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Boundary Control for 2 × 2 Elliptic Systems with Conjugation Conditions

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ABSTRACT

In this paper, we consider 2 × 2 non-cooperative elliptic system involving Laplace operator defined on bounded, continuous and strictly Lipschitz domain of R^n. First we prove the existence and uniqueness for the state of the system under conjugation conditions; then we discuss the existence of the optimal control of boundary type with Neumann conditions, and we find the set of equations and inequalities that characterize it.

Keywords: Boundary Control; Elliptic Systems; Conjugation Conditions; Non-Cooperative Systems; Neumann Conditions

1. Introduction

So many optimal control problems governed by partial differential equations have been studied as in [1-3].

Systems governed by elliptic, parabolic, and hyperbolic operators have been considered, some of which are of distributed type as in [4-12], while some others are of boundary type as in [13-17].

Boundary control problems for non-cooperative $n \times n$ elliptic systems involving Laplace operator have been discussed in [17].

Here, using the theory of [3], we study the boundary control problem for 2 × 2 non-cooperative elliptic systems involving Laplace operator but under conjugation conditions.

Let us consider the following elliptic equations:

$$\begin{cases} -\Delta y_1 + y_1 - y_2 = f_1 & x \in \Omega_1 \cup \Omega_2, \\ -\Delta y_2 + y_1 + y_2 = f_2 & x \in \Omega_1 \cup \Omega_2, \end{cases} \tag{1.1}$$

the heterogeneous boundary Neumann conditions:

$$\frac{\partial y_1}{\partial \nu_A} = g_1, \ \frac{\partial y_2}{\partial \nu_A} = g_2, \ x \in \Gamma, \tag{1.2}$$

and the conjugation conditions:

$$\left[\frac{\partial y_1}{\partial \nu_A}\right] = 0, \ \left[\frac{\partial y_2}{\partial \nu_A}\right] = 0, \ x \in \gamma, \tag{1.3}$$

$$\left\{\frac{\partial y_1}{\partial \nu_A}\right\}^{\pm} = r[y_1], \ \left\{\frac{\partial y_2}{\partial \nu_A}\right\}^{\pm} = r[y_2], \ x \in \gamma, \tag{1.4}$$

where we have the following notations:

Ω is a domain that consists of two open, non-intersecting and strictly Lipschitz domains Ω_1 and Ω_2 from an n-dimensional real linear space R^n *i.e.* $\Omega_1, \ \Omega_2 \subset R^n$ are bounded, continuous, and strictly Lipschitz domains such that

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \phi \ \text{ and } \ \bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2.$$

Furthermore, $\Gamma = (\partial\Omega_1 \cup \partial\Omega_2) \setminus \gamma$ is a boundary of a domain $\bar{\Omega}$, $\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi$, and $\partial\Omega_i$ is a boundary of a domain Ω_i, i = 1, 2.

In addition,

$$g_i \in L^2(\Gamma), \ f_i \in L^2(\Omega) \quad (i = 1, 2),$$

$$0 \le r = r(x) \le r_1 < \infty, r \in C(\gamma),$$

r_1 = constant, and ν is an ort of an outer normal to Γ.

Finally, $[\varphi] = \varphi^+ - \varphi^-$,

$$\varphi^+ = \{\varphi\}^+ = \varphi(x) \ \text{ under } x \in \partial\Omega_2 \cap \gamma,$$

$$\varphi^- = \{\varphi\}^- = \varphi(x) \ \text{ under } x \in \partial\Omega_1 \cap \gamma.$$

The model of system (1) is given by:

$$Ay(x) = A(y_1, y_2) = (-\Delta y_1 + y_1 - y_2, -\Delta y_2 + y_1 + y_2),$$

$$A:\left(H^1(\Omega)\right)^2 \to \left(L^2(\Omega)\right)^2$$

system (1) is called non-cooperative, since the coefficients took the previous form.

We first prove the existence and uniqueness for the state of system (1), then we formulate the control problem. We also prove the existence and uniqueness of the optimal control of boundary type, and we discuss the necessary and sufficient conditions of the optimality.

2. The Existence and Uniqueness for the State of System (1)

Since $H^1(\Omega) \subseteq L^2(\Omega) \subseteq \left(H^1(\Omega)\right)'$, then by Cartesian product we have the following chain [1]:

$$\left(H^1(\Omega)\right)^2 \subseteq \left(L^2(\Omega)\right)^2 \subseteq \left(\left(H^1(\Omega)\right)'\right)^2.$$

On $\left(H^1(\Omega)\right)^2 \times \left(H^1(\Omega)\right)^2$, we define the following bilinear form:

$$\begin{aligned} a(y,\psi) &= \int_\Omega \nabla y_1 \nabla \psi_1 \mathrm{d}x + \int_\Omega \nabla y_2 \nabla \psi_2 \mathrm{d}x \\ &+ \int_\Omega \left(y_1\psi_1 - y_2\psi_1 + y_1\psi_2 + y_2\psi_2\right)\mathrm{d}x \\ &+ \int_\gamma r[y_1][\psi_1]\mathrm{d}\gamma + \int_\gamma r[y_2][\psi_2]\mathrm{d}\gamma, \end{aligned} \tag{2}$$

The bilinear form (2) is continuous, since:

$$\begin{aligned} |a(y,\psi)| &\le \|y_1\|_{H^1(\Omega)}\|\psi_1\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|\psi_2\|_{H^1(\Omega)} \\ &+ \Big(\|y_1\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)} + \|y_2\|_{L^2(\Omega)}\|\psi_2\|_{L^2(\Omega)} \\ &+ \|y_1\|_{L^2(\Omega)}\|\psi_2\|_{L^2(\Omega)} + \|y_2\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)}\Big) \\ &+ r\|[y_1]\|_{L^2(\gamma)}\|[\psi_1]\|_{L^2(\gamma)} + r\|[y_2]\|_{L^2(\gamma)}\|[\psi_2]\|_{L^2(\gamma)}, \end{aligned}$$

since the inequalities $\|y\|_{L^2(\Omega)} \le c\|y\|_{H^1(\Omega)}$ and $\|y\|_{L^2(\gamma)} \le c_1\|y\|_{H^1(\Omega)}$ are true [3]. Then we have:

$$\begin{aligned} &|a(y,\psi)| \\ &\le K\Big(\|y_1\|_{H^1(\Omega)}\|\psi_1\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|\psi_2\|_{H^1(\Omega)} \\ &+ \|y_1\|_{H^1(\Omega)}\|\psi_2\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|\psi_1\|_{H^1(\Omega)}\Big) \\ &\le K\Big(\|y_1\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\Big)\Big(\|\psi_1\|_{H^1(\Omega)} + \|\psi_2\|_{H^1(\Omega)}\Big) \\ &\le K\|y\|_{(H^1(\Omega))^2}\|\psi\|_{(H^1(\Omega))^2}, \quad K \text{ is constant.} \end{aligned}$$

Now, we have the following lemma:

Lemma 1:

The bilinear form (2) is coercive on $\left(H^1(\Omega)\right)^2$, that is, there exists $\lambda \in R$, such that:

$$a(y,y) \ge K_1\|y\|^2_{(H^1(\Omega))^2}, \quad K_1 > 0$$

Proof:

$$\begin{aligned} a(y,y) &= \int_\Omega (\nabla y_1)^2 + y_1^2 \,\mathrm{d}x + \int_\Omega (\nabla y_2)^2 + y_2^2 \,\mathrm{d}x \\ &+ \int_\gamma r\left([y_1]^2 + [y_2]^2\right)\mathrm{d}\gamma \\ &\ge \|y_1\|^2_{H^1(\Omega)} + \|y_2\|^2_{H^1(\Omega)}, \qquad (\text{since } r \ge 0) \end{aligned}$$

hence

$$a(y,y) \ge \|y\|^2_{(H^1(\Omega))^2}, \tag{3}$$

which proves the coerciveness condition of the bilinear form (2). Then we have the following theorem:

Theorem 1:

For a given $f = (f_1, f_2) \in \left(L^2(\Omega)\right)^2$, there exists a unique solution $y = (y_1, y_2) \in \left(H^1(\Omega)\right)^2$ for system (1).

Proof:

Since (3) is hold, then by Lax-Milgram lemma, there exists a unique element

$$y = (y_1, y_2) \quad y = (y_1, y_2) \in \left(H^1(\Omega)\right)^2$$

such that

$$a(y,\psi) = L(\psi) \quad \forall \psi = (\psi_1, \psi_2) \in \left(H^1(\Omega)\right)^2, \tag{4}$$

where $L(\psi)$ is defined by:

$$L(\psi) = \int_\Omega f_1\psi_1 \mathrm{d}x + \int_\Omega f_2\psi_2 \mathrm{d}x + \int_\Gamma g_1\psi_1 \mathrm{d}\Gamma + \int_\Gamma g_2\psi_2 \mathrm{d}\Gamma,$$

$$\forall \psi = (\psi_1, \psi_2) \in \left(H^1(\Omega)\right)^2. \tag{5}$$

The linear form (5) is continuous, since:

$$\begin{aligned} |L(\psi)| &= \|f_1\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)}\|\psi_2\|_{L^2(\Omega)} \\ &+ \|g_1\|_{L^2(\Gamma)}\|\psi_1\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)}\|\psi_2\|_{L^2(\Gamma)}, \end{aligned}$$

since the inequalities $\|y\|_{L^2(\Omega)} \le c\|y\|_{H^1(\Omega)}$ and $\|y\|_{L^2(\Gamma)} \le c_2\|y\|_{H^1(\Omega)}$ are true [3], then:

$$\begin{aligned} |L(\psi)| &\le c\|f_1\|_{L^2(\Omega)}\|\psi_1\|_{H^1(\Omega)} + c\|f_2\|_{L^2(\Omega)}\|\psi_2\|_{H^1(\Omega)} \\ &+ c_2\|g_1\|_{L^2(\Gamma)}\|\psi_1\|_{H^1(\Omega)} + c_2\|g_2\|_{L^2(\Gamma)}\|\psi_2\|_{H^1(\Omega)} \\ &\le \left(c\|f_1\|_{L^2(\Omega)} + c_2\|g_1\|_{L^2(\Gamma)}\right)\|\psi_1\|_{H^1(\Omega)} \\ &+ \left(c\|f_2\|_{L^2(\Omega)} + c_2\|g_2\|_{L^2(\Gamma)}\right)\|\psi_2\|_{H^1(\Omega)}, \end{aligned}$$

hence

$$\begin{aligned} |L(\psi)| &\le K_2\left(\|\psi_1\|_{H^1(\Omega)} + \|\psi_2\|_{H^1(\Omega)}\right) \\ &= K_2\|\psi\|_{(H^1(\Omega))^2}. \qquad (K_2 \text{ is constant}) \end{aligned}$$

Now, let us multiply both sides of first equation of (1.1) by $\psi_1(x)$, and the second equation by $\psi_2(x)$ then integration over Ω, we have:

$$\int_\Omega (-\Delta y_1 + y_1 - y_2)\psi_1 \,dx = \int_\Omega f_1 \psi_1 \,dx ,$$

$$\int_\Omega (-\Delta y_2 + y_1 + y_2)\psi_2 \,dx = \int_\Omega f_2 \psi_2 \,dx .$$

By applying Green's formula:

$$\int_\Omega \nabla y_1 \nabla \psi_1 dx - \int_\Gamma \psi_1 \frac{\partial y_1}{\partial \nu_A} d\Gamma + \int_\Omega (y_1 - y_2)\psi_1 \,dx = \int_\Omega f_1 \psi_1 \,dx ,$$

$$\int_\Omega \nabla y_2 \nabla \psi_2 dx - \int_\Gamma \psi_2 \frac{\partial y_2}{\partial \nu_A} d\Gamma + \int_\Omega (y_1 + y_2)\psi_2 \,dx = \int_\Omega f_2 \psi_2 \,dx ,$$

by sum the two equations, then comparing the summation with (2), (4) and (5) we obtain:

$$\int_\Omega \nabla y_1 \nabla \psi_1 dx + \int_\Omega \nabla y_2 \nabla \psi_2 dx + \int_\Omega (y_1 - y_2)\psi_1 \,dx + \int_\Omega (y_1 + y_2)\psi_2 \,dx = \int_\Omega f_1 \psi_1 \,dx + \int_\Omega f_2 \psi_2 \,dx + \int_\Gamma \psi_1 \frac{\partial y_1}{\partial \nu_A} d\Gamma + \int_\Gamma \psi_2 \frac{\partial y_2}{\partial \nu_A} d\Gamma,$$

then we deduce (1.2), which completes the proof.

3. Formulation of the Control Problem

The space $(L^2(\Gamma))^2$ is the space of controls. For a control $u = (u_1, u_2) \in (L^2(\Gamma))^2$, the state $y(u) = (y_1(u), y_2(u)) \in (H^1(\Omega))^2$ of system (1) is given by the solution of the following systems:

$$\begin{cases} -\Delta y_1(u) + y_1(u) - y_2(u) = f_1 & x \in \Omega_1 \cup \Omega_2, \\ -\Delta y_2(u) + y_1(u) + y_2(u) = f_2 & x \in \Omega_1 \cup \Omega_2, \end{cases} \quad (6.1)$$

$$\frac{\partial y_1}{\partial \nu_A} = g_1 + u_1 , \ \frac{\partial y_2}{\partial \nu_A} = g_2 + u_2 , \ x \in \Gamma, \quad (6.2)$$

and the conjugation conditions:

$$\left[\frac{\partial y_1(u)}{\partial \nu_A}\right] = 0, \ \left[\frac{\partial y_2(u)}{\partial \nu_A}\right] = 0, \ x \in \gamma, \quad (6.3)$$

$$\left\{\frac{\partial y_1(u)}{\partial \nu_A}\right\}^{\pm} = r[y_1], \ \left\{\frac{\partial y_2(u)}{\partial \nu_A}\right\}^{\pm} = r[y_2], \ x \in \gamma, \quad (6.4)$$

Since there exists a generalized solution $y(u) \in (H^1(\Omega))^2$ to the boundary value problem (6), then such solution is reasonable on Γ of $\bar{\Omega}$, and

$$\|y(u)\|_{(L^2(\Gamma))^2} \prec \infty . \quad (7)$$

The observation equation is given by:

$$Z(u) = (Z_1(u), Z_2(u)) = Cy(u) = C(y_1(u), y_2(u)),$$

where $C \in \mathcal{L}\left((L^2(\Gamma))^2 ; (L^2(\Gamma))^2\right)$, namely:

$$Cy(u) = C(y_1(u), y_2(u)) = (y_1(u), y_2(u))$$

i.e.

$$Z(u) = (Z_1(u), Z_2(u)) = y(u) = (y_1(u), y_2(u)), \quad (8)$$

For a given $z_g = (z_{g1}, z_{g2}) \in (L^2(\Gamma))^2$, the cost function is given by

$$J(u) = \|y_1(u) - z_{g1}\|^2_{L^2(\Gamma)} + \|y_2(u) - z_{g2}\|^2_{L^2(\Gamma)} + (Nu, u)_{(L^2(\Gamma))^2} , \quad (9)$$

where $Nu = \bar{a}(x)u, \ 0 \prec a_0 \le \bar{a}(x) \le a_1 \prec \infty.$

The function $y(u) \in (H^1(\Omega))^2$ is specified on the domain $\bar{\Omega}_1 \cup \bar{\Omega}_2$, minimizes the energy functional:

$$\Phi(\psi) = \int_\Omega (\nabla \psi_1)^2 + \psi_1^2 \,dx + \int_\Omega (\nabla \psi_2)^2 + \psi_2^2 \,dx + \int_\gamma r\left([\psi_1]^2 + [\psi_2]^2\right) d\gamma - 2\int_\Omega f_1 \psi_1 dx - 2\int_\Omega f_2 \psi_2 dx - 2\int_\Gamma g_1 \psi_1 d\Gamma - 2\int_\Gamma g_2 \psi_2 d\Gamma - 2\int_\Gamma u_1 \psi_1 d\Gamma - 2\int_\Gamma u_2 \psi_2 d\Gamma \quad (10)$$

on $(H^1(\Omega))^2$, and it is the unique solution in $(H^1(\Omega))^2$ to the weakly stated problem of finding an element $y(u) \in (H^1(\Omega))^2$ that meets the following integral equation:

$$\int_\Omega \nabla y_1 \nabla \psi_1 dx + \int_\Omega \nabla y_2 \nabla \psi_2 dx + \int_\Omega y_1 \psi_1 - y_2 \psi_1 + y_1 \psi_2 + y_2 \psi_2 \,dx + \int_\gamma r[y_1][\psi_1] d\gamma + \int_\gamma r[y_2][\psi_2] d\gamma = \int_\Omega f_1 \psi_1 dx + \int_\Omega f_2 \psi_2 dx + \int_\Gamma g_1 \psi_1 d\Gamma + \int_\Gamma g_2 \psi_2 d\Gamma + \int_\Gamma u_1 \psi_1 d\Gamma + \int_\Gamma u_2 \psi_2 d\Gamma$$

$$\forall \psi(u) = (\psi_1(u), \psi_2(u)) \in (H^1(\Omega))^2 . \quad (11)$$

The control problem then is to find $u=(u_1,\ u_2)\in U_{ad}$ such that $J(u)\le J(v)$, where U_{ad} is a closed convex subset of $\left(L^2(\Gamma)\right)^2$.

The cost function (9) can be written as (see [1]):

$$J(u)=\pi(u,u)-2L(u)+\left\|y_1(0)-z_{g1}\right\|^2_{L^2(\Gamma)} +\left\|y_2(0)-z_{g2}\right\|^2_{L^2(\Gamma)}. \tag{12}$$

In this case, the bilinear form $\pi(.,.)$ and the linear form $L(.)$ are expressed as:

$$\begin{aligned}\pi(u,v)&=\left(y_1(u)-y_1(0),y_1(v)-y_1(0)\right)_{L^2(\Gamma)}\\&\quad+\left(y_2(u)-y_2(0),y_2(v)-y_2(0)\right)_{L^2(\Gamma)}\\&\quad+(\bar{a}u,v)_{\left(L^2(\Gamma)\right)^2},\end{aligned} \tag{13}$$

$$\begin{aligned}L(v)&=\left(z_{g1}-y_1(0),y_1(v)-y_1(0)\right)_{L^2(\Gamma)}\\&\quad+\left(z_{g2}-y_2(0),y_2(v)-y_2(0)\right)_{L^2(\Gamma)}.\end{aligned} \tag{14}$$

Now, we prove the continuity of $\pi(u,v)$ and $L(v)$ on $\left(L^2(\Gamma)\right)^2$ as follows [3]:

Let $\tilde{y}'=\tilde{y}(u')$ and $\tilde{y}''=\tilde{y}(u'')$ be solutions from $H^1(\Omega)$ to problem (11) under f = 0 and g = 0. Then from the bilinear form $a(.,.)$ which is given by (2), we can derive the following inequality:

$$\begin{aligned}&\left\|\tilde{y}'-\tilde{y}''\right\|^2_{\left(H^1(\Omega)\right)^2}\\&\le\mu\, a\left(\tilde{y}'-\tilde{y}'',\tilde{y}'-\tilde{y}''\right)\\&\le\mu\left\|u'-u''\right\|_{\left(L^2(\Gamma)\right)^2}\left\|\tilde{y}'-\tilde{y}''\right\|_{\left(L^2(\Gamma)\right)^2},\end{aligned}$$

Since $\|y\|_{L^2(\Gamma)}\le c_2\|y\|_{H^1(\Omega)}$, then:

$$\begin{aligned}&\left\|\tilde{y}'-\tilde{y}''\right\|^2_{\left(L^2(\Gamma)\right)^2}\\&\le c_2\left\|\tilde{y}'-\tilde{y}''\right\|^2_{H^1(\Omega)^2}\\&\le c_2\mu\left\|u'-u''\right\|_{\left(L^2(\Gamma)\right)^2}\left\|\tilde{y}'-\tilde{y}''\right\|_{\left(L^2(\Gamma)\right)^2},\end{aligned}$$

thus, we have:

$$\left\|\tilde{y}'-\tilde{y}''\right\|_{\left(L^2(\Gamma)\right)^2}\le c_2\mu\left\|u'-u''\right\|_{\left(L^2(\Gamma)\right)^2}.$$

i.e. the function $\tilde{y}(u)$ is continuously dependent on u.

Then the continuity of $\pi(u,v)$ and $L(v)$ on $\left(L^2(\Gamma)\right)^2$ is proved.

The bilinear form $\pi(u,v)$ is coercive on $\left(L^2(\Gamma)\right)^2$ since $(\bar{a}u,v)=\left(\sqrt{\bar{a}}u,\sqrt{\bar{a}}v\right)$,

Thus:

$$\begin{aligned}\pi(u,u)&=\left(y_1(u)-y_1(0),y_1(u)-y_1(0)\right)_{L^2(\Gamma)}\\&\quad+\left(y_2(u)-y_2(0),y_2(u)-y_2(0)\right)_{L^2(\Gamma)}\\&\quad+\left(\sqrt{\bar{a}}u,\sqrt{\bar{a}}u\right)_{\left(L^2(\Gamma)\right)^2}\\&\ge a_0(u,u)_{\left(L^2(\Gamma)\right)^2}.\end{aligned}$$

Then by Lax Milgram lemma, the following theorem is proved. Moreover, it gives the necessary and sufficient conditions of optimality.

Theorem 2:

Assume that (3) holds, there exists a unique optimal control $u=(u_1,u_2)\in U_{ad}$ that is closed convex subset of $\left(L^2(\Gamma)\right)^2$ and it is then characterized by the following equations and inequalities:

$$\begin{cases}-\Delta p_1(u)+p_1(u)+p_2(u)=0,\\-\Delta p_2(u)-p_1(u)+p_2(u)=0,\end{cases}\quad x\in\Omega_1\cup\Omega_2, \tag{15.1}$$

$$\frac{\partial p_1}{\partial\nu_{A^*}}=y_1(u)-z_{g1},\ \frac{\partial p_2}{\partial\nu_{A^*}}=y_2(u)-z_{g2},\quad x\in\Gamma, \tag{15.2}$$

$$\left[\frac{\partial p_1(u)}{\partial\nu_{A^*}}\right]=0,\ \left[\frac{\partial p_2(u)}{\partial\nu_{A^*}}\right]=0,\ x\in\gamma, \tag{15.3}$$

$$\left\{\frac{\partial p_1(u)}{\partial\nu}\right\}^{\pm}=r\left[p_1\right],\ \left\{\frac{\partial p_2(u)}{\partial\nu}\right\}^{\pm}=r\left[p_2\right],\quad x\in\gamma, \tag{15.4}$$

$$\left(p(u)+\bar{a}u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\ge 0, \tag{16}$$

together with (6), where $p(u)=\left(p_1(u),p_2(u)\right)\in\left(H^1(\Omega)\right)^2$ is the adjoint state.

Proof:

The optimal control $u=(u_1,u_2)\in\left(L^2(\Gamma)\right)^2$ is characterized by (see [1])

$$\pi(u,v-u)\ge L(v-u)\quad\forall\, v\in U_{ad}, \tag{17}$$

by (13), and (14):

$$\begin{aligned}&\pi(u,v-u)-L(v-u)\\&=\left(y(u)-z_g,y(v-u)-y(0)\right)_{\left(L^2(\Gamma)\right)^2}\\&\quad+\left(\bar{a}\,u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\ge 0,\end{aligned}$$

thus:

$$\begin{aligned}&\left(y(u)-z_g,y(v)-y(u)\right)_{\left(L^2(\Gamma)\right)^2}\\&+\left(\bar{a}\,u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\ge 0,\end{aligned}$$

this inequality can be written as

$$\left(y_1(u)-z_{g1},y_1(v)-y_1(u)\right)_{L^2(\Gamma)} +\left(y_2(u)-z_{g2},y_2(v)-y_2(u)\right)_{L^2(\Gamma)} +\left(\bar{a}\,u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\geq 0. \quad (18)$$

Now, since: $\left(A^* p,y\right)=\left(p,Ay\right)$, then:

$$\left(p,Ay\right)_{\left(L^2(\Omega)\right)^2} =\left(p_1(u),-\Delta y_1(u)+y_1(u)-y_2(u)\right)_{L^2(\Omega)} +\left(p_2(u),-\Delta y_2(u)+y_1(u)+y_2(u)\right)_{L^2(\Omega)},$$

by using Green's formula, we obtain:

$$\begin{aligned}\left(p,Ay\right)_{\left(L^2(\Omega)\right)^2} &=\left(-\Delta p_1(u)+p_1(u)+p_2(u),y_1(u)\right)_{L^2(\Omega)} -\left(p_1,\frac{\partial y_1}{\partial \nu_A}\right)_{L^2(\Gamma)}+\left(y_1,\frac{\partial p_1}{\partial \nu_{A^*}}\right)_{L^2(\Gamma)}\\ &+\left(-\Delta p_2(u)-p_1(u)+p_2(u),y_2(u)\right)_{L^2(\Omega)} -\left(p_2,\frac{\partial y_2}{\partial \nu_A}\right)_{L^2(\Gamma)}+\left(y_2,\frac{\partial p_2}{\partial \nu_{A^*}}\right)_{L^2(\Gamma)}\\ &=\left(A^* p,y\right)_{\left(L^2(\Omega)\right)^2},\end{aligned}$$

then

$$A^* p=A^*\left(p_1,p_2\right) =\left(-\Delta p_1+p_1+p_2\,,-\Delta p_2-p_1+p_2\right), \quad x\in\Omega_1\cup\Omega_2, \quad (19)$$

Since the adjoint system takes the form [3]:

$$A^* p(v)=0,\ x\in\Omega, \quad (20.1)$$

$$\frac{\partial p}{\partial \nu_{A^*}}=y(v)-z_g,\ x\in\Gamma, \quad (20.2)$$

$$\left[\frac{\partial p}{\partial \nu_{A^*}}\right]=0 \text{ and } \left\{\frac{\partial p}{\partial \nu_{A^*}}\right\}^{\pm}=r[p],\ x\in\gamma, \quad (20.3)$$

and by using (19), system (15) is proved .

From Green's formula the following equations are true:

$$\left(-\Delta p_1(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)} =\left(\nabla p_1(u),\nabla\left(y_1(v)-y_1(u)\right)\right)_{L^2(\Omega)} -\left(\frac{\partial p_1}{\partial \nu_{A^*}},y_1(v)-y_1(u)\right)_{L^2(\Gamma)}, \quad (21)$$

$$\left(-\Delta p_2(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)} =\left(\nabla p_2(u),\nabla\left(y_2(v)-y_2(u)\right)\right)_{L^2(\Omega)} -\left(\frac{\partial p_2}{\partial \nu_{A^*}},y_2(v)-y_2(u)\right)_{L^2(\Gamma)}, \quad (22)$$

by adding

$$\left(p_1(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)},$$

$$\left(p_2(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)}$$

to the both sides of Equation (21), and

$$\left(-p_1(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)},$$

$$\left(p_2(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)}$$

to the both sides of Equation (22), then by (15) we obtain:

$$\left(\frac{\partial p_1}{\partial \nu_{A^*}},y_1(v)-y_1(u)\right)_{L^2(\Gamma)} =\left(\nabla p_1(u),\nabla\left(y_1(v)-y_1(u)\right)\right)_{L^2(\Omega)} +\left(p_1(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)} +\left(p_2(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)}, \quad (23)$$

and

$$\left(\frac{\partial p_2}{\partial \nu_{A^*}},y_2(v)-y_2(u)\right)_{L^2(\Gamma)} =\left(\nabla p_2(u),\nabla\left(y_2(v)-y_2(u)\right)\right)_{L^2(\Omega)} +\left(-p_1(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)} +\left(p_2(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)}. \quad (24)$$

Now, we transform (18) by using (15) as follows:

$$\left(\frac{\partial p_1}{\partial \nu_{A^*}},y_1(v)-y_1(u)\right)_{L^2(\Gamma)} +\left(\frac{\partial p_2}{\partial \nu_{A^*}},y_2(v)-y_2(u)\right)_{L^2(\Gamma)} +\left(\bar{a}u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\geq 0, \quad (25)$$

by (23) and (24), we have:

$$\left(\nabla p_1(u),\nabla\left(y_1(v)-y_1(u)\right)\right)_{L^2(\Omega)}$$
$$+\left(p_1(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)}$$
$$+\left(p_2(u),y_1(v)-y_1(u)\right)_{L^2(\Omega)}$$
$$+\left(\nabla p_2(u),\nabla\left(y_2(v)-y_2(u)\right)\right)_{L^2(\Omega)}$$
$$-\left(p_1(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)}$$
$$+\left(p_2(u),y_2(v)-y_2(u)\right)_{L^2(\Omega)}$$
$$+\left(\bar{a}\,u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\geq 0,$$

by using (2):

$$a\left(y_1(v)-y_1(u),p_1(u)\right)-\int_\gamma r\left[y_1(v)-y_1(u)\right]\left[p_1\right]\mathrm{d}\gamma$$
$$+a\left(y_2(v)-y_2(u),p_2(u)\right)-\int_\gamma r\left[y_2(v)-y_2(u)\right]\left[p_2\right]\mathrm{d}\gamma$$
$$+\left(\bar{a}\,u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\geq 0,$$

from (2), and using Green's formula:

$$\left(p_1(u),A\left(y_1(v)-y_1(u)\right)\right)_{L^2(\Omega)}$$
$$+\int_\Gamma p_1\frac{\partial\left(y_1(v)-y_1(u)\right)}{\partial\nu_A}\mathrm{d}\Gamma$$
$$-\int_\gamma r\left[y_1(v)-y_1(u)\right]\left[p_1\right]\mathrm{d}\gamma$$
$$+\int_\gamma r\left[y_1(v)-y_1(u)\right]\left[p_1\right]\mathrm{d}\gamma$$
$$+\left(p_2(u),A\left(y_2(v)-y_2(u)\right)\right)_{L^2(\Omega)}$$
$$+\int_\Gamma p_2\frac{\partial\left(y_2(v)-y_2(u)\right)}{\partial\nu_A}\mathrm{d}\Gamma$$
$$-\int_\gamma r\left[y_2(v)-y_2(u)\right]\left[p_2\right]\mathrm{d}\gamma$$
$$+\int_\gamma r\left[y_2(v)-y_2(u)\right]\left[p_1\right]\mathrm{d}\gamma$$
$$+\left(\bar{a}\,u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\geq 0,$$

from (6), we obtain:

$$\int_\Gamma p_1\left(v_1-u_1\right)\mathrm{d}\Gamma+\int_\Gamma p_2\left(v_2-u_2\right)\mathrm{d}\Gamma$$
$$+\left(\bar{a}u,v-u\right)_{\left(L^2(\Gamma)\right)^2}\geq 0,$$

which proves (16).

Remark:

If the constraints are absent, *i.e.* when $U_{ad}=\left(L^2(\Gamma)\right)^2$, then the equality:

$p(u)+\bar{a}u=0,\ x\in\Gamma,$ follows from condition (16). Hence

$$u_1=-\frac{p_1}{\bar{a}}\quad\text{and}\quad u_2=-\frac{p_2}{\bar{a}},\quad x\in\Gamma. \tag{26}$$

4. Conclusions

The main result of the paper contains necessary and sufficient conditions of optimality (of Pontryagin's type) for 2 × 2 elliptic systems under Neumann conjugation conditions involving Laplace operator defined on bounded, continuous and strictly Lipschitz domain of R^n, that give characterization of optimal control.

We can consider boundary control problems for 2 × 2 and $n \times n$ elliptic distributed systems with Dirichlet conjugation boundary conditions. Also we can consider boundary control problems for parabolic and hyperbolic distributed systems with Dirichlet and Neumann conjugation boundary conditions. The ideas mentioned above will be developed in forthcoming papers.

Also it is evident that by modifying:

- the boundary conditions,
- the nature of the control (distributed, boundary),
- the nature of the observation,
- the initial differential system,

many of variations on the above problem are possible to study with the help of Lions formalism.

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