilde{\mathbf{H}}^j(\mathfrak{w}_P, E))).
\end{aligned}$$

NB—The non-trivial terms in the above are for $p \leq 0$.

Proof. We use Kostant's theorem, again applied to P (4.4.7). This enables us to use the Kunneth formula to determine that

$$(4.4.10.1) \quad \widetilde{E}_1^{p,q}(P) = \bigoplus_{a;w\in W^P} H^a(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)}) \otimes H_c^{p+q-a-l(w)}(X(\Gamma_{\ell,P}), \widetilde{\mathbf{E}}_{\mu(\ell,w)}).$$

We now recombine the terms:

$$\begin{aligned}
\widetilde{E}_1^{p,q}(P) &= \bigoplus_{a;w\in W^P} H^a(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)} \otimes H_c^{p+q-a-l(w)}(X(\Gamma_{\ell,P}), \widetilde{\mathbf{E}}_{\mu(\ell,w)})) \\
&= \bigoplus_{a;w\in W^P} H^a(M_P, \widetilde{\mathbf{H}}_c^{p+q-a-l(w)}(X(\Gamma_{\ell,P}), \widetilde{\mathbf{E}}_{\mu(\ell,w)} \otimes \widetilde{\mathbf{E}}_{\mu(h,w)})) \\
&= \bigoplus_{a+j+k=p+q} H^a(M_P, \widetilde{\mathbf{H}}_c^k(X(\Gamma_{\ell,P}), \widetilde{\mathbf{H}}^j(\mathfrak{w}_P, E))),
\end{aligned}$$

which is the first formula for $\widetilde{E}_1^{p,q}(P)$ above. The second formula is equivalent, and the third is proved similarly.

We wish to elaborate on the relation between the filtration by holomorphic rank and the Baily-Borel compactification M^* of $M = M_\Gamma$ that was mentioned before (4.4.9). Let $f : \overline{M} \to M^*$ be the natural map. Our first observation is that, by degeneration of the relevant spectral sequences, one has the following and its corollary:

Proposition (4.4.11). $(R^i f_! \widetilde{\mathbf{E}})|_{M_P} \simeq \bigoplus_{j+k=i} \widetilde{\mathbf{H}}_c^k(X(\Gamma_{\ell,P}), \widetilde{\mathbf{H}}^j(\mathfrak{w}_P, E))$.

As we are using a mixture of compact and closed supports, we introduce the symbol $H^\bullet_{\ell,c}$ to refer to that. Then

Corollary (4.4.12). *We have the formula for $\widetilde{E}_1^{p,q}(P)$ of (4.4.10):*

$$\widetilde{E}_1^{p,q}(P) \simeq \bigoplus_{a+i=p+q} H^a(M_P, R^i f_! \widetilde{\mathbf{E}}) \simeq H^{p+q}_{\ell,c}(e'(P), \widetilde{\mathbf{E}}).$$

We consider ∂M^* as a filtered space. Specifically, one has

$$\partial M^* = \bigsqcup_{P \text{ maximal}} M_P$$

(as a set), and we put for each integer t

$$Y_t = \bigsqcup_{hol\, rk\, P \le t} M_P. \tag{4.4.13}$$

This defines a finite increasing filtration of ∂M^*, which we also call Y_∞, by closed subspaces. Note that $Rf_! \widetilde{\mathbf{E}}$ is constructible with respect to the stratification of M^* induced from (4.4.13), and we have for all P:

$$\mathbb{H}^\bullet(M_P, Rf_! \widetilde{\mathbf{E}}) \simeq H^\bullet_{\ell,c}(e'(P), \widetilde{\mathbf{E}}). \tag{4.4.14}$$

We now assert:

Proposition (4.4.15). *The spectral sequence for the filtration by holomorphic rank (see (4.4.10)) coincides with that of the filtered space ∂M^*, with filtration given in (4.4.13), and with coefficients in $\mathcal{F} = Rf_! \widetilde{\mathbf{E}}$:*

$$E_1^{p,q} = H^{p+q}(Y_{-p}, Y_{-p-1}; \mathcal{F}) \Rightarrow H^{p+q}(Y_\infty; \mathcal{F}).$$

Proof. Let $\overline{Y}_t = f^{-1} Y_t$, a closed subset of $\partial \overline{M}$ that consists of a union of maximal faces. As such, it is topologically a manifold-with-boundary. Note that the boundary of a maximal face $\overline{e'(P)}$ consists of faces $\overline{e'(R)}$ with $R \subset P$; then $f(\overline{e'(R)}) \subset M_P$ if and only if R is subordinate to P. It follows that $\overline{Y}_t$ is the union of those $\overline{e'(R)}$ for which $\rho_h(R) \le t$. From (4.4.9, i), one sees that the cochain complex $\mathcal{R}_t^h \overline{K}^\bullet$ is quasi-isomorphic to

$$C^\bullet(\overline{Y}_t, \partial \overline{Y}_t; \widetilde{\mathbf{E}}) \approx C_c^\bullet(\overline{Y}_t - \partial \overline{Y}_t; \widetilde{\mathbf{E}}) \approx C^\bullet(Y_t; Rf_! \widetilde{\mathbf{E}}).$$

This proves our assertion.

The following elementary fact, which is essentially (3.3.4), is fundamental in the determination of the differential d_1 in the holomorphic rank spectral sequence:

Lemma (4.4.16). *Let R be a parabolic subordinate to P, and let*

$$L_R = G_{\ell,R} \cdot G_{h,P}$$

be the associated decomposition of its Levi subgroup. Let P' be a maximal parabolic with $P' \prec P$, and put $R' = R \cap P'$. Then the decomposition of $L_{R'}$ is

$$L_{R'} = G_{\ell,R'} \cdot G_{h,P'}, \qquad \textit{with} \qquad G_{\ell,R'} = G_{\ell,R} \cdot (G_{\ell,P'} \cap G_{h,P}).$$

In particular, when P' and P are consecutive, $G_{\ell,R'} = G_{\ell,R}$.

When P' and P are consecutive, we thus have that

$$R' = L_{R'} \cdot W_{R'} = G_{\ell,R} \cdot G_{h,P'} \cdot (W_R \cdot W_{P'}) \tag{4.4.17}$$

is contained in $R = G_{\ell,R} \cdot G_{h,P} \cdot W_R$. If Γ is sufficiently small, one has the corresponding decomposition of $\Gamma_{R'}$. One can elect to factor out W_P instead; it is just a question of whether one is viewing R' as a parabolic subgroup of P or of P'.

At bottom, d^h is given by $R \mapsto R' = R \cap P'$ whenever $\Pi(R) = P$ and $P' \prec P$ (which entails a shift of rank by one), and the inclusions

$$\overline{e'(P)} \supset \overline{e'(R)} \hookleftarrow e'(R') \hookrightarrow \overline{e'(P')}, \tag{4.4.18}$$

This is the same situation that was faced in (3.5.11), so d_1 is the composition of a restriction mapping and a connecting homomorphism. In the topological setting, the most efficient way to formulate this is without the Kostant decompositions, though the latter may be needed for calculations. We obtain the following formula:

Proposition (4.4.19). *When P is conjugate to $P_{\rho+p}$ and P' is conjugate to $P_{\rho+p+1}$ and satisfies $P' \prec P$,*

$$d_1 : E_1^{p,q}(\overline{K}^\bullet, \mathcal{R}^h) \to E_1^{p+1,q}(\overline{K}^\bullet, \mathcal{R}^h)$$

in the spectral sequence for $\mathcal{R}^h$, is the direct sum of maps $\widetilde{E}_1^{p,q}(P) \to \widetilde{E}_1^{p+1,q}(P')$ given by

$$H^i_{\ell,c}(e'(P), \widetilde{\mathbf{E}}) \xrightarrow{r} H^i_{\ell,c}(e'(P \cap P'), \widetilde{\mathbf{E}}) \xrightarrow{\delta} H^{i+1}_{\ell,c}(e'(P'), \widetilde{\mathbf{E}}),$$

where r is restriction and δ a connecting homomorphism.

We quickly compare the nerve and holomorphic rank spectral sequences (see (4.4.6) and (4.4.10) resp.) in the least complicated non-trivial situation, namely when the $\mathbb{Q}$-rank of G equals two. In that case, one has standard parabolics, maximal: $P_1 \succ P_2$, and minimal $P_0 = P_1 \cap P_2$. Both spectral sequences have only one non-trivial differential, so they both degenerate at E_2. *For the sake of simplicity of notation, we will omit the symbol for summation over* Γ*-conjugacy classes in both cases, though it is important not to lose sight of this;* said summation is explicit only in our formula for the terms of the holomorphic rank spectral sequence, but it is present in the other one as well.

The respective differentials d_1 look like:

$$\text{(4.4.20.N)} \qquad \begin{array}{ccc} H^q(\overline{e'(P_1)},\, \widetilde{\mathbf{E}}) & \xrightarrow{r} & H^q(e'(P_{12}),\, \widetilde{\mathbf{E}}) \\ \oplus & & \| \\ H^q(\overline{e'(P_2)},\, \widetilde{\mathbf{E}}) & \xrightarrow{r} & H^q(e'(P_{12}),\, \widetilde{\mathbf{E}}) \end{array}$$

$$\text{(4.4.20.h)} \qquad H^q_{\ell,c}(e'(P_1),\, \widetilde{\mathbf{E}}) \xrightarrow{r} H^q_{\ell,c}(e'(P_{12}),\, \widetilde{\mathbf{E}}) \xrightarrow{\delta} H^{q+1}_{\ell,c}(e'(P_2),\, \widetilde{\mathbf{E}}).$$

The arithmetic quotients for G_{ℓ,P_1} and G_{h,P_2} are compact; the nilmanifolds coming from the unipotent radicals are always compact. Thus, (4.4.20.h) simplifies to:

$$\text{(4.4.21.h)} \quad H^q(e'(P_1),\, \widetilde{\mathbf{E}}) \xrightarrow{r} H^q(e'(P_{12}),\, \widetilde{\mathbf{E}}) \xrightarrow{\delta} H^{q+1}_c(e'(P_2),\, \widetilde{\mathbf{E}}) \simeq H^{q+1}(\overline{e'(P_2)},\, \partial\overline{e'(P_2)};\, \widetilde{\mathbf{E}}).$$

Since $\mathcal{R}$ is finer than $\mathcal{R}^h$, we have

$$\text{(4.4.22)} \qquad E_2^{0,q}(\overline{K}, \mathcal{R}) \twoheadrightarrow E_2^{0,q}(\overline{K}, \mathcal{R}^h), \qquad E_2^{1,q}(\overline{K}, \mathcal{R}) \hookrightarrow E_2^{1,q}(\overline{K}, \mathcal{R}^h)$$

(these relations can also be deduced directly from (4.4.20.N) and (4.4.21.h)). By reason of degree, the spectral sequence degenerates at E_2 – this need not be true in higher rank – so the total dimension of E_2 in each degree is the same for both.

(4.5) *The mixed Hodge complex filtered by holomorphic rank.* Given the isomorphism of deleted neighborhood cohomology for $\partial\overline{M}$ and ∂M_Σ provided by [HZ2, (3.5.5)], we begin by replacing $\overline{K}^\bullet$ from (4.4.4) by any compatible system of cohomological mixed Hodge complexes (e.g., the de Rham complexes of the corresponding mixed Hodge modules), and put

$$\widetilde{K}^\bullet(R) = C^\bullet(i_R^*\widetilde{\mathbf{E}}).$$

We will see that the double complex $\widetilde{K}^\bullet$, with its Hodge and weight filtrations (one can see F via (4.1) here), is a mixed Hodge complex *filtered by* $\mathcal{R}^h$. The last notion is a technical condition, determined by El Zein, that implies that the spectral sequence for $\mathcal{R}^h$ is one of mixed Hodge structures. What the latter means is spelled out after [HZ2, (5.5.2)], and it includes the assertion that the differentials of the spectral sequence are morphisms of mixed Hodge structure. A useful criterion for that, which is easy to check in practice, is that $\mathcal{R}^h$ *be a convolutant of the weight filtration* W, which in turn implies that $\mathcal{R}^h$ splits on $\mathrm{Gr}^W \widetilde{K}^\bullet$ (see [Z3, (3.6)]).

We already know that $\mathcal{R}$ is a convolutant of W, with $W = \mathcal{R} * W^H$, where W^H is the usual Hodge weight filtration on the individual summands of the complex. We also have that $\mathcal{R} = \mathcal{R}^h * \mathcal{R}^\ell$ by (3.5.9). However, this fact in general is *not* enough to give

$$W = R^h * (R^\ell * W^H); \tag{4.5.1}$$

one would need to check that the three filtrations W^H, R^h and R^ℓ form a distributive family in the sense of [Ka, 1.7]. In this case, however, it is evident that (4.5.1) holds, and we leave it to the reader to verify this. We therefore assert:

Proposition (4.5.2). *The complex $\widetilde{K}^\bullet$ is a mixed Hodge complex filtered by $\mathcal{R}^h$ (as well as by $\mathcal{R}$) that determines the mixed Hodge structure of $\partial\overline{M}$ with coefficients in $\widetilde{\mathbf{E}}$.*

For emphasis, we restate a consequence of (4.5.2) that was mentioned above:

Corollary (4.5.3). *The holomorphic rank spectral sequence in the topological setting is a spectral sequence of mixed Hodge structures.*

With (4.5.3) established, we make a Hodge theoretic comparison of the E_1-terms of the [topological] nerve and holomorphic rank spectral sequences. For that, we examine the respective formulas, (4.4.8) and (4.4.10.1), for the E_1-term. One thing stands out: they both involve the same variations of Hodge structure on M_P, viz., those coming from the representation of $G_{h,P}$ on the $E_{\mu(h,w)}$'s. Where they differ is in the finite-dimensional vector spaces (with the trivial Hodge structure) that the cohomology groups $H^a(M_P, \widetilde{\mathbf{E}}_{\mu(h,w)})$ are tensored with. As in [Z2], an "undesirable" term in a formula can disappear because it gets tensored with 0.

Finally, we assert the conclusion towards which we have been heading:

Theorem (4.5.4). *The holomorphic rank (resp. nerve) spectral sequence for coherent cohomology abutting to $H^\bullet(Z_\Sigma, i^*\mathrm{Gr}^p_F\{\mathcal{DR}(M, \widetilde{\mathbf{E}}^{can}\})$ is canonically isomorphic to the Gr^p_F of the topological holomorphic rank (resp. nerve) spectral sequence abutting to $H^\bullet(\partial\overline{M}, \widetilde{\mathbf{E}}) \simeq H^\bullet_{dn}(Z_\Sigma, \widetilde{\mathbf{E}})$.*

(4.5.5) *Remark.* The same holds for all $F^p/F^{p'}$ (with $p' > p$).

(4.6) *Ghost classes.* We set up the question of the existence of ghosts in its natural settings. The reader should expect by now that there are two parallel notions, one for the topological setting (local systems) and one for the coherent setting (vector bundles).

Let Y be a manifold-with-corners. Topologically, it is just a manifold-with-boundary [BS, App.], but the boundary has designated differentiable corner structure. Let $\{E_\alpha\}$ denote the set of closed faces of codimension one in ∂Y, and let $\widetilde{\mathbf{V}}$ be a local system on Y. We mention that Y and its interior are homotopically indistinguishable. The faces of ∂Y are themselves manifolds-with-corners. If we define the *rank* $\rho(Y)$ of Y to be the largest number of maximal (codimension-one) faces of ∂Y having non-empty intersection, or equivalently, the highest codimension of a boundary face, then, a face E of codimension m in ∂Y has rank at most $\rho(Y)-m$, with equality if and only if E contains a face of highest codimension in ∂Y. Note that this notion of rank is consistent with the intrinsic rank of parabolic subgroups when $Y = \overline{M}$.

A *ghost class* in Y with coefficients in $\widetilde{\mathbf{V}}$ is an element in $H^\bullet(Y, \widetilde{\mathbf{V}})$ with an non-zero image in

$$(4.6.1) \qquad Gh^\bullet(Y, \widetilde{\mathbf{V}}) = \frac{ker\{H^\bullet(Y, \widetilde{\mathbf{V}}) \to \bigoplus_\alpha H^\bullet(E_\alpha, \widetilde{\mathbf{V}})\}}{ker\{H^\bullet(Y, \widetilde{\mathbf{V}}) \to H^\bullet(\partial Y, \widetilde{\mathbf{V}})\}} = \frac{ker\{H^\bullet(Y, \widetilde{\mathbf{V}}) \to \bigoplus_\alpha H^\bullet(E_\alpha, \widetilde{\mathbf{V}})\}}{im\{H^\bullet(Y, \partial Y; \widetilde{\mathbf{V}}) \to H^\bullet(Y, \widetilde{\mathbf{V}})\}}$$

(it is denoted $Spect^\bullet(Y, \widetilde{\mathbf{V}})$ in [Z5]). The above is isomorphic to its image under restriction to ∂Y:

$$(4.6.2) \qquad im\{H^\bullet(Y, \widetilde{\mathbf{V}}) \to H^\bullet(\partial Y, \widetilde{\mathbf{V}})\} \cap ker\{H^\bullet(\partial Y, \widetilde{\mathbf{V}}) \to \bigoplus_\alpha H^\bullet(E_\alpha, \widetilde{\mathbf{V}})\}.$$

We call $Gh^\bullet(Y, \widetilde{\mathbf{V}})$ the *ghost group* of $(Y, \widetilde{\mathbf{V}})$. Note that the definition depends on the corner structure of Y. In the case of $Y = \overline{M}$, we have

$$(4.6.3) \qquad Gh^\bullet(\overline{M}, \widetilde{\mathbf{E}}) = \frac{ker\{H^\bullet(\overline{M}, \widetilde{\mathbf{E}}) \to \bigoplus_P H^\bullet(\overline{e'(P)}, \widetilde{\mathbf{E}})\}}{im\{H^\bullet(\overline{M}, \partial\overline{M}; \widetilde{\mathbf{E}}) \to H^\bullet(\overline{M}, \widetilde{\mathbf{E}})\}},$$

where P runs over maximal parabolics.

Next, let Y be instead a complex manifold, and D an SNC divisor on Y. Let $\mathcal{F}$ be a locally-free sheaf on Y. We assume given a partition of the set of irreducible components of D, which decomposes D into a union $\bigcup_i D_i$ (cf. (2.3)). Then a *ghost class* in Y with coefficients in $\mathcal{F}$, relative to $\{D_i\}$, is an element of $H^\bullet(Y, \mathcal{F})$ that has non-zero image in

$$(4.6.4) \qquad Gh^\bullet(Y, \{D_i\}; \mathcal{F}) = \frac{ker\{H^\bullet(Y, \mathcal{F}) \to \bigoplus_i H^\bullet(D_i, \mathcal{F} \otimes \mathcal{O}_{D_i})\}}{ker\{H^\bullet(Y, \mathcal{F}) \to H^\bullet(D, \mathcal{F} \otimes \mathcal{O}_D)\}},$$

or equivalently:

(4.6.5)

$$im\{H^\bullet(Y,\mathcal{F}) \to H^\bullet(D,\mathcal{F}\otimes\mathcal{O}_D)\} \cap ker\{H^\bullet(D,\mathcal{F}\otimes\mathcal{O}_D) \to \bigoplus_i H^\bullet(D_i,\mathcal{F}\otimes\mathcal{O}_{D_i})\}.$$

Of course, we are most interested in the case where $Y = M_\Sigma$ is a toroidal compactification of a connected component of a Shimura variety, i stands for P, and $D_i = Z_\Sigma(P)$; in that case, (4.6.4) becomes:

(4.6.6)

$$Gh^\bullet(M_\Sigma,\{Z_\Sigma(P)\};\mathcal{F}) = \frac{ker\{H^\bullet(M_\Sigma,\mathcal{F}) \to \bigoplus_P H^\bullet(Z_\Sigma(P),\mathcal{F}\otimes\mathcal{O}_{Z_\Sigma(P)})\}}{ker\{H^\bullet(M_\Sigma,\mathcal{F}) \to H^\bullet(Z_\Sigma,\mathcal{F}\otimes\mathcal{O}_{Z_\Sigma})\}}.$$

The following is an immediate consequence of what we have developed in (4.1.7) and [HZ2, (5.6)]:

(4.6.7) Proposition. *There is a natural decomposition*

$$\mathrm{Gr}^\bullet_F Gh^\bullet(\overline{M},\widetilde{\mathbf{E}}) \simeq Gh^\bullet(M_\Sigma,\{Z_\Sigma(P)\};\mathrm{Gr}^\bullet_F\mathcal{DR}(M,\widetilde{\mathbf{E}})^{can}).$$

(4.6.8) *Remark.* Contained in the above assertion is the fact that the Hodge components of a ghost class are themselves ghost classes. It follows that the existence of ghost classes for the topological cohomology imply the existence of the same for at least one of the vector bundles on the right-hand side and conversely.

Though we could state the essential content of the Hodge-theoretic criterion (4.6.14) now, we first elect to formulate (4.6.3) in terms of a single filtered mixed Hodge complex. Let

(4.6.9) $$\overline{\mathfrak{C}^\bullet} = \mathrm{Cone}\{C^\bullet(\overline{M},\widetilde{\mathbf{E}}) \to C^\bullet(\partial\overline{M},\widetilde{\mathbf{E}})\}[-1],$$

and $\widetilde{\mathfrak{C}}$ its Hodge-theoretic quasi-isomorph:

(4.6.10) $$\widetilde{\mathfrak{C}}^\bullet = \mathrm{Cone}\{C^\bullet(M_\Sigma,Rj_*\widetilde{\mathbf{E}}) \to C^\bullet_{\mathrm{dn}}(\partial M_\Sigma,\widetilde{\mathbf{E}})\}[-1].$$

We have

(4.6.11) $$H^\bullet(\widetilde{\mathfrak{C}}^\bullet) \simeq H^\bullet(\overline{\mathfrak{C}^\bullet}) \simeq H^\bullet(\overline{M},\partial\overline{M};\widetilde{\mathbf{E}})).$$

As the cone of a morphism of mixed Hodge complexes, (4.6.10) is a mixed Hodge complex in the standard way (see [E, p. 76]). We extend the simplicial filtration from the boundary complex by simply setting $\widetilde{\mathfrak{C}}^\bullet = S_1$, and likewise for $\overline{\mathfrak{C}^\bullet}$. We now assert:

(4.6.12) Proposition. *In terms of the filtered mixed Hodge complex* $(\widetilde{\mathfrak{C}}^\bullet, S)$*, the ghost group* $Gh^\bullet(\overline{M}, \widetilde{\mathbf{E}})$ *equals*

$$\frac{im\{H^\bullet((S_1/S_{-1})\widetilde{\mathfrak{C}}^\bullet) \to H^\bullet((S_1/S_0)\widetilde{\mathfrak{C}}^\bullet)\}}{im\{H^\bullet(S_1\widetilde{\mathfrak{C}}^\bullet) \to H^\bullet((S_1/S_0)\widetilde{\mathfrak{C}}^\bullet)\}}.$$

Of course, we may replace $\widetilde{\mathfrak{C}}^\bullet$ in the above by $\overline{\mathfrak{C}^\bullet}$.

The "denominator" in (4.6.12) (see (4.6.3)) is, in degree i,

$$im\{H^i_c(M, \widetilde{\mathbf{E}}) \to H^i(M, \widetilde{\mathbf{E}})\}. \qquad (4.6.13)$$

For E an irreducible representation of G of weight e, the weights occuring in the left-hand side are $\leq i+e$, and those occurring in the right-hand side are $\geq i+e$. It is immediate, then, that the mixed Hodge structure of (4.6.13) is actually pure of weight $i+e$. We obtain the following criterion for the existence of ghosts.

(4.6.14) Criterion. *Assume that* $\widetilde{\mathbf{E}}$ *is pure of weight* e*. If*

$$im\{H^i((S_1/S_{-1})\widetilde{\mathfrak{C}}^\bullet) \to H^i((S_1/S_0)\widetilde{\mathfrak{C}^\bullet})\} \simeq ker\{H^\bullet(\overline{M}, \widetilde{\mathbf{E}}) \to \bigoplus_P H^\bullet(\overline{e'(P)}, \widetilde{\mathbf{E}})\}$$

is not *pure of weight* $i+e$*, then there are ghost classes in* $H^i(\overline{M}, \widetilde{\mathbf{E}})$*.*

Remark. Of course, (4.6.14) makes no statement about the possibility that the weight $i+e$ summand of $Gh^i(\overline{M}, \widetilde{\mathbf{E}})$ be non-zero. The method of [KR] finds ghosts for $\widetilde{\mathbf{E}} = \mathbb{C}$ ($e = 0$) that are in the image of the Borel map (i.e., among the cohomology classes of the invariant forms). These are of weight i. Thus, the above criterion is independent of theirs.

There are other ways to write the ghost group, and each emphasizes a different Hodge-theoretic relation. From the point of view in (4.6.2), we have the equivalent formulas:

(4.6.15) *i)*
$$\begin{aligned} Gh^\bullet(\overline{M}, \widetilde{\mathbf{E}}) &\simeq ker\{H^\bullet(S_0\widetilde{\mathfrak{C}}^\bullet) \to H^\bullet(S_1\widetilde{\mathfrak{C}}^\bullet) \oplus H^\bullet((S_0/S_{-1})\widetilde{\mathfrak{C}}^\bullet)\} \\ &\simeq ker\{H^\bullet(\partial\overline{M}, \widetilde{\mathbf{E}}) \longrightarrow (H^\bullet(\overline{M}, \partial\overline{M}; \widetilde{\mathbf{E}})[1] \oplus \bigoplus_P H^\bullet(e'(P), \widetilde{\mathbf{E}}))\} \end{aligned}$$

ii)
$$\begin{aligned} Gh^\bullet(\overline{M}, \widetilde{\mathbf{E}}) &\simeq ker\{H^\bullet(S_0\widetilde{\mathfrak{C}}^\bullet) \to H^\bullet(S_1\widetilde{\mathfrak{C}}^\bullet)\} \cap im\{H^\bullet(S_{-1}\widetilde{\mathfrak{C}}^\bullet) \to H^\bullet(S_0\widetilde{\mathfrak{C}}^\bullet)\} \\ &\simeq ker\{H^\bullet(\partial\overline{M}, \widetilde{\mathbf{E}}) \to H^\bullet(\overline{M}, \partial\overline{M}; \widetilde{\mathbf{E}})[1]\} \cap im\{H^\bullet(S_{-1}\widetilde{\mathfrak{C}}^\bullet) \to H^\bullet(\partial\overline{M}, \widetilde{\mathbf{E}})\}. \end{aligned}$$

The apparent difference between (4.6.15) and (4.6.14) is that the former has one looking at

(4.6.16)

$$H^\bullet(\partial\overline{M}, \widetilde{\mathbf{E}}) \longrightarrow \bigoplus H^\bullet(e'(P), \widetilde{\mathbf{E}}) \simeq \bigoplus H^\bullet(M_P, \widetilde{\mathbf{H}}^\bullet(\mathfrak{n}_P, E)) \oplus \bigoplus H^\bullet(\mathfrak{n}_P, E),$$

whereas (4.6.14) emphasizes

(4.6.17) $$H^\bullet(\overline{M}, \widetilde{\mathbf{E}}) \longrightarrow \bigoplus_P H^\bullet(e'(P), \widetilde{\mathbf{E}}).$$

(4.6.18) *Remark.* The cohomology $H^i(S_{-1}\widetilde{\mathfrak{C}}^\bullet)$ is that of the codimension-two skeleton of $\partial\overline{M}$, viz.,

$$H^{i-1}(\bigcup_{r(Q)=2} \overline{e'(Q)}, \widetilde{\mathbf{E}}\ ).$$

The above union is a non-vacuous disjoint union if and only if G is of $\mathbb{Q}$-rank two, in which case it equals

$$\bigoplus_{r(Q)=2} H^{i-1}(e'(Q), \widetilde{\mathbf{E}}).$$

In effect, the method used in [Z5, App. A] to rule out ghosts for $G = GSp(4)$ (4×4 matrices), was based on (4.6.15, ii). The main step was to examine the weight structure of $H^\bullet(S_{-1}\widetilde{\mathfrak{C}}^\bullet)/\operatorname{im} H^\bullet(S_0/S_{-1}\widetilde{\mathfrak{C}}^\bullet)$, viewing $H^\bullet(S_{-1}\widetilde{\mathfrak{C}}^\bullet)$ as the iterated deleted neighborhood cohomology associated to $e'(Q) \subset \overline{e'(P)} \subset M$ for P of rank one and containing Q. Note that when $G = GSp(4)$, M_P is a modular curve (uncompactified) if $P \sim P_1$ and is a point if $P \sim P_2$.

The local system $\widetilde{\mathbf{E}}$ on M gets its weights from the weight homomorphism w_G, whereas the local systems on M_P get theirs from w_P. The two are related by the formula $w_P = m_P \cdot w_G$ (see [HZ1, (1.2.2.1)]).

5. On the comparison of Hodge structures.

It has been more than ten years since the "Zucker Conjecture" (see [Z2, §6]) was proved. We quickly recall the statement in (5.1) below. Let M^* be the Baily-Borel Satake compactification of a connected component M of a Shimura variety.

Theorem (5.1). ([L],[SS]). *Let G be the group in the Shimura datum giving rise to M, E a rational representation space for G, and $\widetilde{\mathbf{E}}$ the associated local system on M. Then there is a quasi-isomorphism*

$$\mathcal{L}^\bullet_{(2)}(M^*, \widetilde{\mathbf{E}}) \approx \mathcal{IC}^\bullet(M^*, \widetilde{\mathbf{E}}),$$

between the sheaves of L^2 $\widetilde{\mathbf{E}}$-valued differential forms and the $\widetilde{\mathbf{E}}$-valued intersection cochains on M^*.

This has the standard consequence:

Corollary (5.2). *For all k, $H^k_{(2)}(M,\widetilde{\mathbf{E}}) \simeq IH^k(M^*,\widetilde{\mathbf{E}})$.*

Upon seeing (5.2), it is natural to expect more. Recall that each side of the isomorphism comes with an associated Hodge structure. For the L^2-cohomology (the left-hand side), it comes from the L^2 harmonic forms (see [SZ]); for the intersection cohomology, it comes via Morihiko Saito's mixed Hodge modules [Sa1]. A priori, the two Hodge structures need not correspond under the isomorphism. However, there is an inevitable conjecture, the Hodge theoretic version of (5.2):

Conjecture (5.3). *The isomorphism in (5.2) is an isomorphism of Hodge structures.*

This conjecture remains unresolved. A few cases of (5.3), in which M^* has only isolated singular points, are covered in [Z4]. We remind the reader that one cannot even be sure a priori that the *Hodge numbers* (dim $H^{p,q}$) coincide.

In the direction of (5.3), we now make the following improvement on [H5, 3.3.9]:

Theorem (5.4). *For all k, the mapping*

$$r_k : H^k_{(2)}(M,\widetilde{\mathbf{E}}) \simeq IH^k(M^*,\widetilde{\mathbf{E}}) \to H^k(M,\widetilde{\mathbf{E}})$$

is morphism of mixed Hodge structures.

(5.5) *Remarks.* i) The image of r_k is the lowest non-zero weight level in the mixed Hodge structure of $H^k(M,\widetilde{\mathbf{E}})$. This can be deduced from the decomposition theorem: $IH^k(M^*,\widetilde{\mathbf{E}})$ is seen to have the same image in $H^k(M,\widetilde{\mathbf{E}})$ as $IH^k(M_\Sigma,\widetilde{\mathbf{E}})$ has, and the latter determines the lowest non-zero weight level. Thus, (5.4) asserts that the Hodge structures in (5.2) have a given common Hodge-theoretic quotient. By semi-simplicity, this quotient can be embedded as a common substructure.

ii) Let c denote the codimension of the singular locus in M^*. Included in (5.2) is the assertion that r_k is an isomorphism whenever $k < c$, and is injective for $k = c$. Thus, from (5.4) it follows that (5.3) is true in degrees $k \leq c$.

We will be referring to the following commutative diagram:

(5.6)
$$\begin{array}{ccccc} \overline{M} & \xleftarrow{k} & M & \xrightarrow{j} & M_\Sigma \\ {\scriptstyle \overline{p}}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_\Sigma} \\ M^* & = & M^* & = & M^* \end{array}$$

As in (4.1.8), let $\mathcal{DR}(M,\widetilde{\mathbf{E}})$ be the holomorphic de Rham complex of M with values in $\widetilde{\mathbf{E}}$. We have the explicit formula on M_Σ:

(5.7) $$\mathcal{DR}(M,\widetilde{\mathbf{E}})^{\mathrm{can}} = \Omega^\bullet_{M_\Sigma}(\log Z_\Sigma) \otimes \mathcal{E}^{\mathrm{can}}$$

a complex quasi-isomorphic to $Rj_*\widetilde{\mathbf{E}}$. We write $\mathcal{A}^\bullet(\mathcal{DR}(M,\widetilde{\mathbf{E}})^{\mathrm{can}})$ to denote the Dolbeault resolution of (5.7); this (or more precisely, its associated single complex) is also quasi-isomorphic to $Rj_*\widetilde{\mathbf{E}}$. We also let

$$A^\bullet = A^\bullet(\mathcal{DR}(M,\widetilde{\mathbf{E}})^{\mathrm{can}})$$

denote the complex of its global sections. Then $H^\bullet(A^\bullet) = H^\bullet(M,\widetilde{\mathbf{E}})$.

It is clear that $A^\bullet$ is a subcomplex of

$$A^\bullet_{sia} = A^\bullet_{sia}(M,\widetilde{\mathbf{E}}),$$

the global sections of $\mathcal{A}^\bullet_{sia}(M_\Sigma, Z_\Sigma)\otimes\Omega^\bullet_{M_{\Gamma,\Sigma}}(\log Z_{\Gamma,\Sigma})\otimes\mathcal{E}^{\mathrm{can}}$, which is equivalently (by (2.6.1)) the global sections of the de Rham complex of forms with moderate growth $\mathcal{A}^\bullet_{sia}(\overline{M},\widetilde{\mathbf{E}})$ on $\overline{M}$. We know that $\mathcal{A}^\bullet_{sia}(\overline{M},\widetilde{\mathbf{E}})$ is quasi-isomorphic to $Rk_*\widetilde{\mathbf{E}}$ [B2, 7.4]. There is a tautological extension of $\widetilde{\mathbf{E}}$ to a local system on $\overline{M}_\Gamma$, as the latter is a manifold-with-corners, and $Rk_*\widetilde{\mathbf{E}}$ is quasi-isomorphic to that. Thus, $H^\bullet(A^\bullet_{sia})$ is also $H^\bullet(M,\widetilde{\mathbf{E}})$, so the inclusion of $A^\bullet$ in $A^\bullet_{sia}$ is a quasi-isomorphism.

Both $A^\bullet$ and $A^\bullet_{sia}$ inherit a filtration F from the usual Hodge filtration on $A^\bullet(M,\widetilde{\mathbf{E}})$, which is given in (4.1.2). Note that $(A^\bullet, F)$ underlies the standard mixed Hodge complex for $H^\bullet(M,\widetilde{\mathbf{E}})$.

Proposition (5.8). *The inclusion of filtered complexes $(A^\bullet, F)\hookrightarrow(A^\bullet_{sia}, F)$ is a filtered quasi-isomorphism.*

Proof. We must show that for all p,

$$\mathrm{Gr}^p_F A^\bullet \to \mathrm{Gr}^p_F A^\bullet_{sia} \tag{5.8.1}$$

is a quasi-isomorphism. Because of the way the filtration F is given, we can rewrite (5.8.1) as

$$A^\bullet\{\mathrm{Gr}^p_F(\mathcal{DR}(M,\widetilde{\mathbf{E}})^{\mathrm{can}})\} \to A^\bullet_{sia}\{\mathrm{Gr}^p_F(\mathcal{DR}(M,\widetilde{\mathbf{E}})^{\mathrm{can}})\}. \tag{5.8.2}$$

Now, as we mentioned in §4, $\mathrm{Gr}^p_F(\mathcal{DR}(M,\widetilde{\mathbf{E}})^{\mathrm{can}})$ is a complex of $\mathcal{O}_{M_{\Gamma,\Sigma}}$-modules. That (5.8.2) is a quasi-isomorphism follows by applying (2.2.5) to each term.[8]

Corollary (5.9). *The filtration F on $A^\bullet_{sia}$ induces the Hodge filtration on $H^\bullet(M,\widetilde{\mathbf{E}})$.*

We can now prove (5.4). Let $\mathfrak{h}^k$ denote the space of $\widetilde{\mathbf{E}}$-valued harmonic k-forms on M. According to [BG], $\mathfrak{h}^k$ consists of forms of moderate growth. Then, the inclusion

$$\mathfrak{h}^\bullet \hookrightarrow A^\bullet_{sia}(M,\widetilde{\mathbf{E}}), \tag{5.9.1}$$

---

[8] Alternatively it is possible to invoke (4.1.12) at this point.

which is for trivial reasons compatible with F, induces the morphisms r_k. We may invoke (5.5, i) and the strictness principle for Hodge structures: a morphism of filtered vector spaces $(H, F) \to (H', F')$ in which F and F' define Hodge structures is a morphism of Hodge structures. This completes the proof of (5.4).

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