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# BOUNDARY CONDITIONS MATTER: ON THE SPECTRUM OF INFINITE QUANTUM GRAPHS

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Abstract. We develop a comprehensive spectral geometric theory for two distinguished self-adjoint realisations of the Laplacian, the so-called *Friedrichs* and *Neumann extensions*, on infinite metric graphs. We present a new criterion to determine whether these extensions have compact resolvent or not, leading to concrete examples where this depends on the chosen extension. In the case of discrete spectrum, under additional metric assumptions, we also extend known upper and lower bounds on Laplacian eigenvalues to metric graphs that are merely locally finite. Some of these bounds are new even on compact graphs.

Contents



## 1. Introduction

This article is about spectral properties of different self-adjoint realisations of the Laplace operator on infinite metric graphs. Recently, it was observed in [28] that on infinite metric graphs there can be points in their closure which do not belong to the graph itself (so-called *graph ends*), and that two natural realisations of the Laplacian, namely the *Friedrichs* and the *Neumann extension*, can be understood to impose Dirichlet or Neumann conditions, respectively, at those ends. This raises the question of what effect the choice of the extensions will actually have on the spectrum. The answer is, roughly speaking, that the choice of the extension is quite relevant and can have more dramatic consequences on infinite quantum graphs than it has on their finite counterparts.

We develop qualitative criteria for the absence or existence of essential spectrum, encountering surprises such as a phase transition in the spectrum of the Neumann extension. In the case of purely discrete spectrum,

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we also prove quantitative estimates on eigenvalues, some of which extend estimates previously only known for finite quantum graphs, and some of which are completely new.

Quantum graphs are Schrödinger operators acting on functions supported on metric graphs. In the case of *compact metric graphs*, that is, metric graphs consisting of a finite set of edges, each of finite length, they were introduced to the mathematical literature in the early 1980s [32, 33]. Shortly afterwards it was observed that they become self-adjoint upon imposing appropriate transmission conditions in the vertices. First properties of their spectrum were discussed in [41]. In particular, for the Laplacian with standard (continuity and Kirchhoff-type) vertex conditions the spectrum is purely discrete and 0 is an eigenvalue with multiplicity equal to the number of connected components of the metric graph. This elementary but noteworthy result was historically perhaps the first step towards the development of spectral geometry of quantum graphs. Since then, describing spectral properties of quantum graphs in terms of "geometric" properties of the underlying metric graph, and vice versa, has become an increasingly popular topic, especially in the last decade, see [8] and references therein.

Infinite quantum graphs – and in particular their spectral properties – have been studied since the late 1990s in [10, 12, 6, 9] and more recently among others in [31, 4]: in all these articles the relevant operator is essentially self-adjoint. The case where different self-adjoint extensions may exist was first discussed in the pioneering papers [11, 39], and then more extensively in [16, 29]: in all these papers, the spectral properties of only the Friedrichs extension of the (general realisations of) Laplacians on infinite quantum graphs was discussed explicitly. First investigations relied on a transference principle relating the spectrum of a quantum graph to the spectrum of an infinite matrix: this idea goes back to von Below's early analysis in [5] and relies fundamentally upon the assumption that *all edges have the same length*. If this assumption is dropped, infinite quantum graphs will generally *not* be essentially self-adjoint: the operator domain may then have to include appropriate *boundary conditions at infinity*, which in turn will influence the spectrum.

As intimated above, the notion of infinite quantum graph we are going to discuss relies on the notion of *ends*, a classical concept from graph theory (see, e.g., [15, Chapter 8] and references therein) which was recently adapted to quantum graphs in [28]. It turns out that if an end is *thin enough*, in a sense to be explained below, then a boundary condition has to be imposed on it in order to ensure self-adjointness of a realisation of the Laplacian. One can identify two important types of boundary conditions, namely of Dirichlet or Neumann type, which we will treat in this paper.

Let us also point out parallels of this paper to the spectral geometry on domains and manifolds – a well developed field, see, e.g. [19] for a survey. If $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, then its Dirichlet and Neumann Laplacians both have purely discrete spectrum, whereas on unbounded or non-smooth domains, the situation is more subtle, see, e.g., [1, Chapter 6] or [14]. Likewise, it is known that infinite quantum graphs may or may not have purely discrete spectrum [48, 20, 29, 18].

This paper is organised as follows: Section 2 contains definitions of metric graphs, Sobolev spaces, and extensions of the Laplace operator. In Section 3, we provide criteria on discreteness or non-discreteness of the spectrum of these Laplacian extensions, beyond the one based on finite total length, which has been known since [11]. Another criterion for discreteness of the Friedrichs extension on infinite trees from [29] is shown not to generalise to the Neumann extension and a new Kolmogorov–Riesz-type criterion on discreteness of the spectrum is proved in Theorem 3.5. Subsection 3.3 puts these criteria to use by investigating the example of a parameter-dependent family $\mathcal{G}_\alpha$ (the *diagonal combs*) on which the spectra of the Neumann extensions exhibit a noteworthy phase transition from purely discrete to nonempty essential spectrum, despite all having infinite total length – a phenomenon which somewhat recalls Neumann Laplacians on (bounded) cusp domains but which has no parallels on domains of infinite volume.

Having determined criteria for discreteness of the spectrum, the next natural question is to investigate which bounds on eigenvalues from finite metric graphs remain valid on infinite metric graphs. Section 4 is about lower bounds on eigenvalues. Subsection 4.1 introduces a symmetrisation technique and uses it to prove isoperimetric inequalities on infinite quantum graphs. Subsection 4.2 complements this with further lower bounds in terms of diameter and inradius, and recollects some known Cheeger-type inequalities. In particular, we call attention on Theorem 4.17, a quantum graph analogue of the Hersch–Makai inequality on the lowest eigenvalue of the Dirichlet Laplacian on simply connected domains in terms of the inradius. Finally, Section 5 complements the lower bounds in Section 4 by upper bounds in terms of diameter, total length and the first Betti number. We mention in particular Theorem 5.2, a bound featuring diameter and the first Betti number that is also new for compact graphs.

## 2. Infinite metric and quantum graphs

2.1. **Notation and elementary properties.** We will work with unoriented metric graphs with finite or countably infinite sets of vertices $\mathsf{V}$ and edges $\mathsf{E}$. We follow the formalism in [38], referring to that article for further details.

Let $\mathsf{E}$ be a finite or countably infinite set and $(\ell_\mathsf{e})_{\mathsf{e}\in\mathsf{E}} \in (0,\infty)^{\#\mathsf{E}}$, a vector of edge lengths. Each $\ell_\mathsf{e}$ is finite and positive but we neither assume an upper bound on $\ell_\mathsf{e}$, nor any positive lower bound. One then constructs a metric graph based on the disjoint union

$$\mathcal{E} := \bigsqcup_{\mathsf{e}\in\mathsf{E}} [0,\ell_\mathsf{e}] \ .$$

For now, we use the standard notation $(x,\mathsf{e})$ for the element of $\mathcal{E}$ with $x \in [0,\ell_\mathsf{e}]$ and $\mathsf{e} \in \mathsf{E}$, although in practice, later we will mostly write $x$ instead of $(x,\mathsf{e})$.

To turn this collection of intervals into a network-like structure, observe that any equivalence relation $\equiv$ on the set

$$\mathcal{V} := \bigsqcup_{\mathsf{e}\in\mathsf{E}} \{0,\ell_\mathsf{e}\}$$

of endpoints of $\mathcal{E}$ canonically extends to an equivalence relation on $\mathcal{E}$, again denoted by $\equiv$. Any quotient space $\mathcal{G} := {}^{\mathcal{E}}\!/_{\equiv}$ constructed this way is called a *metric graph* and $\mathsf{V} := {}^{\mathcal{V}}\!/_{\equiv}$ its *set of vertices*, while $\mathcal{E}$ is its *set of metric edges*. We sometimes stress this construction by writing $\mathcal{G} = \mathcal{G}(\mathsf{V},\mathcal{E},\equiv)$.

An edge $\mathsf{e} \simeq [0,\ell_\mathsf{e}] \in \mathsf{E}$ is called *incident* to a vertex $\mathsf{v} \in \mathsf{V}$ (and vice versa) if $(0,\mathsf{e}) \in \mathsf{v}$ or $(\ell_\mathsf{e},\mathsf{e}) \in \mathsf{v}$, and two vertices are *adjacent* if there is an edge incident to both. In this case we also write $\mathsf{v} \sim \mathsf{w}$, where $\mathsf{e} = \mathsf{e}_{\mathsf{v},\mathsf{w}} \in \mathsf{E}$ is the common edge, and call $\mathsf{v},\mathsf{w}$ *adjacent*. Likewise, $\mathsf{e}$ and $\mathsf{f}$ are *adjacent* if they are both incident to a common vertex $\mathsf{v}$. We denote by $\mathsf{E}_\mathsf{v} = \{\mathsf{e} \in \mathsf{E} : \mathsf{e} = \mathsf{e}_{\mathsf{v},\mathsf{w}} \text{ for some } \mathsf{w} \in \mathsf{V}\}$ the set of all edges incident to $\mathsf{v}$. We explicitly allow *loops* (edges incident to only one vertex) and *parallel edges*.

The *degree* of a vertex $\mathsf{v} \in \mathsf{V}$, $\deg(\mathsf{v})$, is the number of edges incident to it, where loops are counted twice. A vertex $\mathsf{v}$ is called a *dummy vertex* if $\deg(\mathsf{v}) = 2$. The deletion (or insertion) of a finite number of dummy vertices, where two incident edges are replaced by one of the same total length (or vice versa), does not affect the the topology or metric structure of the graph nor does it affect any of the Laplace-type operators defined on it we are going to consider in this article. [1] We say that $\mathcal{G}$ is *locally finite* if $\deg(\mathsf{v}) < \infty$ for all $\mathsf{v} \in \mathsf{V}$, and *compact* if its edge set $\mathsf{E}$ is finite.

The *total length* of $\mathcal{G}$ is

$$L(\mathcal{G}) := \|\ell_\mathsf{e}\|_1$$

and we emphasise that it is independent of the equivalence relation $\equiv$, that is, rewiring a metric graph does not affect its total length; it is also independent of the insertion or deletion of dummy vertices. In particular, while this definition requires all edges to have finite length, this is no restriction, since edges of infinite length can be broken up into a countably infinite set of edges of finite length by inserting dummy vertices.

A metric graph is, indeed, a metric space with respect to the metric $\mathrm{dist}_\mathcal{G}$ induced, via the equivalence relation $\equiv$, by the distance on $\mathcal{E}$ defined by

$$\mathrm{dist}_\mathcal{E}\big((x,\mathsf{e}),(y,\mathsf{f})\big) := \begin{cases} |x-y|, & \text{if } \mathsf{e}=\mathsf{f} \text{ and } x,y \in [0,\ell_\mathsf{e}], \\ \infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

More precisely, we let

$$\mathrm{dist}_\mathcal{G}(\xi,\theta) := \inf \sum_{i=1}^{k} \mathrm{dist}_\mathcal{E}(\xi_i,\theta_i), \qquad \xi,\theta \in \mathcal{G}, \tag{2.2}$$

where the infimum is taken over all $k \in \mathbb{N}$ and all $k$-tuples $(\xi_1,\ldots,\xi_k),(\theta_1,\ldots,\theta_k) \subset \mathcal{G}$ with $\xi = \xi_1$, $\theta = \theta_k$, $\xi_i \neq \xi_j$, and $\theta_i \equiv \xi_{i+1}$ for all $i,j = 1,\ldots,k-1$.

[1] In the case of other vertex conditions, this might be more subtle. See, for example, [46].

A metric graph $\mathcal{G}$ is called *connected* if $\mathrm{dist}_{\mathcal{G}}(x,y) < \infty$ for any two $\xi, \theta \in \mathcal{G}$. In this paper we exclusively work with connected metric graphs. The *diameter* of $\mathcal{G}$ is

$$D(\mathcal{G}) := \sup_{x,y\in\mathcal{G}} \mathrm{dist}_{\mathcal{G}}(x,y),$$

and we note that clearly $D(\mathcal{G}) \leq L(\mathcal{G})$. Finally, a simple argument shows that a locally finite! connected metric graph $\mathcal{G}$ is a compact graph if and only if it is compact as a metric space.

*Example* 2.1. Take the interval $(0,1]$ and place a vertex at the points $(n^{-1})_{n\in\mathbb{N}}$. This yields a metric graph $\mathcal{G}$ with

$$L(\mathcal{G}) = D(\mathcal{G}) = 1.$$

A metric graph is also, in a canonical way, a measure space, endowed with the direct sum of Lebesgue measures on each interval $(0, \ell_{\mathsf{e}})$.

**Definition 2.2.** A *subgraph* of $\mathcal{G}(\mathsf{V}, \mathcal{E}, \equiv)$ is a metric graph $\mathcal{G}' = \mathcal{G}'(\mathsf{V}', \mathcal{E}', \equiv')$ with $\mathsf{V}' \subset \mathsf{V}$, $\mathcal{E}' \subset \mathcal{E}$ and $\equiv' \subset \equiv$.

Any such subgraph is called an *induced subgraph* of $\mathcal{G}$ if for any vertices $\mathsf{v}, \mathsf{w}$ in $\mathcal{G}'$, the set of edges $\mathsf{e}_{\mathsf{v},\mathsf{w}}$ connecting them in $\mathcal{G}'$ is equal to the set of edges connecting them in $\mathcal{G}$. In this case we denote by $\partial\mathcal{G}'$ the set of points in $\mathsf{V}'$ which form the topological boundary of $\mathcal{G}'$ as a subset of $\mathcal{G}$.

We now state the general condition on the metric graph $\mathcal{G}$ we are going to impose throughout this article.

**Assumption 2.3.** The metric graph $\mathcal{G}$ is locally finite, that is, $\deg(\mathsf{v}) < \infty$ for all $\mathsf{v} \in \mathsf{V}$, and connected.

Assumption 2.3 implies that $\mathcal{G}$ is a *metric measure space*, cf. [49, Section 3], in the sense that for all $x \in \mathcal{G}$, there is $r_0 > 0$ such that the ball with centre $x$ and radius $r_0$ has finite volume.

We will be mostly interested in *infinite* metric graphs, i.e., for which $\mathsf{E}$ is a (countably) infinite set. Note that due to Assumption 2.3, this is equivalent to $\mathsf{V}$ being infinite. We will also use the following notions:

**Definition 2.4.** Let $\mathcal{G}$ be a locally finite, connected metric graph.

(1) A *walk* is the image of a continuous map $c\colon [0,1] \to \mathcal{G}$.
(2) A *path* is an injective walk.
(3) A *cycle* $C \subset \mathcal{G}$ is a compact subset of $\mathcal{G}$ such that, for all $x, y \in C$, $C \setminus \{x,y\}$ consists of precisely two disjoint paths in $\mathcal{G}$ connecting $x$ and $y$.
(4) The *(first) Betti number* $\beta \in \mathbb{N}_0 \cup \{\infty\}$ of $\mathcal{G}$ is the cardinality of any basis of the cycles of $\mathcal{G}$, that is, the cardinality of any minimal set of cycles of $\mathcal{G}$ whose union, treated as a subset of $\mathcal{G}$, contains all cycles in $\mathcal{G}$.
(5) We call $\mathcal{G}$ a *tree* if it contains no cycles as induced subgraphs.
(6) We call $\mathcal{G}$ *doubly (path) connected* if, for all $x, y \in \mathcal{G}$, there exist two paths $P_1, P_2 \subset \mathcal{G}$ connecting $x$ and $y$, such that $P_1$ and $P_2$ intersect at at most finitely many vertices.

*Remark* 2.5. Note that every locally finite metric graph $\mathcal{G}$ is a *length space*, i.e., for any two points $x, y \in \mathcal{G}$, one has

$$\mathrm{dist}_{\mathcal{G}}(x,y) = \inf L(c) \tag{2.3}$$

where the infimum in (2.3) is taken over all rectifiable curves $c$ in $\mathcal{G}$ connecting $x$ and $y$ and $L(c)$ denotes the length of $c$. Clearly, for $x \neq y$, it suffices in (2.3) to infimize over all *injective* rectifiable curves.

Finally, note that, unlike compact graphs, infinite metric graphs need not be *geodesic spaces*, i.e., given $x, y \in \mathcal{G}$, there may be no *geodesic curve* yielding the infimum in (2.3).

*Remark* 2.6. If $\beta < \infty$, then $\mathcal{G}$ in essence consists of a compact core (subgraph) to which finitely many tree subgraphs are attached. More precisely, since the union of all its (necessarily only finitely many) cycles will be compact, there exists a (generally non-unique) compact, connected subgraph $\mathcal{K}$ of $\mathcal{G}$ which contains them all; thus, $\mathcal{G} \setminus \mathcal{K}$ is a disjoint union of finitely many trees, each of finite or infinite diameter, and each attached to $\mathcal{K}$ at a single vertex. In particular, $\beta = 0$ if and only if $\mathcal{G}$ is itself a tree.

Whenever $\mathsf{E}$ is infinite, the *ends* of $\mathcal{G}$ can be defined as in [28]:

**Definition 2.7.** A *ray* is a sequence of distinct vertices $(\mathsf{v}_n)_{n\in\mathbb{N}}$ such that $\mathsf{v}_n \sim \mathsf{v}_{n+1}$ for all $n \in \mathbb{N}$. Two rays $\mathcal{R}_1$ and $\mathcal{R}_2$ are *equivalent* if there exists a third ray containing infinitely many vertices of both $\mathcal{R}_1$ and $\mathcal{R}_2$. An equivalence class $\omega$ of rays is called a *graph end*. The set of all graph ends of $\mathcal{G}$ is denoted by $\Omega(\mathcal{G})$. We will write $\mathcal{R} \in \omega$ whenever $\mathcal{R}$ is a ray belonging to the end $\omega \in \Omega(\mathcal{G})$.

*Remark* 2.8. At the risk of redundancy, let us stress that *ends are not vertices*. In particular, an end cannot be an endpoint of an edge. For this reason a subset or subgraph of $\mathcal{G}$ is compact if and only if it is closed in $\mathcal{G}$ and intersects a finite number of edges of $\mathcal{G}$.

An important feature of graph ends is their relation to topological ends of a metric graph $\mathcal{G}$. The following two definitions and theorem are from [28].

**Definition 2.9** (Topological end)**.** Consider a sequence $\mathcal{U} = (U_n)$ of nonempty open connected subsets of $\mathcal{G}$ with compact boundaries and such that $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} \overline{U_n} = \emptyset$. Two such sequences $\mathcal{U} = (U_n)$ and $\mathcal{U}' = (U'_n)$ are called *equivalent* if for all $n \in \mathbb{N}$ there exist $j, k$, such that $U'_j \subset U_n$ and $U_k \subset U'_n$. An equivalence class $\gamma$ of sequences is called a topological end of $\mathcal{G}$ and $\mathfrak{C}(\mathcal{G})$ denotes the set of all topological ends of $\mathcal{G}$.

Denote by $\overline{\mathcal{G}}$ the union of $\mathcal{G}$ with its topological ends, a complete metric space when equipped with the canonical metric. Note that if $L(\mathcal{G}) < \infty$, then $\overline{\mathcal{G}}$ is a compact metric space. The converse holds if, additionally, $\mathcal{G}$ is locally finite, i.e., if $\deg(\mathsf{v}) < \infty$ for all $\mathsf{v} \in \mathsf{V}$.

**Theorem 2.10.** *For each topological graph end $\gamma$ of a locally finite graph $\mathcal{G}$ there exists a graph end $\omega_\gamma \in \Omega(\mathcal{G})$ such that for any sequence $\mathcal{U} = (U_n)$, representing $\gamma$ each $U_n$ contains a ray $\mathcal{R}$ of $\omega_\gamma$. The mapping $\gamma \mapsto \omega_\gamma$ defines a bijection between the set of graph ends of $\mathcal{G}$ and the set of topological ends of $\mathcal{G}$.*

In particular, Theorem 2.10 implies that under Assumption 2.3 the notions of topological end and graph end coincide.

**Definition 2.11.** An end $\gamma \in \mathfrak{C}(\mathcal{G})$ has *finite volume* if there is a sequence $\mathcal{U} = (U_n)$ representing $\gamma$ such that its total length $L(U_n)$ is finite for some $n$.

Here and throughout we will denote by $\mathfrak{C}(\mathcal{G})$ and $\mathfrak{C}_0(\mathcal{G})$ the set of ends of $\mathcal{G}$ and the set of finite volume ends of $\mathcal{G}$, respectively.

*Example* 2.12. (1) Consider the vertex set $\mathbb{Z}$, let at each vertex $n$ the vertices $n \pm 1$ be the only adjacent vertices, and let each edge have unit length. This is an infinite metric graph with two ends, which we can identify with $\pm\infty$. An analogous construction based on $\mathbb{Z}^d$, for any $d \geq 2$, leads to an infinite metric graph with only one end.
(2) Consider a rooted binary tree, i.e., a tree each of whose vertices (except for the root) has degree 3. Depending on the lengths we assign to its edges, the graph can have finite total diameter and even finite total length. At the same time, it always has uncountably many ends, which are in a bijective relation with the interval $[0, 1]$.
(3) Take the interval $[-1, 1]$ and place a vertex at the points $(n^{-1})_{n\in\mathbb{N}}$. Intuitively, we are attaching an interval of length 1 to the only end in the metric graph from Example 2.1. However, this is *not* a metric graph in the sense defined above since the point 0 cannot be both an end and a vertex.

2.2. **Compact exhaustions.** For an infinite graph $\mathcal{G}$, we are going to consider appropriate sequences of compact subgraphs $\mathcal{G}_n$ which approximate $\mathcal{G}$ in a suitable way. This will allow us to generalise many properties of $\mathcal{G}_n$ directly to $\mathcal{G}$, in particular some eigenvalue bounds.

**Definition 2.13.** Let $\mathcal{G}$ be a locally finite, connected metric graph. A sequence of induced subgraphs $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is called a *compact exhaustion* of $\mathcal{G}$ if the following conditions are satisfied:

(1) $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots \subset \mathcal{G}$;
(2) for all $\mathsf{e} \in \mathsf{E}$ there exists $n \in \mathbb{N}$ such that $\mathsf{e} \subset \mathcal{G}_n$;
(3) each $\mathcal{G}_n$ is connected;
(4) for all $n \in \mathbb{N}$, $\mathcal{G}_n$ is a compact subgraph of $\mathcal{G}$.

Recall from Remark 2.8 that compact subgraphs of $\mathcal{G}$ are exactly those which intersect only a finite number of edges of $\mathcal{G}$. Properties (1) and (2) imply in particular $\bigcup_{n\in\mathbb{N}} \mathcal{G}_n = \mathcal{G}$ and that, if $\mathcal{G}$ is compact, then every compact exhaustion of $\mathcal{G}$ will eventually become stationary.

*Example* 2.14. We will often use the following construction, which yields a compact exhaustion in any locally finite, connected graph $\mathcal{G}$:

Fix a vertex $\mathsf{v} \in \mathsf{V}$. Then, any $\mathsf{w} \in \mathsf{V}$ has a well-defined combinatorial distance to $\mathsf{v}$. Set

$$\mathcal{G}_{\mathsf{v},n}$$

to be the induced subgraph of $\mathcal{G}$ containing all vertices of $\mathcal{G}$ of combinatorial distance at most $n \in \mathbb{N}$ to $\mathsf{v}$. Recall from Definition 2.2 that the adjacency relations in $\mathcal{G}_{\mathsf{v},n}$ are chosen to mirror those in $\mathcal{G}$, see Figure 2.1 for an illustration.

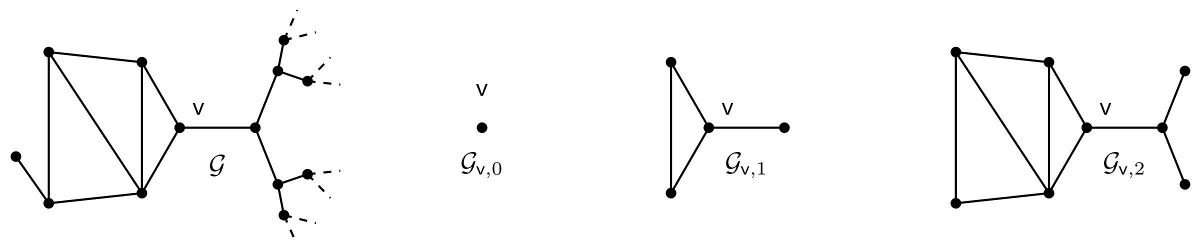


FIGURE 2.1. A metric graph $\mathcal{G}$ and its approximations $\mathcal{G}_{\mathsf{v},0}$, $\mathcal{G}_{\mathsf{v},1}$ and $\mathcal{G}_{\mathsf{v},2}$, as in Example 2.14.

Then by construction $(\mathcal{G}_{\mathsf{v},n})_{n\in\mathbb{N}}$ is a compact exhaustion of $\mathcal{G}$. In fact, more is true: if $\mathcal{G}$ has an infinite edge set, then $\mathcal{G}_{\mathsf{v},n}$ is *compactly* contained in a proper open subset of $\mathcal{G}_{\mathsf{v},n+1}$ for all $n \in \mathbb{N}$.

The following Lemma 2.15, which states that all compact exhaustions are equivalent, follows directly from Definition 2.13. In particular, it justifies that we always use the compact exhaustions of Example 2.14.

**Lemma 2.15.** *Let $\mathcal{G}$ be a locally finite, connected metric graph, let $\mathsf{v} \in \mathsf{V}$ be any vertex of $\mathcal{G}$, let $(\mathcal{G}_{\mathsf{v},n})_{n\in\mathbb{N}}$ be the compact exhaustion from Example 2.14, and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be any other compact exhaustion of $\mathcal{G}$. Then for all sufficiently large $n \in \mathbb{N}$ there exist positive integers $k_1 = k_1(n)$ and $k_2 = k_2(n)$ such that*

$$\mathcal{G}_{\mathsf{v},k_1} \subset \mathcal{G}_n \subset \mathcal{G}_{\mathsf{v},k_2}. \tag{2.4}$$

*Moreover, $k_2 \to \infty$, and $k_1$ can be chosen to tend to $\infty$, as $n \to \infty$.*

For compact $\mathcal{G}$ there are two equivalent ways to define the first Betti number $\beta$: Either as the number of independent cycles in $\mathcal{G}$, as in Definition 2.4, or as $\beta = \#\mathsf{E} - \#\mathsf{V} + 1$. Obviously, the second definition no longer has a meaning on infinite graphs. However, it can be obtained via compact exhaustions, as stated in the following proposition.

**Proposition 2.16.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with Betti number $\beta = \beta(\mathcal{G})$, and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a compact exhaustion of $\mathcal{G}$.*

*(1) If $\beta < \infty$, then $\beta(\mathcal{G}_n)$ is eventually constant and equal to $\beta$.*
*(2) If $\beta = \infty$, then $\beta(\mathcal{G}_n) \to \infty$.*

In particular, for infinite graphs, $\beta$ can alternatively be defined as

$$\lim_{n\to\infty} \beta(\mathcal{G}_n) = \#\mathsf{E}(\mathcal{G}_n) - \#\mathsf{V}(\mathcal{G}_n) + 1,$$

independently of the choice of the compact exhaustion $(\mathcal{G}_n)_{n\in\mathbb{N}}$.

*Proof.* Without loss of generality, we may assume that $\mathcal{G}$ is infinite. Note that (1) $\beta(\mathcal{G}_n) \in \mathbb{N}_0$ for all $n \in \mathbb{N}$; and (2) every cycle in $\mathcal{G}_n$ is also a cycle in $\mathcal{G}$, so $\beta(\mathcal{G}_n) \leq \beta$ for all $n \in \mathbb{N}$. Fix $\mathsf{v} \in \mathsf{V}$ arbitrary and let $(\mathcal{G}_{\mathsf{v},n})_{n\in\mathbb{N}}$ be the compact exhaustion from Example 2.14. Then, (2.4) and the fact that the $\mathcal{G}_{\mathsf{v},n}$ as induced subgraphs have maximal connectivity imply that (with $k_1$ and $k_2$ as in (2.4))

$$\beta(\mathcal{G}_{\mathsf{v},k_1}) \leq \beta(\mathcal{G}_n) \leq \beta(\mathcal{G}_{\mathsf{v},k_2}), \tag{2.5}$$

for sufficiently large $n$.

Thus it suffices to prove the proposition for the sequence $(\mathcal{G}_{\mathsf{v},n})_{n\in\mathbb{N}}$. Note that the maximal connectivity also implies that $\beta(\mathcal{G}_{\mathsf{v},n})$ is a non-decreasing sequence in $n \in \mathbb{N}$. If $\beta < \infty$, there is a finite basis of cycles of $\mathcal{G}$. Since every point on every cycle in this basis may be reached from $\mathsf{v}$ via a path consisting of a finite number of edges, all of these cycles will eventually be contained in $\mathcal{G}_{\mathsf{v},n}$. In particular, $\beta(\mathcal{G}_{\mathsf{v},n}) = \beta$ for all sufficiently large $n \in \mathbb{N}$.

If $\beta = \infty$, then for every $m \in \mathbb{N}$ one finds $m$ independent cycles which contain finite sets of edges and vertices. Thus, for sufficiently large $n$, they will be completely contained in $\mathcal{G}_{\mathsf{v},n}$, and so $\beta(\mathcal{G}_{\mathsf{v},n}) \geq m$. Since $m$ was arbitrary, this shows the claim. □

We turn to the diameter of compact exhaustions. The assumption that they are connected ensures that they all have finite diameter.

**Proposition 2.17.** *Let $\mathcal{G}$ be a locally finite, connected metric graph and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be any compact exhaustion of $\mathcal{G}$. If $\mathcal{G}$ has finite diameter $D(\mathcal{G})$, then*

$$D(\mathcal{G}) \leq \liminf_{n\to\infty} D(\mathcal{G}_n).$$

*If the diameter of $\mathcal{G}$ is infinite, then*

$$\lim_{n\to\infty} D(\mathcal{G}_n) = D(\mathcal{G}) = \infty. \tag{2.6}$$

*Proof.* Let $x_k, y_k \in \mathcal{G}$ be such that $\operatorname{dist}_{\mathcal{G}}(x_k, y_k) \to D(\mathcal{G}) \in [0,\infty]$. Since $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a compact exhaustion, by properties (1) and (2), for each $k \in \mathbb{N}$ there exists some $n_k \in \mathbb{N}$ such that $x_k, y_k \in \mathcal{G}_n$ for all $n \geq n_k$. Now since $\mathcal{G}_n \subset \mathcal{G}$ we clearly have $\operatorname{dist}_{\mathcal{G}_n}(x,y) \geq \operatorname{dist}_{\mathcal{G}}(x,y)$ for all $x, y \in \mathcal{G}_n$ and all $n \in \mathbb{N}$; putting all this together, it follows that

$$\operatorname{dist}_{\mathcal{G}}(x_k, y_k) \leq \operatorname{dist}_{\mathcal{G}_n}(x_k, y_k) \leq D(\mathcal{G}_n)$$

for all $n \geq n_k$. Since $\operatorname{dist}_{\mathcal{G}}(x_k, y_k) \to D(\mathcal{G})$, the claim follows. □

**Proposition 2.18.** *Let $\mathcal{G}$ be a locally finite, connected metric graph. If $D(\mathcal{G}) < \infty$ assume in addition that the Betti number $\beta$ is finite. Then there exists a compact exhaustion $(\mathcal{G}_n)_{n\in\mathbb{N}}$ such that $D(\mathcal{G}_n) \to D(\mathcal{G})$.*

We again recall that, by Remark 2.8, finiteness of the Betti number is equivalent to the condition that there is a compact subgraph of $\mathcal{G}$ containing all cycles. In the case that $D(\mathcal{G}) < \infty$, our proof will show that $D(\mathcal{G}_n) \leq D(\mathcal{G})$ for sufficiently large $n$.

*Proof.* It suffices to consider the case that $\mathcal{G}$ is infinite and, due to (2.6), that $D(\mathcal{G}) < \infty$. We fix $\mathsf{v} \in \mathsf{V}$, and consider the compact exhaustion $\mathcal{G}_{\mathsf{v},n}$ of Example 2.14. For $n \in \mathbb{N}$ large enough that all cycles of $\mathcal{G}$ are contained in $\mathcal{G}_{\mathsf{v},n}$, as noted in Remark 2.6, $\mathcal{G} \setminus \mathcal{G}_{\mathsf{v},n}$ is a disjoint union of trees, each attached to $\mathcal{G}_{\mathsf{v},n}$ at a single vertex.

We claim that, for such $n$,

$$\operatorname{dist}_{\mathcal{G}_{\mathsf{v},n}}(x,y) = \operatorname{dist}_{\mathcal{G}}(x,y) \qquad \text{for all } x, y \in \mathcal{G}_{\mathsf{v},n}. \tag{2.7}$$

Indeed, let $P$ be a walk in $\mathcal{G}$ from $x$ to $y$, see Definition 2.4. It suffices to find a walk $P_n$ in $\mathcal{G}_{\mathsf{v},n}$ connecting $x$ and $y$ which is no longer than $P$, since (2.7) will then follow immediately. In fact, if the walk $P$ is *not* contained in $\mathcal{G}_{\mathsf{v},n}$, then part of it must lie in one or more of the trees of which $\mathcal{G} \setminus \mathcal{G}_{\mathsf{v},n}$ consists. Suppose this tree is attached to $\mathcal{G}_{\mathsf{v},n}$ at a single vertex $\mathsf{w}$; then $P$ must pass through $\mathsf{w}$ twice. Cut out the part of $P$ beyond $\mathsf{w}$, glue the two remaining parts of the walk together, and repeat for every connected component of $\mathcal{G} \setminus \mathcal{G}_{\mathsf{v},n}$ through which $P$ passes. The new walk $P_n$ lies in $\mathcal{G}_{\mathsf{v},n}$, still connects $x$ and $y$, and has shorter length. This proves the claim.

It follows from (2.7) and the fact that $\mathcal{G}_{\mathsf{v},n} \subset \mathcal{G}$ that, for all $n$ for which (2.7) holds,

$$D(\mathcal{G}_{\mathsf{v},n}) = \sup_{x,y\in\mathcal{G}_{\mathsf{v},n}} \operatorname{dist}_{\mathcal{G}_{\mathsf{v},n}}(x,y) = \sup_{x,y\in\mathcal{G}_{\mathsf{v},n}} \operatorname{dist}_{\mathcal{G}}(x,y) \leq \sup_{x,y\in\mathcal{G}} \operatorname{dist}_{\mathcal{G}}(x,y) = D(\mathcal{G}).$$

Combining this with the result of Proposition 2.17 yields the conclusion. □

2.3. **Lebesgue and Sobolev spaces.** We next introduce function spaces on a metric graph $\mathcal{G}$, embedding theorems for which will be discussed in Section 3.

Any metric graph $\mathcal{G}$ has both a topological and a measure theoretical structure. This immediately defines the space of continuous functions and Lebesgue spaces. However, for clarity, we will recall the definitions and important properties. Here and throughout, if $f : \mathcal{G} \to \mathbb{R}$ is a function and $\mathsf{e} \in \mathsf{E}$ an edge of $\mathcal{G}$, then $f_\mathsf{e}$ will denote the restriction of $f$ to $\mathsf{e}$.

**Definition 2.19.** Define

$$L^2(\mathcal{G}) := \left\{ f \in \bigoplus_{\mathsf{e}\in\mathsf{E}} L^2(\mathsf{e}) : \sum_{\mathsf{e}\in\mathsf{E}} \|f_\mathsf{e}\|^2_{L^2(\mathsf{e})} < \infty \right\},$$

equipped with the canonical inner product and norm, and define

$$L^2_c(\mathcal{G}) := \left\{ f \in L^2(\mathcal{G}) : f(x) = 0 \text{ almost everywhere outside a compact subset of } \mathcal{G} \right\}.$$

The other $L^p$-spaces, $p \in [1, \infty]$, are defined analogously. In view of Remark 2.8, any function in $L^2_c(\mathcal{G})$ is identically zero outside of a finite set of edges of $\mathcal{G}$.

**Definition 2.20.** Denote by $C(\mathcal{G})$ the set of all functions $f : \mathcal{G} \to \mathbb{R}$ which are continuous with respect to the canonical metric on $\mathcal{G}$, and by

$$C_c(\mathcal{G}) := \{f \in C(\mathcal{G}) : f(x) = 0 \text{ outside a compact subset of } \mathcal{G}\}$$

the set of continuous functions of compact support.

Again, $C_c(\mathcal{G})$-functions are necessarily supported on a finite set of edges, and, if $\mathcal{G}$ has any ends, then it is not a closed space. Denoting by $\overline{\mathcal{G}}$ the metric space which is the union of $\mathcal{G}$ with its ends as in Section 2.1, we also write $C(\overline{\mathcal{G}})$ for the complete space of all continuous functions on $\overline{\mathcal{G}}$. It is canonically identified with the space of continuous functions on $\mathcal{G}$ that can be continuously extended to the ends of the graph.

**Definition 2.21.** Denote by $H^1(\mathcal{G})$ the Sobolev space

$$H^1(\mathcal{G}) := \left\{ f \in C(\mathcal{G}) : f_\mathsf{e} \in H^1(\mathsf{e}) \text{ for all } \mathsf{e} \in \mathsf{E}, \text{ and } \sum_{\mathsf{e}\in\mathsf{E}} \|f_\mathsf{e}\|^2_{H^1(\mathsf{e})} < \infty \right\},$$

equipped with the canonical inner product and norm. Here, the space $H^1(\mathsf{e})$ consists exactly of the absolutely continuous functions on $\mathsf{e}$ whose distributional derivative is in $L^2(\mathsf{e})$.

Replacing $L^2$-spaces by $L^p$-spaces in the definition, we also obtain the Sobolev spaces $W^{1,p}(\mathcal{G})$ for $p \geq 1$; in particular, $W^{1,2}(\mathcal{G}) = H^1(\mathcal{G})$.

It is not hard to show that, for any $f \in W^{1,p}(\mathcal{G})$, there is a canonical extension of $f$ to the ends of $\mathcal{G}$ (see [28, Definition 3.3]). In particular, $f \in W^{1,p}(\mathcal{G})$ can be identified with a function on $\overline{\mathcal{G}}$; up to a canonical identification, we have $W^{1,p}(\mathcal{G}) \subset C(\overline{\mathcal{G}})$ (see also Lemma 3.1).

We also define the (not necessarily closed) subspace of all $H^1$-functions of compact support,

$$H^1_c(\mathcal{G}) := \left\{ f \in H^1(\mathcal{G}) : f(x) = 0 \text{ outside a compact subset of } \mathcal{G} \right\}$$

and its completion in $H^1$,

$$H^1_0(\mathcal{G}) := \overline{H^1_c(\mathcal{G})}^{\|\cdot\|_{H^1(\mathcal{G})}}.$$

The issue whether $H^1_0(\mathcal{G}) = H^1(\mathcal{G})$ is intimately related to the essential self-adjointness of the Laplacian $\Delta_{\mathcal{G}}|_{L^2_c}$. This will be briefly discussed in Section 2.4, and is addressed in more detail in [28].

*Remark* 2.22. (1) We have already remarked that inserting or deleting a dummy vertex does not change the metric structure of $\mathcal{G}$. Indeed, it is easy to see (and well known) that, up to isometric isomorphism, the spaces $H^1(\mathcal{G})$ and $L^2(\mathcal{G})$ are not modified, either, upon inserting or deleting dummy vertices.

(2) Similarly to (1), we observe that formally modifying $\mathcal{G}$ by identifying two or more Dirichlet vertices does not affect the spaces $H^1_0(\mathcal{G})$ or $L^2(\mathcal{G})$, either as sets or in their topological properties.

(3) One has

$$H^1_0(\mathcal{G}) = \{f \in H^1(\mathcal{G}) : f(\gamma) = 0 \text{ for all } \gamma \in \mathfrak{C}(\mathcal{G})\},$$

that is, $H^1_0(\mathcal{G})$-functions satisfy a Dirichlet (zero) condition at every end, see [28, Theorem 3.12]. This will provide a unified way to prescribe *Dirichlet conditions at vertices*: Indeed, to impose a Dirichlet condition on a vertex $\mathsf{v}$, simply remove $\mathsf{v}$ and appropriately add infinitely many dummy vertices on all edges incident to it. This way, $\mathsf{v}$ is replaced by $\deg(\mathsf{v})$ many ends $\gamma$, and every $f \in H^1_0(\mathcal{G})$ must vanish at $\mathsf{v}$.

(4) Correspondingly, it will also be interesting to consider Sobolev spaces associated with $\mathcal{G}$ where we impose Dirichlet conditions on a subset of $\mathfrak{C}(\mathcal{G}) \cup \mathsf{V}$; for some subset $\mathfrak{V} \subset \mathfrak{C}(\mathcal{G}) \cup \mathsf{V}$ we define the space

$$H^1_0(\mathcal{G};\mathfrak{V}) := \{f \in H^1(\mathcal{G}) : f(\mathsf{v}) = 0 \text{ for all } \mathsf{v} \in \mathfrak{V}\}.$$

We regard this as imposing a Dirichlet condition at each vertex and each end belonging to $\mathfrak{V}$.

2.4. **Laplace-type operators on infinite graphs.** Here we recall several natural realisations of the Laplacian on a locally finite graph $\mathcal{G}$. We start by defining the maximal Laplacian, which does not see the topological structure of the graph.

**Definition 2.23.** For an arbitrary edge $\mathsf{e}$ of $\mathcal{G}$ we denote by

$$\mathcal{H}_{\mathsf{e},\max} = -\frac{\mathrm{d}^2}{\mathrm{d}x_{\mathsf{e}}^2}, \qquad \operatorname{dom}(\mathcal{H}_{\mathsf{e},\max}) = H^2(\mathsf{e})$$

the Laplacian on $\mathsf{e}$ on its maximal domain $H^2(\mathsf{e})$, that is, the space of $L^2(\mathsf{e})$ functions with their distributional derivative in $H^1(\mathsf{e})$. The maximal Laplacian is then defined by

$$\mathcal{H}_{\max} = \bigoplus_{\mathsf{e}\in\mathsf{E}} \mathcal{H}_{\mathsf{e},\max}, \qquad \operatorname{dom}(\mathcal{H}_{\max}) = \bigoplus_{\mathsf{e}\in\mathsf{E}} H^2(\mathsf{e}).$$

In practice we will always restrict this operator to spaces satisfying standard vertex conditions: firstly, any function $f$ in the operator domain should be in $C(\mathcal{G})$; secondly, it should satisfy the Kirchhoff condition, where the sum of the normal derivatives of $f$ at every vertex should be zero:

$$\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{v}}} \frac{\partial}{\partial\nu_{\mathsf{e}}} f(\mathsf{v}) = 0 \qquad \text{for all } \mathsf{v} \in \mathsf{V}.$$

Imposing only these two conditions leads to the *maximal Kirchhoff Laplacian* on $\mathcal{G}$:

**Definition 2.24.** Call the operator $\mathcal{H}$ on $L^2(\mathcal{G})$ defined by

$$\begin{aligned}\operatorname{dom}(\mathcal{H}) &= \left\{ f \in \operatorname{dom}(\mathcal{H}_{\max}) \cap L^2(\mathcal{G}) : f \in C(\mathcal{G}) \text{ and } \sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{v}}} \frac{\partial}{\partial\nu_{\mathsf{e}}} f(\mathsf{v}) = 0 \text{ for all } \mathsf{v} \in \mathsf{V} \right\}\\ \mathcal{H}f &= \mathcal{H}_{\max} f\end{aligned}$$

the *maximal Kirchhoff Laplacian* on $\mathcal{G}$.

Alternatively, we can restrict to the (closure of) the space of functions of compact support to obtain the *minimal Kirchhoff Laplacian*.

**Definition 2.25.** Define an operator $\mathcal{H}^0_0$ on $L^2(\mathcal{G})$ by

$$\begin{aligned}\operatorname{dom}(\mathcal{H}^0_0) &:= \left\{ f \in \operatorname{dom}(\mathcal{H}_{\max}) \cap L^2_c(\mathcal{G}) : f \in C(\mathcal{G}) \text{ and } \sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{v}}} \frac{\partial}{\partial\nu_{\mathsf{e}}} f(\mathsf{v}) = 0 \text{ for all } \mathsf{v} \in \mathsf{V} \right\},\\ \mathcal{H}^0_0 f &:= \mathcal{H}_{\max} f\end{aligned}$$

and call its closure

$$\mathcal{H}_0 := \overline{\mathcal{H}^0_0}^{L^2(\mathcal{G})}$$

the *minimal Kirchhoff Laplacian* on $\mathcal{G}$.

While $\mathcal{H}_0$ is symmetric, it is not necessarily self-adjoint. In fact, [29, Lemma 2.7] states that on any locally finite metric graph $\mathcal{G}$ we have $\mathcal{H}^*_0 = \mathcal{H}$ : in particular, $\mathcal{H}_0$ will never be self-adjoint on any locally finite graph $\mathcal{G}$ if there is a graph end of finite volume.

In order to construct self-adjoint extensions of these operators, it is natural to resort to quadratic forms:

$$\mathfrak{t}_{\mathrm{N}}[f] := \int_{\mathcal{G}} |f'(x)|^2 \,\mathrm{d}x, \qquad \mathrm{dom}[t_{\mathrm{N}}] := H^1(\mathcal{G}), \tag{2.8}$$

$$\mathfrak{t}_{\mathrm{F}}[f] := \int_{\mathcal{G}} |f'(x)|^2 \,\mathrm{d}x, \qquad \mathrm{dom}[t_{\mathrm{F}}] := H^1_0(\mathcal{G}). \tag{2.9}$$

**Definition 2.26.** The *Neumann extension* $\mathcal{H}_{\mathrm{N}}$ is the self-adjoint operator on $L^2(\mathcal{G})$ associated with $\mathfrak{t}_{\mathrm{N}}$, and the *Friedrichs extension* $\mathcal{H}_{\mathrm{F}}$ is the self-adjoint operator on $L^2(\mathcal{G})$ associated with $\mathfrak{t}_{\mathrm{F}}$.

It can be shown, see [28, Corollary 6.7], that on metric graphs with a finite number of ends of finite volume, $\mathcal{H}_{\mathrm{N}}$ corresponds to imposing Neumann boundary conditions at all ends while the Friedrichs extension $\mathcal{H}_{\mathrm{F}}$ imposes Dirichlet boundary conditions at all ends.

*Remark* 2.27. We can define, and some of our results in the following sections extend to, mixed versions of the Neumann and Friedrichs extensions: given $\mathfrak{V} \subset \mathsf{V} \cup \mathfrak{C}(\mathcal{G})$, take the quadratic form given by

$$\mathfrak{t}_{\mathfrak{V}}[f] := \int_{\mathcal{G}} |f'(x)|^2 \,\mathrm{d}x, \qquad \mathrm{dom}[\mathfrak{t}_{\mathfrak{V}}] := H^1_0(\mathcal{G};\mathfrak{V}).$$

Since $H^1_0(\mathcal{G};\mathfrak{V})$ is a closed subspace of $H^1(\mathcal{G})$ that is dense in $L^2(\mathcal{G})$, the form $\mathfrak{t}_{\mathfrak{V}}$ induces a self-adjoint operator $\mathcal{H}_{\mathfrak{V}}$ in $L^2(\mathcal{G})$. Roughly speaking, for all functions in the domain of $\mathcal{H}_{\mathfrak{V}}$ we are imposing Dirichlet-type conditions at *some* ends and/or at *some* vertices, along with standard conditions at all other vertices and Neumann conditions at all other ends. We will sometimes sloppily refer to $\mathcal{H}_{\mathfrak{V}}$ as the *Laplacian with mixed conditions.*

## 3. Embedding theorems and discreteness of the spectrum

3.1. **Embedding theorems for Sobolev spaces.** In this section, we present criteria for discreteness of the spectrum of the operator $\mathcal{H}_{\mathrm{N}}$ and, hence, of all self-adjoint Laplacian extensions dominating $\mathcal{H}_{\mathrm{N}}$ in the sense of forms, including $\mathcal{H}_{\mathrm{F}}$. Note that discreteness of the spectrum of self-adjoint extensions is equivalent to the compactness of the embedding of their form domains in $L^2(\mathcal{G})$. Consequently, this subsection is largely devoted to Sobolev embedding theorems.

Abstract criteria for the compactness of embeddings of Sobolev spaces over metric measure spaces, as introduced in [21], are also known; we refer, e.g., to [24, Theorem 2]. Other sufficient criteria of more geometric flavour have been provided in [29, Corollary 3.5 and Corollary 4.5], [16, Theorem 4.18]. We also mention [48] which provides criteria for the discreteness of the spectrum and investigates the asymptotics of the eigenvalues on so-called *regular trees.*

We start with [28, Lemma 3.2], which will be used in several places below.

**Lemma 3.1.** *Let $\mathcal{G}$ be a locally finite, connected metric graph. Then $H^1(\mathcal{G})$ embeds continuously into $C(\overline{\mathcal{G}}) \cap L^\infty(\mathcal{G})$.*

More can be said if $\mathcal{G}$ has finite total length: the following is [11, Theorem 2.2].

**Theorem 3.2.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite total length. Then $H^1(\mathcal{G})$ embeds compactly into $L^2(\mathcal{G})$.*

We observe that, under the assumptions of Theorem 3.2, it is already known that the resolvent of $\mathcal{H}_{\mathrm{N}}$ is a trace class operator, see [28, Theorem 5.1]: both results imply, of course, that $\mathcal{H}_{\mathrm{N}}$ has compact resolvent and that, accordingly, the embedding of $H^1(\mathcal{G})$ into $L^2(\mathcal{G})$ is compact, see also [29, Corollary 3.5(iv)].

The proof of [28, Theorem 5.1] is based on a relatively technical approach using properties of the integral kernel of the resolvent operator.

*Remark* 3.3. Closely looking at the proof of [11, Theorem 2.2] shows that $H^1(\mathcal{G})$ is actually continuously embedded in the Hölder space $C^{0,\frac{1}{2}}(\overline{\mathcal{G}})$, rather than merely in $C(\overline{\mathcal{G}})$.

Actually, more is true. If $\mathcal{G}$ has finite length, the same proof clearly gives the continuity of the embedding $W^{1,p}(\mathcal{G}) \hookrightarrow C^{0,\frac{1}{q}}(\overline{\mathcal{G}})$, $p \in [1,\infty]$, where $q$ is the dual exponent of $p$. This yields that any $W^{1,p}(\mathcal{G})$, $p \in (1,\infty]$, embeds compactly into any $L^r(\mathcal{G})$, $r \in [1,\infty]$.

An immediate consequence is:

**Corollary 3.4.** *Let $\mathcal{G}$ be locally finite, connected metric graph with finite total length. Then, any Markovian extension $\mathcal{H}$ of $\mathcal{H}_0$ (in particular, $\mathcal{H}_\mathrm{F}$ and $\mathcal{H}_\mathrm{N}$) has purely discrete spectrum.*

Our second criterion is subtler. Inspired by [22], we can provide a Kolmogorov–Riesz-type result that characterises those form domains that are compactly embedded in $L^2(\mathcal{G})$ or those graphs, where certain form domains are compactly embedded.

**Theorem 3.5.** *Let $\mathcal{G}$ be a locally finite, connected metric graph, and let $K$ be a closed subspace of $H^1(\mathcal{G})$. Then the embedding of $K$ into $L^2(\mathcal{G})$ is compact if and only if for all $\varepsilon > 0$ there is a finite subgraph $\mathcal{G}_c$ of $\mathcal{G}$ such that*

$$\|f\|_{L^2\left((\mathcal{G}_c)^\complement\right)} \le \varepsilon \quad \text{for all } f \in K \text{ with } \quad \|f\|_{H^1(\mathcal{G})} \le 1, \tag{3.1}$$

*where $(\mathcal{G}_c)^\complement$ denotes the complement of $\mathcal{G}_c$ within $\mathcal{G}$.*

Theorem 3.5 implies Theorem 3.2, since if $\mathcal{G}$ has finite length, then $H^1(\mathcal{G}) \hookrightarrow L^\infty(\mathcal{G})$, and thus $\|f\|_{L^2((\mathcal{G}_c)^\complement)} \le |(\mathcal{G}_c)^\complement|^{1/2}\|f\|_{L^\infty((\mathcal{G}_c)^\complement)}$ can be made arbitrarily small independently of $f$ by choosing $|(\mathcal{G}_c)^\complement|$ is small enough. The usefulness of Theorem 3.5 will be demonstrated in Theorem 3.10 where compactness of the embedding is proved for a family of infinite metric graphs for which known criteria cannot be used.

*Proof.* ($\Longleftarrow$) Let $\varepsilon > 0$, take a finite subgraph $\mathcal{G}_c \subset \mathcal{G}$ as in the theorem, and denote by $B$ the unit ball within $K$. Construct a new graph $\mathcal{G}^*$ from $\mathcal{G}$ by doubling all edges in $\mathcal{G}_c$, that is, replacing each $\mathsf{e}$ in $\mathcal{G}_c$ with a pair $(\mathsf{e}', \mathsf{e}'')$ of identical edges, and keeping edges in $(\mathcal{G}_c)^\complement$. Let $B^*$ be the set of functions $f^* \in H^1(\mathcal{G}^*)$ such that there exists $f \in B$ with

$$f^*_{|\mathsf{e}'} = f^*_{|\mathsf{e}''} = f_{|\mathsf{e}} \quad \text{for all } \mathsf{e} \subset \mathcal{G}_c, \text{ and } \quad f^*_{(\mathcal{G}_c)^\complement} = f_{(\mathcal{G}_c)^\complement}.$$

The subgraph $\mathcal{G}_c^*$ can be identified with an Eulerian walk of finite length $p$ which allows to identify $f^*_{|\mathcal{G}_c^*} \in B^*$ with a function $\hat{f} \in H^1([0,p])$. On $\mathcal{G}_c^*$ we can then define

$$f^*(x+h) = \begin{cases} \hat{f}(x+h) & \text{if } x \in \mathcal{G}_c^*, \text{ and } x+h \in [0,p], \\ 0 & \text{if } x \in \mathcal{G}_c^*, \text{ and } x+h \notin [0,p]. \end{cases}$$

By uniform continuity of $\hat{f}$ on $[0,p]$ there is $\delta_0 > 0$ so that for each $\delta \le \delta_0$ and $h < \delta$ we have

$$\int_{\mathcal{G}_c^*} |f^*(x+h) - f^*(x)|^2 dx < \varepsilon^2. \tag{3.2}$$

We cover $[0,p]$ with $N$ non-overlapping intervals of length $\delta$,

$$[0,p] = \bigcup_{j=1}^N \overline{I_j}, \quad \text{where} \quad I_j \cap I_k = \emptyset, \quad \text{for } k \ne j, \quad \text{and} \quad |I_j| = \delta \quad \text{for all } j \in \{1, \dots, N\}.$$

This allows us to define an orthogonal projection $P\colon K \to L^2(\mathcal{G}^*)$

$$P(f^*)(x) := \begin{cases} \frac{1}{\delta}\int_{I_j} f^*(x)\mathrm{d}x & \text{if } x \in \mathcal{G}_c^*, \text{ and } x \in I_j \text{ on the Eulerian walk}, \\ 0 & \text{if } x \in (\mathcal{G}_c)^\complement. \end{cases}$$

We have

$$\|f^* - Pf^*\|^2_{L^2(\mathcal{G}^*)} = \int_{\mathcal{G}_c^\complement} |f^* - Pf^*|^2 + \int_{\mathcal{G}_c^*} |f^* - Pf^*|^2, \tag{3.3}$$

and note that the first summand in (3.3) is at most $\varepsilon^2$ by assumption. The second term in (3.3) can be considered as an integral along the Eulerian walk and estimated by

$$\sum_{i=1}^N \int_{I_j} \left| \frac{1}{\delta} \int_{I_j} (f^*(x) - f^*(y))\,\mathrm{d}y \right|^2 \mathrm{d}x \le \sum_{i=1}^N \int_{I_j} \frac{1}{\delta} \int_{I_j} |f^*(y) - f^*(x)|^2\,\mathrm{d}y\ \mathrm{d}x$$

We substitute $h = y - x$ and emphasise that $h \in (-\delta, \delta)$. Thus, (3.2) implies

$$\begin{aligned}\|f^* - Pf^*\|^2_{L^2(\mathcal{G}^*)} &\le \varepsilon^2 + \sum_{i=1}^{N} \int_{I_j} \frac{1}{\delta} \int_{-\delta}^{\delta} |f^*(x+h) - f^*(x)|^2 \,\mathrm{d}h \,\mathrm{d}x \\ &= \varepsilon^2 + \frac{1}{\delta} \int_{-\delta}^{\delta} \int_{\mathcal{G}_c^*} |f^*(x+h) - f^*(x)|^2 \mathrm{d}h \,\mathrm{d}x \le 3\varepsilon^2.\end{aligned}$$

Using the triangle inequality,

$$\|f^*\|_{L^2(\mathcal{G}^*)} \le \sqrt{3}\varepsilon + \|Pf^*\|_{L^2(\mathcal{G}^*)}.$$

Now, if $f^*, g^* \in B^*$ with $\|Pf^* - Pg^*\|_{L^2(\mathcal{G}^*)} < \varepsilon$, then we have

$$\|f^* - g^*\|_{L^2(\mathcal{G}^*)} < (\sqrt{3} + 1)\varepsilon.$$

Furthermore, $P$ is a bounded operator, and $B^*$ is bounded, thus $P(B^*)$ itself is bounded. Finally, since $P$ is of finite rank, it is totally bounded. By [22, Lemma 1], we find that $B^*$ is totally bounded in $L^2(\mathcal{G}^*)$, and thus also $B$ in $L^2(\mathcal{G})$.
($\Longrightarrow$) Conversely, assume that $K$ ist compactly embedded in $L^2(\mathcal{G})$. Then the unit ball $B \subset K$ is totally bounded in $L^2(\mathcal{G})$, and for every $\varepsilon > 0$ there is a finite $\varepsilon$-cover $\{U_1, \dots U_n\}$ with central points $g_1, \dots, g_n \in B$. For each $g_i$ there is a finite subgraph $\mathcal{G}_i$, so that $\|g_i\|_{L^2((\mathcal{G}_c)^\complement)} < \varepsilon$. Thus, there is a finite and connected subgraph $\mathcal{G}_c$ that fulfils the conditions for all $g_i$ with $1 \le i \le n$. Let now $f \in B$. There is $g_i$ with $\|f - g_i\|_{L^2(\mathcal{G})} < \varepsilon$, whence by the triangle inequality

$$\|f\|_{L^2((\mathcal{G}_c)^\complement)} \le \|f - g_i\|_{L^2((\mathcal{G}_c)^\complement)} + \|g_i\|_{L^2((\mathcal{G}_c)^\complement)} < 2\varepsilon. \qquad \square$$

We conclude this subsection with a criterion on compactness of embeddings of Sobolev spaces on *trees*. The following proposition is a generalisation of results in [29], see for instance Lemma 8.1 therein, where infinite trees without degree one vertices are considered. We also mention that the metric condition in Proposition 3.6 is already known to play a role for spectral properties of infinite quantum graphs with $\delta$-couplings, see [16, Theorem 3.5].

**Proposition 3.6.** *Let $\mathcal{G}$ be a locally finite metric tree and*

$$\mathfrak{V} := \mathfrak{C}(\mathcal{G}) \cup \{\mathsf{v} \in \mathsf{V} \mid \deg(\mathsf{v}) = 1\}.$$

*If for all $\varepsilon > 0$ there are only finitely many edges of length larger than $\varepsilon$, then the embedding of $H^1_0(\mathcal{G}, \mathfrak{V})$ into $L^2(\mathcal{G})$ is compact and $\mathcal{H}_{\mathfrak{V}}$ has purely discrete spectrum.*

*Proof.* On infinite trees without degree one vertices ("leaves"), the statement follows from [29], see Section 8.1. therein. It remains to prove the statement in the presence of leaves.

Compactness of the embedding is equivalent to

$$\lim_{k\to\infty} \lambda_k(\mathcal{G}, \mathfrak{V}) = \infty, \quad \text{where} \quad \lambda_k(\mathcal{G}, \mathfrak{V}) = \inf_{\substack{X \subset H^1_0(\mathcal{G},\mathfrak{V}) \\ \dim X = k}} \sup_{\substack{\phi \in X \\ \|\phi\|_{L^2(\mathcal{G})} = 1}} \|\phi'\|_{L^2(\mathcal{G})}.$$

Enumerate the at most countably many leaves by $\mathsf{v}_j$ and turn the tree $\mathcal{G}$ into a leafless tree $\mathcal{G}^+$ by taking copies of a leafless tree of finite diameter, scaling it by $j^{-1}$, and attaching it to $\mathsf{v}_j$. Now, $\mathcal{G}^+$ is a leafless tree satisfying the conditions of the proposition. Furthermore, since $\mathcal{G} \subset \mathcal{G}^+$, we have $H^1_0(\mathcal{G}, \mathfrak{V}) \subset H^1_0(\mathcal{G}^+)$, which implies

$$\infty = \lim_{k\to\infty} \lambda_k(\mathcal{G}^+) \le \lim_{k\to\infty} \lambda_k(\mathcal{G}, \mathfrak{V}). \qquad \square$$

*Remark* 3.7. At the risk of redundancy let us emphasise that the previous proposition does not require the graph to have finite diameter, as can be seen on an infinite regular tree with edge length $n^{-1}$ in the $n$-th generation.

3.2. **Spectral properties of self-adjoint extensions of the Laplacian.** Whenever $\mathcal{H}_{\mathrm{N}}$ and/or $\mathcal{H}_{\mathrm{F}}$ have purely discrete spectrum – or, equivalently, compact resolvent –, we will denote by

$$\mu_1 < \mu_2 \leq \mu_3 \leq \ldots \to \infty$$

the ordered eigenvalues of $\mathcal{H}_{\mathrm{N}}$, repeated according to their multiplicities, and by

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$$

the ordered eigenvalues of $\mathcal{H}_{\mathrm{F}}$. We write $\psi_k$ and $\varphi_k$, respectively, for the eigenfunctions associated with $\mu_k$ and $\lambda_k$, respectively, chosen to form orthonormal bases of $L^2(\mathcal{G})$.

Since $\mathcal{G}$ is connected, and the semigroups generated by both $\mathcal{H}_{\mathrm{F}}$ and $\mathcal{H}_{\mathrm{N}}$ are positive irreducible, standard Perron–Frobenius theory, see e.g. [3, Proposition 4.12], implies that the $\mu_1$ and $\lambda_1$ are necessarily simple. If $\mathcal{G}$ has finite total length, then $\mu_1 = 0$ with corresponding eigenfunctions being the constant functions, while $\lambda_1 > 0$, as we shall see below. In order to emphasise the dependence on $\mathcal{G}$, we will also write $\mu_k(\mathcal{G})$ and $\lambda_k(\mathcal{G})$, whenever convenient.

The $\mu_n$ and $\lambda_n$ admit min-max and max-min variational characterisations in terms of the forms $\mathfrak{t}_{\mathrm{N}}$ and $\mathfrak{t}_{\mathrm{F}}$ from (2.8) and (2.9), respectively. For instance if the total length of $\mathcal{G}$ is finite, we have

$$\mu_2(\mathcal{G}) = \inf \left\{ \frac{\int_{\mathcal{G}} |f'(x)|^2 \,\mathrm{d}x}{\int_{\mathcal{G}} |f(x)|^2 \,\mathrm{d}x} : 0 \neq f \in H^1(\mathcal{G}),\ \int_{\mathcal{G}} f(x)\,\mathrm{d}x = 0 \right\}, \tag{3.4}$$

$$\lambda_1(\mathcal{G}) = \inf \left\{ \frac{\int_{\mathcal{G}} |f'(x)|^2 \,\mathrm{d}x}{\int_{\mathcal{G}} |f(x)|^2 \,\mathrm{d}x} : f \in H^1_0(\mathcal{G}) \right\} \tag{3.5}$$

and more generally

$$\lambda_1(\mathcal{G}; \mathfrak{V}) = \inf \left\{ \frac{\int_{\mathcal{G}} |f'(x)|^2 \,\mathrm{d}x}{\int_{\mathcal{G}} |f(x)|^2 \,\mathrm{d}x} : f \in H^1_0(\mathcal{G}; \mathfrak{V}) \right\} \tag{3.6}$$

for any non-empty subset $\mathfrak{V} \subset \mathsf{V} \cup \mathfrak{C}(\mathcal{G})$, see Remark 2.27, where the infimum in (3.4)–(3.6) is attained if and only if $f$ is a corresponding eigenfunction. As usual, we call the quotient appearing in the above expressions the *Rayleigh quotient* of $f$.

Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be any compact exhaustion of $\mathcal{G}$; we recall (Definition 2.2) that $\partial\mathcal{G}_n$ denotes the topological boundary of $\mathcal{G}_n$ in $\mathcal{G}$. Recalling the notation from Remark 2.22 we use $H^1_0(\mathcal{G}_n; \partial\mathcal{G}_n)$ to denote the subspace of $H^1(\mathcal{G})$ in which all functions vanish on $\mathcal{G} \setminus \mathcal{G}_n$ and denote by $\lambda_k(\mathcal{G}_n)$ and $\varphi_k(\mathcal{G}_n)$ the eigenvalues and corresponding eigenfunctions of the Laplacian $\mathcal{H}_{\partial\mathcal{G}_n}$ on $L^2(\mathcal{G}_n)$ (recall Remark 2.27), that is, Dirichlet conditions are satisfied at $\partial\mathcal{G}_n$, and continuity–Kirchhoff conditions are satisfied at all other vertices of $\mathcal{G}_n$. We denote by $\mu_k(\mathcal{G}_n)$ and $\psi_k(\mathcal{G})$ the eigenvalues and eigenfunctions of the Laplacian with continuity–Kirchhoff conditions at all vertices of $\mathcal{G}_n$, respectively; the associated form is $\mathfrak{t}_{\mathrm{N}}$ on $H^1(\mathcal{G}_n)$.

We observe that we may identify any $H^1_0(\mathcal{G}_n; \partial\mathcal{G}_n)$ with a subspace of $H^1_0(\mathcal{G}_m; \partial\mathcal{G}_n)$ whenever $m > n$, as well as with a subspace of $H^1_c(\mathcal{G})$, upon extension by zero of the functions in $H^1_0(\mathcal{G}_n)$. In fact, it follows directly from the definition of compact approximations and $H^1_c(\mathcal{G})$ that, with this identification,

$$H^1_c(\mathcal{G}) = \bigcup_{n\in\mathbb{N}} H^1_0(\mathcal{G}_n; \partial\mathcal{G}_n). \tag{3.7}$$

The following approximation result will be used repeatedly in order to prove a large number of our spectral estimates in the following sections.

**Lemma 3.8.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite length and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be any compact exhaustion of $\mathcal{G}$. Then the following assertions hold.*

*(1) For all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$ we have $\lambda_k(\mathcal{G}_n) \geq \lambda_k(\mathcal{G})$, and $\lambda_k(\mathcal{G}_n) \to \lambda_k(\mathcal{G})$ as $n \to \infty$.*

*(2) For all $k \in \mathbb{N}$ we also have*

$$\mu_k(\mathcal{G}) \geq \limsup_{n\to\infty} \mu_k(\mathcal{G}_n). \tag{3.8}$$

*(3) If in addition $\mathcal{G}$ has finite Betti number, then for every $k \in \mathbb{N}$ we have $\mu_k(\mathcal{G}_n) \geq \mu_k(\mathcal{G})$ for all $n \in \mathbb{N}$ sufficiently large.*

In particular, if $\mathcal{G}$ has finite Betti number, then $\mu_k(\mathcal{G}_n) \to \mu_k(\mathcal{G})$ as $n \to \infty$, for any compact exhaustion.

The proof of part (1) is closely related to the notion of Mosco convergence [37], part (2) involves an elementary argument using the restriction of the eigenfunctions on $\mathcal{G}$ to $\mathcal{G}_n$, and part (3) follows from a surgery-type argument. In the general case it does not seem obvious whether there is actually convergence $\mu_k(\mathcal{G}_n) \to \mu_k(\mathcal{G})$ for $k \geq 2$ (the case $k = 1$ is trivial).

*Proof.* (1) The inequality $\lambda_k(\mathcal{G}_n) \geq \lambda_k(\mathcal{G})$ for all $k, n \in \mathbb{N}$ is an immediate consequence of the identification of every element of $H^1_0(\mathcal{G}_n)$ with an element of $H^1_c(\mathcal{G})$, and hence of $H^1_0(\mathcal{G})$, via extension by zero, together with the min-max principle for $\lambda_k$.

On the other hand, given any $u \in H^1_0(\mathcal{G})$, by (3.7) and the definition of $H^1_0(\mathcal{G})$ as the closure in $H^1$ of $H^1_c(\mathcal{G})$, there exists a sequence of functions $u_n \in H^1_0(\mathcal{G}_n)$ such that $u_n \to u$ in $H^1_0(\mathcal{G})$. Fix $k \geq 1$, write $\varphi_j := \varphi_j(\mathcal{G}) \in H^1_0(\mathcal{G})$ for the normalised eigenfunctions for $\lambda_j(\mathcal{G})$, $j = 1, \ldots, k$, and for each $j$ choose $\varphi_{j,n} \in H^1_0(\mathcal{G}_n)$ such that $\varphi_{j,n} \to \varphi_j$ in $H^1_0(\mathcal{G})$ (and hence also in $C(\overline{\mathcal{G}})$ by Theorem 3.2). Note that, by the dominated convergence theorem (using, e.g., $2|\varphi_i\varphi_j| \in L^\infty(\mathcal{G})$ as a dominating function, cf. Lemma 3.1), for all $i \neq j$,

$$\int_{\mathcal{G}} \varphi_{i,n}(x)\varphi_{j,n}(x)\ \mathrm{d}x \longrightarrow \int_{\mathcal{G}} \varphi_i(x)\varphi_j(x)\ \mathrm{d}x = 0.$$

If we now consider the following renormalised test functions, mutually orthogonal in $L^2(\mathcal{G})$,

$$\tilde{\varphi}_{j,n} := \varphi_{j,n} - \sum_{i=1}^{j-1} \langle \varphi_{i,n}, \varphi_{j,n} \rangle_{L^2(\mathcal{G})} \varphi_{i,n},$$

then the above convergence results imply that $\tilde{\varphi}_{j,n}$ is nonzero for $n$ sufficiently large, and an elementary computation yields

$$\lambda_k(\mathcal{G}_n) \leq \max_{j=1,\ldots,k} \frac{\int_{\mathcal{G}} |\tilde{\varphi}_{j,n}'(x)|^2\ \mathrm{d}x}{\int_{\mathcal{G}} |\tilde{\varphi}_{j,n}(x)|^2\ \mathrm{d}x} \longrightarrow \max_{j=1,\ldots,k} \frac{\int_{\mathcal{G}} |\varphi_j'(x)|^2\ \mathrm{d}x}{\int_{\mathcal{G}} |\varphi_j(x)|^2\ \mathrm{d}x} = \lambda_k(\mathcal{G}), \tag{3.9}$$

as $n \to \infty$, where for the first inequality we have used the min-max characterisation of $\lambda_k(\mathcal{G}_n)$. Hence $\limsup_{n\to\infty} \lambda_k(\mathcal{G}_n) \leq \lambda_k(\mathcal{G})$.

(2) We use the respective restrictions of $\psi_j := \psi_j(\mathcal{G}) \in H^1(\mathcal{G})$ to $H^1(\mathcal{G}_n)$, $j = 1, \ldots, k$, as test functions on $\mathcal{G}_n$. Firstly, since the measure of $\mathcal{G} \setminus \mathcal{G}_n$ tends to zero, by the monotone convergence theorem,

$$\|\psi_j\|_{L^2(\mathcal{G}_n)} \to \|\psi_j\|_{L^2(\mathcal{G})}, \qquad \|\psi_j'\|_{L^2(\mathcal{G}_n)} \to \|\psi_j'\|_{L^2(\mathcal{G})}$$

for all $j = 1, \ldots, k$; moreover, a further application of the dominated convergence theorem (using, e.g., $|\psi_i\psi_j| \in L^\infty(\mathcal{G})$ as a dominating function) shows that

$$\int_{\mathcal{G}_n} \psi_i(x)\psi_j(x)\ \mathrm{d}x \to 0.$$

On $\mathcal{G}_n$ we thus consider the $k$-dimensional space spanned by the orthogonal functions

$$\tilde{\psi}_{j,n} := \psi_j - \sum_{i=1}^{j-1} \langle \psi_i, \psi_j \rangle_{L^2(\mathcal{G}_n)} \psi_i.$$

An argument entirely analogous to the one in part (1) now shows that, for each $j = 1, \ldots, k$, $\tilde{\psi}_{j,n}$ is nonzero for $n \in \mathbb{N}$ sufficiently large, and (3.9) holds with $\mu_k$ in place of $\lambda_k$, and $\tilde{\psi}_{j,n}$, $\tilde{\psi}_j$ in place of $\tilde{\varphi}_{j,n}$, $\tilde{\varphi}_j$. This proves the claim.

(3) Here we will use in an essential way that, for all $n$ sufficiently large, $\mathcal{G} \setminus \mathcal{G}_n$ consists of a disjoint union of a finite number of pairwise disjoint trees, each of which is attached to $\mathcal{G}_n$ at a single vertex, cf. Remark 2.6. Fix such an $n$ and denote by $\mathcal{T}_1, \ldots, \mathcal{T}_j$ these trees, and by $\mathsf{v}_1, \ldots, \mathsf{v}_j$ the corresponding vertices of attachment.

It suffices to show that $\mu_k(\mathcal{G}) \leq \mu_k(\mathcal{G} \setminus \mathcal{T}_1)$ for all $k \in \mathbb{N}$. But this, in turn, follows from an argument completely analogous to the proof of [8, Theorem 3.10(1)] (with $r = 1$): we first observe that clearly

$$\mu_{k+1}(\mathcal{T}_1 \dot\cup (\mathcal{G} \setminus \mathcal{T}_1)) \leq \mu_k(\mathcal{G} \setminus \mathcal{T}_1) \tag{3.10}$$

for all $k \in \mathbb{N}$, since the spectrum of the disjoint union of $\mathcal{T}_1$ and $\mathcal{G} \setminus \mathcal{T}_1$ is equal to the union of their spectra (counting multiplicities), and $\mu_1(\mathcal{T}_1) = 0 \leq \mu_k(\mathcal{G} \setminus \mathcal{T}_1)$ for all $k \in \mathbb{N}$. Next observe that $\mathcal{G}$ is formed from

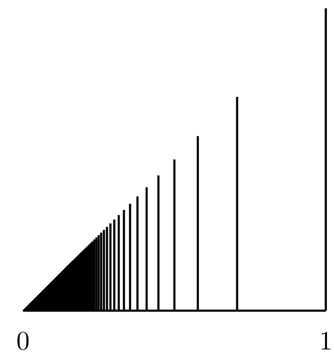

FIGURE 3.1. The diagonal comb graph $\mathcal{G}_{\frac{1}{2}}$

$\mathcal{T}_1\dot\cup(\mathcal{G}\setminus\mathcal{T}_1)$ by gluing the two graphs together at the vertex $\mathsf{v}_1$; at the level of $H^1$-spaces, we have that $H^1(\mathcal{G})$ may be identified with the codimension one subspace of $H^1(\mathcal{T}_1\dot\cup(\mathcal{G}\setminus\mathcal{T}_1))$ consisting of those functions whose values satisfy an additional continuity condition at $\mathsf{v}_1$ (or rather its preimage vertices in $\mathcal{T}_1$ and $\mathcal{G}\setminus\mathcal{T}_1$ before gluing). The proof of [8, Theorem 3.4] may be repeated verbatim to give

$$\mu_k(\mathcal{G}) \le \mu_{k+1}(\mathcal{T}_1\dot\cup(\mathcal{G}\setminus\mathcal{T}_1)).$$

Combining this with (3.10) yields the conclusion. □

3.3. **An example: The diagonal comb.** In contrast to finite metric graphs, on infinite metric graphs even the type of spectrum can depend in a subtle way on the boundary conditions. In this subsection, we illustrate this phenomenon on a parameter-dependent family of metric graphs $(\mathcal{G}_\alpha)_{\alpha>0}$ (the "diagonal combs"). While $\mathcal{H}_{\mathrm{F}}$ has always purely discrete spectrum, the spectrum of $\mathcal{H}_{\mathrm{N}}$ experiences a phase transition from purely discrete to nonempty essential spectrum at $\alpha = 1/2$. As far as we are aware, such a phenomenon has not been described before. We comment on it and compare it to the Laplacian on so-called horn-shaped domains in Remark 3.11 below.

**Definition 3.9.** For $\alpha > 0$, we define the "diagonal comb" metric graph $\mathcal{G}_\alpha$ by taking the interval $(0, 1]$ ("horizontal shaft"), putting on every point $\frac{1}{n^\alpha}$, $n \in \mathbb{N}$, a vertex, and attaching to it an edge ("tooth") of length $\frac{1}{n^\alpha}$.

Figure 3.1 contains an illustration in the case $\alpha = \frac{1}{2}$. Note that the larger $\alpha$, the sparser the teeth become. In particular, if $\alpha > 1$, the graph has finite total length and, by Corollary 3.4, $\mathcal{H}_{\mathrm{N}}$ (or any Markovian extension of $\mathcal{H}$) will have purely discrete spectrum.

**Theorem 3.10.** *For the diagonal comb metric graph $\mathcal{G}_\alpha$ we have:*

*(1) For all $\alpha > \frac{1}{2}$, $H^1(\mathcal{G}_\alpha)$ is compactly embedded in $L^2(\mathcal{G}_\alpha)$. In particular, $\mathcal{H}_{\mathrm{N}}$ has compact resolvent and purely discrete spectrum.*

*(2) If $\alpha \in (0, \frac{1}{2}]$, then there is an $L^2(\mathcal{G}_\alpha)$-orthonormal sequence of $H^1(\mathcal{G}_\alpha)$-functions with uniformly bounded $H^1(\mathcal{G}_\alpha)$ norm. In particular, $\mathcal{H}_{\mathrm{N}}$ has nonempty essential spectrum.*

*(3) For all $\alpha > 0$, if we set $\mathfrak{V}$ to be the union of $\{0\}$ with the set of all tips of the teeth, then $H^1_0(\mathcal{G}_\alpha, \mathfrak{V})$ is compactly embedded into $L^2(\mathcal{G}_\alpha)$. Consequently, $\mathcal{H}_{\mathfrak{V}}$ has compact resolvent and purely discrete spectrum.*

Note that the phase transition happens at $\alpha = 1/2$, that is, among graphs of infinite total length (the transition from infinite to finite total length being at $\alpha = 1$). Also, all $\mathcal{G}_\alpha$ are trees with only a finite number of edges of length larger than $\varepsilon$ for any $\varepsilon > 0$ – the criterion for discreteness of the spectrum of the Friedrichs extension in Proposition 3.6. Thus, Theorem 3.10 demonstrates in particular that this criterium cannot be extended to the Kirchhoff extension.

*Remark* 3.11. Let us compare Theorem 3.10 to results on the Laplacian on *horn shaped domains* $\Omega \subset \mathbb{R}^n$, that are connected domains, bounded in $x_1$-direction, with

$$\lim_{t \to \pm\infty} \operatorname{diam}\{x \in \Omega \colon x_1 = t\} = 0.$$

It is known that on any such domain, the spectrum of the *Dirichlet Laplacian* is purely discrete [44, 50]. Theorem 3.10.(3) or more generally the compactness criterion of Proposition 3.6 seem to be the corresponding analogues on metric graphs.

However, for the *Neumann Laplacian* on domains, the situation is different. Indeed, on connected domains of infinite volume, the essential spectrum of the Neumann Laplacian is always nonempty, see the appendix of [14], which also provides examples of horn-shaped domains of finite volume with nonempty spectrum.

Thus, Theorem 3.10.(1)–(2), which describes a phase transition from purely discrete to nonempty essential spectrum for the Neumann extension among metric graphs of infinite volume, seems to present a new, metric-graph specific phenomenon which has no obvious equivalent among domains.

*Proof of Theorem 3.10.* (1) We will use the compactness criterion of Theorem 3.5. Fix $\varepsilon > 0$ and let $f \in H^1(\mathcal{G}_\alpha)$ with $\|f\|_{H^1(\mathcal{G}_\alpha)} \leq 1$. Writing $x$ for the point on the shaft identified with $[0,1]$, we have $f(0) = 0$. Now, by an application of the fundamental theorem of calculus (and a suitable approximation argument) as well as Cauchy–Schwarz,

$$|f(x)| = \left| \int_0^x f'(t)\,\mathrm{d}t \right| \leq \sqrt{x}\|f\|_{H^1(\mathcal{G}_\alpha)} = \sqrt{x}. \tag{3.11}$$

An analogous calculation yields that at any point $y \in \mathcal{G}_\alpha$ such that $\operatorname{dist}(0,y) \leq \delta$ we have $|f(y)| \leq \sqrt{\delta}$.

Denote by $\mathsf{e}_k$ the $k$-th tooth (counted from 1 to 0, i.e. from right to left in the orientation of Figure 3.1), and note that all points in $\mathsf{e}_k$ are at most at distance $2k^{-\alpha}$ to 0; hence

$$\|f\|^2_{L^2(\mathsf{e}_k)} \leq |\mathsf{e}_k| \cdot \max_{y \in \mathsf{e}_k} |f(y)|^2 \leq k^{-\alpha} \cdot 2k^{-\alpha} = \frac{2}{k^{2\alpha}}.$$

Since $\alpha > \frac{1}{2}$, the sum of upper bounds on all teeth converges. Hence, summing the respective tails of these series, we obtain that, for sufficiently large $k_0 \in \mathbb{N}$,

$$\sum_{k=k_0}^{\infty} \|f\|^2_{L^2(\mathsf{e}_k)} \leq \frac{\varepsilon}{2},$$

for any $f \in H^1(\mathcal{G}_\alpha)$ with $\|f\|_{H^1(\mathcal{G}_\alpha)} \leq 1$. Likewise, (3.11) yields that on the shaft, possibly for a larger choice of $k_0$,

$$\|f\|^2_{L^2([0,k_0^{-\alpha}])} \leq \frac{\varepsilon}{2}.$$

Thus, taking $\mathcal{G}_c$ to be the subgraph of the comb cut off at the $k_0$-th tooth, we have shown precisely that (3.1) holds for this $\varepsilon$. The compactness of the embedding of $H^1(\mathcal{G}_\alpha)$ in $L^2(\mathcal{G}_\alpha)$ now follows from Theorem 3.5.

(2) We will only give a proof for the critical parameter $\alpha = \frac{1}{2}$ since the case $\alpha \in (0, \frac{1}{2})$ follows by a completely analogous argument. Hence, we consider $\mathcal{G} := \mathcal{G}_{\frac{1}{2}}$ throughout.

We construct functions $\phi_n$ as follows: For $n \in \mathbb{N}$, let $\phi_n \in C_c(\mathcal{G})$ be

$$\begin{cases} \text{linearly rising from 0 to 1 in } [\frac{1}{\sqrt{2n}}, \frac{1}{\sqrt{n}}] \text{ on the horizontal shaft } \mathsf{e}, \\ \text{linearly falling from 1 to 0 in } [\frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}} - \frac{1}{\sqrt{2n}}] \text{ on the horizontal shaft}, \\ \text{constant on the teeth and and zero everywhere else.} \end{cases}$$

It is clear that all $\phi_n$ are in $H^1(\mathcal{G})$, and we calculate

$$\int_{\mathcal{G}} |\phi_n'(x)|^2\,\mathrm{d}x = 2\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{2n}}\right)^{-1} = \frac{2}{1 - \sqrt{1/2}}\sqrt{n},$$

and

$$\int_{\mathcal{G}} |\phi_n(x)|^2\,\mathrm{d}x = \frac{2}{3}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{2n}}\right) + \sum_k \frac{1}{\sqrt{k}} \left|\phi_n\left(\frac{1}{\sqrt{k}}\right)\right|^2, \tag{3.12}$$

where $\phi_n(x)$ denotes the value of $\phi_n$ at the point of the horizontal shaft $\mathsf{e}$ that is identified with $x \in (0,1]$. Now, note that $\left|\phi_n\left(\frac{1}{\sqrt{k}}\right)\right|^2$ is certainly greater than $\frac{1}{4}$ if

$$\frac{1}{\sqrt{k}} \in \left[\frac{1}{\sqrt{2n}} + \frac{1}{2}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{2n}}\right), \frac{1}{\sqrt{n}}\right], \quad \text{equivalently,} \quad k \in \left[n, \frac{8}{(\sqrt{2}+1)^2}n\right] \supset \left[n, \frac{4}{3}n\right].$$

Thus, the sum in (3.12) contains (for sufficiently large $n$) at least $n/4$ terms, each of which is at least $\frac{1}{4\sqrt{2n}}$, and we obtain the lower bound

$$\int_{\mathcal{G}} |\phi(x)|^2 \,\mathrm{d}x \geq \frac{n}{16}\frac{1}{\sqrt{2n}} = \frac{\sqrt{n}}{16\sqrt{2}}.$$

We have found that, for sufficiently large $n$,

$$\frac{\int_{\mathcal{G}} |\phi_n'(x)|^2 \,\mathrm{d}x}{\int_{\mathcal{G}} |\phi(x)|^2 \,\mathrm{d}x} \leq \frac{64}{\sqrt{2}-1}.$$

Since, after passing to a subsequence, we can make the $\phi_n$ have mutually disjoint supports, up to renormalisation they yield an orthornormal sequence of $H^1(\mathcal{G})$ functions with uniformly bounded $H^1(\mathcal{G})$-norm.

(3) This is a direct consequence of Proposition 3.6. □

*Remark* 3.12. Theorem 3.10 (2) asserts that the essential spectrum is nonempty but provides no further information on its structure. However, a closer look at the proof shows that, for $\alpha$ strictly below the critical threshold $\alpha = \frac{1}{2}$, the infimum of the spectrum is zero, since the test functions constructed in the proof will then have Rayleigh quotients converging to zero. It would be interesting to understand $\inf \sigma(\mathcal{H}_{\mathrm{N}})$ in the critical case $\alpha = \frac{1}{2}$.

There also is an obvious generalisation of the argument used in the proof of Theorem 3.10 (1) which provides a criterion to determine whether in the presence of (finitely many) graph ends of infinite volume purely discrete spectrum persists.

For this purpose, for any end $\gamma \in \overline{\mathcal{G}}$, and $0 < r < R$, denote by $A_{r,R}(\gamma) := \{x \in \mathcal{G} : r \leq \operatorname{dist}(x,\gamma) \leq R\}$ be the (closed) annulus in $\mathcal{G}$ about $\gamma$ of inner radius $r$ and outer radius $R$, and by $|A_{r,R}|$ its volume. We then have

**Proposition 3.13.** *Let a locally finite, connected metric graph graph $\mathcal{G}$ of finite diameter have only finitely many ends $\gamma_1, \ldots, \gamma_n$ of infinite volume. Assume that for each $\gamma_i$ there exists some $\alpha_i > 0$ such that $\mathcal{G}$ satisfies the following volume growth estimate near $\gamma_i$:*

$$|A_{\frac{1}{k+1},\frac{1}{k}}(\gamma_i)| \leq \frac{1}{k^{\alpha_i}}$$

*for all $k \in \mathbb{N}$ sufficiently large. Then $H^1(\mathcal{G})$ embeds compactly in $L^2(\mathcal{G})$.*

The proof of Proposition 3.13 follows along the lines of the proof of Theorem 3.10 (2), using the estimate $|f(x)| \leq \sqrt{\operatorname{dist}(x,\gamma_i)}$ for all $x \in \mathcal{G}$ and all ends $\gamma_i$, valid for any $H^1(\mathcal{G})$-function $f$ with norm 1, to obtain the bound

$$\|f\|^2_{L^2\left(A_{\frac{1}{k+1},\frac{1}{k}}(\gamma_i)\right)} \leq \frac{1}{k^{1+\alpha}}$$

for all $k$ large enough, implying that $\|f\|^2_{L^2(B_{1/k}(\gamma_i))} \to 0$ as $k \to \infty$. In fact, a slight generalisation of Proposition 3.13 is even possible using the same argument: for each end $\gamma_i$ it suffices that for some sequence $r_k \to 0$ the series $\sum_k r_k |A_{r_{k-1},r_k}(\gamma_i)|$ converges.

## 4. Lower bounds on the lowest positive eigenvalue

4.1. **Symmetrisation techniques for isoperimetric inequalities.** This section is devoted to generalising the "principal" isoperimetric inequalities for the eigenvalues of the Laplacian on a compact metric graph, namely, the inequalities due to Nicaise [40], Friedlander [17], and Band–Lévy [2], to general locally finite graphs with finite length. While the inequalities themselves (see below) could be obtained via an approximation argument using surgery principles generalised to infinite graphs, a suitable compact exhaustion and Lemma 3.8, we will give a proof based on symmetrisation and the coarea formula. This more closely resembles the proofs of Friedlander and Band–Lévy; the principal advantage is that this way we can characterise the respective cases of equality, which the approximation argument would not permit.

**Theorem 4.1** (Nicaise–Friedlander for infinite graphs: Neumann extension)**.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite length $L > 0$. Then for the $k$-th eigenvalue $\mu_k(\mathcal{G})$, $k \geq 2$, of the Neumann extension $\mathcal{H}_{\mathrm{N}}$ on $\mathcal{G}$,*

$$\mu_k(\mathcal{G}) \geq \frac{\pi^2 k^2}{4L^2}. \tag{4.1}$$

*There is equality if and only $\mathcal{G}$ is a star consisting of $k$ (finite or infinite) path subgraphs of length $L/k$ each, glued together at a common vertex.*

By an infinite path subgraph we mean a path graph which consists of an infinite number of edges but having finite total length; these are isometrically isomorphic to intervals which are open at one endpoint and closed at the other. Thus, in the case of the star graph, the endpoints of one or more of the "rays" of finite length will not belong to the graph if the corresponding "ray" contains an infinite number of edges (any "missing" endpoints are the ends of the graph). But the total length of each "ray" must always be $L/k$. If $k = 2$, the corresponding graph is isometrically isomorphic to an interval of length $L$ with zero, one or two ends.

For the case of the Friedrichs extension we have the equivalent of the Dirichlet version of the theorems of Nicaise and Friedlander, see [40, Théorème 3.2] and [17, Lemma 3], respectively. Here and in the sequel, we let $\mathfrak{V} \subset \overline{\mathcal{G}} \subset \mathsf{V} \cap \mathfrak{C}(\mathcal{G})$, and in place of the Friedrichs extension $\mathcal{H}_{\mathrm{F}}$ consider the more general $\mathcal{H}_{\mathfrak{V}}$ of Remark 2.27; we recall that this imposes a Dirichlet conditon on the (essentially arbitrary) subset $\mathfrak{V}$ of the union of the set of all vertices and the set of all ends of $\mathcal{G}$.

**Theorem 4.2** (Nicaise–Friedlander for infinite graphs: Friedrichs extension)**.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite length $L > 0$ and assume $\mathfrak{V} \neq \emptyset$. Then the lowest eigenvalue $\lambda_1(\mathcal{G}, \mathfrak{V})$ of the mixed Friedrichs–Neumann extension $\mathcal{H}_{\mathfrak{V}}$ on $\mathcal{G}$ satisfies*

$$\lambda_1(\mathcal{G}, \mathfrak{V}) \geq \frac{\pi^2}{4L^2}. \tag{4.2}$$

*Equality in* (4.2) *is attained if and only if $\mathcal{G}$ is isometrically isomorphic to an interval of length $L$ with mixed Dirichlet–Neumann conditions.*

In the above theorem, the case $\mathfrak{V} = \mathfrak{C}(\mathcal{G})$ describes the Friedrichs extension $\mathcal{H}_{\mathrm{F}}$.

Finally, for doubly path connected graphs, we have the following improvements of (4.1) (for the first nontrivial eigenvalue $\mu_2$) and (4.2): their counterparts for finite metric graphs are [2, Theorem 2.1] and [7, Lemma 4.3], respectively.

**Theorem 4.3** (Band–Lévy for infinite graphs)**.** *Let $\mathcal{G}$ be a locally finite, doubly connected metric graph with finite length $L > 0$. Then the first nontrivial eigenvalue $\mu_2(\mathcal{G})$ of the Neumann extension $\mathcal{H}_{\mathrm{N}}$ on $\mathcal{G}$ satisfies*

$$\mu_2(\mathcal{G}) \geq \frac{4\pi^2}{L^2}. \tag{4.3}$$

*Equality in* (4.3) *holds if and only if $\mathcal{G}$ is a symmetric necklace.*

Recall that a *symmetric necklace*, see [2, Example 1.7], is a metric graph obtained concatenating (a finite or countably infinite number of) pumpkin graphs, each on two equally long edges. We emphasise that in the infinite case it is allowed to have two ends.

**Theorem 4.4** (Berkolaiko–Kennedy–Kurasov–Mugnolo for infinite graphs)**.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite length $L > 0$. Take $\emptyset \neq \mathfrak{V} \subset \mathsf{V} \cap \mathfrak{C}_{\mathcal{G}}$ to contain all degree one vertices of $\mathcal{G}$ and suppose that for all $x \in \mathcal{G}$ there are at least two edge-disjoint paths connecting $x$ and $\mathfrak{V}$.*

*Then the first nontrivial eigenvalue $\lambda_1(\mathcal{G}, \mathfrak{V})$ of $\mathcal{H}_{\mathfrak{V}}$ on $\mathcal{G}$ satisfies*

$$\lambda_1(\mathcal{G}, \mathfrak{V}) \geq \frac{\pi^2}{L^2}. \tag{4.4}$$

*Equality in* (4.4) *implies that $\mathcal{G}$ is a symmetric necklace such that either (1) $\mathfrak{V}$ consists of one end, equipped with the Friedrichs extension, or (2) the necklace is finite with a single Dirichlet condition imposed at one*

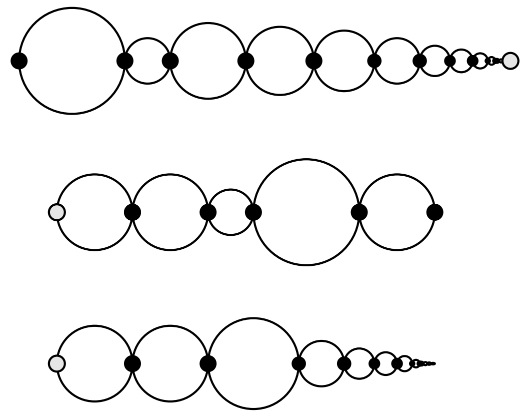

FIGURE 4.1. The three cases where equality in (4.4) holds. Dirichlet conditions (at vertices or ends) are coloured light grey, standard/Neumann conditions are depicted as filled black circles. (1) In the top graph the Friedrichs extension leads to a formal Dirichlet condition at the end on the right side of the infinite necklace; (2) the middle graph is a compact necklace with a Dirichlet condition at one end and a standard condition at the other; (3) in the bottom row there is a Dirichlet vertex of degree two at the finite end of the necklace and a Neumann condition at the end.

*of its extremities, or (3) $\mathfrak{V}$ consists of one end, equipped with the Neumann extension, but there is a single Dirichlet condition at the other extremity of the necklace.*

See also Figure 4.1 for an illustration. Here it is convenient to think of the two vertices as being glued together (in opposition to our usual approach) due to the parallels with the minimising graphs in Theorem 4.3.

The topological assumption on $\mathcal{G}$ in Theorem 4.4 is equivalent to requiring that the metric space obtained from $\mathcal{G}$ upon identifying all points in $\mathfrak{V}$, must be doubly connected.

The proofs will make use of the following technical result, which is a consequence of the coarea formula (cf. [17, Proof of Lemma 3], [2, Proof of Theorem 2.1] and [7, Proof of Theorem 3.4]). Notationally, given a graph $\mathcal{G}$ and a measurable function $f : \mathcal{G} \to \mathbb{R}$, we denote by

$$S_t = S_t(f) := \{x \in \mathcal{G} : f(x) = t\} \subset \mathcal{G}$$

the level surface of $f$ and by

$$m_f(t) := |\{x \in \mathcal{G} : f(x) < t\}|$$

the Lebesgue measure of its sublevel set, for any $t \in \mathbb{R}$.

**Lemma 4.5.** *Let $\mathcal{G}$ be a locally finite metric graph of finite total length, and let $f \in H^1(\mathcal{G}) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} C^1([0, \ell_{\mathsf{e}}])$. Then, with the notation just introduced, $m_f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous; moreover, for almost every $t \in \mathbb{R}$, $S_t$ has finite cardinality and*

$$m'_f(t) = \sum_{x\in S_t} \frac{1}{|f'(x)|}; \tag{4.5}$$

*finally,*

$$\int_{\mathcal{G}} |f'(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \sum_{x\in S_t} |f'(x)| \, \mathrm{d}t. \tag{4.6}$$

Since the key ingredient of the proof is the coarea formula, which to the best of our knowledge has not been explicitly formulated for infinite metric graphs, we first give a version of the latter; we remark that, at least in the case of compact graphs, a generalisation of the following result to functions of bounded variations was given very recently in [36].

**Lemma 4.6** (Coarea formula for infinite graphs). *Let $\mathcal{G}$ be any metric graph on a countable edge set, let $f \in C(\mathcal{G}) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} C^1([0,\ell_\mathsf{e}])$ and let $\varphi \in L^1_{loc}(\mathcal{G})$ be nonnegative. Then, with the notation introduced above,*

$$\int_{\mathcal{G}} \varphi(x)|f'(x)| \,\mathrm{d}x = \int_{\mathbb{R}} \sum_{x\in S_t} \varphi(x) \,\mathrm{d}t. \tag{4.7}$$

*Proof.* Fix $f \in C(\mathcal{G}) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} C^1([0,\ell_\mathsf{e}])$ and fix $\varphi$. We restrict to the regular values $t \in \mathbb{R}$ of $f$, that is, those values of $t \in \mathbb{R}$ such that $S_t \cap \mathsf{V} = \emptyset$ and $S_t \cap \{x \in \mathcal{G} \setminus \mathsf{V} : f'(x) = 0\} = \emptyset$. Denote this set of regular values by $\mathfrak{R}(f)$. Note that the set of exceptional values $\mathbb{R} \setminus \mathfrak{R}(f)$ is countable, since $\mathsf{V}$ and $\mathsf{E}$ are countable and since $\{x \in \mathsf{e} : f'(x) = 0\}$ is countable for all $\mathsf{e} \in \mathsf{E}$ by Sard's theorem [47] under our regularity assumptions on $f$.

Let $\{\mathsf{e}_i\}_{i\in I} = \mathsf{E}$ be an enumeration of the edges of $\mathcal{G}$. Then we may apply the coarea formula, valid for any interval, for each $i \in I$, so that the statement holds for $f_i := f_{\mathsf{e}_i}$ and $\varphi_i := \varphi|_{\mathsf{e}_i}$ on $\mathsf{e}_i$. In particular, and since by the monotone convergence theorem

$$\int_{\mathbb{R}} \sum_{x\in S_t\cap\bigcup_{i=1}^k \mathsf{e}_i} \varphi(x) \,\mathrm{d}t = \sum_{i=1}^k \int_{\mathbb{R}} \sum_{x\in S_t\cap\mathsf{e}_i} \varphi(x) \,\mathrm{d}t = \sum_{i=1}^k \int_{\mathsf{e}_i} \varphi(x)|f'(x)| \,\mathrm{d}x \longrightarrow \int_{\mathcal{G}} \varphi(x)|f'(x)| \,\mathrm{d}x,$$

we obtain (4.7). □

*Proof of Lemma 4.5.* Take $f$ as in the statement of the lemma and apply (4.7) to $f$ (with $\varphi = f$ as well); this immediately yields (4.6). Since under our current assumptions $\mathcal{G}$ has finite length, we also have $f \in W^{1,1}(\mathcal{G})$. Repeating the argument but taking $\varphi = 1$ shows that

$$\int_{\mathbb{R}} \sum_{x\in S_t} 1 \,dt = \int_{\mathcal{G}} |f'(x)| \,\mathrm{d}x < \infty,$$

which in particular implies that $S_t$ is finite for almost all $t \in \mathbb{R}$. We now restrict to the set of regular values $t \in \mathfrak{R}(f)$ for which additionally $S_t$ is finite; this possibly smaller set still has full measure in $\mathbb{R}$. Now by construction, for all such $t$ we have

$$S_t = \bigcup_{i\in I} S_t \cap \mathsf{e}_i, \qquad m_f(t) = \sum_{i\in I} m_{f_i} := \sum_{i\in I} |\{x \in \mathsf{e}_i : f(x) < t\}|,$$

and

$$m_f'(t) = \sum_{i\in I} m_{f_i}'(t) = \sum_{i\in I} \sum_{x\in S_t\cap\mathsf{e}_i} \frac{1}{|f'(x)|} = \sum_{x\in S_t} \frac{1}{|f'(x)|},$$

where the finiteness of $S_t$ implies the finiteness of all the sums involved, and hence the validity of the identities, as well as the absolute continuity of $m_f$. This completes the proof. □

With the help of Lemma 4.5, the key symmetrisation argument used in the compact case can be generalised directly to infinite graphs. For this we need to introduce some more notation: given a graph $\mathcal{G}$ of length $L > 0$ and a function $f \in H^1(\mathcal{G}) \hookrightarrow C(\overline{\mathcal{G}}) \hookrightarrow L^\infty(\mathcal{G})$, we define its symmetrisation (or decreasing rearrangement) $f^* \in C([0,L])$ via the level set property

$$|\{x \in (0,L) : f^*(x) < t\}| = m_f(t) \qquad \text{for all } t \in \mathbb{R}$$

(and extension by continuity to $x = 0, L$). Taking $f \in H^1(\mathcal{G})$ to be fixed, we also set $n(t) := \#S_t = \#\{x \in \mathcal{G} : f(x) = t\}$, which is a nonnegative integer for almost all $t \in \mathbb{R}$ by Lemma 4.5; and in fact is (again, for almost all $t \in \mathbb{R}$) at least 2 under the assumptions of Theorems 4.3 and 4.4, by an argument essentially based on Menger's Theorem as in [2, 7]. The following result is now standard; its proof, using the properties established in Lemma 4.5, follows the proof of, e.g., [2, Theorem 2.1] essentially verbatim (see in particular Eq. (3.13) there). We therefore omit it.

**Lemma 4.7.** *Let $\mathcal{G}$ be a locally finite, connected metric graph of total length $L > 0$, let $f \in H^1(\mathcal{G}) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} C^1([0,\ell_\mathsf{e}])$, with minimum $m$ and maximum $M \geq m$ in $\overline{\mathcal{G}}$, respectively, and let $f^*$ be its increasing rearrangement, as just described. Then $f^* \in H^1(0,L)$,*

$$\int_{\mathcal{G}} |f(x)|^2 \,\mathrm{d}x = \int_0^L |f^*(x)|^2 \,\mathrm{d}x, \tag{4.8}$$

*and*

$$\int_{\mathcal{G}} |f'(x)|^2 \, \mathrm{d}x \geq \operatorname{ess\,inf}_{t\in[m,M]} n(t)^2 \int_0^L |(f^*)'(x)|^2 \, \mathrm{d}x. \tag{4.9}$$

*Equality in* (4.9) *implies that* $f'(x)$ *takes on a common value at all* $x \in S_t$*, for almost all* $t \in [m, M]$*.*

As a consequence we obtain the functional inequality of Friedlander [17, Lemma 3] for infinite graphs. We set this up to work simultaneously for both $H^1$-functions vanishing at at least one point in $\mathcal{G}$ and functions in $H^1_0(\mathcal{G})$; the result for the latter will immediately imply Theorem 4.2. Recall that $H^1_0(\mathcal{G}; \{y\}) \subset H^1(\mathcal{G})$ denotes the subspace of $H^1$-functions that vanish at $y$, whether $y$ is a vertex or an end.

**Lemma 4.8.** *Let* $\mathcal{G}$ *be a locally finite, connected metric graph with finite length* $L > 0$ *and let* $y \in \overline{\mathcal{G}}$*. Then*

$$\int_{\mathcal{G}} |f'(x)|^2 \, \mathrm{d}x \geq \frac{\pi^2}{4L^2} \int_{\mathcal{G}} |f(x)|^2 \, \mathrm{d}x \tag{4.10}$$

*for all* $f \in H^1_0(\mathcal{G}; \{y\}) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} C^1([0, \ell_\mathsf{e}])$*. In either case, equality in* (4.10) *can occur for a nonzero function* $f$ *if and only if* $\mathcal{G}$ *is isometrically isomorphic to an interval, with* $y$ *being one of its endpoints, and* $f$ *is proportional to* $\sin(\pi s/2L)$*, where* $s$ *is the distance to* $y$*.*

*Proof.* In either case, by replacing $f$ by $|f|$ we may assume that $f$ is nonnegative. If $y$ is a vertex, then $f \in H^1_0(\mathcal{G}; \{y\})$, since $f$ is continuous and has a zero in $\mathcal{G}$, $f^*(0) = 0$; it follows from Lemma 4.7, the fact that $n(t) \geq 1$ for all $t \in (m, M)$ by continuity of $f$, and the usual one-dimensional inequality (cf. [17, Eq. (2.7)]) that

$$\int_{\mathcal{G}} |f'(x)|^2 \, \mathrm{d}x \geq \int_0^L |(f^*)'(x)|^2 \, \mathrm{d}x \geq \frac{\pi^2}{4L^2} \int_0^L |f^*(x)|^2 \, \mathrm{d}x = \frac{\pi^2}{4L^2} \int_{\mathcal{G}} |f(x)|^2 \, \mathrm{d}x. \tag{4.11}$$

If $y$ is an end, then $f(x) \to 0$ at the end $y$, necessarily $\operatorname{ess\,inf}_{x\in\mathcal{G}} f(x) = 0$ and so $f^*(0) = 0$, from which (4.11) follows.

If there is equality in (4.10), and hence in every step of (4.11), then $n(t) = 1$ for almost all $t \in (m, M) = (0, M)$ and hence for *all* $t \in (0, M)$; moreover, since up to scalar multiples and reflections $f^*(x) = \sin(\pi x/2L)$, $f$ can only vanish on a set of measure zero, and hence at most at a single point. That is, $f$ can take on any value between 0 and its maximum on $\overline{\mathcal{G}}$ only once. A standard continuity argument shows that $\mathcal{G}$ must be a path graph, with $f$ vanishing at exactly one of its endpoints. The fact that $f$, considered as a function on $[0, L]$, minimises the Rayleigh quotient of the second derivative on $(0, L)$ with a Dirichlet condition at 0, means that it must be proportional to the first eigenfunction $\sin(\frac{\pi x}{2L})$. □

We can now give the proof of Theorems 4.2, 4.3 and 4.4; the proof of Theorem 4.1 requires a further intermediate result and will be given afterwards.

*Proof of Theorem 4.2.* This follows immediately from Lemma 4.8applied to the first eigenfunction of $\mathcal{H}_\mathfrak{V}$, together with the variational characterisation (3.6) of $\lambda_1(\mathcal{G}; \mathfrak{V})$. □

*Proof of Theorem 4.3.* Let $\psi_2 \in H^1(\mathcal{G})$ be any eigenfunction for $\mu_2(\mathcal{G})$; then it necessarily changes sign on $\mathcal{G}$, being orthogonal to the constant functions. We next claim that, if $m = \min_{x\in\overline{\mathcal{G}}} \psi_2(x)$ and $M = \max_{x\in\overline{\mathcal{G}}} \psi_2(x)$, then $n(t) = 2$ for almost all $t \in (m, M)$: indeed, choose $x_n, y_n \in \mathcal{G}$ such that $\psi_2(x_n) =: m_n \to m$ and $\psi_2(y_n) =: M_n \to M$. Fix $n \in \mathbb{N}$. Then the assumption that $\mathcal{G}$ is doubly connected means that there are two paths $P_1$ and $P_2$ connecting $x_n$ and $y_n$, intersecting at a null set of vertices. Since $\psi_2$ is certainly continuous, it takes on every value between $m_n$ and $M_n$ at least once on each paths; hence $n(t) = 2$ for all $t \in (m_n, M_n)$ except possibly at the finite set of values $\psi_2$ takes at the vertices. Now let $n \to \infty$; since the countable union of finite sets is certainly a null set, we have proved the claim.

We next let $\mathcal{G}_0$ be any *nodal domain* of $\psi_2$, that is, the closure of a connected component of $\{x \in \mathcal{G} : \psi_2 \neq 0\}$. Let $\eta \in H^1(\mathcal{G})$ be the restriction of $\psi_2$ to $\mathcal{G}_0$, extended by zero on $\mathcal{G} \setminus \mathcal{G}_0$; then taking $\eta$ as a test function in the weak formulation of the eigenvalue problem

$$\int_{\mathcal{G}} \psi_2'(x) v'(x) \, \mathrm{d}x = \mu_2(\mathcal{G}) \int_{\mathcal{G}} \psi_2(x) v(x) \, \mathrm{d}x \qquad \text{for all } v \in H^1(\mathcal{G})$$

leads to

$$\mu_2(\mathcal{G}) = \frac{\int_{\mathcal{G}} |\eta'(x)|^2 \, \mathrm{d}x}{\int_{\mathcal{G}} |\eta(x)|^2 \, \mathrm{d}x} = \frac{\int_{\mathcal{G}_0} |\psi_2'(x)|^2 \, \mathrm{d}x}{\int_{\mathcal{G}_0} |\psi_2(x)|^2 \, \mathrm{d}x}, \tag{4.12}$$

just as holds in the compact case. Using that $\psi_2(x) = 0$ for at least one $x \in \mathcal{G}_0$ and that $n(t) = 2$ for a.e. $t \in [0, \max \psi_2]$, Lemma 4.7 together with an argument analogous to (4.11) implies that

$$\mu_2(\mathcal{G}) \geq \frac{\pi^2}{L(\mathcal{G}_0)^2}.$$

Now this holds for every nodal domain $\mathcal{G}_0$. Since there are at least two, as noted above, at least one has total length at most $L/2$, which leads to the estimate $\mu_2(\mathcal{G}) \geq 4\pi^2/L^2$.

To conclude the proof, let us observe that equality in (4.3) clearly holds if $\mathcal{G}$ is a symmetric necklace, as it becomes clear considering the Rayleigh quotient of such a graph, which by homogeneity reduces to that of an individual interval of half length.

Conversely, let for a given eigenfunction $\psi_2$ equality in (4.3) hold: up to shortening $\mathcal{G}$ we can without loss of generality assume $\psi_2$ to vanish only on a Lebesgue zero set. Then the support $\mathcal{G}_\pm$ of the positive/negative part of $\psi_2$ satisfies $|\mathcal{G}_\pm| = L/2$. Now, because the increasing rearrangement $\psi_2^*$ of $\psi_2$ satisfies

$$\frac{\int_{\mathcal{G}_+} |\psi_2'|^2 \, \mathrm{d}x}{\int_{\mathcal{G}_+} |\psi_2|^2 \, \mathrm{d}x} = \frac{\int_{\mathcal{G}_+^*} |(\psi_2^*)'|^2 \, \mathrm{d}x}{\int_{\mathcal{G}_+^*} |\psi_2^*|^2 \, \mathrm{d}x},$$

we deduce as above that $n(t) = \eta = 2$ for a.e. $t \in [0, \max \psi]$. Thus, up to an exceptional (Lebesgue null) set, $\mathcal{G}$ must consist of two paths representing the pre-images of the set $(\min \psi_2, \max \psi_2)$. This is only possible if $\mathcal{G}$ is a (possibly degenerate) symmetric necklace. Furthermore, any two parallel edges must have equal length: as we have already remarked in Lemma 4.7, because we are assuming equality in (4.9) we necessarily have $|\psi_2'(x)| = |\psi_2'(y)|$ for any $x$ and $y$ in the same level set $\psi_2^{-1}(t)$. We conclude that $\psi_2$ is identical along the two paths, hence the paths have the same length. □

*Proof of Theorem 4.4.* Since $\mathcal{G}$ has finite total length, $\mathcal{H}_{\mathfrak{V}}$ has compact resolvent and there exists a positive ground state $\psi_1 \in H_0^1(\mathcal{G}) \cap C(\overline{\mathcal{G}}) \cap \oplus_{\mathsf{e} \in \mathsf{E}} C^1([0, \ell_{\mathsf{e}}])$. In particular, the function $\psi_1$ can only vanish at $\mathfrak{V}$, where it attains its minimum $m$. Therefore, $\psi_1$ is an $H^1(\mathcal{G})$-function which takes all values in $[0, M]$ in $\overline{\mathcal{G}}$, where $M := \max_{x \in \overline{\mathcal{G}}} \psi_1(x)$. Lemma 4.7 implies that the Rayleigh quotient of $\psi_1$ is at least $\operatorname{ess\,inf}_{t \in [0,M]} n(t)^2$ times the Rayleigh quotient of its increasing rearrangement $\psi_1^*$, which in turn is no smaller than $\left(\frac{\pi}{2L}\right)^2$, that is, the lowest eigenvalue of the Laplacian on $[0, L]$ with mixed Dirichlet/Neumann boundary conditions.

Since $\overline{\mathcal{G}}$ is compact and $\psi_1(x) \to 0$ whenever $x \to \mathfrak{V}$, every value in $(0, M]$ with $M := \max_{x \in \mathcal{G}} \psi_1(x)$ will be attained at least once by $\psi_1$. We claim that every such value will in fact be attained at least twice, with the possible exception of a countable set. Indeed, if for some $t \in (0, M]$ there exists exactly one $x$ such that $\psi_1(x) = t$, then cutting the graph at $x$ disconnects $\mathcal{G}$ into an upper and a lower level set of $\psi_1$. By our topological assumptions on $\mathcal{G}$, this is only possible if $x$ is a vertex, of which there are only countably many. Thus $\operatorname{ess\,inf}_{t \in [0,M]} n(t) \geq 2$, which concludes the proof of (4.3).

To discuss the case of equality in (4.4), observe that the symmetrisation process yields equality if and only if $n(t) = 2$ for a.e. $t$, which in turn implies that $\mathcal{G}$ is a symmetric necklace, with the minimum of $\psi_1$ attained at precisely one of its extremities – either a vertex or an end, by assumption. Observe that in the former case, the necklace may still be either infinite (and in this case $\mathcal{H}_{\mathfrak{V}}$ would be a mixed Friedrichs/Neumann extension) or finite. □

To complete the proof of Theorem 4.1, we next observe that, as in the compact case, it suffices to consider the case where $\mathcal{G}$ is a (now possibly infinite) tree; this follows from a basic surgery argument.

**Lemma 4.9.** *Let $\mathcal{G}$ be a locally finite metric graph of finite length. If $\mathcal{G}'$ is any graph obtained from $\mathcal{G}$ by cutting through $\mathcal{G}$ countably many times, then*

$$\mu_k(\mathcal{G}) \geq \mu_k(\mathcal{G}')$$

*for all $k \in \mathbb{N}$. If $\mathcal{G}'$ is obtained from $\mathcal{G}$ by cutting through $j \in \mathbb{N}$ times, then*

$$\mu_k(\mathcal{G}) \leq \mu_{k+j}(\mathcal{G}')$$

*for all* $k \in \mathbb{N}$.

We refer to [8, Definition 3.2] for the definition of cutting through a vertex; we note that the definition of *cuts* originally given for compact metric graphs makes equal sense in the infinite case, since cutting through vertex is a local graph operation and the cutting procedure in Lemma 4.9 is iterative, i.e., we do not need to cut through infinitely many vertices simultaneously. The proof of Lemma 4.9 is analogous to the one of [8, Theorem 3.4] and therefore omitted.

Note that it is always possible to obtain a tree from any locally finite graph $\mathcal{G}$, upon cutting a countable number of times. We will thus assume for the rest of the section that $\mathcal{G}$ is a (locally finite, connected) tree. In what follows, we will need the following notation. For an arbitrary point $x \in \mathcal{G}$ the graph $\mathcal{G}\backslash\{x\}$ consists of $p_x \in \mathbb{N}_0$ connected components. We denote the closure of these $p_x$ components by $\left\{\mathcal{G}^1(x), \dots \mathcal{G}^{p_x}(x)\right\}$; these graphs are also trees. Inductively, we can remove a finite number of points $\{x_1, \dots x_n\}$. The set of the resulting graphs will be denoted by $\mathcal{G}(x_1, \dots, x_n)$, and an element of this set with $k(x_1, \dots, x_n)$ elements by $\mathcal{G}^j(x_1, \dots x_n)$ with $1 \leq j \leq k(x_1, \dots, x_n)$.

The next lemma is a generalisation of [17, Lemma 4].

**Lemma 4.10.** *Let $\mathcal{G}$ be a locally finite metric tree with length $L(\mathcal{G}) < \infty$. For every $0 < l < L(\mathcal{G})$ there exists some $x \in \mathcal{G}$ such that, for the subgraphs $\left\{\mathcal{G}^1(x), \dots, \mathcal{G}^{p_x}(x)\right\}$ and an appropriate indexing, we have*

$$L(\mathcal{G}^1(x)) \leq L(\mathcal{G}) - l \quad \textit{and} \quad L(\mathcal{G}^i(x)) \leq l \quad \textit{for all} \quad 2 \leq i \leq p_x.$$

*Proof.* We take $\varepsilon > 0$ with $\min(L(\mathcal{G}) - l, l) < L(\mathcal{G}) - \varepsilon$. We take a compact exhaustion $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$ and fix $n$ large enough that $L(\mathcal{G}\backslash\mathcal{G}_n) < \varepsilon$. Now consider $\mathcal{G}_n$. This graph is compact, and $\mathcal{G} \setminus \mathcal{G}_n$ consists of at most finitely many (possibly infinite) graphs $\widehat{\mathcal{G}}_k$, $k = 1, \dots, m$, attached as pendants to $\mathcal{G}_n$ at finitely many vertices (cf. the discussion after Proposition 2.16). The sum of the lengths of these $\widehat{\mathcal{G}}_k$ does not exceed $\varepsilon$. We replace each of these $m$ graphs $\widehat{\mathcal{G}}_k$ by an edge $\mathsf{e}_k$ of length $L(\widehat{\mathcal{G}}_k)$. The resulting graph, call it $\widehat{\mathcal{G}}_n$, satisfies the conditions of [17, Lemma 4]. Thus, we can choose the point $x$ from this lemma. This point $x$ cannot be an element of the interior of an edge $\mathsf{e}_k$, since otherwise one of the resulting graphs would have length exceeding $L(\mathcal{G}) - \varepsilon$. Since $x$ cannot be a vertex of degree one, we conclude that $x \in \mathcal{G}_n$. Per construction $x$ also has the desired properties in the graph $\mathcal{G}$. $\square$

We can now obtain a generalisation of [17, Lemma 2]; the proof in the compact case may be repeated verbatim and is thus omitted.

**Lemma 4.11.** *Let $\mathcal{G}$ be a locally finite metric tree with finite length $L(\mathcal{G}) > 0$ and let $n \in \mathbb{N}$ with $n \geq 2$. Then there exist $n-1$ points $x_1, \dots x_{n-1} \in \mathcal{G}$ such that, for any element of the set $\mathcal{G}(x_1, \dots, x_{n-1})$, we have*

$$L\left(\mathcal{G}^j(x_1, \dots x_{n-1})\right) \leq \frac{L(\mathcal{G})}{n} \qquad 1 \leq j \leq k(x_1, \dots, x_{n-1}).$$

We can finally give the proof of Theorem 4.1, which closely follows that of [17, Theorem 1].

*Proof of Theorem 4.1.* Let $\psi_j$ be any eigenfunction associated with $\mu_j$, $j \in \mathbb{N}$, and fix $k \geq 2$. Then for any $x_1, \dots, x_{k-1} \in \mathcal{G}$ there exists a nontrivial linear combination $\psi$ of the $k$ eigenfunctions $\psi_1, \dots, \psi_k$ such that $\psi(x_i) = 0$ for all $i = 1, \dots, k$ (note that the existence of such a $\psi$ only requires the linear independence of the $\psi_j$; it does not require any properties of their nodal counts). We choose the points $x_i$ according to Lemma 4.10, that is, in such a way that (keeping the notation of the previous lemmata) for every subgraph $\mathcal{G}^j(x_1, \dots, x_k)$, with $1 \leq j \leq k(x_1, \dots, x_k)$ belonging to the set $\mathcal{G}(x_1, \dots x_k)$,

$$L\left(\mathcal{G}^j(x_1, \dots, x_{k-1})\right) \leq \frac{L(\mathcal{G})}{k} \qquad 1 \leq j \leq k(x_1, \dots, x_k).$$

Now since $\psi$ is nontrivial there exists at least one subgraph, call it $\mathcal{G}^1$, on which $\psi$ does not vanish identically. An argument similar to the one used to obtain (4.12) shows that

$$\int_{\mathcal{G}^1} |\psi'(x)|^2 \, \mathrm{d}x \leq \mu_k(\mathcal{G}) \int_{\mathcal{G}^1} |\varphi(x)|^2 \, \mathrm{d}x. \tag{4.13}$$

Since $\psi$ satisfies a Dirichlet condition at at least one point of $\mathcal{G}^1$ (namely whichever of the $x_1, \dots, x_n$ belong(s) to the boundary of $\partial\mathcal{G}^1$), Lemma 4.1 implies

$$\int_{\mathcal{G}^1} |\psi'(x)|^2 \, \mathrm{d}x \geq \frac{\pi^2}{4L(\mathcal{G}^1)^2} \int_{\mathcal{G}^1} |\varphi(x)|^2 \, \mathrm{d}x. \tag{4.14}$$

The fact that $L(\mathcal{G}^1) \leq L(\mathcal{G})/k$ leads to (4.1).

We still have to show the case of equality. The only difference to the proof for the finite case is that instead of segments with a Dirichlet endpoint $y$ we may also have infinite path graphs of finite length, with a Dirichlet endpoint $y$. Note that at the graph end of any such path the same Neumann condition holds as for a leaf of a finite path graph. Applying in particular Lemma 4.11, the rest of the proof carries over verbatim from [17, Section 2], so we omit it. □

### 4.2. **Further lower bounds.**

4.2.1. *Diameter.* Apart from the total length/volume, a quantity in spectral geometry is the diameter. It was proved in [27, Section 5] that neither lower nor upper bounds on the spectral gap are generally possible in terms of diameter alone. However, estimates can be obtained if the diameter is complemented by total length, or if the graph has a special topological structure. In what follows we extend such estimates to infinite graphs.

We start with the bound in terms of length and diameter. In the compact case the following theorem may be found in [42, Theorem 4.4.6]; see also [26, Theorem 1.1] and [27, Theorem 7.2] for earlier iterations.

**Proposition 4.12.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite length $L > 0$ and diameter $D > 0$, and finite Betti number. Then*

$$\mu_2(\mathcal{G}) \geq \frac{2}{LD}. \tag{4.15}$$

*Remark* 4.13. Proposition 4.12 holds whenever one can find a compact exhaustion $(\mathcal{G}_n)_n$ such that $\operatorname{diam} \mathcal{G}_n$ converges to $\operatorname{diam} \mathcal{G} = D$. One such example would be an infinite or semi-infinite ladder.

*Proof.* Choose any compact exhaustion of $\mathcal{G}$, and apply [42, Theorem 4.4.6]. Proposition 2.18 and Lemma 3.8(2) then yield (4.15) on $\mathcal{G}$. □

One could, similarly, generalise the result of [26, Theorem 1.2], which gives a lower bound on $\mu_k$ in terms of $L$ and $D$, to locally finite graphs with finite Betti number. This estimate can be greatly improved if $\mathcal{G}$ is a (finite or infinite) tree. In this case, we obtain:

**Proposition 4.14.** *Let $\mathcal{G}$ be a locally finite metric tree and let*

$$\mathfrak{V} = \mathfrak{C}(\mathcal{G}) \cup \{\mathsf{v} \in \mathsf{V} \mid \deg(\mathsf{v}) = 1\}.$$

*If $\mathcal{G}$ has finite diameter $D$, then the lowest eigenvalue $\lambda_1(\mathcal{G}, \mathfrak{V})$ of $\mathcal{H}_{\mathfrak{V}}$ satisfies*

$$\lambda_1(\mathcal{G}, \mathfrak{V}) \geq \frac{\pi^2}{D^2}. \tag{4.16}$$

Note that due to Proposition 3.6, the assumptions of Proposition 4.14 imply that the spectrum of $\mathcal{H}_{\mathfrak{V}}$ is purely discrete. The proof of Proposition 4.14 follows precisely along the lines of [7, Lemma 4.6] and is therefore omitted.

*Remark* 4.15. Other lower bounds on $\lambda_1(\mathcal{G})$ are obtained in [48, Corollary 3.6 and Theorem 4.1.(i)] in terms of both diameter and further related quantities (the *height* and *reduced height* of the tree), but under the additional assumption that the tree is *radially symmetric* and a Dirichlet condition is imposed on its root.

The discussion following [11, Proposition 2.4] shows that the assumption that $\mathcal{G}$ is a tree cannot generally be dropped.

4.2.2. *Inradius.* Another natural geometric quantity is the *inradius* $\mathrm{Inr}(\mathcal{G},\mathfrak{V})$, that is, the supremum of radii of closed balls within $\overline{\mathcal{G}}$ that do not intersect the Dirichlet set $\mathfrak{V}\subset\mathfrak{C}(\mathcal{G})\cup\mathsf{V}$. In $\mathbb{R}^2$, the *Makai inequality* [34] (see also [23]) states that there exists an absolute constant $C>0$ such that for all bounded and simply connected domains $\Omega\subset\mathbb{R}^2$ the lowest eigenvalue $\lambda_1(\Omega)$ of the Dirichlet Laplacian on $\Omega$ satisfies

$$\lambda_1(\Omega)\geq\frac{C}{\mathrm{Inr}(\Omega)^2},$$

where $\mathrm{Inr}(\Omega)$ is the inradius of $\Omega$, $\mathrm{Inr}(\Omega)=\sup\{r>0:\ \text{there exists } x\in\Omega \text{ such that } B_r(x)\subset\Omega\}$.

It is fairly easy to see that a metric graph analogue of a Makai inequality can only hold on trees (as the Makai inequality itself holds only on simply connected domains) with a Dirichlet condition imposed on all ends and on all degree one vertices. The next proposition proves a Makai inequality for a special class of such trees, namely trees with a *centre point.*

**Definition 4.16.** Suppose $\mathcal{G}$ is a locally finite metric tree of finite inradius. Let $\mathsf{v_c}$ be a vertex of $\mathcal{G}$ and let

$$\mathfrak{V}:=\mathfrak{C}(\mathcal{G})\cup\{\mathsf{v}\in\mathsf{V}\mid\ \deg(\mathsf{v})=1,\ \mathsf{v}\neq\mathsf{v_c}\}. \tag{4.17}$$

We say that $\mathsf{v_c}$ is a *centre vertex* of $\mathcal{G}$ if $\mathrm{dist}(\mathsf{v_c},\gamma_1)=\mathrm{dist}(\mathsf{v_c},\gamma_2)$ holds for any $\gamma_1,\gamma_2\in\mathfrak{V}$. In that case we call $\mathcal{G}$ a *centred tree.*

In particular, in the trees we consider, there is at most one vertex of degree one (namely $\mathsf{v_c}$, if it is of degree one) where standard vertex conditions are imposed, whereas on all all other vertices, we impose Dirichlet conditions. Note that, if $\mathsf{v_c}$ is a centre vertex of $\mathcal{G}$, then $\mathrm{Inr}(\mathcal{G};\mathfrak{V})=\mathrm{dist}(\mathsf{v_c},\gamma)$ for all $\gamma\in\mathfrak{V}$.

**Theorem 4.17** (Hersch–Makai on trees with a centre point)**.** *Suppose $\mathcal{G}$ is a locally finite metric tree of inradius with a centre vertex $\mathsf{v_c}$ and let $\mathfrak{V}$ be as in* (4.17)*. Then the lowest eigenvalue $\lambda_1(\mathcal{G},\mathfrak{V})$ of $\mathcal{H}_{\mathfrak{V}}$ admits the lower bound*

$$\lambda_1(\mathcal{G},\mathfrak{V})\geq\frac{\pi^2}{4\,\mathrm{Inr}(\mathcal{G},\mathfrak{V})^2}. \tag{4.18}$$

Let us emphasise that the existence of a centre vertex is strictly weaker than radial symmetry as investigated in [48], as can be seen on the the diagonal comb graphs of Section 3.3. Example 4.20 below shows that the assumption on the existence of a centre vertex in Theorem 4.17 cannot be dropped.

Theorem 4.17 is a direct consequence of Proposition 4.14 since under the assumption of the existence of a centre vertex of degree greater than 1, the diameter is twice the inradius; while for trees with a centre vertex of degree 1 the inequality follows by mirroring the tree at the centre vertex. However, we will provide a different proof here using the following surgery principle, which we believe to be interesting in its own right.

**Lemma 4.18.** *Let $\mathcal{G}$ be a locally finite, connected metric graph, and $\mathfrak{V}\subset\{\mathsf{v}\in\mathsf{V}\mid\ \deg(\mathsf{v})=1\}\cap\mathfrak{C}(\mathcal{G})$. Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two closed subgraphs of $\mathcal{G}$ such that $(\mathcal{G}_1,\mathcal{G}_2)$ is a partition of $\mathcal{G}$, i.e., $\mathcal{G}=\mathcal{G}_1\cup\mathcal{G}_2$ holds, and $\mathcal{G}_1\cap\mathcal{G}_2$ consists of finitely many vertices of $\mathcal{G}$. Let $\psi$ denote a nonnegative and nontrivial eigenfunction corresponding to $\lambda_1(\mathcal{G},\mathfrak{V})$ and let $\mathfrak{V}_1:=\mathcal{G}_1\cap\mathfrak{V}$. If, for each vertex $\mathsf{v}\in\mathcal{G}_1\cap\mathcal{G}_2$ and each edge $\mathsf{e}$ connecting $\mathsf{v}$ with a vertex $\mathsf{w}$ in $\mathcal{G}_2$, the derivative of $\psi$ at $\mathsf{v}$ pointing into $\mathsf{e}$, then the inequality*

$$\lambda_1(\mathcal{G},\mathfrak{V})\geq\lambda_1(\mathcal{G}_1,\mathfrak{V}_1) \tag{4.19}$$

*holds. The inequality is strict if and only if at least one of the above-mentioned derivatives is strictly negative.*

*Proof.* Let $\mathsf{E}_1$ denote the edge set of $\mathcal{G}_1$ and let $\psi_1$ denote the restriction of $\psi$ to $\mathcal{G}_1$. For a given vertex $\mathsf{v}\in\mathcal{G}_1\cap\mathcal{G}_2$ let $\mathsf{E}_{2,\mathsf{v}}$ denote the set of edges in $\mathcal{G}_2$ that are incident to $\mathsf{v}$, and suppose that for each such edge $\mathsf{e}\simeq[0,\ell_\mathsf{e}]$ the vertex $\mathsf{v}$ is identified with 0. Then, our assumption states that $\psi_\mathsf{e}'(0)\leq 0$ for all $\mathsf{e}\in\mathsf{E}_{2,\mathsf{v}}$. A direct calculation using integration by parts yields

$$\begin{aligned}\|\psi_1'\|^2_{L^2(\mathcal{G}_1)}&=\sum_{\mathsf{e}\in\mathsf{E}_1}\int_0^{\ell_\mathsf{e}}|\psi_\mathsf{e}'(x)|^2\mathrm{d}x=\sum_{\mathsf{e}\in\mathsf{E}_1}\varphi_\mathsf{e}\psi_\mathsf{e}'\,\big|_0^{\ell_\mathsf{e}}-\sum_{\mathsf{e}\in\mathsf{E}_1}\int_0^{\ell_\mathsf{e}}\psi_\mathsf{e}(x)\psi_\mathsf{e}''(x)\mathrm{d}x\\&=\sum_{\mathsf{v}\in\mathcal{G}_1\cap\mathcal{G}_2}\psi(\mathsf{v})\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{v},2}}\psi_\mathsf{e}'(0)+\lambda_1(\mathcal{G},\mathfrak{V})\|\psi_1\|^2_{L^2(\mathcal{G}_1)}\\&\leq\lambda_1(\mathcal{G},\mathfrak{V})\|\psi\|^2_{L^2(\mathcal{G}_1)},\end{aligned}$$


FIGURE 4.2. A partition of a graph $\mathcal{G}$ into two subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$. To apply Lemma 4.18 the derivatives of the eigenfunction $\psi$ in the vertices $\mathsf{v}_1$ and $\mathsf{v}_2$ pointing into the respective edges $\mathsf{e}_1$, $\mathsf{e}_2$ and $\mathsf{e}_3$ have to be nonpositive.

whence

$$\frac{\|\psi_1'\|^2_{L^2(\mathcal{G}_1)}}{\|\psi_1\|^2_{L^2(\mathcal{G}_1)}} \leq \lambda_1(\mathcal{G},\mathfrak{V}).$$

Noting that $\mathsf{v} \in \mathcal{G}_1 \cap \mathcal{G}_2$ implies $\deg(\mathsf{v}) \geq 2$ and hence $\mathsf{v} \notin \mathfrak{V}$, it follows (e.g. by the same reasoning as in [30]) that $\psi(\mathsf{v}) > 0$, and thus the inequality is strict if and only if $\psi_{\mathsf{e}}'(0) < 0$ for some $\mathsf{v} \in \mathcal{G}_1 \cap \mathcal{G}_2$ and some $\mathsf{e} \in \mathsf{E}_{2,\mathsf{v}}$. The inequality (4.19) is an immediate consequence of the variational principle (3.6) – with strict inequality if $\psi_{\mathsf{e}}'(0) < 0$ for some $\mathsf{v} \in \mathcal{G}_1 \cap \mathcal{G}_2$ and some $\mathsf{e} \in \mathsf{E}_{2,\mathsf{v}}$. It remains to show that we have equality in (4.19) if $\psi_{\mathsf{e}}'(0) = 0$ for all $\mathsf{v} \in \mathcal{G}_1 \cap \mathcal{G}_2$ and all $\mathsf{e} \in \mathsf{E}_{2,\mathsf{v}}$. Indeed, in that case the restricted function $\psi_1 = \psi|_{\mathcal{G}_1}$ satisfies Kirchhoff conditions on $\mathcal{G}_1$ in the vertices in $\mathcal{G}_1 \cap \mathcal{G}_2$ and, thus, $\psi_1$ is an eigenfunction of $\mathcal{H}_{\mathfrak{V}_1}$ with corresponding eigenvalue $\lambda_1(\mathcal{G},\mathfrak{V})$. Since $\psi_1$ is strictly positive except at the Dirichlet vertices, it must be associated with the first eigenvalue of $\mathcal{H}_{\mathfrak{V}_1}$, which implies equality in (4.19). $\square$

Note that we will only use Lemma 4.18 in its simplest form, where $\mathcal{G}_1 \cap \mathcal{G}_2$ consists of exactly one vertex $\mathsf{v}$ of $\mathcal{G}$ and where there is exactly one edge $\mathsf{e}$ of $\mathcal{G}$ that connects $\mathsf{v}$ with a vertex $\mathsf{w} \in \mathcal{G}_2$. In that case, one only needs to check the sign of one derivative of $\psi$ to apply Lemma 4.18.

*Alternative proof of Theorem 4.17.* We prove the statement in three steps.

*Step 1:* Suppose first that $\mathcal{G}$ is a compact tree and $\deg(\mathsf{v}_\mathsf{c}) = 1$. If $|\mathfrak{V}| = 1$, then $\mathcal{G}$ is isometrically isomorphic to an interval with mixed Dirichlet/Neumann conditions in the degree one vertices and therefore $\lambda_1(\mathcal{G},\mathfrak{V}) = \frac{\pi^2}{4\,\mathrm{Inr}(\mathcal{G},\mathfrak{V})^2}$. Next assume that $|\mathfrak{V}| \geq 2$. Let $\psi$ denote a nonnegative eigenfunction corresponding to $\lambda_1(\mathcal{G},\mathfrak{V})$. Using induction and the Kirchhoff condition, it can be shown that there exists a path $\mathcal{P}$ in $\mathcal{G}$ connecting the centre point $\mathsf{v}_\mathsf{c}$ and a vertex $\mathsf{v} \in \mathfrak{V}$ such that $\psi$ is decreasing along $\mathcal{P}$. Since $|\mathfrak{V}| \geq 2$ holds, $\mathcal{P}$ passes at least one vertex other than $\mathsf{v}, \mathsf{v}_\mathsf{c}$. Let $\mathsf{w}$ denote the unique vertex in $\mathsf{V} \setminus \mathfrak{V}$ that is adjacent to $\mathsf{v}$, let $\mathcal{G}'$ denote the graph obtained after removing the edge $\mathsf{vw}$ from $\mathcal{G}$ and let $\mathfrak{V}' := \mathfrak{V} \setminus \{\mathsf{v}\}$. Since $\psi$ is decreasing on the edge $\mathsf{vw}$, we may apply Lemma 4.18 to obtain $\lambda_1(\mathcal{G},\mathfrak{V}) \geq \lambda_1(\mathcal{G}',\mathfrak{V}')$. Repeating this argument inductively, after a finite number of steps we are reduced to $|\mathfrak{V}| = 1$, which proves the estimate (4.18) for all compact trees whose centre vertex has degree 1.

*Step 2:* Suppose now that $\mathcal{G}$ is a compact tree and $\mathsf{v}_\mathsf{c}$ has degree $d > 1$. Again let $\psi$ denote a nonnegative eigenfunction corresponding to $\lambda_1(\mathcal{G},\mathfrak{V})$. Since $\psi$ satisfies Kirchhoff conditions in $\mathsf{v}_\mathsf{c}$, there exists an edge $\mathsf{e}$ incident to $\mathsf{v}_\mathsf{c}$ such that $\psi$ has nonpositive derivative on $\mathsf{e}$ at $\mathsf{v}_\mathsf{c}$. Now, cutting through the centre vertex $d-1$ times yields $d$ disjoint trees; for each of these, $\mathsf{v}_\mathsf{c}$ continues to be the centre vertex. From these trees, let $\mathcal{G}_2$ denote the tree containing the edge $\mathsf{e}$ and let $\mathcal{G}_1$ denote its complement in $\mathcal{G}$, the (restored) union of the other $d-1$ trees. In $\mathcal{G}_1$ the centre vertex $\mathsf{v}_\mathsf{c}$ has degree $d-1$. Setting $\mathfrak{V}_1 := \mathfrak{V} \cap \mathcal{G}_1$ and applying Lemma 4.18 we obtain $\lambda_1(\mathcal{G},\mathfrak{V}) \geq \lambda_1(\mathcal{G}_1,\mathfrak{V}_1)$. An induction argument together with Step 1 now yields (4.18) for all compact trees.

*Step 3:* Finally, we suppose that $\mathcal{G}$ is an arbitrary (infinite) tree satisfying the assumptions of Proposition 4.17. We consider the compact exhaustion $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$ with

$$\mathcal{G}_n := \left\{ x \in \mathcal{G} \;\middle|\; \mathrm{dist}_{\mathcal{G}}(x,\mathsf{v}_\mathsf{c}) \leq \left(1 - \frac{1}{2n}\right)\mathrm{Inr}(\mathcal{G},\mathfrak{V}) \right\}, \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, the graph $\mathcal{G}_n$ is a (compact) tree graph with centre vertex $\mathsf{v_c}$ and inradius

$$\operatorname{Inr}(\mathcal{G}_n, \partial\mathcal{G}_n) = \left(1 - \frac{1}{2n}\right) \operatorname{Inr}(\mathcal{G}, \mathfrak{V}).$$

Therefore, using (4.18) in the compact case, we find

$$\lambda_1(\mathcal{G}_n, \partial\mathcal{G}_n) \geq \frac{\pi^2}{4 \operatorname{Inr}(\mathcal{G}_n, \partial\mathcal{G}_n)^2}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using Lemma 3.8(1), we obtain (4.18) in the infinite case. □

*Remark* 4.19. The estimate (4.18) is sharp, as can be seen on equilateral star graphs. In fact, a careful analysis of our proof of Proposition 4.17 and the statement about equality in Lemma 4.18 show that equality holds in (4.18) if and only if the graph is an equilateral star graph.

We now show that without the existence of a centre point, (4.18) need not hold.

*Example* 4.20. Take $\mathcal{G}_T$ to be the 3-star graph depicted in Figure 4.3, consisting of two edges of length $\ell_1$ and one edge of length $\ell_2 < \ell_1$, that meet at a single vertex $\mathsf{v}$. Take $\mathfrak{V}$ to be the set of the three degree one vertices. We claim that

$$\lambda_1(\mathcal{G}_T, \mathfrak{V}) < \frac{\pi^2}{4 \operatorname{Inr}(\mathcal{G}_T)^2}.$$

Indeed, call $\mathcal{G}_T^+$ the graph obtained by adding another edge of length $\ell_2$ to $\mathsf{v}$ and call $\mathfrak{V}^+$ the set of its degree one vertices. The graph $\mathcal{G}_T^+$ can be understood as two Dirichlet intervals $[0, \ell_1 + \ell_2]$, glued at one point. By symmetry, the restriction of any ground state $\psi$ (i.e. eigenfunction for $\lambda_1(\mathcal{G}_T, \mathfrak{V})$) must coincide with the corresponding Dirichlet ground states on each interval. On the one hand, this implies

$$\lambda_1(\mathcal{G}_T^+, \mathfrak{V}^+) = \frac{\pi^2}{(\ell_1 + \ell_2)^2} = \frac{\pi^2}{4 \operatorname{Inr}(\mathcal{G})^2},$$

on the other hand, the outgoing derivatives of the (nonnegative) ground state at $\mathsf{v}$ on the two shorter edges are strictly decreasing. Consequently, by Lemma 4.18, removing one of those edges will strictly decrease $\lambda_1$, which is the claim.

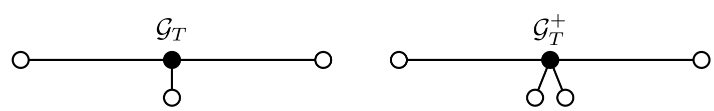


FIGURE 4.3. The metric graphs $\mathcal{G}_T$ and $\mathcal{G}_T^+$ from Example 4.20

*Remark* 4.21. Note that Example 4.20 does not exclude *per se* the validity of a Makai inequality for the Friedrichs extension on metric trees, but it shows that it cannot hold with constant $\pi^2/4$. Let us also emphasise that on two-dimensional domains, the optimal constant in the Makai inequality is unknown, Makai himself having proved it with $C = 1$. However, Hersch [23] proved before Makai that the optimal constant on *convex* two-dimensional domains is $\pi^2/4$; we may possibly regard Theorem 4.17 as an analogue of Hersch's result.

4.2.3. *Isoperimetric constant.* We conclude this section with a brief reminder of a class of estimates based on a third geometric object, Cheeger's isoperimetric constant.

A celebrated result due to Cheeger relates the lowest positive eigenvalue of the Laplace–Beltrami operator on manifolds $M$ with a so-called *isoperimetric constant* that measures how easily $M$ can be split into two pieces [13]. Counterparts of these inequalities are known for combinatorial and metric graphs.

For compact metric graphs $\mathcal{G}$, the relevant version of this quantity was introduced by Nicaise in [40] as

$$h(\mathcal{G}) := \inf \frac{|\partial\tilde{\mathcal{G}}|}{\min\{|\tilde{\mathcal{G}}|, |\tilde{\mathcal{G}}^{\complement}|\}}, \tag{4.20}$$

the infimum being taken over all nonempty, open subgraphs $\tilde{\mathcal{G}} \subsetneq \mathcal{G}$. Now, an adaptation of Cheeger's classical result for manifolds shows [40, Théorème 3.2] that

$$\mu_2(\mathcal{G}) \geq \frac{h(\mathcal{G})^2}{4}.$$

Further Cheeger-type estimates for the Laplacian on compact metric graphs have been obtained, for example, in [43, 35].

The definition of the above isoperimetric constant can be extended to the case of infinite metric graphs of finite length, but it does generally not make sense any more if $\mathcal{G}$ fails to be compact: borrowing some ideas from [25, § 3], one can introduce

$$h^{(0)}(\mathcal{G}) := \inf \frac{|\partial\tilde{\mathcal{G}}|}{|\tilde{\mathcal{G}}|}, \tag{4.21}$$

the infimum being taken over all nonempty, open (but not necessarily connected!) subgraphs $\tilde{\mathcal{G}} \subset \mathcal{G}$ such that $\overline{\tilde{\mathcal{G}}}$ is compact. Then, the estimates

$$\lambda_1(\mathcal{G}) \geq \frac{h^{(0)}(\mathcal{G})^2}{4} \quad \text{and} \quad \mu_2(\mathcal{G}) \geq \frac{h^{(1)}(\mathcal{G})^2}{4} \tag{4.22}$$

are known: the latter was proved in [29, Theorem 3.4]. The former can be proved exactly as in [43, Theorem 6.1] provided a coarea formula for infinite graphs is available, which is indeed the case: see Lemma 4.6 above.

## 5. Upper bounds on the eigenvalues

We finish with two upper bounds, which complement some of the lower bounds of Section 4. The first, a bound on $\mu_2$ in terms of length and diameter, follows more or less directly from the corresponding known inequality in the compact case (see [27, Section 7]) via a compact exhaustion; the second, which gives an upper bound on $\mu_k$ in terms of the diameter and Betti number of the graph (Theorem 5.2), is new also in the case of compact graphs.

**Proposition 5.1.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite length $L > 0$ and diameter $D > 0$ and finite Betti number $\beta \in \mathbb{N}_0$. Then $\mathcal{G}$ satisfies*

$$\mu_2(\mathcal{G}) \leq \frac{\pi^2}{D^2}\frac{4L - 3D}{D}. \tag{5.1}$$

Analogously to Remark 4.13, the conclusion of the theorem holds whenever one can find a compact exhaustion with diameter converging to $D$.

*Proof.* The proof follows directly from combining [27, Theorem 7.1] (which gives (5.1) on any compact graph) applied to any compact exhaustion of $\mathcal{G}$, together with Proposition 2.17 and Lemma 3.8(3). □

We finish with the estimate mentioned above which is also new for compact graphs.

**Theorem 5.2.** *Let $\mathcal{G}$ be a locally finite, connected metric graph with finite diameter $D > 0$, finite Betti number $\beta \in \mathbb{N}_0$, and for which the Neumann extension $\mathcal{H}_{\mathrm{N}}$ has compact resolvent. Then for all $k \geq 2$ we have*

$$\mu_k(\mathcal{G}) \leq (k + \beta - 1)^2 \frac{\pi^2}{D^2}. \tag{5.2}$$

In the case of a compact tree graph (i.e., for which $\beta = 0$) we recover a theorem of Rohleder [45, Theorem 3.4]; for general compact graphs it improves the upper bound on $\mu_2$ in [27, Remark 6.3], $\mu_2(\mathcal{G}) \leq \frac{4\pi^2|\mathcal{E}|^2}{D^2}$, and provides a corresponding bound on the higher eigenvalues $\mu_k$, $k \geq 3$, for the first time.

*Proof.* Fix $k \geq 2$ and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a compact exhaustion of $\mathcal{G}$ for which $D(\mathcal{G}_n) \to D(\mathcal{G})$ (see Proposition 2.18); we suppose $n$ to be large enough that all cycles of $\mathcal{G}$ are contained in $\mathcal{G}_n$, so that $\beta(\mathcal{G}_n) = \beta$ (see Proposition 2.16). We will prove (5.2) for $\mathcal{G}_n$; the statement for $\mathcal{G}$ then follows from Lemma 3.8(3).

Since $\mathcal{G}_n$ has Betti number $\beta$, it is possible to cut it $\beta$ times to produce a tree $\mathcal{T}$, which obviously satisfies $D(\mathcal{T}) \geq D(\mathcal{G}_n)$. By Lemma 4.9, we have $\mu_k(\mathcal{G}_n) \leq \mu_{k+\beta}(\mathcal{T})$. Now [45, Theorem 3.4] implies that

$$\mu_k(\mathcal{G}_n) \leq \mu_{k+\beta}(\mathcal{T}) \leq \frac{\pi^2(k+\beta-1)^2}{D(\mathcal{T})^2} \leq \frac{\pi^2(k+\beta-1)^2}{D(\mathcal{G}_n)^2},$$

which completes the proof. □

Observe that, for $\beta = 0$, (5.2) is sharp for all $k$ (simply take $\mathcal{G}$ to be an interval), but it is not clear what happens for higher $\beta$. Simple examples such as a loop graph suggest that it might be rougher.

## References


[1] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces.* Elsevier, Amsterdam, 2003.

[2] R. Band and G. Lévy. Quantum graphs which optimize the spectral gap. *Ann. Henri Poincaré*, 18:3269–3323, 2017.

[3] A. Bátkai, M. Kramar Fijavž, and A. Rhandi. *Positive Operator Semigroups*, volume 257 of *Operator Theory: Advances and Applications.* Birkhäuser, Cham, 2017.

[4] S. Becker, F. Gregorio, and D. Mugnolo. Schrödinger and polyharmonic operators on infinite graphs: Parabolic well-posedness and $p$-independence of spectra. *J. Math. Anal. Appl.*, 495:124748, 2021.

[5] J. von Below. A characteristic equation associated with an eigenvalue problem on $c^2$-networks. *Lin. Algebra Appl.*, 71:309–325, 1985.

[6] J. von Below and J.A. Lubary. Harmonic functions on locally finite networks. *Result. Math.*, 45:1–20, 2004.

[7] G. Berkolaiko, J.B. Kennedy, P. Kurasov, and D. Mugnolo. Edge connectivity and the spectral gap of combinatorial and quantum graphs. *J. Phys. A*, 50:365201, 2017.

[8] G. Berkolaiko, J.B. Kennedy, P. Kurasov, and D. Mugnolo. Surgery principles for the spectral analysis of quantum graphs. *Trans. Amer. Math. Soc.*, 372:5153–5197, 2019.

[9] J. Brüning, V. Geyler, and K. Pankrashkin. Spectra of self-adjoint extensions and applications to solvable Schrödinger operators. *Rev. Math. Phys.*, 20:1–70, 2008.

[10] R. Carlson. Hill's equation for a homogeneous tree. *Electronic J. Differ. Equ.*, 23:1–30, 1997.

[11] R. Carlson. Nonclassical Sturm–Liouville problems and Schrödinger operators on radial trees. *Electronic J. Differ. Equ.*, 71:1–24, 2000.

[12] C. Cattaneo. The spectrum of the continuous Laplacian on a graph. *Monats. Math.*, 124:124–215, 1997.

[13] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In R.C. Gunning, editor, *Problems in Analysis*, pages 195–199, Princeton, NJ, 1970. Princeton Univ. Press.

[14] E.B. Davies and B. Simon. Spectral properties of Neumann Laplacian of horns. *GAFA, Geom. Funct. Anal.*, 2:105–117, 1992.

[15] R. Diestel. *Graph Theory – 5th edition*, volume 173 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, 2017.

[16] P. Exner, A. Kostenko, M. Malamud, and H. Neidhardt. Spectral theory of infinite quantum graphs. *Ann. Henri Poincaré*, 19:3457–3510, 2018.

[17] L. Friedlander. Extremal properties of eigenvalues for a metric graph. *Ann. Inst. Fourier*, 55:199–212, 2005.

[18] H. Gernandt and C. Trunk. Locally finite extensions and Gesztesy–Šeba realizations for the Dirac operator on a metric graph. In A. Jeribi, editor, *Operator Theory (Proc. Hammamet 2018)*, De Gruyter Proceedings in Mathematics, pages 25–54, Warsaw, 2021. De Gruyter.

[19] D.S. Grebenkov and B.-T. Nguyen. Geometrical structure of Laplacian eigenfunctions. *SIAM Review*, 55:601–667, 2013.

[20] S. Haeseler. *Analysis of Dirichlet forms on graphs.* PhD thesis, Friedrich Schiller Universität Jena, 2014.

[21] P. Hajłasz. Sobolev spaces on an arbitrary metric space. *Potential Analysis*, 5:403–415, 1996.

[22] H. Hanche-Olsen and H. Holden. The Kolmogorov–Riesz compactness theorem. *Expos. Mathematicae*, 28:385–394, 2010.

[23] J. Hersch. Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum. *Z. Angew. Math. Phys.*, 11(5):387–413, 1960.

[24] A. Kałamajska. On compactness of embedding for Sobolev spaces defined on metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 24:123–132, 1999.

[25] M. Keller and D. Mugnolo. General Cheeger inequalities for $p$-Laplacians on graphs. *Nonlinear Anal.*, 147:80–95, 2016.

[26] J.B. Kennedy. A family of diameter-based eigenvalue bounds for quantum graphs. In F.M. Atay, P.B. Kurasov, and D. Mugnolo, editors, *Discrete and Continuous Models in the Theory of Networks (Proc. Bielefeld 2017)*, volume 281 of *Oper. Theory Adv. Appl.*, pages 213–240, Basel, 2020. Birkhäuser.

[27] J.B. Kennedy, P. Kurasov, G. Malenová, and D. Mugnolo. On the spectral gap of a quantum graph. *Ann. Henri Poincaré*, 17:2439–2473, 2016.

[28] A. Kostenko, D. Mugnolo, and N. Nicolussi. Self-adjoint and Markovian extensions of infinite quantum graphs. *J. London Math. Soc.*, 105:1262–1313, 2022.

[29] A. Kostenko and N. Nicolussi. Spectral estimates for infinite quantum graphs. *Calc. Var.*, 58:15, 2019.

[30] P. Kurasov. On the ground state for quantum graphs. *Lett. Math. Phys.*, 109:2491–2512, 2019.

[31] D. Lenz and K. Pankrashkin. New relations between discrete and continuous transition operators on (metric) graphs. *Int. Equations Oper. Theory*, 84:151–181, 2016.

[32] G. Lumer. Connecting of local operators and evolution equations on networks. In F. Hirsch, editor, *Potential Theory (Proc. Copenhagen 1979)*, pages 230–243, Berlin, 1980. Springer-Verlag.
[33] G. Lumer. Espaces ramifiés et diffusion sur les réseaux topologiques. *C.R. Acad. Sc. Paris Sér. A*, 291:627–630, 1980.
[34] E. Makai. A lower estimation of the principal frequencies of simply connected membranes. *Acta Math. Acad. Sci. Hung.*, 16:319–327, 1965.
[35] J.M. Mazón. The Cheeger cut and Cheeger problem in metric graphs. arXiv:2203.07760, 2022.
[36] J.M. Mazón. The total variation flow in metric graphs. *Maths. Engineering*, 5:1–38, 2023.
[37] U. Mosco. Approximation of the solutions of some variational inequalities. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., III Serie*, 21:373–394, 1967.
[38] D. Mugnolo. What is actually a metric graph? arXiv:1912.07549.
[39] K. Naimark and M. Solomyak. Eigenvalue estimates for the weighted Laplacian on metric trees. *Proc. London Math. Soc.*, 80:690–724, 2000.
[40] S. Nicaise. Spectre des réseaux topologiques finis. *Bull. Sci. Math., II. Sér.*, 111:401–413, 1987.
[41] B.S. Pavlov and M. D. Faddeev. Model of free electrons and the scattering problem. *Theor. Math. Phys.*, 55:485–492, 1983.
[42] M. Plümer. *Contributions to the Spectral Theory of Quantum Graphs*. PhD thesis, FernUniversität in Hagen, 2022.
[43] O. Post. Spectral analysis of metric graphs and related spaces. In G. Arzhantseva and A. Valette, editors, *Limits of Graphs in Group Theory and Computer Science*, pages 109–140. EPFL – CRC, Boca Raton, FL, 2009.
[44] F. Rellich. Das Eigenwertproblem von $\Delta\lambda + \lambda\mu = 0$ in Halbröhren. In *Studies and Essays*, pages 329–344. Interscience, New York, 1948.
[45] J. Rohleder. Eigenvalue estimates for the Laplacian on a metric tree. *Proc. Amer. Math. Soc.*, 145:2119–2129, 2017.
[46] J. Rohleder and C. Seifert. Spectral monotonicity for Schrödinger operators on metric graphs. In F.M. Atay, P.B. Kurasov, and D. Mugnolo, editors, *Discrete and Continuous Models in the Theory of Networks (Proc. Bielefeld 2017)*, volume 281 of *Oper. Theory Adv. Appl.*, pages 291–310. Birkhäuser, Basel, 2020.
[47] A. Sard. The measure of the critical values of differentiable maps. *Bull. Amer. Math. Soc.*, 48:883–890, 1942.
[48] M. Solomyak. On the spectrum of the Laplacian on regular metric trees. *Waves Random Media*, 14:155–171, 2004.
[49] K.-T. Sturm. On the geometry of metric measure spaces. *Acta Math.*, 196:65–131, 2006.
[50] M. van den Berg. On the spectrum of the Dirichlet Laplacian for horn-shaped regions in $R^n$ with infinite volume. *J. Funct. Anal.*, 58:150–156, 1984.

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