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BOUNDARY CONDITIONS FOR THE DIFFUSION SOLUTION OF COUPLED CONDUCTION-RADIATION PROBLEMS

by Marvin Goldstein and John R. Howell

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ABSTRACT

The boundary condition for use with the diffusion solution for coupled radiation and conduction energy transfer is derived. An effective slip coefficient is presented as a function of the conduction-radiation parameter. Comparison to exact numerical solutions is good for the geometry of infinite parallel black plates. In the course of the analysis, a uniformly valid asymptotic expansion for the combined conduction and radiation problem in a nongray medium in the optically thick regime, and an exact analytical solution to the radiation-conduction transport equation in the boundary regime are obtained.

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SUMMARY

The diffusion solution for radiative energy transfer in gases contains the assumption that only local conditions contribute to the radiative flux at a point. When this assumption is justified, the diffusion solution is straightforward and accurate. For problems involving coupled radiation and conduction, the diffusion solution is probably the simplest approach that can be taken to determine temperature distributions. It is well known that diffusion methods for radiation are inaccurate near bounding surfaces, however, because energy from the boundaries, rather than the local gas volume, is a major factor.

In this report, the correct boundary condition for the combined conduction- and radiation-diffusion solution is derived. The condition at the boundary is in the form of an effective temperature discontinuity that corrects for the error introduced in the diffusion solution by the presence of the boundary. This condition is a mathematical artifice only, because physically no discontinuity can exist in the presence of conduction. The effective slip is derived by the use of matched asymptotic expansions of the exact equation of transfer. The entire temperature distribution in the gas can be determined from the diffusion solution with a slip boundary condition and a correction near the wall found from the linearized exact equations. The linearized equations have been solved exactly by the methods of singular integral equations. Often the diffusion solution itself will provide the portion of the temperature distribution that is of interest.

INTRODUCTION

It is well known that the mathematical formulation of energy-transfer problems in enclosures containing absorbing-emitting gases in which radiation is the only mode considered can lead to a prediction of a temperature discontinuity at the gas-boundary interface (refs. 1 to 3). The discontinuity arises because a gas element immediately adjacent to the boundary receives only a portion of the total absorbed energy from the boundary, while the remainder comes from surrounding gas and boundaries at different temperatures. The element then reaches an equilibrium temperature somewhere between the temperature of the adjacent boundary and the surrounding gas. This equilibrium temperature may be quite far from the boundary temperature.

Heaslet and Warming (ref. 2) have presented predictions of the slip in the absence of conduction (radiative equilibrium) for a gray gas between infinite parallel plates. These authors used Chandrasekhar's tabulated X- and Y-functions (ref. 3) to provide useful exact solutions.

Probstein (ref. 4) and Deissler (ref. 5) included the temperature discontinuity in formulating boundary conditions for use with diffusion solutions of radiative transfer.

Although the temperature slip condition has proved useful in radiative transfer, it must be realized that an actual discontinuity in temperature arises only mathematically. Physically, some conduction of energy between the gas and the boundary will occur, except perhaps in certain extreme situations, and any conduction will remove the discontinuity. (In extreme cases, such as for an extremely rarified gas, other assumptions used in the equations of radiative transfer also become invalid, chiefly the assumption of local thermodynamic equilibrium.) Thus, in any radiation problem involving a diffusion analysis or the typical transport analyses, no physical temperature discontinuity can exist. However, for small conduction, very strong temperature gradients might be expected near the boundary.

The purpose of this report is to analyze the region of the gas near a bounding surface in which these strong gradients exist, and to determine the effect of thermal conduction on the gradients. An effective slip coefficient for use in the steady-state diffusion solution for combined radiation and conduction problems is then derived for use with hot or cold boundaries. The utilization of the coefficients presented herein effectively extends the usefulness of the diffusion solution by specifying accurate boundary conditions for its formulation.

SYMBOLS

A_ν	function arising from separation of variable solution (eqs. (76) and (77))
a	gas linear absorption coefficient
B	intensity of emission
B_ν	function given by eq. (87)
b	intensity of emission in inner region
C_ν	function defined by eq. (83)
C_1, C_2, C_3, C_4	integration constants

D characteristic length

E_v function defined in eq. (103)

G function defined by eq. (97)

g function defined by eq. (98)

H_v function arising from separation of variables solution for h (eq. (76))

h $\pi i_c^{(1)}/T_w^{*4}$

I radiation intensity

i intensity in inner region when subscripted

K constant defined by eq. (31)

k_v function defined in eq. (103)

L curve defined in fig. 2

N radiation-conduction parameter, $\bar{a}\lambda_k/4\sigma T_r^3$

q energy flux

T absolute temperature

t temperature in inner region

u dummy variable of integration

v separation variable

X function defined by eqs. (101)

x distance from bounding surface

x^+ dimensionless coordinate, $x^*/\delta(\epsilon)$

Z complex variable for eqs. (111) and those that follow

α function defined by eq. (29)

β defined by eq. (124a)

Γ dimensionless energy flux, $q/\sigma\left(T_{w1}^4 - T_{w2}^4\right)$

γ ratio of wall temperatures, $T_{w,2}^*/T_{w,1}^*$

Δ temperature difference by extrapolation of diffusion solution (fig. 1)

$\delta(\epsilon)$ intermediate or gage parameter

ϵ expansion parameter, $1/\bar{a}D$

Θ_v function arising from separation of variables solution for $t_{cl}^{(1)}$ (eq. (77))

θ angle measured from normal of boundary

λ_k conductivity of gas

μ $\cos\theta$

ξ temperature gradient at $x^* \to 0$; $dT^*/dx^*|_{x^*\to 0}$

σ Stefan-Boltzmann constant

τ optical thickness, $\bar{a}x$

Φ function defined by eq. (93)

Ψ radiation slip coefficient, $\sigma\left[T_w^4 - T_g^4(x = 0)\right]/q_R$

Ω solid angle

ω frequency

Subscripts:

c complementary solution

D diffusion solution

g gas

k kinetic conduction

R radiation

r reference

w surface or wall temperature

ω	frequency dependent	+, -	approaching boundary from inside or outside domain
1, 2	at surface 1 or 2	*	dimensionless property
Superscripts:		—	mean value
(0), (1), (2)	terms in expansion		

ANALYSIS

Consider a black wall at temperature T_w that bounds a gas extending far from the wall (fig. 1). The mean free path for radiation $1/a_\omega$ is assumed to be much longer at all frequencies ω than the gas kinetic mean free path, so that the gas can be assumed in local thermodynamic equilibrium. Scattering in the gas is neglected.

The analysis, briefly outlined, is as follows: At regions far from the wall (i. e., for $a_\omega x \to \infty$), a diffusion solution becomes exact to order $(1/Da_\omega)^2$. At regions very near

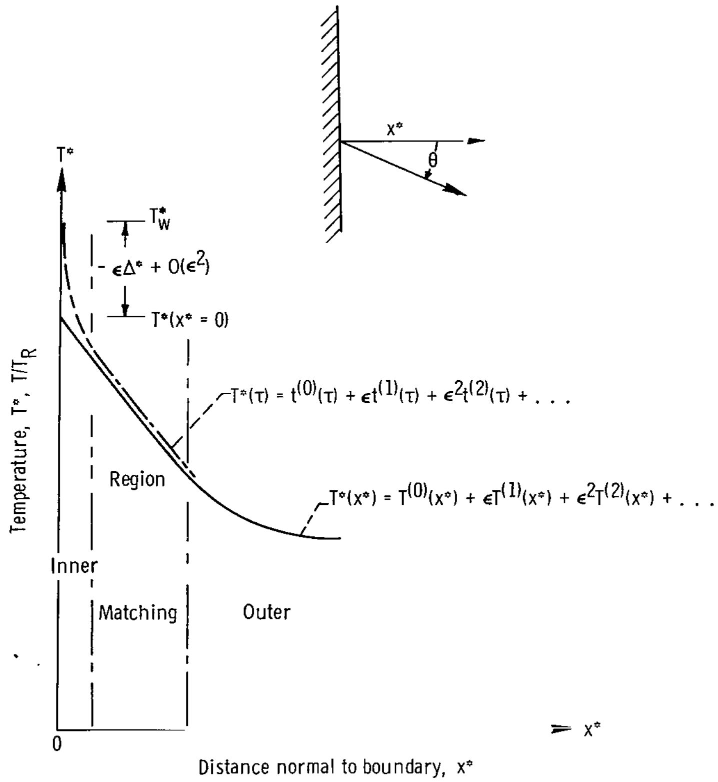

Figure 1. - Definition of regions in gas near boundary.

the wall ($a_\omega x \sim 1$), a linearized solution for the gas is shown to hold. The solutions for these two regions are then matched for intermediate distances from the wall. Up to terms of order $(1/Da_\omega)^2$ this expansion agrees with the usual diffusion solutions in regions not too near solid boundaries. On the basis of this correct expansion, the boundary conditions for the solutions to the diffusion equation must be modified by introducing an apparent discontinuity in temperature Δ at $x = 0$ to account for the boundary region where the diffusion solution breaks down. The values of Δ depend on the usual conduction-radiation parameter N and on the limiting value of the outer solution temperature gradient.

The problem is now examined in detail. To determine the slip coefficient to first order, the only changes necessary to consider are those in the direction x normal to the wall. The results will be valid, however, for two- and three-dimensional problems.

At any arbitrary position in the medium, the equation of transfer for radiation (ref. 3) can be written (scattering neglected) as

$$\frac{\cos \theta}{a_\omega} \frac{\partial I_\omega}{\partial x} = B_\omega(T) - I_\omega \tag{1a}$$

where I_ω is the local spectral radiation intensity, and B_ω is the blackbody spectral intensity. The equation denoting the conservation of energy between the conduction and radiation processes is, for conditions of steady state and no flow,

$$\frac{d}{dx} \lambda_k \frac{dT}{dx} = \frac{d}{dx} \int_\Omega \int_\omega \cos \theta \, I_\omega \, d\omega \, d\Omega \tag{1b}$$

At a solid black boundary,

$$\left.\begin{aligned} T &= T_w \\ I_\omega &= B_\omega(T_w) \end{aligned}\right\} \quad 0 \leq \cos \theta \leq 1 \tag{2}$$

These equations can be nondimensionalized with the introduction of

$$\mu = \cos\theta$$

$$a^*_\omega = \frac{a_\omega}{\bar{a}}$$

$$x^* = \frac{x}{D}$$

$$T^* = \frac{T}{T_r}$$

$$I^*_\omega = \frac{I_\omega}{4\sigma T_r^4}$$

$$B^*_\omega = \frac{B_\omega}{4\sigma T_r^4}$$

$$N = \frac{\bar{a}\lambda_k}{4\sigma T_r^3}$$

where $\bar{a}$, D, and T_r are arbitrarily chosen reference values of the absorption coefficient, a characteristic length, and a temperature, respectively. These reference values are chosen in such a way as to make the nondimensional terms of order one (at least in regions far from the wall). Typically, $\bar{a}$ would be an appropriate frequency-averaged absorption coefficient, T_r an average radiating temperature, and D a length of the same order as that over which the temperature changes significantly in the outer region. The general equations governing the conduction-radiation process then become

$$\frac{\mu}{a^*_\omega D\bar{a}} \frac{\partial I^*_\omega}{\partial x^*} = B^*_\omega - I^*_\omega \qquad (3)$$

and

$$\frac{1}{\bar{a}D} \frac{d}{dx^*} N \frac{dT^*}{dx^*} = \frac{d}{dx^*} \int_{\omega=0}^{\infty} \int_{\Omega=4\pi} \mu I_{\omega}^* \, d\Omega \, d\omega \tag{4}$$

Expansion for Outer Region

For regions far from the wall, the analysis proceeds as follows: Let the quantity ϵ be defined as

$$\epsilon = \frac{1}{\bar{a}D}$$

By definition, an optically thick gas is one in which ϵ is small as compared with one. Now equations (3) and (4) can be written

$$\epsilon \frac{\mu}{a_{\omega}^*} \frac{\partial I_{\omega}^*}{\partial x^*} = B_{\omega}^* - I_{\omega}^* \tag{5}$$

$$\epsilon \frac{d}{dx^*} N \frac{dT^*}{dx^*} = \frac{d}{dx^*} \int_{\omega=0}^{\infty} \int_{\Omega=4\pi} \mu I_{\omega}^* \, d\omega \, d\Omega \tag{6}$$

For $\epsilon << 1$, T^*, I_{ω}^*, and B_{ω}^* can be expanded:

$$I_{\omega}^* = I_{\omega}^{(0)} + \epsilon I_{\omega}^{(1)} + \epsilon^2 I_{\omega}^{(2)} + \ldots \tag{7}$$

$$T^*(x^*) = T^{(0)}(x^*) + \epsilon T^{(1)}(x^*) + \epsilon^2 T^{(2)}(x^*) + \ldots \tag{8}$$

$$B_{\omega}^*(T^*) = B_{\omega}^{(0)} + \epsilon B_{\omega}^{(1)} + \epsilon^2 B_{\omega}^{(2)} + \ldots \tag{9}$$

The relation for $B_{\omega}^*(T^*)$ can also be expanded in a Taylor series about $T^{(0)}$:

$$B_{\omega}^*(T^*) = B_{\omega}^*(T^{(0)}) + \frac{\partial B_{\omega}^*(T^{(0)})}{\partial T^{(0)}} (T^* - T^{(0)}) + \frac{1}{2!} \frac{\partial^2 B_{\omega}^*(T^{(0)})}{\partial (T^{(0)})^2} (T^* - T^{(0)})^2 + \ldots \tag{10}$$

Then, substituting equation (8) into equation (10) and gathering terms yield

$$B^*_\omega(T^*) = B^*_\omega(T^{(0)}) + \epsilon \frac{\partial B^*_\omega(T^{(0)})}{\partial T^{(0)}} T^{(1)} + \epsilon^2 \left[\frac{\partial B^*_\omega(T^{(0)})}{\partial T^{(0)}} T^{(2)} + \frac{1}{2!} \frac{\partial^2 B^*_\omega(T^{(0)})}{\partial (T^{(0)})^2} (T^{(1)})^2 \right] + \ldots \tag{11}$$

Comparison of equations (9) and (11) provides the relations

$$B^{(0)}_\omega = B^*_\omega(T^{(0)}) \tag{12a}$$

$$B^{(1)}_\omega = \frac{\partial B^*_\omega(T^{(0)})}{\partial T^{(0)}} T^{(1)} \tag{12b}$$

$$B^{(2)}_\omega = \frac{\partial B^*_\omega(T^{(0)})}{\partial T^{(0)}} T^{(2)} + \frac{1}{2!} \frac{\partial^2 B^*_\omega(T^{(0)})}{\partial (T^{(0)})^2} (T^{(1)})^2 \tag{12c}$$

. . .

Equations (7) to (9) and (12) can now be substituted into equation (5). This substitution and equating like powers of ϵ result in

$$I^{(0)}_\omega = B^{(0)}_\omega \tag{13a}$$

$$I^{(1)}_\omega = B^{(1)}_\omega - \frac{\mu}{a^*_\omega} \frac{\partial B^{(0)}_\omega}{\partial x^*} \tag{13b}$$

$$I^{(2)}_\omega = B^{(2)}_\omega - \frac{\mu}{a^*_\omega} \frac{\partial I^{(1)}_\omega}{\partial x^*} \tag{13c}$$

. . .

$$I^{(n)}_\omega = B^{(n)}_\omega - \frac{\mu}{a^*_\omega} \frac{\partial I^{(n-1)}_\omega}{\partial x^*} \tag{13n}$$

. . .

Similarly, substituting equations (7) to (9) and (12) into equation (6) and equating like powers of ϵ give (after use of eq. (13))

$$\frac{d}{dx^*} N \frac{dT^{(0)}}{dx^*} = -\frac{d}{dx^*} \int_{\omega=0}^{\infty} \frac{d\omega}{a_\omega^*} \frac{\partial B_\omega^{(0)}}{\partial x^*} \int_{\Omega=4\pi} \mu^2 \, d\Omega$$

$$= -\frac{d^2}{dx^{*2}} \int_0^{\infty} \frac{B_\omega^{(0)} \, d\omega}{a_\omega^*} \int_{\Omega=4\pi} \mu^2 \, d\Omega \tag{14a}$$

$$\frac{d}{dx^*} N \frac{dT^{(1)}}{dx^*} = -\frac{d^2}{dx^{*2}} T^{(1)} \int_0^{\infty} \frac{1}{a_\omega^*} \frac{\partial B_\omega^{(0)}}{\partial T^{(0)}} \, d\omega \int_{\Omega=4\pi} \mu^2 \, d\Omega \tag{14b}$$

Note that the zeroth power terms make no contribution because, since $B_\omega^{(0)}$ is independent of μ, $\int_{\Omega=4\pi} B_\omega^{(0)} \mu \, d\Omega = 0$. Now define a dimensionless radiative conductivity as

$$\lambda^*(T^*) \equiv \int_0^{\infty} \frac{1}{a_\omega^*} \frac{\partial B_\omega^*}{\partial T^*} \, d\omega \int_{\Omega=4\pi} \mu^2 \, d\Omega \tag{15}$$

Then equation (14) becomes

$$\frac{d}{dx^*} \left[N \frac{dT^{(0)}}{dx^*} + \lambda^*(T^{(0)}) \frac{dT^{(0)}}{dx^*} \right] = 0 \tag{16a}$$

$$\frac{d}{dx^*} \left\{ N \frac{dT^{(1)}}{dx^*} + \frac{d}{dx^*} \left[\lambda^*(T^{(0)}) T^{(1)} \right] \right\} = 0 \tag{16b}$$

The term $\lambda^*(T^*)$ can be expanded about $T^{(0)}$ as was done for $B_\omega^*(T^*)$ in equations (9) to (12), and the result is the expansion

$$\lambda^*(T^*) = \lambda^*(T^{(0)}) + \epsilon \frac{d}{dT^{(0)}} [\lambda^*(T^{(0)})]T^{(1)} + O(\epsilon^2)$$

As before,

$$\lambda^*(T^*) = \lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \ldots$$

Hence,

$$\lambda^{(0)} = \lambda^*(T^{(0)}) \tag{16c}$$

$$\lambda^{(1)} = \frac{d\lambda^*(T^{(0)})}{dT^{(0)}} T^{(1)} \tag{16d}$$

Multiplying equation (16b) by ϵ and adding the result to equation (16a) give

$$\frac{d}{dx^*}\left\{N \frac{d}{dx^*}(T^{(0)} + \epsilon T^{(1)}) + \lambda^*(T^{(0)}) \frac{d}{dx^*}(T^{(0)} + \epsilon T^{(1)}) + \epsilon T^{(1)} \frac{d}{dx^*}\left[\lambda^*(T^{(0)})\right]\right\} = 0 \tag{17a}$$

Substituting the relation for $\lambda^*(T^*)$ into equation (17a) and using equation (16c) and (16d) result in

$$\frac{d}{dx^*}\left[N \frac{d}{dx^*}(T^{(0)} + \epsilon T^{(1)}) + (\lambda^{(0)} + \epsilon\lambda^{(1)}) \frac{d}{dx^*}(T^{(0)} + \epsilon T^{(1)}) + O(\epsilon^2)\right] = 0 \tag{17b}$$

or using equation (8) results in

$$\frac{d}{dx^*}\left[N \frac{dT^*}{dx^*} + \lambda^*(T^*) \frac{dT^*}{dx^*}\right] + O(\epsilon^2) = 0 \tag{17c}$$

which is the same expression that is found by the usual diffusion approximation. The diffusion result can be obtained by expanding I^* but not T^*, and then neglecting terms of order ϵ^2. Such a procedure, however, does not allow the matching of inner and outer expansions to all orders in ϵ to obtain a uniformly valid expansion, as is now shown.

However, equation (17c) is shown to be correct to order ϵ^2 in regions not too close to solid boundaries. Equations (13) and (16) are the governing equations for the outer region.

Expansion for the Inner Region

Because the small parameter ϵ multiplies the highest order derivative appearing in equations (5) and (6), the analysis deals with a singular perturbation problem wherein the formal asymptotic expressions (7) to (9) cannot satisfy all the boundary conditions of the original problem. This situation indicates the presence of a "boundary layer" in the region near the wall within which steep gradients are expected (i.e., large values of $\partial I_\omega^*/\partial x^*$ and dT^*/dx^* in eqs. (5) and (6)). Thus the formal asymptotic expansion (eq. (17)) must break down in the boundary region, although it is expected to remain valid in the region far from the boundary.

To obtain a valid expansion in the boundary region, the procedure continues with the introduction of a "stretched" coordinate (or optical depth) τ:

$$\tau = \frac{x^*}{\epsilon} = \bar{a}x \tag{18}$$

Introducing this new variable into equations (3) and (4) and maintaining the other dimensionless quantities the same as in the outer expansion give

$$\frac{\mu}{a_\omega^*} \frac{\partial I_\omega^*}{\partial \tau} = B_\omega^* - I_\omega^* \tag{19}$$

and

$$\frac{d}{d\tau} N \frac{dT^*}{d\tau} = \frac{d}{d\tau} \int_{\omega=0}^{\infty} \int_{\Omega=4\pi} \mu I_\omega^* \, d\Omega \, d\omega \tag{20}$$

All terms in equations (19) and (20) are expected to be of order one in the inner region, since the highest order derivatives are no longer multiplied by ϵ. When the coordinate is stretched, the gradients in the inner region are of order one. Therefore, an attempt can be made to expand the solutions to these equations in terms of the small parameter ϵ as

$$T^*(\tau) = t^{(0)}(\tau) + \epsilon t^{(1)}(\tau) + \epsilon^2 t^{(2)}(\tau) + \ldots \tag{21}$$

$$I^*_\omega = i^{(0)}_\omega + \epsilon i^{(1)}_\omega + \epsilon^2 i^{(2)}_\omega + \ldots \tag{22}$$

$$B^*_\omega = b^{(0)}_\omega + \epsilon b^{(1)}_\omega + \epsilon^2 b^{(2)}_\omega + \ldots \tag{23}$$

Expanding B^*_ω in a Taylor series about $t^{(0)}$ and substituting equation (21), and then equating like powers of ϵ between the resulting equation and equation (23) give

$$b^{(0)}_\omega = B^*_\omega(t^{(0)}) \tag{24a}$$

$$b^{(1)}_\omega = \left[\frac{\partial B^*_\omega(t)}{\partial t}\right]_{t=t^{(0)}} t^{(1)} \tag{24b}$$

$$b^{(2)}_\omega = t^{(2)}\left[\frac{\partial B^*_\omega(t)}{\partial t}\right]_{t=t^{(0)}} + \frac{\left(t^{(1)}\right)^2}{2!}\left[\frac{\partial^2 B^*_\omega(t)}{\partial (t)^2}\right]_{t=t^{(0)}} \tag{24c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Substituting equations (21) to (23) and (24) into equations (19) and (20) gives the linearized equations for the inner region as

$$\frac{\mu}{a^*_\omega}\frac{\partial i^{(n)}}{\partial \tau} = b^{(n)}_\omega - i^{(n)}_\omega \qquad n = 0,\ 1,\ 2,\ \ldots \tag{25}$$

$$\frac{d}{d\tau} N \frac{dt^{(n)}}{d\tau} = \frac{d}{d\tau}\int_{\omega=0}^{\infty}\int_{\Omega=4\pi} \mu i^{(n)}_\omega \, d\Omega \, d\omega \qquad n = 0,\ 1,\ 2,\ \ldots \tag{26}$$

The boundary conditions (eq. (2)) at $\tau = 0$ must now be satisfied by the solutions to equations (25) and (26). By equating like powers of ϵ in equations (21) and (22) at $\tau = 0$:

$$t^{(0)} = T_w^* \tag{27a}$$

$$t^{(n)} = 0 \qquad n > 0 \tag{27b}$$

$$i_\omega^{(0)} = B_\omega^*(T_w) \qquad 0 \leq \mu \leq 1 \tag{27c}$$

$$i_\omega^{(n)} = 0;\ n > 0 \qquad 0 \leq \mu \leq 1 \tag{27d}$$

Note that the conditions given by equations (27) are not sufficient to determine completely the solution of equations (25) and (26), and the conditions at large τ must be further specified.

Matching of Inner and Outer Solutions

The formal expansions for the outer region (eqs. (7) to (9)) and those for the inner region (eqs. (21) to (23)) are still not completely determined. At this point, boundary conditions for the outer solutions cannot be specified because, as already noted, they break down near the boundary. The inner solutions are not specified completely because their behavior for large τ has not been determined. This indeterminancy is removed by the requirement that the full expansion (inner and outer) be uniformly valid. In the present context, this uniformity means that, in a certain sense to be specified more precisely, the inner and outer expansions must merge smoothly in some intermediate region, for example, $1/\bar{a} << x << D$. In this region, the procedure requires that both the inner expansions (eqs. (21) to (23)) and the outer expansions (eqs. (7) to (9)) be valid asymptotic representations of the true solutions.

For convenience, two order symbols are defined. First,

$$\varphi(z) = O[\Psi(z)]$$

means that there exists a positive $\mathscr{K}$ such that

$$\lim_{z \to 0} \frac{|\varphi|}{|\Psi|} < \mathscr{K}$$

and second,

$$\varphi(z) = o[\Psi(z)]$$

means that

$$\lim_{z \to 0} \frac{|\varphi|}{|\Psi|} = 0$$

The matching of the inner and outer expansions proceeds. It is well known (ref. 6) that to accomplish the matching a sufficient requirement is that an intermediate variable

$$x^+ = \frac{x^*}{\delta(\epsilon)}$$

exist with $\epsilon << \delta(\epsilon) << 1$ and $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ (e. g. , $\delta(\epsilon) = \epsilon^{3/4}$) such that, if j terms are retained in the expansions, the inner and outer expansions are expressed in terms of x^+ and all terms of $o(\epsilon^N)$ are neglected with x^+ held fixed, then the two expansions become identical. That is, it is required that

$$\lim_{\epsilon \to 0} \left\{ \frac{T^{(0)}[x^+\delta(\epsilon)] + \ldots \epsilon^j T^{(j)}[x^+\delta(\epsilon)] - t^{(0)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] - \ldots - \epsilon^j t^{(j)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right]}{\epsilon^j} \right\} = 0 \tag{28}$$

with similar expressions for the expansions of I^*_ω and B^*_ω. The conditions on the inner and outer expansions must be found that will make them satisfy equation (28).

First, define

$$\alpha^*(T^{(0)}) \equiv \int \lambda^*(T^{(0)})dT^{(0)} \tag{29}$$

Then, substituting equation (29) into equation (16a) after integrating gives, for the outer region,

$$NT^{(0)} + \alpha^*(T^{(0)}) = C_1 + C_2x^* \tag{30}$$

From this equation, it is clear that

$$\lim_{\epsilon \to 0} T^{(0)}[x^+\delta(\epsilon)] = \text{constant} \equiv K = T^{(0)}(x^* = 0) \tag{31}$$

From equations (12) and (13), it follows that

$$\lim_{\epsilon \to 0} I_\omega^{(0)}[x^+\delta(\epsilon)] = B_\omega^*(K) \tag{32a}$$

and

$$\lim_{\epsilon \to 0} B_\omega^{(0)}[x^+\delta(\epsilon)] = B_\omega^*(K) \tag{32b}$$

When noting equations (21) to (23) and that as $\epsilon \to 0$, with x^+ fixed, $\tau \to \infty$, it follows that for the zero-order inner and outer expansions to match,

$$\lim_{\epsilon \to 0} t^{(0)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] = \lim_{\tau \to \infty} t^{(0)}(\tau) = K \tag{33a}$$

$$\lim_{\epsilon \to 0} i_\omega^{(0)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] = \lim_{\tau \to \infty} i_\omega^{(0)}(\tau) = B_\omega^*(K) \tag{33b}$$

$$\lim_{\epsilon \to 0} b_\omega^{(0)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] = \lim_{\tau \to \infty} b_\omega^{(0)}(\tau) = B_\omega^*(K) \tag{33c}$$

To zero order, the inner equations (25) and (26) are

$$\frac{\mu}{a_\omega^*} \frac{\partial i_\omega^{(0)}}{\partial \tau} = b_\omega^{(0)} - i_\omega^{(0)} \tag{34}$$

and

$$\frac{d}{d\tau} N \frac{dt^{(0)}}{d\tau} = \frac{d}{d\tau} \int_{\omega=0}^{\infty} \int_{\Omega=4\pi} \mu i_\omega^{(0)} \, d\Omega \, d\omega \tag{35}$$

The zero-order inner equations (34) and (35), with boundary conditions (33) for $\tau \to \infty$ and boundary conditions (27a) and (27c) for $\tau = 0$, have the solutions

$$K = T_w^* \tag{36a}$$

$$t^{(0)} = T_w^* \tag{36b}$$

$$b_\omega^{(0)} = B_\omega^*(T_w^*) \tag{36c}$$

$$i_\omega^{(0)} = B_\omega^*(T_w^*) \qquad -1 \le \mu \le 1 \tag{36d}$$

To match the first-order terms requires the expanding of $\alpha^*(T^{(0)})$ in a Taylor series around K as

$$\alpha^*\left\{T^{(0)}[x^+\delta(\epsilon)]\right\} = \alpha^*(K) + (T^{(0)} - K)\left[\frac{d\alpha^*}{dT^{(0)}}\right]_{T^{(0)}=K} + O\left[(T^{(0)} - K)^2\right] \tag{37}$$

Note that, at $x^* = 0$, equation (30) becomes

$$NK + \alpha^*(K) = C_1 \tag{38a}$$

and also

$$\left[\frac{d\alpha^*}{dT^{(0)}}\right]_{T^{(0)}=K} = \lambda^*(K) \tag{38b}$$

Substituting equation (37) into equation (30), using equations (38) and rearranging terms give

$$T^{(0)} - K = \frac{C_2\delta(\epsilon)x^+}{N + \lambda^*(K)} + O\left[(T^{(0)} - K)^2\right] \tag{39}$$

Squaring equation (39) shows that

$$(T^{(0)} - K)^2 = O[\delta^2(\epsilon)] \tag{40}$$

Now, $\delta(\epsilon)$ is chosen such that

$$\lim_{\epsilon \to 0}\left[\frac{\delta^2(\epsilon)}{\epsilon}\right] = 0 \tag{41}$$

(For example, $\delta(\epsilon) = \epsilon^{\gamma}$ where $1/2 < \gamma < 1$ satisfies this relation.) In view of equations (40) and (41), equation (39) becomes

$$T^{(0)} - K = \frac{C_2 \delta(\epsilon) x^+}{N + \lambda^*(K)} + o(\epsilon) \tag{42}$$

After equation (30) is differentiated once and evaluated at $x^* = 0$,

$$C_2 = [N + \lambda^*(K)]\left[\frac{dT^{(0)}}{dx^*}\right]_{x^*=0} \tag{43}$$

Substituting equation (43) into equation (42) gives

$$\begin{aligned} T^{(0)} - K &= \delta(\epsilon) x^+ \left[\frac{dT^{(0)}}{dx^*}\right]_{x^*=0} + o(\epsilon) \\ &= \delta(\epsilon) x^+ \left[\frac{dT^*}{dx^*}\right]_{x^*=0} + o(\epsilon) \end{aligned} \tag{44}$$

Equation (16b) is integrated to obtain

$$T^{(1)} = \frac{C_3}{N + \lambda^*(T^{(0)})} + \frac{C_4 x^*}{N + \lambda^*(T^{(0)})}$$

When this is rewritten in terms of the intermediate variable and a new constant Δ^* is defined by

$$\Delta^* = T^{(1)}(x^* = 0) \tag{45}$$

the following equation is obtained:

$$T^{(1)}[\delta(\epsilon)x^+] \equiv \frac{C_3}{N + \lambda^*(T^{(0)})} + O[\delta(\epsilon)]$$

$$\equiv \Delta^* + O[\delta(\epsilon)] \tag{46}$$

The limit equation (28) is now written for $j = 1$ and equations (44), (46), and (36b) are substituted for $T^{(0)}$, $T^{(1)}$, and $t^{(0)}$, respectively:

$$\lim_{\epsilon \to 0} \left\{ \frac{T^{(0)}[x^+\delta(\epsilon)] + \epsilon T^{(1)}[x^+\delta(\epsilon)] - t^{(0)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] - \epsilon t^{(1)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right]}{\epsilon} \right\}$$

$$= \lim_{\epsilon \to 0} \left(\frac{\left[K + \delta(\epsilon)x^+ \left.\frac{dT^{(0)}}{dx^*}\right|_{x^*=0} + o(\epsilon)\right] + \epsilon\left\{\Delta^* + O[\delta(\epsilon)]\right\} - T_W^* - \epsilon t^{(1)}\left[\frac{x^*\delta(\epsilon)}{\epsilon}\right]}{\epsilon} \right) = 0 \tag{47}$$

Since $K = T_W^*$, equation (47) becomes

$$\lim_{\epsilon \to 0} \left\{ -\frac{\delta(\epsilon)x^+}{\epsilon} \left.\frac{dT^{(0)}}{dx^*}\right|_{x^*=0} - \Delta^* + t^{(1)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] \right\} = \lim_{\epsilon \to 0} \left\{ \frac{o(\epsilon)}{\epsilon} + O[\delta(\epsilon)] \right\} = 0 \tag{48}$$

Because $\tau = x^+\delta(\epsilon)/\epsilon$ and

$$\lim_{\substack{\epsilon \to 0 \\ x^+ = \text{constant}}} (\tau) = \lim_{\epsilon \to 0} \left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] = \infty \tag{49}$$

it can be concluded from equation (48) that

$$\lim_{\tau \to \infty} t^{(1)}(\tau) \sim \Delta^* + \tau\left(\frac{dT^*}{dx^*}\right)_{x^*=0} \tag{50}$$

if the outer solution is to be correctly matched. Equation (50) then provides the boundary condition of the inner solution for $t^{(1)}$ at large τ.

To determine the behavior of $i^{(1)}$ at large τ, expand equation (13a) in a Taylor series about $T^{(0)} = T_w^*$:

$$I_\omega^{(0)}[x^+\delta(\epsilon)] = B_\omega^{(0)}[x^+\delta(\epsilon)] = B^{(0)}(K) + (T^{(0)} - K)\left(\frac{\partial B^{(0)}}{\partial T^{(0)}}\right)_{T^{(0)}=K} + O\left[(T^{(0)} - K)^2\right] \tag{51}$$

Then, the substitution of equations (40) and (44) into equation (51) gives (note that $K = T_w^*$)

$$I_\omega^{(0)}[x^+\delta(\epsilon)] = B_\omega^{(0)}(T_w) + \left(\frac{\partial B^{(0)}}{\partial T^{(0)}}\right)_{T^{(0)}=T_w^*} \cdot \delta(\epsilon)x^+ \left.\frac{dT^*}{dx^*}\right|_{x^*=0} + o(\epsilon) \tag{52}$$

Substituting equation (12b) into equation (13b) gives

$$I_\omega^{(1)}[x^+\delta(\epsilon)] = \frac{\partial B_\omega^{(0)}}{\partial T^{(0)}} T^{(1)}[x^+\delta(\epsilon)] - \frac{\mu}{a_\omega^*} \frac{\partial B_\omega^{(0)}}{\partial x^*} \tag{53}$$

or noting that

$$\frac{\partial B_\omega^{(0)}}{\partial T^{(0)}} [x^+\delta(\epsilon)] = \left.\frac{\partial B^{(0)}}{\partial T^{(0)}}\right|_{T^{(0)}=T_w^*} + O[\delta(\epsilon)]$$

$$\frac{dT^{(0)}}{dx^*} [x^+\delta(\epsilon)] = \left.\frac{dT^{(0)}}{dx^*}\right|_{x^*=0} + O[\delta(\epsilon)]$$

$$= \left.\frac{dT^*}{dx^*}\right|_{x^*=0} + O[\delta(\epsilon)]$$

and then substituting equation (46) into equation (53) to eliminate $T^{(1)}$ give

$$I_\omega^{(1)}[x^+\delta(\epsilon)] = \Delta^*\left[\frac{\partial B_\omega^{(0)}}{\partial T^{(0)}}\right]_{T^{(0)}=T_w^*} - \frac{\mu}{a_\omega^*}\left.\frac{\partial B_\omega^{(0)}}{\partial T^{(0)}}\right|_{T^{(0)}=T_w^*} \cdot \left.\frac{dT^*}{dx^*}\right|_{x^*=0} + O[\delta(\epsilon)] \tag{54}$$

The matching condition for intensity is analogous to that for temperature (eq. (47)). Substituting equations (54), (52), and (36d) into the matching condition gives

$$\lim_{\epsilon\to 0}\left\{\frac{I_\omega^{(0)}[x^+\delta(\epsilon)] + \epsilon I_\omega^{(1)}[x^+\delta(\epsilon)] - i_\omega^{(0)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right] - \epsilon i_\omega^{(1)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right]}{\epsilon}\right\}$$

$$= \lim_{\epsilon\to 0}\left\{\frac{B_\omega^{(0)}(K) + \left(\frac{\partial B^{(0)}}{\partial T^{(0)}}\right)_{T^{(0)}=T_w^*} \cdot \delta(\epsilon)x^+ \left.\frac{dT^*}{dx^*}\right|_{x^*=0} + o(\epsilon) + \epsilon\,\Delta^*\left[\frac{\partial B_\omega^{(0)}}{\partial T^{(0)}}\right]_{T^{(0)}=T_w^*} - \frac{\epsilon\mu}{a_\omega^*}\left.\frac{\partial B_\omega^{(0)}}{\partial T^{(0)}}\right|_{T^{(0)}=T_w^*} \cdot \left.\frac{dT^*}{dx^*}\right|_{x^*=0} + \epsilon O[\delta(\epsilon)] - B_\omega^*(T_w^*) - \epsilon i_\omega^{(1)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right]}{\epsilon}\right\} = 0 \tag{55}$$

Note from equation (12a) that $B_\omega^*(T) = B_\omega^{(0)}(T)$; thus, equation (55) becomes

$$\lim_{\epsilon\to 0}\left\{\left.\frac{\partial B^{(0)}}{\partial T^{(0)}}\right|_{T_0=T_w^*}\left[\frac{\delta(\epsilon)x^+}{\epsilon}\left.\frac{dT^*}{dx^*}\right|_{x^*=0} + \Delta^* - \frac{\mu}{a_\omega^*}\left.\frac{dT^*}{dx^*}\right|_{x^*=0}\right] - i_\omega^{(1)}\left[\frac{x^+\delta(\epsilon)}{\epsilon}\right]\right\} = \lim_{\epsilon\to 0}\left\{\frac{o(\epsilon)}{\epsilon} + O[\delta(\epsilon)]\right\} = 0 \tag{56}$$

or, finally, it can be concluded that

$$\lim_{\tau\to\infty} i_\omega^{(1)}(\tau) \sim \left[\Delta^* + \left(\tau - \frac{\mu}{a_\omega^*}\right)\left.\frac{dT^*}{dx^*}\right|_{x^*=0}\right]\left.\frac{\partial B_\omega^*}{\partial T^*}\right|_{T^*=T_w^*} \tag{57}$$

The boundary value problem for the first-order terms of the inner expansion is now completely specified, and its parts are now collected in one place. From equations (25) and (26) with $n = 1$, equations (24b) and (27a) are

$$\frac{\mu}{a^*_\omega} \frac{\partial i^{(1)}_\omega}{\partial \tau} = \left[\frac{\partial B^*_\omega}{\partial t}(t)\right]_{t=T^*_w} t^{(1)} - i^{(1)}_\omega \tag{58}$$

and

$$\frac{d}{d\tau} N \frac{dt^{(1)}}{d\tau} = \frac{d}{d\tau} \int_{\omega=0}^{\infty} \int_{\Omega=4\pi} \mu\, i^{(1)}_\omega \, d\Omega \, d\omega \tag{59}$$

From equations (27b), (27d), (50), and (57),

$$\left.\begin{aligned}
i^{(1)}_\omega &= 0 && \tau = 0,\ 0 \leq \mu \leq 1 \\
t^{(1)} &= 0 && \tau = 0 \\
t^{(1)} &\to \Delta^* + \tau\left(\frac{dT^*}{dx^*}\right)_{x^*=0} && \tau \to \infty \\
i^{(1)}_\omega &\to \left[\Delta^* + \left(\tau - \frac{\mu}{a^*_\omega}\right) \left.\frac{dT^*}{dx^*}\right|_{x^*=0}\right]\left(\frac{\partial B^*_\omega}{\partial T^*}\right)_{T^*=T^*_w} && \tau \to \infty
\end{aligned}\right\} \tag{60}$$

Now, set

$$i^{(1)}_\omega = i^{(1)}_{\omega,p} + i^{(1)}_{\omega,c} \tag{61}$$

and

$$t^{(1)} = t^{(1)}_p + t^{(1)}_c \tag{62}$$

where

$$t_p^{(1)} = \Delta^* + \tau \left.\frac{dT^*}{dx^*}\right|_{x^*=0}$$

$$i_{\omega,p}^{(1)} = \left[\Delta^* + \left(\tau - \frac{\mu}{a_\omega^*}\right)\left.\frac{dT^*}{dx^*}\right|_{x^*=0}\right]\left.\frac{\partial B_\omega^*}{\partial T^*}\right|_{T^*=T_w^*}$$

The direct substitution of expressions (61) and (62) into equations (58) to (60) then shows that $i_{\omega,c}^{(1)}$ and $t_c^{(1)}$ must satisfy the following boundary value problem:

$$\frac{\mu}{a_\omega^*}\frac{\partial i_{\omega,c}^{(1)}}{\partial \tau} = \left[\left(\frac{\partial B_\omega^*}{\partial T^*}\right)_{T^*=T_w^*} \cdot t_c^{(1)}\right] - i_{\omega,c}^{(1)} \tag{63}$$

$$N\frac{dt_c^{(1)}}{d\tau} = \int_{\omega=0}^{\infty}\int_{\Omega=4\pi} \mu\, i_{\omega,c}^{(1)}\, d\Omega\, d\omega \tag{64}$$

For $0 \le \mu \le 1$,

$$\left.\begin{aligned} i_{\omega,c}^{(1)} &= \left(\frac{\mu}{a_\omega^*}\left.\frac{dT^*}{dx^*}\right|_{x^*=0} - \Delta^*\right)\left(\frac{dB_\omega^*}{dT^*}\right)_{T^*=T_w^*} & \tau = 0 \\ t_c^{(1)} &= -\Delta^* & \tau = 0 \end{aligned}\right\} \tag{65}$$

$$\left.\begin{aligned} i_{\omega,c}^{(1)} &\to 0 & \tau \to \infty \\ t_c^{(1)} &\to 0 & \tau \to \infty \end{aligned}\right\} \tag{66}$$

Equation (63) can be integrated with respect to ω to give

$$\mu\frac{d}{d\tau}\int_0^{\infty}\frac{1}{a_\omega^*}\, i_{\omega,c}^{(1)}\, d\omega = t_c^{(1)}\int_0^{\infty}\left(\frac{\partial B_\omega^*}{\partial T^*}\right)_{T^*=T_w^*} \cdot d\omega - \int_0^{\infty} i_{\omega,c}^{(1)}\, d\omega \tag{67}$$

To proceed from this point, it is convenient, although not necessary, to assume that a^*_ω is independent of ω; that is, the analysis deals with a gray gas. This approximation is made here only to eliminate a discussion of the effects of the spectral absorption coefficient, which adds no new features. Under this assumption, $a^* = a^*_\omega$. Defining

$$i_c^{(1)} \equiv \int_0^\infty i_{\omega, c}^{(1)} \, d\omega$$

and using the expression

$$\frac{T^{*3}}{\pi} = \int_0^\infty \frac{\partial B^*_\omega}{\partial T^*} \, d\omega \tag{68}$$

equation (67) becomes

$$\frac{\mu}{a^*} \frac{di_c^{(1)}}{d\tau} = \frac{T_w^{*3}}{\pi} t_c^{(1)} - i_c^{(1)} \tag{69}$$

Equation (64), under the gray gas restriction, is

$$N \frac{dt_c^{(1)}}{d\tau} = 2\pi \int_{-1}^{1} i_c^{(1)} \mu \, d\mu \tag{70}$$

and the boundary conditions (eqs. (65)) become

$$\left.\begin{aligned} i_c^{(1)} &= \left(\frac{\mu}{a^*} \left. \frac{dT^*}{dx^*} \right|_{x^*=0} - \Delta \right) \frac{T_w^{*3}}{\pi} & & 0 \leq \mu \leq 1, \ \tau = 0 \\ t_c^{(1)} &= -\Delta & & \tau = 0 \end{aligned}\right\} \tag{71}$$

Defining

$$\left.\begin{aligned} h &\equiv \frac{\pi i_c^{(1)}}{T_w^{*4}} \\ t &\equiv \frac{t_c^{(1)}}{T_w^*} \\ \tau^* &\equiv a^* \tau^* \\ \xi &\equiv \frac{1}{T_w^* a^*} \left.\frac{dT^*}{dx^*}\right|_{x^*=0} \\ \Delta &\equiv \frac{\Delta^*}{T_w^*} \\ N_w &\equiv \frac{N}{T_w^{*3}} \end{aligned}\right\} \tag{72}$$

equations (66), (69), (70), and (71) become

$$\mu \frac{\partial h}{\partial \tau^*} = t - h \tag{73}$$

$$N_w \frac{dt}{d\tau^*} = 2 \int_{-1}^{1} h\mu \, d\mu \tag{74}$$

$$\left.\begin{aligned} h &= \mu\xi - \Delta \qquad \text{for } 0 \leq \mu \leq 1 \\ t &= -\Delta \qquad \text{for } 0 \leq \mu \leq 1 \end{aligned}\right\} \tau^* = 0 \tag{75a}$$

$$h \rightarrow 0; \; t \rightarrow 0 \quad \text{for} \quad \tau^* \rightarrow \infty \tag{75b}$$

and solution of the boundary value problem specified by equations (73) to (75) will complete the solution of the entire problem. The method employed is similar to that used by Ferziger and Simmons (ref. 7) for a different radiation transport problem and is based on the work of Case (ref. 8).

Solution of the Boundary Value Problem

A separation of variables solution is assumed valid for the problem specified by equations (73) to (75) and is of the form

$$h = \int H_v(\mu) e^{-\tau^*/v} A_v \, dv \tag{76}$$

$$t = \int A_v \Theta_v e^{-\tau^*/v} \, dv \tag{77}$$

Examination of the boundary condition, equation (75b), shows that

$$R_e(v) > 0 \tag{78}$$

The trivial solution $h = t_c^{(1)} = \text{constant}$ exists, but it is neglected because of boundary condition (75b).

Substitution of equations (76) and (77) into equations (73) and (74) gives

$$-\left(\frac{\mu}{v} - 1\right) H_v = \Theta_v \tag{79}$$

$$-\frac{N_w}{v} \Theta_v = 2 \int_{-1}^{1} H_v(\mu) \mu \, d\mu \tag{80}$$

Eliminating Θ_v from equation (79) by substitution in equation (80) and then rewriting equation (80) give

$$(\mu - v) H_v = \frac{2v^2}{N_w} \int_{-1}^{1} H_v(\mu) \mu \, d\mu \tag{81}$$

and

$$\Theta_v = -\frac{2v}{N_w} \int_{-1}^{1} H_v(\mu) \mu \, d\mu \tag{82}$$

In general, v may be a complex number. However, if v is assumed complex in equation (81), then for the imaginary part of v to be nonzero, N_w must be negative. However, N_w is a parameter that can take on only positive values, so that for this problem $\mathrm{Im}(v) = 0$, and v is taken as a positive real number by this argument and equation (78). It is sufficient to restrict v to the range $0 < v < 1$. With the function C_v defined as

$$C_v \equiv \int_{-1}^{1} H_v(\mu)\mu \, d\mu \tag{83}$$

equation (81) becomes

$$H_v = \frac{2C_v v^2}{N_w} \text{ P. V. } \frac{1}{\mu - v} + \delta(\mu - v)B_v \tag{84}$$

where P. V. denotes that the Cauchy principal values of the integrals are to be used.

Multiplying equation (84) by $\mu \, d\mu$ and integrating from -1 to 1 give

$$C_v = \frac{2v^2}{N_w} C_v\left[2 + v \ln\left(\frac{1 - v}{1 + v}\right)\right] + vB_v \tag{85}$$

The function H_v can be arbitrarily normalized so that

$$\frac{2}{N_w} C_v = v \tag{86}$$

Equation (85) can then be written

$$B_v = \frac{N_w}{2} - 2v^2 - v^3 \ln\left(\frac{1 - v}{1 + v}\right) \tag{87}$$

With equation (87) B_v is completely specified.

Equations (84) and (82) become, with the substitution of equation (86),

$$H_v = \text{P. V. } \frac{v^3}{\mu - v} + B_v\delta(\mu - v) \tag{88}$$

and

$$\Theta_v = -v^2 \tag{89}$$

Now that H_v and Θ_v are known functions of v, they may be substituted into equations (76) and (77) to yield

$$\left.\begin{aligned} h &= \text{P.V.} \int_0^1 \frac{v^3 e^{-\tau^*/v}}{\mu - v} A_v \, dv + B_\mu A_\mu e^{-\tau^*/\mu} & \mu \geq 0 \\ h &= \text{P.V.} \int_0^1 \frac{v^3 e^{-\tau^*/v}}{\mu - v} A_v \, dv & \mu < 0 \end{aligned}\right\} \tag{90}$$

and

$$t = -\int_0^1 v^2 A_v e^{-\tau^*/v} \, dv \tag{91}$$

The value of the remaining function A_v is found by the substitution of equation (90) into the boundary condition (75a) to obtain

$$\text{P.V.} \int_0^1 \frac{v^3 A_v}{\mu - v} \, dv + B_\mu A_\mu = \xi\mu - \Delta \tag{92}$$

This singular integral equation may be solved for A_μ as follows: Consider the domain D (fig. 2) bounded by the closed curve L + L' in the complex plane. Define the function $\Phi(Z)$ as

$$\Phi(Z) = -\frac{1}{2\pi i} \int_0^1 \frac{v^3 A_v}{Z - v} \, dv \tag{93}$$

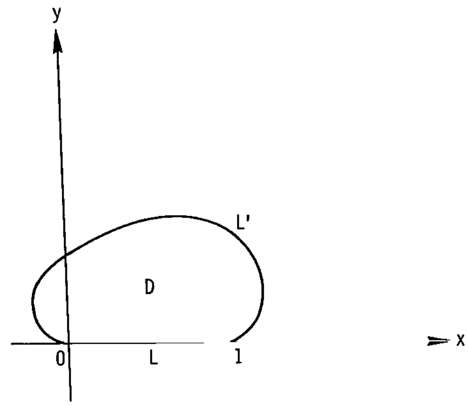

Figure 2. - Domain in complex plane bounded by closed curve.

As long as $v^3 A_v$ is p-integrable (in the Lebesgue sense) on (0, 1) where $p > 1$, then $\Phi(Z)$ is analytic everywhere except on L. For such a function, the Plemelj formulas apply almost everywhere (refs. 9 to 11):

$$\Phi^+(v) - \Phi^-(v) = v^3 A_v \qquad \text{for } 0 < v < 1 \tag{94}$$

and

$$\Phi^+(v) + \Phi^-(v) = \text{P. V. } \frac{1}{\pi i} \int_0^1 \frac{u^3 A_u}{u - v} du \tag{95}$$

where $\epsilon > 0$ and

$$\Phi^{\pm}(v) \equiv \lim_{\epsilon \to 0^+} \Phi(v \pm i\epsilon)$$

Equations (94) and (95) are used in equation (92) to find that

$$\Phi^+(v) = G(v)\Phi^-(v) + g(v) \tag{96}$$

where the function G(v) is defined as

$$G(v) \equiv \frac{B_v + \pi i v^3}{B_v - \pi i v^3} = \frac{N_w - 4v^2 - 2v^3\left[\ln\left(\frac{1 - v}{1 + v}\right) - \pi i\right]}{N_w - 4v^2 - 2v^3\left[\ln\left(\frac{1 - v}{1 + v}\right) + \pi i\right]} \tag{97}$$

and

$$g(v) \equiv \frac{\xi v^4 - v^3 \Delta}{B_v - \pi i v^3} = \frac{2(\xi v^4 - v^3 \Delta)}{N_w - 4v^2 - 2v^3\left[\ln\left(\frac{1 - v}{1 + v}\right) + \pi i\right]} \tag{98}$$

Note that $G(v)$ is continuous and nonvanishing on L and that $g(v)$ is p-integrable in the range $0 < v < 1$. The functions $g_1(v)$ and $G_1(v)$ on $L + L'$ are defined as

$$g_1 = 0, \quad G_1 = 1 \quad \text{on } L'$$

$$g_1 = g, \quad G_1 = G \quad \text{on } L$$

The behavior of G_1 will now be examined. For all values of N_w, the modulus of $G_1(v)$ is 1. For the case of chief interest here, that is, for $N_w \neq 0$, the argument of $G_1(v)$ varies continuously in the range from 0 to less than 2π and back to 0 as v goes from 0 to 1 on L. The argument of G_1 is 0 on L'; hence, G_1 is single-valued on $L + L'$, and the index of the boundary value problem (refs. 9 or 10) is 0.

For the special case of $N_w = 0$ (no conduction), however, the argument of $G_1(v)$ changes continuously from 0 to -2π as v varies from 0 to 1; the index for this boundary value problem is -1.

For an index of 0, a solution to the boundary value problem always exists for an arbitrary Δ and ξ (ref. 10). For an index of -1, however, an auxiliary condition must be imposed to relate Δ and ξ. This condition eliminates the possibility of satisfying the boundary condition $T(x = 0) = T_0$. The possibility of a physical slip in temperature at the wall for $N_w = 0$ then arises. This case will not be pursued further. Note that

$$\lim_{\epsilon \to 0} g(0 + \epsilon) = 0 \qquad \epsilon > 0$$

$$\lim_{\epsilon \to 0} g(1 - \epsilon) = 0 \qquad \epsilon > 0$$

and that G_1 and g_1 are therefore continuous and single valued on $L + L'$ for $N_W \neq 0$. The only discontinuity of $\Phi(Z)$ is on L so that equation (96) can be rewritten as

$$\Phi^+(v) = G_1(v)\Phi^-(v) + g_1(v) \qquad v \text{ contained in } L + L' \tag{99}$$

where the superscript (+) or (-) is understood to refer to the function approaching $L + L'$ from inside or outside D, respectively.

The functions $G_1(v)$ and $g_1(v)$ for any two arbitrary points except $v = 1$ satisfy the Holder condition

$$|G(v_1) - G(v_2)| < R|v_1 - v_2|^{\alpha}$$

where R is a positive number and $0 < \alpha \leq 1$. Until recently, the Holder condition was required of both $g(v)$ and $G(v)$ before solution of equation (96) could be shown to exist. However, it has recently been established (refs. 10 to 12) that, as long as $g(v)$ is p-integrable where $p > 1$ and $G(v)$ is continuous as is the case herein, a solution for A_v in the class of p-integrable functions can be found that satisfies the boundary value problem almost everywhere (except on a set of measure zero); this solution is obtained (ref. 10) by setting

$$\Phi^{\pm}(v) = X^{\pm}(v)\left[\pm\frac{1}{2}\frac{g(v)}{X^+(v)} + \text{P. V. } \frac{1}{2\pi i}\int_0^1 \frac{g(u)du}{X^+(u)(u - v)}\right] \tag{100}$$

where

$$X(Z) = \exp\left[\frac{1}{2\pi i}\int_0^1 \frac{\ln G(v)}{v - Z}\, dv\right] \tag{101a}$$

and

$$\left.\begin{aligned} X^+(v) &= G_1(v)X^-(v) \\ X^+(v) &= \exp\left[\frac{1}{2}\ln G(v) + \text{P. V. } \frac{1}{2\pi i}\int_0^1 \frac{\ln G(u)du}{u - v}\right] \end{aligned}\right\} \tag{101b}$$

With an appropriate choice of the branch of the logarithm,

$$\ln G(v) = 2i \tan^{-1} \frac{\pi v^3}{B_v} \tag{102}$$

Define

$$\left.\begin{aligned} k_v &\equiv \frac{B_v}{\pi v^3} \\ E_v &\equiv \exp\left[\frac{1}{\pi} \text{ P. V.} \int_0^1 \frac{\tan^{-1}\left(\frac{1}{k_u}\right)du}{u - v}\right] \end{aligned}\right\} \tag{103}$$

With the substitution of equation (100) into equation (94) and with the use of equation (101) and the definitions given by equations (102) and (103), the result may be written

$$A_v v^3 = \frac{1}{\pi\left(k_v^2 + 1\right)}\left[(\xi v - \Delta)k_v + \left(k_v^2 + 1\right)^{1/2} E_v \frac{1}{\pi} \text{ P. V.} \int_0^1 \frac{(\xi u - \Delta)du}{E_u\left(k_u^2 + 1\right)^{1/2}(u - v)}\right] \tag{104}$$

If equation (91) is combined with equation (75a), the boundary condition becomes

$$\int_0^1 v^2 A_v \, dv = \Delta \tag{105}$$

Substituting equation (104) into equation (105) gives a relation between Δ/ξ and N_W. Instead of proceeding directly in this manner, however, it is simpler to proceed as follows: Note that

$$\lim_{\mu \to 0} \text{ P. V.} \int_0^1 \frac{v^3 A_v}{\mu - v} dv = -\int_0^1 v^2 A_v \, dv \tag{106}$$

because $\lim_{v \to 0} v^2 A_v$ is finite by equation (104). If the limit $\mu \to 0$ in equation (92) is taken, the following expression is obtained:

$$-\int_0^1 v^2 A_v \, dv + \lim_{\mu \to 0} B_\mu A_\mu = -\Delta \tag{107}$$

Comparison with the boundary condition (eq. (105)) gives

$$\lim_{\mu \to 0} B_\mu A_\mu = 0$$

Substituting equation (104) now gives

$$-\Delta + \lim_{v \to 0} E_v \frac{1}{\pi} \text{P. V.} \int_0^1 \frac{du(\xi u - \Delta)}{E_u \left(k_u^2 + 1\right)^{1/2} (u - v)} = 0 \tag{108}$$

Since $(\xi u - \Delta) \Big/ \left[u E_u \left(k_u^2 + 1\right)\right]$ is bounded at $u = 0$, and $\lim_{v \to 0} E_v$ is finite, the limit in equation (108) exists and reduces to

$$\frac{\Delta}{\xi}\left[\frac{1}{\pi}\int_0^1 \frac{du}{u E_u \left(k_u^2 + 1\right)^{1/2}} + \frac{1}{E_0}\right] = \frac{1}{\pi}\int_0^1 \frac{du}{E_u \left(k_u^2 + 1\right)^{1/2}} \tag{109}$$

where

$$E_0 = \lim_{v \to 0} E_v$$

Equation (109) provides a relation for Δ/ξ as a function of N. This result can be further simplified as follows:

Equation (101) is written as

$$\frac{1}{X^+(v)} - \frac{1}{X^-(v)} = -\frac{2\pi i v^3}{B_v - \pi i v^3} \frac{1}{X^+(v)} \qquad \text{where } v \in L \tag{110}$$

The coefficient of $X^+(v)$ on the right side goes to 0 at $v = 0$ and $v = 1$, and it follows from equation (101) that $X(Z)$ never vanishes.

The application of the Plemelj formulas and the boundary condition

$$X(Z) \to 1 \text{ as } Z \to \infty$$

gives

$$\frac{1}{X(Z)} = -\frac{1}{\pi}\int_0^1 \frac{dv}{\left(k_v^2 + 1\right)^{1/2}(v - Z)E_v} + 1 \tag{111}$$

Expanding equation (111) in an asymptotic series near $Z = \infty$ yields

$$\frac{1}{X(Z)} = 1 + \frac{1}{Z}\frac{1}{\pi}\int_0^1 \frac{dv}{E_v\left(k_v^2 + 1\right)^{1/2}} - \ldots \tag{112}$$

A similar expansion of equation (101a) gives

$$\frac{1}{X(Z)} = 1 + \frac{1}{Z}\frac{1}{2\pi i}\int_0^1 \ln G(v)dv - \ldots \tag{113}$$

and equating coefficients of $1/Z$ results in

$$\int_0^1 \frac{dv}{E_v\left(k_v^2 + 1\right)^{1/2}} = \frac{1}{2i}\int_0^1 \ln G(v)dv = \int_0^1 \tan^{-1}\left(\frac{1}{k_v}\right)dv \tag{114}$$

Since at $v = 0$, $1\Big/\left[v\left(k_v^2 + 1\right)^{1/2}E_v\right]$ is bounded, equation (111) becomes in the limit $Z \to 0$

$$\lim_{Z \to 0} \frac{1}{X(Z)} = -\frac{1}{\pi} \int_0^1 \frac{dv}{\left(k_v^2 + 1\right)^{1/2} vE_v} + 1 \tag{115}$$

It follows from equation (101a), since $\ln G(v) = 0$ at $v = 0$, that

$$\lim_{Z \to 0} \frac{1}{X(Z)} = \lim_{u \to 0} \exp\left[-\frac{1}{2\pi i} \text{ P. V. } \int_0^1 \frac{\ln G(v)}{v - u} dv\right] = \frac{1}{E_0} \tag{116}$$

With the use of equations (114) to (116) in equation (109) finally results in

$$\frac{\Delta}{\xi} = \frac{1}{\pi} \int_0^1 \tan^{-1}\left(\frac{1}{k_v}\right) dv \tag{117}$$

From equations (8), (31), and (45), it follows that

$$T^*(x^* = 0) - T_w^* = \epsilon \Delta^* + O(\epsilon^2) \tag{118}$$

that is, up to terms of $O(\epsilon^2)$, $\epsilon \Delta^* T_r$ is the apparent discontinuity in temperature obtained by extrapolating the outer solution to the wall.

As already shown by equation (17c), in the outer region the temperature satisfies the usual diffusion approximation up to terms of $O(\epsilon^2)$; hence, equation (118) gives the proper boundary condition for solutions to the diffusion equation, with the apparent discontinuity in the wall temperature appearing to account for the region near the wall where the diffusion solution breaks down. If the radiation slip coefficient is defined as

$$\psi(N_w) = \frac{3}{4} \frac{1}{\pi} \int_0^1 \tan^{-1}\left(\frac{1}{k_v}\right) dv$$

then equations (117), (118), and (72) show that

$$T^*(x^* = 0) - T_w^* = \epsilon T_w^* \frac{4}{3} \psi(N_w)\xi + O(\epsilon^2)$$

$$= \epsilon \frac{1}{a^*} \left.\frac{dT^*}{dx^*}\right|_{x^*=0} \frac{4}{3} \psi(N_w) + O(\epsilon^2)$$

or, written in terms of the dimensional quantities,

$$\frac{3[T_D(x = 0) - T_w]}{4 \frac{1}{a} \left.\frac{dT_D}{dx}\right|_{x=0}} = \psi(N_w) + O(\epsilon^2)$$

where T_D is the temperature obtained from the outer solution, which is equivalent to the diffusion solution to order ϵ^2. Because

$$4T_w^{*3}[T^*(x^* = 0) - T_w^*] = T^{*4}(x^* = 0) - T_w^{*4} + O(\epsilon^2)$$

the expression for the slip coefficient can also be written

$$\frac{T_D^4(x = 0) - T_w^4}{\frac{16}{3a} T_w^3 \left.\frac{dT_D}{dx}\right|_{x=0}} = \psi(N_w) + O(\epsilon^2)$$

Alternatively, define as usual

$$q_r = -16 \frac{\sigma}{3a} T_w^3 \left.\frac{dT_D}{dx}\right|_{x=0}$$

which is the radiation contribution to the wall heat flux in the diffusion approximation. Then

$$\frac{\sigma\left[T_D^4(x = 0) - T_w^4\right]}{-q_r} = \psi(N_w) + O(\epsilon^2) \qquad (119a)$$

APPLICATIONS

Equation (117) provides a relation for the linearized radiation slip coefficient Δ/ξ in terms of a simple integral that depends on the single parameter N_w, the usual conduction-radiation parameter. Examination of the equation (118) shows that the radiation slip coefficient ψ is related to Δ/ξ by

$$\psi = \frac{\sigma\left[T_w^4 - T_D^4(x = 0)\right]}{q_r} = \frac{3}{4}\frac{\Delta}{\xi} \tag{119b}$$

where $T_D(x = 0)$ is the temperature in the gas at the boundary to be used in the diffusion approximation. Deissler (ref. 5) predicts an approximate radiation slip on the basis of a second-order radiation diffusion solution, and his result for $N_w = 0$ is

$$\psi(N_w = 0) = 0.5$$

The results from the present analysis, shown in figure 3, closely approach Deissler's result for small N_w.

Prediction of Energy Transfer

The slip coefficient derived herein can be readily incorporated into the diffusion solution for predicting energy transfer. For the case of infinite parallel plates containing a gray gas and separated by a distance D, exact numerical solutions are available in the literature (ref. 13) for determining the accuracy of results. For this geometry, the diffusion solution (eq. (17c)) after one integration becomes

$$q^* = -N \frac{dT^*}{dx^*} - \lambda^*(T^*) \frac{dT^*}{dx^*} = \frac{q}{4\sigma T_r^4} \tag{120}$$

in which the constant of integration is $q^* = q_r^* + q_c^*$ where q^* is the dimensionless total energy flux and q_r^* and q_c^* are the dimensionless radiative and conductive fluxes, respectively. For no sources or sinks, q^* is constant with x^*, and equation (120) can be integrated directly.

The slip coefficients ψ_1 and ψ_2 are then used to eliminate the temperatures at

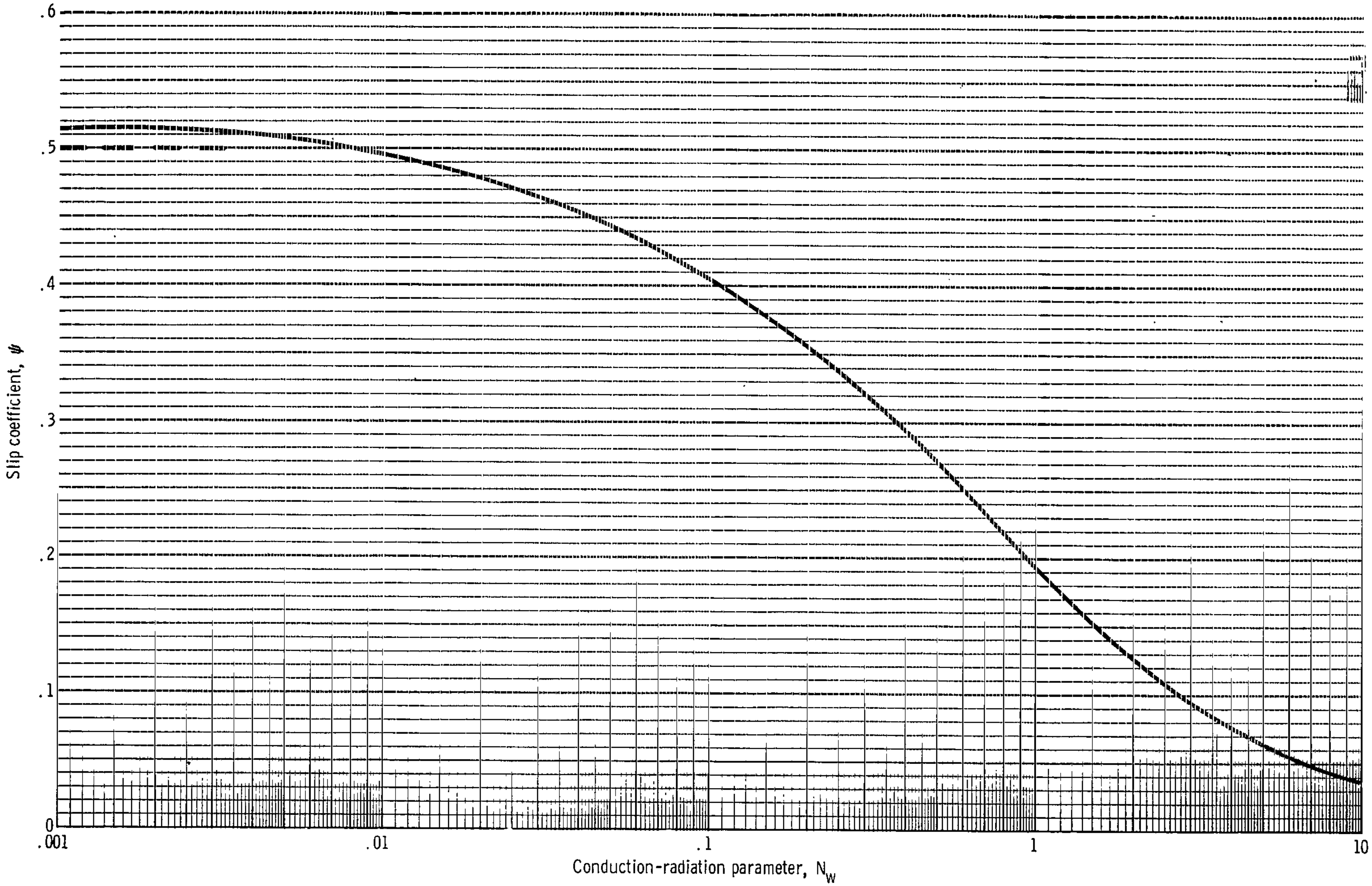

Figure 3. - Apparent slip coefficient for use with diffusion solution as function of conduction-radiation parameter.

each boundary. From equations (119) and (120),

$$\frac{(T_w^*)_i^4 - (T_i^*)^4}{\psi_i} = \mp \frac{4q^*}{1 + \frac{3}{4} N_{w,i}} \tag{121}$$

where the sign of q^* depends on the direction of the normal to the boundary and will be positive for $i = 2$ and negative for $i = 1$. Linearizing and substituting into the integrated equation (120) to eliminate T_1^* and T_2^* give

$$\Gamma = \frac{q}{\sigma\left(T_{w,1}^4 - T_{w,2}^4\right)} = \frac{1}{\frac{3\tau_D}{4} + \psi_1 + \psi_2}\left[1 + \frac{3N_{w,1}(1 - \gamma)}{(1 - \gamma^4)}\right] \tag{122}$$

where γ is the ratio of the wall temperatures $T_{w,2}^*/T_{w,1}^*$ and τ_D is the optical thickness of the gas aD.

Predictions based on equation (122) are shown in table I and are compared with the numerical results of references 13 to 15 and with the simple additive solution of the exact uncoupled radiative and conductive fluxes as proposed by many authors. Agreement by both approximations is good, and the present formulation is slightly more accurate. An

TABLE I. - COMPARISON OF APPROXIMATE AND EXACT SOLUTIONS FOR ENERGY TRANSFER THROUGH GRAY GAS BETWEEN INFINITE PARALLEL PLATES

Optical thickness of diffusion solution, τ_D	Ratio of wall temperatures, γ	Radiation-conduction parameter at surface 1, N_1	Solution					
			Exact		Present		Additive	
			Dimensionless energy flux, Γ	Reference	Dimensionless energy flux, Γ	Error, percent	Dimensionless energy flux, Γ	Error, percent
10	0.5	1	0.336	1	0.336	0	0.323	-3.9
10	.5	.1	.140	1	.140	0	.130	-7.2
10	.5	.02916	.133	14	.126	-5.3	.125	-6.0
3	.2	.208	.583	15	.567	-2.9	.552	-6.5
1	.5	.1	.863	13	.850	-1.5	.773	-10.4
1	.5	.01	.647	13	.612	-5.4	.581	-10.2

advantage of the present analysis is that temperature distributions in the medium are easily obtained by the integration of equation (120), but these cannot be obtained from the additive technique.

More General Solutions to Optically Thick Situations

For the purpose of discussion, consider the supposition that both the absorption coefficient a and the thermal conductivity λ_K are functions of the local gas temperature. A well-known fact is that, for $a = a(T)$, pure radiation solutions of both energy transfer and temperature distributions may be greatly changed from the solutions for a assumed constant (ref. 16). For $a = a(T)$, then, the additive solution for heat flux in combined radiation-conduction problems may be poor because of the influence of conduction on the temperature profile, which may change the radiative flux greatly from the radiative flux present with no conduction.

If terms of $O(\epsilon^2)$ are neglected and the dimensional forms are substituted for all terms, equation (17c) can be written

$$\frac{d}{dx}\left\{\left[\lambda_k(T) + \frac{16\sigma T^3}{3a(T)}\right]\frac{dT}{dx}\right\} = 0 \tag{123}$$

Now define

$$\beta \equiv \int^T \lambda_k(T)dT + \frac{16\sigma}{3}\int^T \frac{T^3}{a(T)}\,dT \tag{124a}$$

or

$$\beta \equiv \int^T \lambda_k(T)dT + \frac{4\pi}{3}\int^T \frac{1}{a(T)}\frac{dB}{dT}\,dT \tag{124b}$$

Equation (123) can then be written

$$\nabla^2\beta = 0 \tag{125}$$

while the boundary condition (eq. (119)) is, with the use of the diffusion relation for q_r,

$$\beta(x = 0) - \beta_w = \frac{4}{3a(T_{x=0})} \psi \left.\frac{\partial\beta}{\partial n}\right|_{x=0} \tag{126}$$

where $\partial\beta/\partial n$ is the rate of change in β with direction normal to the boundary. Equations (125) and (126) are the general formulation of the diffusion equation for combined conduction and radiation with the extrapolated slip boundary condition.

CONCLUDING REMARKS

The correct boundary condition for use with the diffusion approximation for combined conduction-radiation problems was derived. The boundary condition is given in terms of a slip coefficient ψ, where

$$\psi = \frac{\sigma\left[T_w^4 - T_D^4(x = 0)\right]}{q_r}$$

Values of ψ as a function of the conduction-radiation parameter N_w are presented in graphical form. Results obtained with the use of the diffusion approximation and the slip-coefficient boundary condition compare well with exact solutions for the case of a gray gas contained between infinite parallel black plates.

The entire temperature distribution in the gas may be found by applying the solution of the linearized exact solution near the wall and the diffusion solution away from the wall. The procedure used herein guarantees that the solutions will match in the intermediate region and will give a uniformly valid representation of the entire temperature field.

Lewis Research Center,
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