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TOPICAL REVIEW

Boundary control in reconstruction of manifolds and metrics (the BC method)

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Abstract. One of the approaches to inverse problems based upon their relations to boundary control theory (the so-called BC method) is presented. The method gives an efficient way to reconstruct a Riemannian manifold via its response operator (dynamical Dirichlet-to-Neumann map) or spectral data (a spectrum of the Beltrami–Laplace operator and traces of normal derivatives of the eigenfunctions). The approach is applied to the problem of recovering a density, including the case of inverse data given on part of a boundary. The results of the numerical testing are demonstrated.

Introduction

The goal of this paper is to present one of the approaches to boundary-value inverse problems (IPs) based upon their relations to boundary control theory. We are dealing with the so-called BC method proposed by the author in 1986 (see Belishev 1987a); its modernized version (Belishev 1990b) lies as a basis of this paper.

To demonstrate the opportunities of the method we choose, perhaps, the most impressive of its achievements: that is a reconstruction of Riemannian manifolds. Moreover, the problem of recovering a density is considered; this is the problem which the BC method was created to solve. Let us describe the main results.

(i) Let (Ω, g) be a smooth compact Riemannian manifold with a border Γ; consider the dynamical system

$$u_{tt} - \Delta_g u = 0 \qquad \text{in } \Omega \times (0, T) \tag{1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \tag{2}$$

$$u|_{\Gamma\times[0,T]} = f. \tag{3}$$

Let $u = u^f(x, t)$ be its solution (wave) initiated by a boundary control f. The response operator (dynamical Dirichlet-to-Neumann map) is defined as the map $R^T : f \to \partial u^f/\partial\nu|_{\Gamma\times[0,T]}$ (ν being an outward normal). At the final moment $t = T$ the waves moving from Γ fill the subdomain $\Omega^T = \{x \in \Omega|\, \mathrm{dist}(x, \Gamma) < T\}$. By virtue of a hyperbolicity of problem (1)–(3) the operator R^{2T} is determined by the submanifold (Ω^T, g). The remarkable fact is that the opposite turns out to be true: *we show that the operator R^{2T} determines (Ω^T, g) up to isometry*.

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(ii) Let $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\varphi_k(\cdot)\}_{k=1}^{\infty}$ be the spectrum and the eigenfunctions of the problem

$$-\Delta_g\varphi = \lambda\varphi \qquad \text{in } \Omega$$
$$\varphi|_\Gamma = 0$$

functions $\{\varphi_k\}$ being orthonormalized in $L_2(\Omega)$; $\psi_k := \partial\varphi_k/\partial\nu$. The set of pairs $\{\lambda_k; \psi_k(\cdot)\}_{k=1}^{\infty}$ is said to be the (Dirichlet) spectral data of a manifold. One of our results is that *spectral data determine* (Ω, g) *up to isometry*.

(iii) Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary Γ; let $\rho > 0$ be a smooth function (density) given in $\overline{\Omega}$. The dynamical system of the form (1)–(3) corresponding to the wave equation $\rho u_{tt} - \Delta u = 0$ determines a response operator and spectral data. We propose *an efficient procedure which recovers* $\rho|_{\Omega^T}$ *via* R^{2T} or $\rho|_\Omega$ *via* $\{\lambda_k; \psi_k\}$. The analogous results are obtained for the case of both kinds of *inverse data given on any open subset of a boundary*.

As an approach, the BC method is of a complex character: it uses geometry, asymptotic methods (propagation of singularities), control theory and functional analysis. The role of the organizing frame is played by the system theory. One reason to call the approach the boundary control method is as follows. One of the central facts which is necessary to justify the method is a property of controllability of system (1)–(3): the reachable set $\mathcal{U}^T = \{u^f(\cdot, T) | f \in L_2(\Gamma \times [0, T])\}$ is dense in $L_2(\Omega^T)$. Furthermore, the use of controllability relates the BC method to an approach based upon the Hilbert uniqueness method (Lions, Puel, Yamamoto and others; see, e.g., Yamamoto (1995)); both approaches exploit the well known principle of system theory: if a system is controllable, it is the observable that gives the possibility of extracting information concerning a reachable part of the system from the corresponding measurements.

The first variant of the BC method (Belishev 1987a) was based upon a transparent physical idea: operating by a boundary control to create in a domain *the waves of a standard shape* (Dirac δ-functions). Later this idea led to a variant of the method using some of the multidimensional analogues of the classical Gelfand–Levitan–Krein's equations (Belishev 1987b, Belishev and Blagovestchenskii 1992). Recently, Rakesh noted that in the one-dimensional case this variant (see Belishev 1996b) is similar to an approach proposed by Sondhi and Gopinath (Gopinath and Sondhi 1971, Sondhi and Gopinath 1971).

As one more analogue and predecessor of our method, the 'local approach' belonging to Blagovestchenskii (1971) has to be mentioned. A dynamical variant of the BC method may be considered as its multidimensional generalization.

The BC method was originated independently and practically simultaneously† with other approaches to the multidimensional IPs (Kohn, Lee, Nachman, Novikov, Sylvester, Uhlmann, Vogelius and others). Comparing it with the known methods the following pecularities should be noted:

(i) the method is of invariant character: it recovers not only coefficients of equations but Riemannian manifolds of *an arbitrary topology* (note that the compactness and C^∞-smoothness of a manifold do not play the central role in reconstruction);

(ii) the BC method gives more than a uniqueness of determination, it proposes the recovering procedures which may be used as a basis of numerical algorithms;

(iii) the method works in the case of data given on part of a boundary; its dynamical variant leads to unimprovable (time optimal) results;

† The paper by Belishev (1987a) was submitted to *Doklady Akad. Nauk SSSR* (presented by L D Faddeev) on 29 April 1986, and published in June 1987. The papers by Belishev and Kurylev (1986, 1987) were written later; the first paper used the scheme identical with that of Belishev (1987a) and referred to this latter work.

(iv) a simple and clear background (integration by parts, controllability plus geometrical optics) makes the method of a rather general character which gives reason to hope for its applications to more complicated systems of elasticity, electrodynamics etc. First steps in this direction have already been taken (Avdonin and Belishev 1996, Belishev 1995, 1996a, Belishev *et al* 1997).

In conclusion, we describe the contents and the structure of the paper.

Section 1 is devoted to the geometrical preliminaries. In section 2 the direct boundary-value initial problems are considered; the geometrical optics relations are presented. Section 3 introduces spaces and operators which describe the dynamical system (1)–(3) in terms of control theory; the visualizing operator V^T appears in section 3.5. Section 4 deals with a property of controllability; a duality 'controllability–observability' is considered. Section 5 plays a central role by demonstrating a way to visualize the waves through the boundary measurements. The operator V^T is represented in the form of an operator integral which is determined by the inverse data. Section 6 deals with a reconstruction itself. We describe a way to obtain an isometrical copy of an original manifold from a picture of waves given by operator V^T. Thus, a reconstruction is realized by the scheme 'inverse data $\Rightarrow$ the visualizing operator $\Rightarrow$ manifold'. In section 7 a simplified variant of the approach is applied to a problem of recovering a density in $\Omega \subset \mathbb{R}^n$. In particular, the case of inverse data given on part of a boundary is considered. Section 7.7 contains results of numerical testing of the algorithms based upon the BC method.

The paper is written so that the reader who prefers applications could ignore the material of theoretical character. To understand how the method recovers a density one can read the paper along the path:

section 1: 1.1; 1.2; 1.3; 1.4, (i)–(iii); 1.5, (i); 1.6
section 2: 2.1; 2.2; 2.4; 2.5
section 3: 3.1; 3.2; 3.3; 3.4
section 4: 4.1; 4.3
section 7: completely.

We use the abbreviations: IP, inverse problem; sgc, semigeodesical coordinates; DS, dynamical system; BCP, boundary control problem; AI, amplitude integral; AF, amplitude formula.

1. Geometry

The geometrical preliminaries are given. The basic object is the semigeodesical coordinates considered 'in the large' on a Riemannian manifold.

1.1. Eikonal and cut locus

Let (Ω, g) be a compact C^∞-smooth Riemannian manifold with a border Γ, $\dim \Omega = n \geqslant 2$ and g a metric tensor on Ω.

The function

$$\tau(x) := \mathrm{dist}(x, \Gamma) \qquad x \in \Omega$$

is called *an eikonal*. Its level sets

$$\Gamma^\xi := \{x \in \Omega | \tau(x) = \xi\} \qquad \xi \geqslant 0$$

are called *equidistant surfaces* of the border Γ; $\Gamma^0 = \Gamma$. A family of subdomain

$$\Omega^\xi := \{x \in \Omega | \tau(x) < \xi\} \qquad \xi > 0$$

extends with respect to ξ. A 'cross size' of a manifold is characterized by the number

$$T_* := \max_\Omega \tau(\cdot) = \inf\{\xi > 0 | \Omega^\xi = \Omega\}.$$

Let ℓ_γ be a geodesic starting from a point $\gamma \in \Gamma$ in the normal direction and $\ell_\gamma[0, s]$ its segment of length $s > 0$. The second end point of the segment is denoted by $x(\gamma, s) \in \ell_\gamma$; for $s = 0$ we set $x(\gamma, 0) = \gamma$. A *critical length* $s = s_*(\gamma)$ is defined by the conditions:

(i) $\tau(x(\gamma, s)) = s$ for $0 \leqslant s \leqslant s_*(\gamma)$;

(ii) $\tau(x(\gamma, s)) < s$ for $s > s_*(\gamma)$.

Thus, if $s \leqslant s_*(\gamma)$, segment $\ell_\gamma[0, s]$ is the shortest geodesic connecting $x(\gamma, s)$ with Γ, whereas for $s > s_*(\gamma)$ the segment does not minimize $\mathrm{dist}(x(\gamma, s), \Gamma)$. Function $s_*(\cdot)$ is continuous on Γ (Gromol *et al* 1968, Hartman 1964).

The point $x(\gamma, s_*(\gamma))$ is called *a separation point* on ℓ_γ. A set of separation points

$$\omega := \bigcup_{\gamma \in \Gamma} x(\gamma, s_*(\gamma))$$

is said to be *a separation set (cut locus)* of a manifold with respect to its border (Gromol *et al* 1968, Hartman 1964). The well known fact is that a cut locus is a closed set of zero volume,

$$\omega = \overline{\omega} \qquad \mathrm{vol}\,\omega = 0 \tag{1.1}$$

which is separated from the border:

$$T_\omega := \mathrm{dist}(\omega, \Gamma) = \min_\Gamma s_*(\cdot) > 0.$$

For $\xi < T_\omega$ the set $\Gamma^\xi \cap \omega$ is empty; if $\xi \geqslant T_\omega$, the part $\Gamma^\xi \setminus \omega$ of an equidistant surface is a smooth $(n-1)$-dimensional manifold (perhaps, unconnected). Thus, the regularity of Γ^ξ may be violated on a cut locus only.

1.2. Geodesic projection

Fix $x \in \Omega$ and define its *geodesic projection* on a border:

$$\mathrm{pr}\,x := \{\gamma \in \Gamma |\, \mathrm{dist}(\gamma, x) = \tau(x)\}.$$

Thus, $\mathrm{pr}\,x$ is a subset on Γ containing all the points being nearest to x.

Fix $\xi \in [0, T_*]$ and introduce the subsets of a border

$$\sigma_+^\xi := \mathrm{pr}(\Gamma^\xi \setminus \omega) \qquad \sigma_\omega^\xi := \mathrm{pr}(\Gamma^\xi \cap \omega) \qquad \sigma_-^\xi = \Gamma \setminus (\sigma_+^\xi \cup \sigma^\xi)$$

which form a partition

$$\Gamma = \sigma_+^\xi \cup \sigma_\omega^\xi \cup \sigma_-^\xi \tag{1.2}$$

and may be characterized in terms of the function $s_*(\cdot)$ as follows:

$$\sigma_+^\xi = \{\gamma \in \Gamma | s_*(\gamma) > \xi\} \qquad \sigma_\omega^\xi = \{\gamma \in \Gamma | s_*(\gamma) = \xi\} \qquad \sigma_-^\xi = \{\gamma \in \Gamma | s_*(\gamma) < \xi\}.$$

By virtue of the continuity of $s_*(\cdot)$, the sets $\sigma_\pm^\xi$ are open on Γ; set s_ω^ξ is closed. Set σ_+^ξ is decreasing, whereas set σ_-^ξ is increasing when ξ varies from 0 to T_*.

Denoting $\Omega_\perp^\xi := \Omega \setminus \Omega^\xi$, one has the relation

$$\mathrm{dist}(\sigma_-^\xi, \Omega_\perp^\xi) > \xi \qquad T_\omega < \xi < T_* \tag{1.3}$$

following easily from the definitions.

Let us remark in addition that the map $\mathrm{pr} : \Omega \to \Gamma$ turns out to be a diffeomorphism between $\Gamma^\xi \setminus \omega$ and σ_+^ξ. If $\xi < T_\omega$, one has $\mathrm{pr}\,\Gamma^\xi = \sigma_+^\xi = \Gamma$.

1.3. Semigeodesical coordinates

A simple fact is that every point outside the cut locus is connected with the border by a unique shortest (normal) geodesic. Therefore, a projection of $x \in \Omega \setminus \omega$ contains only one point $\gamma(x) := \mathrm{pr}\, x \in \Gamma$, whereas a pair $\gamma(x)$, $\tau(x)$ determines x uniquely and may be considered as its coordinates.

In more detail, let us fix point x_0 and its (small) vicinity $B : x_0 \in B \subset \Omega \setminus \omega$ such that $\mathrm{pr}\, B$ is covered by local coordinates $\gamma^1, \ldots, \gamma^{n-1}$ on Γ. System $\gamma^1 \circ \mathrm{pr}, \ldots, \gamma^{n-1} \circ \mathrm{pr}$, $\tau(\cdot)$ is said to be *the semigeodesical coordinates* (sgc) on B. Below we use the same term for the pair $\gamma(\cdot)$, $\tau(\cdot)$ if there is no need to put $\gamma^1, \ldots, \gamma^{n-1}$ in detail.

Property (1.1) gives a remarkable possibility to use sgc 'in the large', i.e. almost everywhere on Ω (Hartman 1964).

We denote by $\mathfrak{g}$ a metric tensor in sgc which has the well known form

$$\mathfrak{g} = \begin{pmatrix} & & & \vdots & 0 \\ & \mathfrak{g}_{\mu\nu} & & \vdots & \\ \ldots & \ldots & \ldots & & 0 \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} \qquad \mu, \nu = 1, \ldots, n-1.$$

The length and volume elements are

$$\mathrm{d}s^2 = \mathfrak{g}_{\mu\nu}\, \mathrm{d}\gamma^\mu\, \mathrm{d}\gamma^\nu + \mathrm{d}\tau^2 \qquad \mathrm{d}\Omega = (\det \mathfrak{g}(\gamma, \tau))^{1/2}\, \mathrm{d}\gamma^1 \ldots \mathrm{d}\gamma^{n-1}\, \mathrm{d}\tau = \beta(\gamma, \tau)\, \mathrm{d}\Gamma\, \mathrm{d}\tau \tag{1.4}$$

where

$$\beta(\gamma, \tau) := \left(\frac{\det \mathfrak{g}(\gamma, \tau)}{\det \mathfrak{g}(\gamma, 0)} \right)^{1/2} \qquad \text{and} \qquad \mathrm{d}\Gamma := (\det \mathfrak{g}(\gamma, 0))^{1/2}\, \mathrm{d}\gamma^1 \ldots \mathrm{d}\gamma^{n-1}$$

is a canonical measure on a border. Note that β and $\mathrm{d}\Gamma$ do not depend on the choice of local coordinates; function $\beta \in C^\infty(\Omega \setminus \omega)$ is positive everywhere.

Recall that in local coordinates $\eta^1, \ldots, \eta^n$ the Laplace operator is written as follows:

$$\Delta_g = (\det g)^{-1/2} \frac{\partial}{\partial \eta^k} (\det g)^{1/2} g^{kl} \frac{\partial}{\partial \eta^l} \tag{1.5}$$

where $g = \{g_{kl}(\eta^1, \ldots, \eta^n)\}_{k,l=1}^n$ is the metric tensor; $\{g^{kl}\} = \{g_{kl}\}^{-1}$. In sgc it takes the form

$$\Delta_{\mathfrak{g}} = (\det \mathfrak{g})^{-1/2} \frac{\partial}{\partial \gamma^\mu} (\det \mathfrak{g})^{1/2} \mathfrak{g}^{\mu\nu} \frac{\partial}{\partial \gamma^\nu} + (\det \mathfrak{g})^{-1/2} \frac{\partial}{\partial \tau} (\det \mathfrak{g})^{1/2} \frac{\partial}{\partial \tau} \tag{1.6}$$

with smooth $\mathfrak{g}^{\mu\nu}$.

1.4. Pattern

Here we introduce a geometrical object which plays the central role in the BC method.

Semigeodesical coordinates induce the map i from $\Omega \setminus \omega$ into the cylinder $\Gamma \times [0, T_*]$

$$i : x \to (\gamma(x), \tau(x)).$$

The image

$$\Theta := i(\Omega \setminus \omega) = \bigcup_{x \in \Omega \setminus \omega} (\gamma(x), \tau(x)) \subset \Gamma \times [0, T_*]$$

is said to be *a pattern* of Ω.

The following facts may be easily checked.

(i) The tensor $\mathfrak{g}$ determines a metric on Θ. The map i transforms $(\Omega\setminus\omega, g)$ onto $(\Theta, \mathfrak{g})$ isometrically; its inverse i^{-1} coincides with classical $\exp_\Gamma$.

(ii) For any $\xi \in [0, T_*)$ a smooth part $\Gamma^\xi \setminus \omega$ of equidistant surface is mapped onto the set $\sigma_+^\xi \times \{\tau = \xi\} \subset \Theta$. Correspondingly, the representation

$$\Omega \setminus \omega = \bigcup_{\xi\in[0,T_*)} \Gamma^\xi \setminus \omega$$

is transformed into

$$\Theta = \bigcup_{\xi\in[0,T_*)} \sigma_+^\xi \times \{\tau = \xi\} \tag{1.7}$$

which may be considered as a 'horizontal' bundle of a pattern.

(iii) An 'upper' border of a pattern

$$\theta := \bigcup_{\gamma\in\Gamma} (\gamma, s_*(\gamma))$$

is said to be *a coast*. The continuity of $s_*(\cdot)$ implies

$$\mathrm{mes}_{\mathrm{d}\Gamma\,\mathrm{d}\tau}\theta = 0 \tag{1.8}$$

on $\Gamma \times [0, T_*]$.

(iv) The inverse map $i^{-1} = \exp_\Gamma$ may be extended from a pattern onto a coast by continuity. Everywhere in the following we suppose the extension to be done, denoting it by the same symbol i^{-1}. An extended map transfers $\Theta \cup \theta$ onto Ω, and θ onto ω, but not injectively. If point $m \in \omega$ is connected with Γ by the shortest geodesics $\ell_{\gamma'}, \ell_{\gamma''}, \ldots$ (so that $s_*(\gamma') = s_*(\gamma'') = \cdots = \tau(m)$)), then one has

$$i^{-1}((\gamma', s_*(\gamma')) = i^{-1}((\gamma', s_*(\gamma'')) = \cdots = m$$

i.e. i^{-1} glues points of a coast.

The objects introduced above are shown in figure 1.

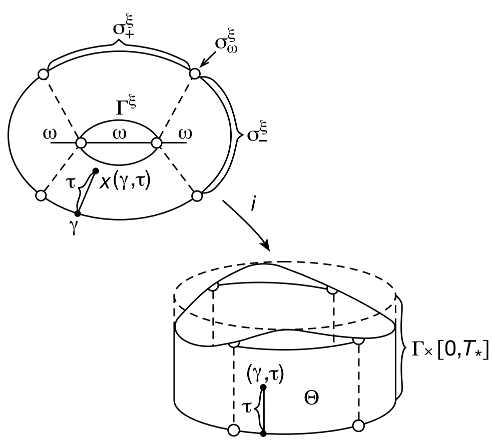

Figure 1. Semigeodesical coordinates and pattern.

In the following, dealing with dynamical problems we shall use reduced patterns. Fix positive $T \leqslant T_*$; the subdomain Ω^T (see 1.1) equipped with the tensor $g|_{\Omega^T}$ may be considered as a separate Riemannian manifold. As such, it has the cut locus

$$\omega^T := \omega \cap \Omega^T$$

and the pattern

$$\Theta^T := i(\Omega^T \setminus \omega^T) \subset \Gamma \times [0, T]$$

with the coast

$$\theta^T := \theta \cap [\Gamma \times (0, T)].$$

Varying T one has an increasing family of patterns Θ^T which exhausts pattern $\Theta = \Theta^{T_*}$.

(v) The following remark would be useful: the map i^{-1} transfers $(\Theta^T, \mathfrak{g})$ onto $(\Omega^T \setminus \omega^T, g)$ isometrically; it glues points of a coast: $i^{-1}(\theta^T) = \omega^T$.

A pattern was first introduced in Belishev (1990b).

1.5. Images

(i) Let us agree to consider β introduced in section 1.3 as a function on a pattern: $\beta \in C^\infty(\Theta)$, $\beta > 0$.

Fix $T \leqslant T_*$; for any function y given on Ω^T we define function $\tilde{y}$ on $\Sigma^T := \Gamma \times [0, T]$ as follows:

$$\tilde{y}(\gamma, \tau) := \begin{cases} \beta^{1/2}(\gamma, \tau) y(x(\gamma, \tau)) & (\gamma, \tau) \in \Theta^T \\ 0 & (\gamma, \tau) \in \Sigma^T \setminus \Theta^T. \end{cases}$$

Function $\tilde{y}$ is said to be *an image* of y, the corresponding map $I^T : y \to \tilde{y}$ being called *an image operator*.

(ii) Introduce the (real) Hilbert space $\mathcal{H}^T := L_2(\Omega^T)$,

$$(y, v)_{\mathcal{H}^T} = \int_{\Omega^T} \mathrm{d}\Omega\, y(x) v(x)$$

and the space $\mathcal{F}^T := L_2(\Sigma^T)$,

$$(f, h)_{\mathcal{F}^T} = \int_{\Sigma^T} \mathrm{d}\Gamma\, \mathrm{d}\tau\, f(\gamma, \tau) h(\gamma, \tau).$$

Let $\mathcal{F}^T_\Theta$ be the subspace of functions localized on a pattern:

$$\mathcal{F}^T_\Theta := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \Theta^T\}.$$

The (orthogonal) projector X^T_Θ in $\mathcal{F}^T$ onto $\mathcal{F}^T_\Theta$ cuts off functions on a pattern:

$$(X^T_\Theta f)(\gamma, \tau) = \begin{cases} f(\gamma, \tau) & (\gamma, \tau) \in \Theta^T \\ 0 & (\gamma, \tau) \in \Sigma^T \setminus \Theta^T. \end{cases}$$

Lemma 1.1. *The image operator acts isometrically from $\mathcal{H}^T$ into $\mathcal{F}^T$, the relations*

$$(I^T y, I^T v)_{\mathcal{F}^T} = (y, v)_{\mathcal{H}^T} \qquad \operatorname{Ran} I^T = \mathcal{F}^T_\Theta \qquad (I^T)^* I^T = \mathbb{1}_{\mathcal{H}^T} \qquad I^T (I^T)^* = X^T_\Theta. \tag{1.9}$$

being valid ($\mathbb{1}$ are identical operators).

Proof. Operator I^T is correctly defined on $\mathcal{H}^T$ by virtue of (1.1). For arbitrary y, $v \in C_0^\infty(\Omega^T \setminus \omega^T)$ one has the equalities

$$(y, v)_{\mathcal{H}^T} = \int_{\Omega^T} \mathrm{d}\Omega\, y(x)v(x) = \text{(see (1.1))} = \int_{\Omega^T \setminus \omega^T} = \text{(see (1.4))}$$

$$= \int_{\Theta^T} \mathrm{d}\Gamma\, \mathrm{d}\tau\, \beta(\gamma, \tau) y(x(\gamma, \tau)) v(x(\gamma, \tau)) = (\tilde{y}, \tilde{v})_{\mathcal{F}^T} = (I^T y, I^T v)_{\mathcal{F}^T}.$$

Thus, I^T is an isometry.

Operator I^T transfers $C_0^\infty(\Omega^T \setminus \omega^T)$ *onto* $C_0^\infty(\Theta^T)$. Indeed, the inclusion $I^T C_0^\infty(\Omega^T \setminus \omega^T) \subset C_0^\infty(\Theta^T)$ is obvious; on the other hand, for any $f \in C_0^\infty(\Theta^T)$ one can find the preimage $(I^T)^{-1} f \in C_0^\infty(\Omega^T \setminus \omega^T)$ as follows:

$$((I^T)^{-1} f)(x) = \beta^{-1/2}(\gamma(x), \tau(x)) f(\gamma(x), \tau(x)) \qquad x \in \Omega^T \setminus \omega^T.$$

A density of sets $C_0^\infty(\Omega^T \setminus \omega^T)$ and $C_0^\infty(\Theta^T)$ in $\mathcal{H}^T$ and $\mathcal{F}_\Theta^T$ implies $\operatorname{Ran} I^T = \mathcal{F}_\Theta^T$. Thus, two relations in (1.9) are established; the rest of (1.9) is just a corollary of the first. The lemma is proved. □

Corollary. *The operator* $(I^T)^* : \mathcal{F}^T \to \mathcal{H}^T$ *acts by the rule*

$$((I^T)^* f)(x) = (\beta^{-1/2} X_\Theta^T f)(\gamma(x), \tau(x)) \qquad x \in \Omega^T \setminus \omega^T.$$

Suppose functions $y \in C^2(\Omega^T)$ and $w \in C(\Omega^T)$ to be connected through the Laplacian,

$$\Delta_g y = w \qquad \text{in } \Omega^T.$$

Let $\tilde{y}$, $\tilde{w}$ be their images on Θ^T. An image operator induces the corresponding relation on a pattern:

$$\tilde{\Delta} \tilde{y} = \tilde{w} \qquad \text{in } \Theta^T \tag{1.10}$$

with operator $\tilde{\Delta} := I^T \Delta (I^T)^{-1}$, which may be represented in the form

$$\tilde{\Delta} = \mathfrak{g}^{\mu\nu} \frac{\partial^2}{\partial \gamma^\mu \partial \gamma^\nu} + \mathfrak{g}^\nu \frac{\partial}{\partial \gamma^\nu} + \mathfrak{g}^0 + \frac{\partial^2}{\partial \tau^2}$$

with smooth coefficients. The representation (1.10) may be checked by a simple calculation.

1.6. Domains of influence

In conclusion of the geometrical preliminaries we introduce one class of the sets used below in dynamical problems.

Denote $Q^T := \Omega \times [0, T]$ $(0 < T \leqslant T_*)$; for any point $(x_0, \tau_0) \in Q^T$ define *future and past cones*

$$K_\pm^T[(x_0, \tau_0)] := \{(x, \tau) \in Q^T \,|\, \operatorname{dist}(x, x_0) \leqslant \pm(\tau - \tau_0)\}.$$

For any $D \subset Q^T$ the sets

$$K_\pm^T[D] := \bigcup_{(x_0, \tau_0) \in D} K_\pm^T[(x_0, \tau_0)]$$

are called *future* and *past domains of influence* of subset D. The following facts may be simply derived from the definitions given above and in section 1.2.

(i) Let

$$\Sigma^{T,\xi} := \Gamma \times [T - \xi, T] \qquad 0 < \xi \leqslant T$$

be part of the lateral surface Σ^T of cylinder Q^T; the representation

$$K_+^T[\Sigma^{T,\xi}] = \{(x,\tau) \in Q^T | \tau \geqslant \tau(x) + (T-\xi)\}$$

is valid. Thus, this domain lies above the characteristic surface

$$\mathcal{X}^{T,\xi} := \{(x,\tau) \in Q^T | \tau = \tau(x) + (T-\xi)\}.$$

(ii) Consider the set $\Omega_\perp^\xi \times \{\tau = T\}$ lying on an upper base of Q^T (recall that $\Omega_\perp^\xi := \Omega \setminus \Omega^\xi$), and the sets $\sigma_\pm^\xi \times \{\tau = T-\xi\}$, $\sigma_\omega^\xi \times \{\tau = T-\xi\}$ which form a partition of the cross section $\Gamma \times \{\tau = T-\xi\}$ of Σ^T. The following relations hold:

$$(\sigma_+^\xi \cup \sigma_\omega^\xi) \times \{\tau = T-\xi\} \subset K_-^T[\Omega_\perp^\xi \times \{\tau = T\}]$$
$$\sigma_-^\xi \times \{\tau = T-\xi\} \not\subset K_-^T[\Omega_\perp^\xi \times \{\tau = T\}] \qquad (1.11)$$

the latter being meaningful only if $T_\omega < \xi < T_*$ (see (1.3)).

Properties (i) and (ii) are illustrated on figure 2 (the part of Σ^T belonging to $K_-^T[\Omega_\perp^\xi \times \{\tau = T\}]$ is shaded on figure 2(b)).

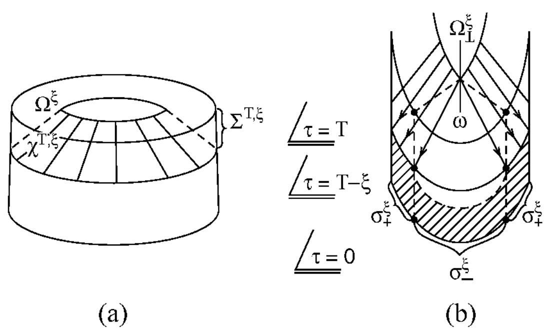

Figure 2. Domains of influence.

2. Waves

Properties of waves initiated into a manifold by boundary sources (controls) are presented. The waves play the role of the main tool used by the external observer who investigates a manifold from its border.

2.1. Boundary-value initial problem

Consider the problem

$$u_{tt} - \Delta_g u = 0 \qquad \text{in Int } Q^T \qquad (2.1)$$
$$u|_{t=0} = u_t|_{t=0} = 0 \qquad (2.2)$$
$$u|_{\Sigma^T} = f \qquad (2.3)$$

with a final moment $t = T > 0$ and function $f = f(\gamma, t)$ which is said to be a *(Dirichlet) boundary control*; let $u = u^f(x,t)$ be its solution (*wave*).

Let us list briefly some known facts concerning waves.

(i) Introduce a set of smooth controls $\mathcal{M}^T := \{f \in C^\infty(\Sigma^T) | (\partial/\partial t)^k f|_{t=0} = 0, k = 0, 1, \ldots\}$; for any $f \in \mathcal{M}^T$ the problem (2.1)–(2.3) has a unique classical solution $u^f \in C^\infty(Q^T)$.

(ii) The map $f \to u^f$ acts continuously from $L_2(\Sigma^T)$ into $C([0,T]; L_2(\Omega))$ (see Lasiecka *et al* 1986) its extension by continuity determines a generalized solution u^f for $f \in L_2(\Sigma^T)$ satisfying (2.1) as a distribution.

(iii) The hyperbolicity of problem (2.1)–(2.3) leads to a property which is interpreted as a finiteness of the speed of wave propagation: for any $f \in L_2(\Sigma^T)$ one has the inclusion

$$\operatorname{supp} u^f \subset K_+^T[\operatorname{supp} f]$$

in Q^T. Since $\operatorname{supp} f \subset \Sigma^T$ and $K_+^T(\Sigma^T) = \{(x,t) \in Q^T | t \geqslant \tau(x)\}$ (see section 1.6) this implies the inclusion

$$\operatorname{supp} u^f(\cdot, \xi) \subset \overline{\Omega}^{\xi} \qquad 0 < \xi \leqslant T \tag{2.4}$$

in Ω. Thus, Ω^ξ may be interpreted as part of a manifold filled by waves up to the moment $t = \xi$, that selects the value $t = T_*$ as a time needed for waves to fill the whole of the manifold.

(iv) An independence of the metric tensor g on time leads to the well known stationary state property. Let $f \in \mathcal{F}^T$ and $f(\,;\,\xi)$ be a *delayed* control,

$$f(\gamma, t; \xi) := \begin{cases} 0 & 0 \leqslant t < T - \xi \\ f(\gamma, t - (T - \xi)) & T - \xi \leqslant t \leqslant T. \end{cases}$$

The relation

$$u^{f(\,;\,\xi)}(\,\cdot\,, T) = u^f(\cdot, \xi) \qquad \text{in } \Omega \tag{2.5}$$

just means that a delay of a control implies the same delay of a wave. As a corollary one can obtain

$$u^{(\partial/\partial t)^k f} = \left(\frac{\partial}{\partial t}\right)^k u^f \qquad \text{in } Q^T \tag{2.6}$$

for $f \in \mathcal{M}^T$, $k = 1, 2, \ldots$.

(v) Let $\nu = \nu(\gamma)$ be an outward normal at point $\gamma \in \Gamma$, $\partial_\nu := \partial/\partial\nu$; the map $f \to \partial_\nu u^f|_{\Sigma^T}$ defined on $\mathcal{M}^T$ acts continuously from the subspace $\{f \in H^1(\Sigma^T) | f|_{t=0} = 0\} \subset H^1(\Sigma^T)$ into $L_2(\Sigma^T)$ ($H^\alpha(\ldots)$ be the Sobolev classes) (Lasiecka *et al* 1986, Lions 1968).

2.2. Dual problem

The boundary-value initial problem

$$v_{tt} - \Delta_g v = 0 \qquad \text{in Int } Q^T \tag{2.7}$$

$$v|_{t=T} = 0 \qquad v_t|_{t=T} = y \tag{2.8}$$

$$v|_{\Sigma^T} = 0 \tag{2.9}$$

is said to be *dual* to problem (2.1)–(2.3). Let $v = v^y(x,t)$ be a solution; the following is a list of its properties.

(i) For any $y \in C_0^\infty(\Omega)$ the problem has a unique classical solution $v^y \in C^\infty(Q^T)$.

(ii) The map $y \to v^y$ is continuous from $L_2(\Omega)$ into $H^1(Q^T) \cap C([0,T]; H_0^1(\Omega))$; this fact permits the definition of a generalized solution for $y \in L_2(\Omega)$ extending the map by a continuity (see Lasiecka *et al* 1986).

(iii) Defined on $C_0^\infty(\Omega)$, the map $y \to \partial_\nu\, v^y|_{\Sigma^T}$ acts continuously from $L_2(\Omega)$ into $L_2(\Sigma^T)$ (see Lasiecka *et al* 1986). Therefore, a trace $\partial_\nu\, v^y|_{\Sigma^T}$ is correctly defined for v^y, $y \in L_2(\Omega)$.

(iv) A hyperbolicity of a dual problem leads to the inclusion

$$\operatorname{supp} v^y \subset K_-^T[\operatorname{supp} y \times \{t = T\}] \tag{2.10}$$

in Q^T.

One reason to call problem (2.7)–(2.9) dual to problem (2.1)–(2.3) is the following relation between their solutions.

Lemma 2.1. *For any $f \in L_2(\Sigma^T)$, $y \in L_2(\Omega)$ the equality*

$$\int_\Omega \mathrm{d}\Omega\, u^f(x,T)y(x) = \int_{\Sigma^T} \mathrm{d}\Gamma\,\mathrm{d}t\, f(\gamma,t)\partial_\nu v^y(\gamma,t) \tag{2.11}$$

is valid.

Proof. For $f \in \mathcal{M}^T$, $y \in C_0^\infty$ and the corresponding classical solutions one has the equalities

$$\begin{aligned}
0 = \int_{Q^T} &\mathrm{d}\Omega\,\mathrm{d}t[u_{tt}^f(x,t) - \Delta_g u^f(x,t)]v^y(x,t)\\
&= \int_\Omega \mathrm{d}\Omega\,[u_t^f(x,t)v^y(x,t) - u^f(x,t)v_t^y(x,t)]|_{t=0}^{t=T}\\
&\quad - \int_0^T \mathrm{d}t \int_\Gamma \mathrm{d}\Gamma\,[\partial_\nu u^f(\gamma,t)v^y(\gamma,t) - u^f(\gamma,t)\partial_\nu v^y(\gamma,t)]\\
&\quad + \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}\Omega\, u^f(x,t)[v_{tt}^y(x,t) - \Delta_g v^y(x,t)]\\
&= \text{(see (2.2), (2.3), (2.7)–(2.9))}\\
&= -\int_\Omega \mathrm{d}\Omega\, u^f(x,T)y(x) + \int_0^T \mathrm{d}t \int_\Gamma \mathrm{d}\Gamma\, f(\gamma,t)\partial_\nu v^y(\gamma,t)
\end{aligned}$$

which implies (2.11) for smooth f, y. Extending the established equality on $f \in L_2(\Sigma^T)$, $y \in L_2(\Omega)$ by continuity, one can obtain the necessary result. The lemma is proved. □

Solution v^y describes a wave produced by the perturbation of the velocity. Such a wave propagates (in inverted time!) into a manifold whose border is rigidly fixed.

2.3. Propagation of wave discontinuities

The well known fact is that discontinuous controls generate discontinuous waves. The discontinuities of waves propagate along bicharacteristic (rays), their amplitudes being calculated by means of geometrical optics.

Choose a smooth control $f \in \mathcal{M}^T$ and fix parameter $\xi : 0 < \xi < T \leqslant T_*$; let

$$f_\xi(\gamma,t) := \theta(t-(T-\xi))f(\gamma,\xi) = \begin{cases} 0 & 0 \leqslant t < T-\xi \\ f(\gamma,t) & T-\xi \leqslant t < T \end{cases}$$

be its cutting-off function ($\theta(\ldots)$ is the Heavyside function), $\operatorname{supp} f_\xi \subset \Sigma^{T,\xi}$. In general, f_ξ is a discontinuous control having a discontinuity at the moment $t = T - \xi$:

$$f_\xi(\gamma,t)|_{t=T-\xi-0}^{t=T-\xi+0} = f(\gamma, T-\xi). \tag{2.12}$$

The corresponding wave u^{f_ξ} is localized in domain $K_+^T[\Sigma^{T,\xi}]$, and our goal is to describe its behaviour near the characteristic surface $\mathcal{X}^{T,\xi}$ which bounds $\operatorname{supp} u^{f_\xi}$ from below (see figure 2(a)).

Let $\omega_\varepsilon := \{x \in \Omega|\,\mathrm{dist}(x,\omega) < \varepsilon\}$ be a vicinity of the cut locus; denote $Q_\varepsilon^T := (\Omega \setminus \omega_\varepsilon) \times [0, T]$; $s_+ := s\theta(s)$.

Lemma 2.2. *For arbitrary (small)* $\varepsilon > 0$ *one can find* $\delta = \delta(\varepsilon) > 0$ *such that the representation*

$$u^{f_\xi}(x,t) = \beta^{-1/2}(\gamma(x),\tau(x))f(\gamma(x),t-\tau(x))\theta(t-\tau(x)-(T-\xi)) \\ +w_\varepsilon^f(x,t;\xi)(t-\tau(x)-(T-\xi))_+ \tag{2.13}$$

is valid for any $(x,t) \in Q_\varepsilon^T$ *satisfying* $t < \tau(x) + (T-\xi) + \delta$ *(i.e. lying under* $\mathcal{X}^{T,\xi-\delta}$*). The function* w_ε^f *is bounded:*

$$|w_\varepsilon^f| \leqslant c(T, f, \varepsilon)$$

uniformly on $\xi \in (0, T)$.

This result is known (see, e.g., Babich and Buldyrev 1991, Wainberg 1982); the reader could find a variant of the proof belonging to Kachalov in Belishev and Kachalov (1994).

Representation (2.13) shows that a discontinuity of the wave moves into Ω from the border Γ with a unit velocity. At the final moment $t = T$ it is localized on the surface Γ^ξ playing the role of a forward front of wave; an amplitude of the discontinuity being calculated as follows:

$$\lim_{\tau\to\xi-0} u^{f_\xi}(x(\gamma,\tau),T) = \beta^{-1/2}(\gamma,\xi)f(\gamma,T-\xi). \tag{2.14}$$

These kind of relations are known as the geometrical optics formulae (see Babich and Bulydrev 1991, Wainberg 1982). Comparing (2.14) with (2.12) one can say that up to the factor $\beta^{-1/2}$ of a geometrical nature the shape of the wave discontinuity repeats the shape of the discontinuity of control.

Let us remark that (2.13) describes the behaviour of a wave only *near* the characteristics $\mathcal{X}^{T,\xi}$ carrying a discontinuity, and *out of a cut locus*. If $T > T_\omega$, far from this area a wave can possess singularities of more complicated structure.

2.4. Discontinuities in the dual problem

As in system (2.1)–(2.3), the same effect is present in the dual one: discontinuous data produce discontinuous waves.

Choose $y \in C^\infty(\Omega)$; let

$$y_\xi(x) := \theta(\tau(x)-\xi)\,y(x) \qquad x \in \Omega$$

be its cutting-off function on the subdomain $\Omega_\perp^\xi = \Omega \setminus \Omega^\xi = \{x \in \Omega|\tau(x) \geqslant \xi\}$. Note that, in general, y_ξ has a discontinuity at surface Γ_ξ. Consider problem (2.7)–(2.9) with data $v|_{t=T} = 0$, $v_t|_{t=T} = y_\xi$; let v^{y_ξ} be the corresponding solution.

Lemma 2.3. *In the case of* $0 < \xi < T \leqslant T_*$*, the relation*

$$\lim_{t\to T-\xi-0} \partial_\nu v^{y_\xi}(\gamma,t) = \begin{cases} \beta^{1/2}(\gamma,\xi)\,y(x(\gamma,\xi)) & (\gamma,\xi) \in \sigma_+^\xi \times \{t = T-\xi\} \\ 0 & (\gamma,\xi) \in \sigma_-^\xi \times \{t = T-\xi\} \end{cases} \tag{2.15}$$

is valid.

Equality (2.15) is dual to (2.14), the duality being known as a reciprocity law. Omitting its proof (see Wainberg 1982), we only give the following 'physical' explanation.

The discontinuous perturbation of velocity y_ξ, $\operatorname{supp} y_\xi \subset \Omega_\perp^\xi$, generates a discontinuity of the wave v^{y_ξ} which propagates (in inverted time) along the rays (see the arrows on figure 2(b)) towards Γ. Reaching a border at the moment $t = T - \xi$, the discontinuity interacts not with the whole of Γ but with the 'illuminated part' $\sigma_+^\xi \cup \sigma_\omega^\xi$ only (see (1.11)). In points of σ_+^ξ an amplitude of interaction may be calculated by means of geometrical optics giving the first line in (2.15).

The part $\sigma_-^\xi \subset \Gamma$ is not covered by the wave at $t = T - \xi$ since σ_-^ξ is placed far from $\Omega_\perp^\xi$ (see (1.11), (2.10)) which explains the second line of (2.15).

Let us remark that geometrical optics is not applicable at points of $\sigma_\omega^\xi \times \{t = T - \xi\}$ lying on a coast of a pattern. Fortunately, in view of (1.8), this will not create problems later.

Considering the right-hand side of (2.15) as a function of (γ, ξ), and comparing it with the definition of images (section 1.5) one can rewrite the relation as follows:

$$\lim_{t \to T-\xi-0} \partial_\nu v^{y_\xi}(\gamma, t) = \tilde{y}(\gamma, \xi) \qquad \text{a.e. on } \Sigma^T. \tag{2.16}$$

This is the formula which motivates us to introduce images. It represents an image of y as a collection of wave discontinuities propagating in a dual system and being detected on a border.

2.5. *Spectral representation*

Here we describe briefly the Fourier method for problems (2.1)–(2.3) and (2.7)–(2.9).

Operator $L : L_2(\Omega) \to L_2(\Omega)$, $\operatorname{Dom} L = H^2(\Omega) \cap H_0^1(\Omega)$,

$$Lu := -\Delta_g u$$

is self-adjoint and positively defined. Let $\{\lambda_k\}_{k=1}^\infty$, $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ be its spectrum and φ_k be the corresponding eigenfunctions ($L\varphi_k = \lambda_k \varphi_k$) normalized by the condition

$$\int_\Omega \mathrm{d}\Omega \, \varphi_k(x)\varphi_l(x) = \delta_{kl}.$$

System $\{\varphi_k\}_{k=1}^\infty$ forms a basis in $L_2(\Omega)$ that gives the possibility of representing waves by a Fourier series. Denote

$$\psi_k := \partial_\nu \varphi_k|_\Gamma \qquad s_k^T = s_k^T(\gamma, t) := -\lambda_k^{-1/2} \sin[\,\lambda_k^{1/2}(T - t)\,]\psi_k(\gamma).$$

Lemma 2.4. *(i) For any $f \in L_2(\Sigma^T)$ the representation*

$$u^f(\cdot, T) = \sum_{k=1}^\infty c_k^f(T)\,\varphi_k \qquad c_k^f(T) = \int_{\Sigma^T} \mathrm{d}\Gamma \, \mathrm{d}t \, s_k^T(\gamma, t) f(\gamma, t) \tag{2.17}$$

is valid, the series converging in $L_2(\Omega)$.

(ii) For any $y \in L_2(\Omega)$ the representations

$$v^y(\cdot, t) = -\sum_{k=1}^\infty y_k \, \lambda_k^{-1/2} \sin[\lambda_k^{1/2}(T - t)]\varphi_k \qquad y_k = \int_\Omega \mathrm{d}\Omega \, y(x)\,\varphi_k(x) \tag{2.18}$$

$$\partial_\nu v^y|_{\Sigma^T} = \sum_{k=1}^\infty y_k \, s_k^T \tag{2.19}$$

are valid, the series converging in $L_2(\Omega)$ and $L_2(\Sigma^T)$ correspondingly.

One can obtain the proof just by integrating by parts (see, e.g., Lions 1968).

The set of pairs $\{\lambda_k; \psi_k(\cdot)\}_{k=1}^{\infty}$ is said to be *(Dirichlet) spectral data* of manifold (Ω, g). In what follows it plays the role of data of the spectral IP. In this connection it would be important to note in advance that the Fourier coefficients in (2.17) are determined by $\{\lambda_k; \psi_k\}$.

3. Dynamics

The boundary-value initial problems introduced previously are equipped with the attributes of dynamical systems (spaces and operators) as is customary in control theory.

3.1. Control operator

We begin to consider problem (2.1)–(2.3) as a dynamical system (DS). As such, it is denoted by α^T throughout what follows.

The Hilbert space of controls (inputs) $\mathcal{F}^T = L_2(\Sigma^T)$ is called *an outer space* of the DS α^T.

By virtue of (ii), section 2.1 waves (states) $u^f(\cdot, t)$ belong to the space $\mathcal{H} = L_2(\Omega)$ which is said to be *an inner space* of the DS α^T.

An 'input $\rightarrow$ state' correspondence in α^T is realized by *the control operator* $W^T : \mathcal{F}^T \rightarrow \mathcal{H}$,

$$W^T f := u^f(\cdot, T)$$

acting continuously from an outer space into an inner space. Let us discuss some of its properties.

Introduce *the delay operator* $\mathcal{T}^{T,\xi} : \mathcal{F}^T \rightarrow \mathcal{F}^T$,

$$\mathcal{T}^{T,\xi} f := f(\,\cdot\,; \xi)$$

(see (iv), section 2.1) where ξ is a parameter, $0 \leqslant \xi \leqslant T$; $\mathcal{T}^{T,0} = \mathbb{O}_{\mathcal{F}^T}$, $\mathcal{T}^{T,T} = \mathbb{1}_{\mathcal{F}^T}$. Note that ξ is an action time of the delayed control $\mathcal{T}^{T,\xi} f$. A stationary state property (see (2.5)) of the DS α^T may be rewritten as follows:

$$W^T \mathcal{T}^{T,\xi} f = u^f(\cdot, \xi) \qquad 0 \leqslant \xi \leqslant T. \tag{3.1}$$

Equality (2.6) for $k = 2$, $t = T$ takes the form

$$u^{(\partial/\partial t)^2 f}(\cdot, T) = u^f_{tt}(\cdot, T)$$

that implies

$$W^T \frac{\partial^2}{\partial t^2} = \Delta_g W^T \qquad \text{on } \mathcal{M}^T. \tag{3.2}$$

The outer space $\mathcal{F}^T$ contains an increasing family of subspaces

$$\mathcal{F}^{T,\xi} := \mathcal{T}^{T,\xi} \mathcal{F}^T = \{f \in \mathcal{F}^T \,|\, \operatorname{supp} f \subset \Sigma^{T,\xi}\} \qquad 0 \leqslant \xi \leqslant T \tag{3.3}$$

$(\Sigma^{T,\xi} := \Gamma \times [T - \xi, T])$ formed by delayed controls. In accordance with (2.4) and (3.1) a control operator maps this family into one of subspaces of the inner space $\mathcal{H}$:

$$\mathcal{H}^\xi := \{u \in \mathcal{H} \,|\, \operatorname{supp} u \subset \overline{\Omega}^\xi\} \qquad 0 \leqslant \xi \leqslant T \tag{3.4}$$

so that the relation

$$W^T \mathcal{F}^{T,\xi} \subset \mathcal{H}^\xi \qquad 0 \leqslant \xi \leqslant T \tag{3.5}$$

holds.

As it was noted in (iii), section 2.1, the value T_* coincides with the time needed for waves moving from a border to fill a manifold. This T_* enters the following important result.

Lemma 3.1. *For $T < T_*$ a control operator is injective:*

$$\mathrm{Ker}\, W^T = \{0\}. \tag{3.6}$$

The proof can be found in Avdonin *et al* (1994) and Belishev (1990a).

3.2. Operator of observation

The problem (2.7)–(2.9) determines a dynamical system which is said to be *dual* to α^T and is denoted by α_*^T.

The operator $O^T : \mathcal{H} \to \mathcal{F}^T$,

$$O^T y := \partial_\nu v^y|_{\Sigma^T}$$

is called *an operator of observation*. It realizes a 'state $\to$ output' correspondence in the DS α_*^T. Due to property (iii), section 2.2, operator O^T is correctly defined.

The following result clarifies finally the meaning of the term 'duality'.

Proposition 3.1. *For any $T > 0$ the equality*

$$O^T = (W^T)^* \tag{3.7}$$

holds.

Indeed, the relation

$$(W^T f, y)_{\mathcal{H}} = (f, O^T y)_{\mathcal{F}^T}$$

is no more than a way to write (2.11).

3.3. Response operator

An 'input $\to$ output' map in the DS α^T is determined by *the response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\mathrm{Dom}\, R^T = \{f \in H^1(\Sigma^T) | f|_{t=0} = 0\}$,

$$R^T f := \partial_\nu u^f|_{\Sigma^T}$$

which is correctly defined by virtue of (v), section 2.1. In contrast to operators of control and observation it is not continuous.

A response operator describes the reply of a dynamical system to an action of a control. It may be identified with information being obtained by an outer observer from dynamical boundary measurements.

The hyperbolicity of system (2.1)–(2.3) implies the following well known fact. Corresponding to double time, the operator R^{2T} is determined by the submanifold (Ω^T, g) being independent on $(\Omega_\perp^T, g)$. In the following the operator R^{2T} will play the role of inverse data, and, as such, it contains information on Ω^T *only*.

3.4. Connecting operator

The operator introduced here is one of the main objects of the BC method.

Let us define the map $C^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$C^T := (W^T)^* W^T = O^T W^T \tag{3.8}$$

which is said to be *the connecting operator* of the DS α^T. This term is explained by the relation

$$(C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}} = (u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}} \tag{3.9}$$

i.e. operator C^T connects metrics of outer and inner spaces.

By its definition, C^T is a continuous non-negative operator in $\mathcal{F}^T$. In view of $\operatorname{Ker} C^T = \operatorname{Ker} W^T$, one has

$$\operatorname{Ker} C^T = \{0\} \qquad T < T_* \tag{3.10}$$

(see (3.6)).

The role of the connecting operator in our approach stands out due to the following remarkable fact: C^T may be expressed in explicit form through the inverse data (in particular, through a response operator). To formulate the result we need some auxiliary operators:

the operator of an odd continuation $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$,

$$(S^T f)(\cdot, t) = \begin{cases} f(\cdot, t) & 0 \leqslant t < T \\ -f(\cdot, 2T - t) & T \leqslant t \leqslant 2T \end{cases}$$

the reducing operator $N^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^T$,

$$N^{2T} g := g|_{\Sigma^T}$$

the operator selecting an odd part of controls $P_-^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$

$$(P_-^{2T} g)(\cdot, t) = \tfrac{1}{2}[g(\cdot, t) - g(\cdot, 2T - t)]$$

the operator of integration $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$,

$$(J^{2T} g)(\cdot, t) = \int_0^t \mathrm{d}s\, g(\cdot, s).$$

One can easily check the relation

$$(S^T)^* = 2N^{2T} P_-^{2T}. \tag{3.11}$$

Let R^{2T} be a response operator of the DS α^{2T} (problem (2.1)–(2.3) with a final moment $t = 2T$).

Theorem 3.1. *For any $T > 0$ and $f \in H^1(\Sigma^T)$ the representation*

$$C^T f = \tfrac{1}{2}(S^T)^* R^{2T} J^{2T} S^T f \tag{3.12}$$

holds.

Proof. Fix $f \in H^1(\Sigma^T)$ and note that control $h := J^{2T} S^T f$ belongs to Dom R^{2T}, being an even function, $h(\cdot, t) = h(\cdot, 2T - t)$. Let u^h be a solution of the problem

$$u_{tt} - \Delta_g u = 0 \qquad \text{in Int } Q^{2T}$$
$$u|_{t=0} = u_t|_{t=0} = 0$$
$$u|_{\Sigma^{2T}} = h.$$

Denoting $w(x, t) := u^h(x, t) - u^h(x, 2T - t)$ one can check the relations

$$w_{tt} - \Delta_g w = 0 \qquad \text{in Int } Q^T \tag{3.13}$$
$$w|_{t=T} = 0 \qquad w_t|_{t=T} = 2u_t^h(\cdot, T) \tag{3.14}$$
$$w|_{\Sigma^T} = 0. \tag{3.15}$$

The derivative in (3.14) may be calculated as follows:

$$w_t|_{t=T} = 2u_t^h(\cdot, T) = (\text{see (2.6)}) = 2u^{h_t}(\cdot, T) = 2u^{S^T f}(\cdot, T) = 2u^f(\cdot, T) = 2W^T f. \tag{3.16}$$

Comparing (3.13)–(3.16) with the dual problem (2.7)–(2.9) we conclude that $w = v^y$ with $y = 2W^T f$. By definition of the operator of observation, this implies

$$\partial_\nu w|_{\Sigma^T} = \partial_\nu v^y|_{\Sigma^T} = O^T y = 2O^T W^T f = 2C^T f. \tag{3.17}$$

On the other hand, calculating the same derivative directly one obtains

$$\partial_\nu w(\cdot, t) = \partial_\nu u^h(\cdot, t) - \partial_\nu u^h(\cdot, 2T - t) = (R^{2T} h)(\cdot, t) - (R^{2T} h)(\cdot, 2T - t)$$

which may be rewritten in the form

$$\partial_\nu w|_{\Sigma^T} = N^{2T} 2P_-^{2T} R^{2T} h = 2N^{2T} P_-^{2T} R^{2T} J^{2T} S^{2T} f = (\text{see (3.11)}) = (S^T)^* R^{2T} J^{2T} S^T f. \tag{3.18}$$

Comparing (3.17) with (3.18) we obtain (3.12). The theorem is proved. □

Unfortunately, representation (3.12) does not hold for arbitrary $f \in \mathcal{F}^T$. The reason is that, in contrast to the one-dimensional case, the multidimensional operator $R^{2T} J^{2T}$ is unbounded (Bardos and Lebeau, private communication). Therefore, to find $C^T f$ in the general case one must invoke a passage to the limit.

The set of pairs $\{\lambda_k; \psi_k(\cdot)\}_{k=1}^\infty$ (see section 2.5) will play the role of data in the spectral IP. The following result shows that the connecting operator is determined by spectral data.

Theorem 3.3. *For any $T > 0$ the representation*

$$C^T = \sum_{k=1}^\infty (\cdot, s_k^T)_{\mathcal{F}^T}\, s_k^T \tag{3.19}$$

is valid, the series converging in a strong operator topology.

Proof. Fix $f \in \mathcal{F}^T$; in accordance with (2.17) one has

$$u^f(\cdot, T) = W^T f = \sum_{k=1}^\infty (f, s_k^T)_{\mathcal{F}^T} \varphi_k.$$

Applying the (continuous) operator O^T, one obtains the representation

$$C^T f = O^T W^T f = \sum_{k=1}^\infty (f, s_k^T) O^T \varphi_k = (\text{see (2.19)}) = \sum_{k=1}^\infty (f, s_k^T) s_k^T$$

in the form of a series converging in $\mathcal{F}^T$. The theorem is proved. □

Operator C^T was first used in dynamical IP in Belishev (1987b). It would be interesting to note that the dual operator $\Lambda = W^T(W^T)^*$ plays a basic role in the Hilbert uniqueness method (Lions 1988).

3.5. Visualizing operator

Completing the list of operators associated with systems α^T and α_*^T, we introduce one more map which connects dynamics with geometry. Recall that the image operator $I^T : y \to \tilde{y}$ was defined in section 1.5.

The map $V^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$V^T := I^T W^T$$

is said to be *the visualizing operator*. Acting by the rule $f \to \tilde{u}^f(\cdot, T)$, it makes the wave images be objects of an outer space.

The meaning and future role of the visualizing operator may be announced as follows. An external observer operating on a border (in $\mathcal{F}^T$) cannot see the waves *into* a manifold. Suppose, that the observer is able to determine V^T from boundary measurements (inverse data). If so, the remarkable possibility of making wave pictures (images) visible on a pattern is obtained. Moreover, applying an image operator to (3.1) one obtains the relation

$$V^T \mathcal{T}^{T,\xi} f = \tilde{u}^f(\cdot, \xi) \qquad 0 \leqslant \xi \leqslant T \tag{3.20}$$

so that an observer could visualize on a pattern the whole of a wave process. Then the observer could extract from the pictures information concerning a manifold. It is a program which will be realized below, when we shall solve the IPs.

Relations (1.10) and (3.2) lead to the equality

$$V^T \frac{\partial^2}{\partial t^2} = \tilde{\Delta} V^T \qquad \text{on } \mathcal{M}^T \tag{3.21}$$

which is required later.

4. Wave shaping

Can one shape a wave by means of a boundary control? In some sense, the answer is positive which leads to important consequences for IPs.

4.1. Boundary control problem. Controllability

Let $y \in \mathcal{H}^T$ be a function given in subdomain Ω^T filled by waves by the moment $t = T$; *the boundary control problem* (BCP) is to find $f \in \mathcal{F}^T$ satisfying

$$u^f(\cdot, T) = y. \tag{4.1}$$

The problem is evidently equivalent to the equation

$$W^T f = y. \tag{4.2}$$

Therefore, lemma 3.1 implies the following.

Proposition 4.1. *For any $T < T_*$ the BCP has no more than one solution.*

The set of all of possible states of the DS α^T

$$\mathcal{U}^T := W^T \mathcal{F}^T = \{u^f(\cdot, T) | f \in \mathcal{F}^T\}$$

is said to be *reachable* (at the moment T). By virtue of hyperbolicity it lies in $\mathcal{H}^T$ (see (3.5)); consequently, to analyse a solvability of the BCP is to study the embedding $\mathcal{U}^T \subset \mathcal{H}^T$. The following result plays the key role in the BC method.

Theorem 4.1. *For any $T > 0$ the equality*

$$\mathrm{clos}_{\mathcal{H}} \mathcal{U}^T = \mathcal{H}^T \tag{4.3}$$

is valid.

Postponing the proof until section 4.4, let us discuss the meaning of the result and some of its useful corollaries.

Relation (4.3) shows that any function $y \in \mathcal{H}^T$ may be approximated by waves u^f arbitrarily closely in L_2-metric. In control theory this property is known as (approximate) *controllability* of the DS α^T.

Turning back to the solvability of the BCP, the following results can be mentioned.

(i) For times $T < T_*$, in spite of its density in $\mathcal{H}^T$ a reachable set is rather poor: for any ball $B_r \subset \mathrm{Int}\, \Omega^T$ one has $C_0^\infty(B_r) \not\subset \mathcal{U}^T$; so that $\mathcal{U}^T \neq \mathcal{H}^T$ and the control operator W^T does not act isomorphically in contrast to the one-dimensional case (see Avdonin *et al* 1994). Due to this fact the BCP turns out to be ill-posed.

(ii) For a sufficiently large time T_0 which is determined by the geometry of Ω one has $\mathcal{U}^{T_0} = \mathcal{H}$, so that the BCP is solvable but not uniquely (see Bardos *et al* 1992 and section 6.8).

4.2. Observability

It is customary in control theory to reformulate a property of controllability in dual terms (see Avdonin and Ivanov 1995, Lions 1968, Russell 1978).

We say that the dual DS α_*^T is *observable* (at time T) if the relation

$$\mathrm{Ker}\, O^T = \mathcal{H} \ominus \mathcal{H}^T \tag{4.4}$$

is fulfilled. As follows from the well known operator relation

$$\mathrm{Ker}\, O^T = \mathrm{Ker}(W^T)^* = \mathcal{H} \ominus \mathrm{clos}_{\mathcal{H}}\, \mathrm{Ran}\, W^T = \mathcal{H} \ominus \mathrm{clos}_{\mathcal{H}} \mathcal{U}^T$$

the observability of α_*^T is equivalent to the controllability of α^T. Thus, by virtue of (4.3) a dual system is observable on any $T > 0$.

Property (4.4) is of interesting physical meaning. If perturbation y, initiating a wave process v^y in the DS α_*^T, satisfies $\mathrm{supp}\, y \cap \Omega^T \neq \{\varnothing\}$ (i.e. y is localized not far from a border), it has to manifest itself on Γ during a time interval $[0, T]$. Moreover, by virtue of the relation

$$\mathcal{H}^T \cap \mathrm{Ker}\, O^T = \{0\} \tag{4.5}$$

the part $y|_{\Omega^T}$ of the perturbation is *uniquely determined* by the trace $\partial_\nu\, v^y|_{\Sigma^T}$. This relates an observability to Huygens's rule known in wave propagation theory (Bardos and Belishev 1995, Belishev 1994): in accordance with the rule, the forward front of wave may be constructed as the envelope of the spheres whose centres belong to the boundary of supp y.

4.3. Wave projectors

Consider the reachable sets

$$\mathcal{U}^\xi := W^T \mathcal{F}^{T,\xi} = (\text{see (3.1)}) = \{u^f(\cdot,\xi) \,|\, f \in \mathcal{F}^T\} \qquad 0 \leqslant \xi \leqslant T \tag{4.6}$$

corresponding to intermediate times. The stationary state property of the DS α^T together with its controllability lead to the relation

$$\mathrm{clos}_{\mathcal{H}} \mathcal{U}^\xi = \mathcal{H}^\xi \qquad 0 \leqslant \xi \leqslant T. \tag{4.7}$$

The (orthogonal) projector P^ξ acting in $\mathrm{clos}\,\mathcal{U}^T$ onto $\mathrm{clos}\,\mathcal{U}^\xi$ is said to be *a wave projector*. It is an intrinsic object of the system α^T.

The projector G^ξ in $\mathcal{H}^T$ onto $\mathcal{H}^\xi$ acts as follows,

$$(G^\xi y)(x) = \theta(\xi - \tau(x))y(x) = \begin{cases} y(x) & x \in \Omega^\xi \\ 0 & x \in \Omega^\xi_\perp \end{cases}$$

cutting off functions on Ω^ξ.

Equality (4.7) implies

$$P^\xi = G^\xi \qquad 0 \leqslant \xi \leqslant T. \tag{4.8}$$

Being of great importance for IPs, this result merits being commented upon in more detail. Certainly, 'geometric' projectors G^ξ as well as wave projectors P^ξ are determined by a Riemannian metric in Ω, but the equality (4.8) is far from being evident. Moreover, it is not a general fact: as a counterexample the so-called two-velocity systems may be mentioned, where a direct analogue of (4.7) and (4.8) does not hold (Belishev *et al* 1997, Belishev and Ivanov 1995).

The personal experience of the author shows that to apply the BC method to a concrete case one has first to clarify how the wave projectors act. For the DS α^T, *due to its controllability*, the answer appears to be simple and explicit: these projectors cut off functions on subdomains filled by waves. As we shall see later, it is a surprise for IPs. Moreover, a lack of controllability in the two-velocity case mentioned above leads to essential difficulties in IPs for this kind of dynamical system.

4.4. Proof of theorem 4.1

(i) In view of the equivalence of controllability and observability, to prove (4.3) is to demonstrate (4.4). Since $\mathcal{H} \ominus \mathcal{H}^T \subset \mathrm{Ker}\, O^T$ by hyperbolicity, it would be enough to establish $\mathrm{Ker}\, O^T \subset \mathcal{H} \ominus \mathcal{H}^T$, i.e. that the inclusion $y \in \mathrm{Ker}\, O^T$ implies $y = 0$ in Ω^T.

(ii) Choose $y \in \mathrm{Ker}\, O^T$; let $v^y \in H^1(Q^T)$ be a solution of (2.9)–(2.11) satisfying

$$\partial_\nu v^y|_{\Sigma^T} = O^T y = 0$$

by the choice of y. As may be easily seen, due to condition $v^y(\cdot, T) = 0$ the odd continuation

$$w(x,t) := \begin{cases} v^y(x,t) & \text{in } \Omega \times [0,T] \\ -v^y(x, 2T - t) & \text{in } \Omega \times [T, 2T] \end{cases}$$

turns out to be a function of the class $H^1(Q^{2T})$ satisfying

$$w_{tt} - \Delta_g w = 0 \qquad \text{in } Q^{2T} \tag{4.9}$$

$$w|_{\Sigma^{2T}} = 0 \qquad \partial_\nu w|_{\Sigma^{2T}} = 0. \tag{4.10}$$

(iii) (*The Holmgren–John–Tataru uniqueness theorem.*) Denote

$$K^{2T} := \{(x,t) \in Q^{2T} \,|\, 0 \leqslant \tau(x) < T - |t - T|\}.$$

Let us show that (4.9) and (4.10) imply

$$w = 0 \qquad \text{in } K^{2T}. \tag{4.11}$$

Indeed, by virtue of Tataru's result (Tataru 1993, 1995) concerning a uniqueness of the (zero) continuation of a solution of the wave equation across a non-characteristic surface, any point (γ, t) lying on *a time-like* surface $\Gamma \times (0, 2T)$ has a vicinity (in Q^{2T}) in which w vanishes identically. Therefore, in any smaller cylinder $Q_\varepsilon^{2T-\varepsilon} := \Omega \times [\varepsilon, 2T - \varepsilon] \subset Q^{2T}$ ($\varepsilon > 0$) the set $\operatorname{supp} w$ is separated from a lateral surface $\sum_\varepsilon^{2T-\varepsilon} := \Gamma \times [\varepsilon, 2T - \varepsilon]$:

$$\operatorname{supp} w \cap \{\Omega^\xi \times [\varepsilon, 2T - \varepsilon]\} = \{\varnothing\}$$

with small enough $\xi = \xi(\varepsilon) > 0$. This is the first step of zero continuation of w from Σ^{2T} into K^{2T}.

(iv) Fix ε and consider a subdomain in $Q_\varepsilon^{2T-\varepsilon}$ of the form

$$K_\varepsilon^{2T-\varepsilon} := \{(x, t) | 0 \leqslant \tau(x) < T - |t - T| - \varepsilon\}$$

bounded by $\Sigma_\varepsilon^{2T-\varepsilon}$ and two characteristics $t = \tau(x) + \varepsilon$, $t = 2T - \tau(x) - \varepsilon$. There exists (see John 1948, Russell 1971) an increasing family of 'lens-shaped' sets $\{\mathcal{L}(\lambda)\}$, $\lambda \in [0, 1)$ such that:

(1) $\mathcal{L}(\lambda') \subset \mathcal{L}(\lambda')$, $\lambda' < \lambda''$; $\mathcal{L}(\lambda) \subset \Omega^\xi \times [\varepsilon, 2T - \varepsilon]$ for $\lambda < \lambda_0$ (with some $\lambda_0 \in (0, 1)$);

(2) for every λ the part $S(\lambda) := [\partial \mathcal{L}(\lambda)] \cap \operatorname{Int} Q_\varepsilon^{2T-\varepsilon}$ of a boundary of $\mathcal{L}(\lambda)$ is a smooth time-like surface;

(3) the family exhausts the subdomain $K_\varepsilon^{2T-\varepsilon} : \bigcup_{\lambda \in [0,1)} \mathcal{L}(\lambda) = K_\varepsilon^{2T-\varepsilon}$.

Assume that

$$\operatorname{supp} w \cap K_\varepsilon^{2T-\varepsilon} \neq \{\varnothing\}. \tag{4.12}$$

Increasing λ from zero one can find $\lambda = \lambda_*$ such that

$$\operatorname{supp} w \cap \mathcal{L}(\lambda) = \{\varnothing\} \qquad \lambda \leqslant \lambda_* \qquad \operatorname{supp} w \cap S(\lambda_*) \neq \{\varnothing\}$$

(i.e. the value λ_* corresponds to the first contact of $\mathcal{L}(\lambda)$ with $\operatorname{supp} w$). Evidently, in some vicinity of point $p \in \operatorname{supp} w \cap S(\lambda_*)$ the uniqueness of the zero continuation of w across $S(\lambda_*)$ is broken. Therefore, assumption (4.12) leads to a contradiction, which implies $w = 0$ in $K_\varepsilon^{2T-\varepsilon}$. In view of an arbitrariness of ε we obtain (4.11).

(v) The equality (4.11) implies $w_t = 0$ in K^{2T}, so that $w_t(\cdot, T) = y = 0$ in $K^{2T} \cap \{t = T\} = \Omega^T$. Therefore (see (i)), one has the inclusion $\operatorname{Ker} O^T \subset \mathcal{H} \ominus \mathcal{H}^T$. The theorem is proved. □

The idea of the proof is taken from Russell's paper (1971) which used the classical work of John (1948).

The reader should note a central role of the uniqueness theorem used in the proof. The theorem has been known for the wave equation with (real) analytical coefficients (John 1948, Russell 1971) for a long time. Its generalization to a non-analytical case has taken much time and effort. Recent progress in this direction was stimulated by Robbiano (1991) and developed by Hörmander (1992). In 1993 it was crowned by a remarkable result of Tataru (1993, 1995) which settled the question for $C^1(\overline{\Omega})$-coefficients.

The first papers devoted to the BC method referred to the formulation of the Holmgren–John theorem declared in Russell (1978, p 685). Unfortunately, private communications found out an absence of the proof. That is why, beginning from Belishev (1990a) we were forced to postulate property (4.3). Thus, during a period 1986–1993 the BC method covered some unclear class of 'controllable' dynamical systems, and it was Tataru's result which permitted us to justify our approach.

5. Visualization of waves

We demonstrate that boundary measurements determine the visualizing operator. The efficient constructions (amplitude integral and amplitude formula) are proposed to represent V^T via R^{2T} or $\{\lambda_k, \psi_k\}$. They are based upon results of sections 2.3 and 2.4 concerning the propagation of discontinuities.

5.1. Inverse problems

Let us begin with the statement of the IPs to be solved in sections 5 and 6:

(i) (*dynamical IP*) given the response operator R^{2T} to recover the manifold (Ω^T, g);

(ii) (*spectral IP*) given the spectral data $\{\lambda_k; \psi(\cdot)\}_{k=1}^{\infty}$ to recover the manifold (Ω, g).

Just for simplicity a metric on a border is assumed to be known. It can be shown that tensor $g|_\Gamma$ is determined by either kind of inverse data.

Speaking about the 'recovering' we mean the determination of a manifold up to isometry.

5.2. Operator sums

We begin to describe an operator construction which solves the IPs.

Let $\{\mathcal{F}^{T,\xi}\}$, $0 \leqslant \xi \leqslant T$ be a family of subspaces in $\mathcal{F}^T$ and $X^{T,\xi}$ be a projector in $\mathcal{F}^T$ onto $\mathcal{F}^{T,\xi}$,

$$(X^{T,\xi} f)(\cdot, t) = \theta(t - (T - \xi)) f(\cdot, t) \qquad 0 \leqslant t \leqslant T.$$

Recall that $\{\mathcal{H}^\xi\}$, $0 \leqslant \xi \leqslant T$, is a family of subspaces in $\mathcal{H}$; G^ξ projects in $\mathcal{H}$ onto $\mathcal{H}^\xi$.

Choose a partition $\Xi = \{\xi_j\}_{j=0}^N$: $0 = \xi_0 < \xi_1 < \cdots < \xi_N = T$ of the interval $0 \leqslant t \leqslant T$; let $r(\Xi) = \max_j \Delta\xi_j$ be its range; $\Delta\xi_j := \xi_j - \xi_{j-1}$. Projectors $\Delta_j X^{T,\xi} := X^{T,\xi_j} - X^{T,\xi_{j-1}}$ act by the rule

$$(\Delta_j X^{T,\xi} f)(\cdot, t) = \begin{cases} f(\cdot, t) & \xi_{j-1} \leqslant t < \xi_j \\ 0 & \text{for other } \ t \in [0, T] \end{cases}$$

and satisfy

$$\sum_{j=1}^N \Delta_j X^{T,\xi} = \mathbb{1}_{\mathcal{F}^T} \qquad \Delta_j X^{T,\xi} \Delta_k X^{T,\xi} = \mathbb{O}_{\mathcal{F}^T} \qquad j \neq k. \tag{5.1}$$

Projectors $\Delta_j G^\xi := G^{\xi_j} - G^{\xi_{j-1}}$ act as follows:

$$(\Delta_j G^\xi y)(x) = \begin{cases} y(x) & x \in \Omega^{\xi_j} \setminus \Omega^{\xi_{j-1}} \\ 0 & \text{for other } \ x \in \Omega \end{cases}$$

the relations

$$\sum_{j=1}^N \Delta_j G^\xi = G^T \qquad \Delta_j G^\xi \Delta_k G^\xi = \mathbb{O}_{\mathcal{H}} \qquad j \neq k \tag{5.2}$$

are valid. An operator sum of the form

$$A_\Xi^T := \sum_{j=1}^N \Delta_j G^\xi W^T \Delta_j X^{T,\xi} \tag{5.3}$$

corresponds to the partition Ξ.

Lemma 5.1. *The sum $A^T_\Xi : \mathcal{F}^T \to \mathcal{H}$ is a continuous operator, the estimate*

$$\|A^T_\Xi\| \leqslant \|W^T\| \tag{5.4}$$

being valid.

Proof. For any $f \in \mathcal{F}^T$ the orthogonality (5.2) and (5.3) imply

$$\|A^T_\Xi f\|^2_{\mathcal{H}} = \sum_{j=1}^{N} \|\Delta_j G^\xi W^T \Delta_j X^{T,\xi} f\|^2_{\mathcal{H}} \leqslant \|W^T\|^2 \sum_{j=1}^{N} \|\Delta_j X^{T,\xi} f\|^2_{\mathcal{F}^T} = \|W^T\|^2 \|f\|^2_{\mathcal{F}^T}$$

that gives (5.4). The lemma is proved. □

5.3. *Amplitude integral*

A convergence of sums (5.3) is established here. Some heuristic considerations are prefaced to a rigorous result to make clear what limit of A^T_Ξ should be expected.

Assume for simplicity $T < T_\omega$, so that a cut locus plays no role. Fix $f \in \mathcal{M}^T$, and consider a separate term of (5.3):

$$\Delta_j G^\xi W^T \Delta_j X^{T,\xi} f = \Delta_j G^\xi W^T X^{T,\xi_j} f - \Delta_j G^\xi W^T X^{T,\xi_{j-1}} f = \Delta_j G^\xi W^T X^{T,\xi_j} f. \tag{5.5}$$

Indeed, the set supp $W^T\ X^{T,\xi_{j-1}}$ lies in $\overline{\Omega}^{\xi_{j-1}}$; therefore, it does not get into a layer $\Omega^{\xi_j} \setminus \overline{\Omega}^{\xi_{j-1}}$ that implies $\Delta_j G^\xi W^T\ X^{T,\xi_{j-1}} f = 0$.

Control $X^{T,\xi_j} f$ coincides with f_{ξ_j} (see section 2.3) and, therefore, wave $W^T f_{\xi_j}$ has a discontinuity on its forward front Γ^{ξ_j}. Projector $\Delta_j G^\xi$ selects a part of this wave lying near the discontinuity, and if $r(\Xi)$ is small enough this part may be described by geometrical optics: relation (2.14) gives the approximate equality

$$\Delta_j C^\xi W^T f_{\xi_j} \approx \begin{cases} \beta^{-1/2}(\gamma(x), \xi_j) f(\gamma(x), T - \xi_j) & x \in \Omega^{\xi_j} \setminus \Omega^{\xi_{j-1}} \\ 0 & \text{for other } \ x \in \Omega. \end{cases}$$

Taking into account the fact that

$$\xi_j \approx \tau(x) \qquad \text{for } x \in \Omega^{\xi_j} \setminus \Omega^{\xi_{j-1}}$$

and summing the terms (5.5) one obtains

$$(A^T_\Xi f)(x) \approx \begin{cases} \beta^{-1/2}(\gamma(x), \tau(x)) f(\gamma(x), T - \tau(x)) & x \in \Omega^T \\ 0 & \text{for other } \ x \in \Omega. \end{cases}$$

Recalling the corollary of lemma 1.1 (with $X^T_\Theta = \mathbb{1}_{\mathcal{F}^T}$ for $T < T_\omega$) one can rewrite

$$A^T_\Xi f \approx (I^T)^* Y^T f$$

where $Y^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$(Y^T f)(\cdot, t) := f(\cdot, T - t) \qquad 0 < t < T.$$

So, the convergence $A^T_\Xi \to (I^T)^* Y^T$ should be expected.

Theorem 5.1. *For any $T : 0 < T \leqslant T_*$ a refinement of a partition leads to the convergence*

$$\lim_{r(\Xi)\to 0} A^T_\Xi = (I^T)^* Y^T \tag{5.6}$$

in a weak operator topology.

Proof. Fix $f \in \mathcal{M}^T$, $\varepsilon > 0$ and choose $y \in C_0^\infty(\Omega^T)$ such that $\operatorname{supp} y \subset \Omega^T \setminus \overline{\omega}_\varepsilon$; find $\delta = \delta(\varepsilon)$ (see lemma 2.2); choose a partition satisfying $r(\Xi) < \delta$.

Applying (2.13) for $t = T$, $\xi = \xi_j$ $(j = 1, \ldots, N)$ one obtains the representation

$$u^{f_{\xi_j}}(x, T) = \beta^{-1/2}(\gamma(x), \tau(x)) f(\gamma(x), T - \tau(x))\theta(\xi_j - \tau(x)) + w^f(x, T; \xi_j)(\xi_j - \tau(x))_+$$

being valid for $x \in \Omega^T \setminus \omega_\varepsilon$, $\tau(x) > \xi_j - \delta \geqslant \xi_j - \xi_{j-1}$, i.e. on the set $[\Omega^{\xi_j} \setminus \overline{\Omega}^{\xi_{j-1}}] \setminus \omega_\varepsilon$ ('thin layer'). All of this may be rewritten as follows:

$$\Delta_j G^\xi [u^{f_{\xi_j}}(\cdot, T) - (I^T)^* Y^T f] = \begin{cases} w^f(\cdot, T; \xi_j)(\xi_j - \tau(\cdot)) & \text{in } [\Omega^{\xi_j} \setminus \overline{\Omega}^{\xi_{j-1}}] \setminus \omega_\varepsilon \\ 0 & \text{for other points of } \Omega^T. \end{cases} \tag{5.7}$$

Representing in the layer

$$u^{f_{\xi_j}}(\cdot, T) = W^T X^{T,\xi_j} f = W^T \Delta_j X^{T,\xi} f$$

and summing terms (5.7) one can obtain

$$[A_\Xi^T - (I^T)^* Y^T] f = \sum_{l=1}^{N} \Delta_j G^\xi w^f(\cdot, T; \xi_j)(\xi_j - \tau(\cdot)) \qquad \text{in } \Omega^T \setminus \omega_\varepsilon$$

that implies

$$\begin{aligned} |([A_\Xi^T - (I^T)^* Y^T] f, y)_{\mathcal{H}}| &= \left| \sum_{j=1}^{N} \int_{\Omega^{\xi_j} \setminus \Omega^{\xi_{j-1}}} \mathrm{d}\Omega\, w^f(x, T; \xi_j)(\xi_j - \tau(x)) y(x) \right| \\ &= \text{(see (1.4), (1.7))} \\ &= \left| \sum_{j=1}^{N} \int_{\xi_{j-1}}^{\xi_j} \mathrm{d}\tau\, (\xi_j - \tau) \int_{\sigma_+^\tau} \mathrm{d}\Gamma\, \beta(\gamma, \tau) w^f(x(\gamma, \tau), T; \xi_j) y(x(\gamma, \tau)) \right| \\ &\leqslant \text{(see (2.13))} \\ &\leqslant \max_{\Omega^T \setminus \omega_\varepsilon} \beta(\gamma(\cdot), \tau(\cdot)) c(T, f, \varepsilon) \max_{\Omega^T} |y| \sum_{j=1}^{N} \frac{(\xi_j - \xi_{j-1})^2}{2} \leqslant c(T, f, \varepsilon, y) r(\Xi). \end{aligned}$$

As a result, we have

$$([A_\Xi^T - (I^T)^* Y^T] f, y)_{\mathcal{H}^T} \underset{r(\Xi) \to 0}{\longrightarrow} 0.$$

Taking into account a density of $C_0^\infty(\Omega^T \setminus \omega)$ in $\mathcal{H}^T$ and a boundedness of sums (see (5.4)) one obtains (5.6). The theorem is proved. □

The heuristic considerations presented before the theorem motivate to call the limit

$$A^T := \lim_{r(\Xi) \to 0} A_\Xi^T =: \int_0^T \mathrm{d}G^\xi\, W^T\, \mathrm{d}X^{T,\xi}$$

an amplitude integral (AI). Indeed, it was constructed by summing amplitudes of wave discontinuities (2.14). The AI was first introduced in Belishev (1990b) and developed in Belishev and Kachalov (1994). Moreover, in the latter paper a *strong* convergence $A_\Xi^T \to A^T$ is established for $T < T_\omega$.

The next relation is a simple corollary of (5.6): the adjoint operator $(A^T)^* : \mathcal{H}^T \to \mathcal{F}^T$ may be represented in the form

$$(A^T)^* = \int_0^T \mathrm{d}X^{T,\xi} (W^T)^* \mathrm{d}G^\xi = Y^T I^T \tag{5.8}$$

with the same kind of convergence (weak).

5.4. Images via AI

The following result represents an image operator in the form of an amplitude integral.

Lemma 5.2. *The representation*

$$I^T = Y^T \int_0^T \mathrm{d}X^{T,\xi} (W^T)^* \, \mathrm{d}P^\xi \tag{5.9}$$

holds.

Proof.

$$I^T = (\text{see } (5.8)) = Y^T (A^T)^* = Y^T \int_0^T \mathrm{d}X^{T,\xi} (W^T)^* \, \mathrm{d}G^\xi = (\text{see } (4.8))$$
$$= Y^T \int_0^T \mathrm{d}X^{T,\xi} (W^T)^* \, \mathrm{d}P^\xi.$$

The lemma is proved. □

A remarkable peculiarity of this result is that it relates geometry and dynamics: being an object of geometric nature, the image operator is represented via intrinsic operators W^T, P^ξ of the DS α^T (and standard operators Y^T, $X^{T,\xi}$). The reader should note the role of controllability.

Introduce the operators $\Pi^{T,\xi} : \mathcal{F}^T \to \mathcal{F}^T$,

$$\Pi^{T,\xi} := (W^T)^* P^\xi W^T = O^T P^\xi W^T \qquad 0 \leqslant \xi \leqslant T.$$

As a simple corollary of (5.9) we obtain the following principal result.

Theorem 5.2. *The visualizing operator may be represented in the form of AI:*

$$V^T = Y^T \int_0^T \mathrm{d}X^{T,\xi} \, \mathrm{d}\Pi^{T,\xi}. \tag{5.10}$$

Proof.

$$V^T = I^T W^T = (\text{see } (5.9)) = Y^T \int_0^T \mathrm{d}X^{T,\xi} (W^T)^* \, \mathrm{d}P^\xi W^T$$
$$= Y^T \int_0^T \mathrm{d}X^{T,\xi} \, \mathrm{d}[(W^T)^* P^\xi W^T] = Y^T \int_0^T \mathrm{d}X^{T,\xi} \, \mathrm{d}\Pi^{T,\xi}.$$

The theorem is proved. □

In conclusion, observe that representation (5.10), as well as the amplitude integral, itself is related to the problem of triangular factorization of operators (Belishev 1990b, Belishev and Pushnitski 1996). That is not surprising in view of the well known and deep connections between IPs and factorization (Belishev 1996b, Faddeev 1974, Gokhberg and Krein 1970, Nizhnik 1991).

5.5. Amplitude formula

Another way to express images via amplitudes of discontinuities is given by formula (2.16).

Theorem 5.3. *For any $f \in \mathcal{M}^T$ the representation*

$$(V^T f)(\gamma, \xi) = (\widetilde{u^f(\cdot, T)})(\gamma, \xi) = \lim_{t \to T-\xi-0} ((W^T)^* P_\perp^\xi W^T f)(\gamma, t) \qquad (5.11)$$

$(P_\perp^\xi := \mathbb{1}_{\mathcal{H}^T} - P^\xi)$ *is valid for almost all* $(\gamma, \xi) \in \Sigma^T$.

Proof. The expression under the limit sign in (2.16) may be transformed as follows,

$$\partial_\nu v^{y_\xi} = O^T y_\xi = O^T G_\perp^\xi y = (\text{see (3.7), (4.8)}) = (W^T)^* P_\perp^\xi y.$$

Taking $y = W^T f$ one obtains (5.11). The theorem is proved. □

To emphasize the dynamical nature of relation (5.11) (see considerations around (2.16)) we call it an amplitude formula (AF). The AF was introduced in Belishev (1990b).

5.6. Models

Investigating a dynamical system through boundary measurements, an external observer looks for its intrinsic structure and properties. As a first step, some kind of a copy (model) of a DS may be constructed. Note that from the point of view of system theory we are speaking about *realizations* of a DS corresponding to the boundary measurements (Kalman *et al* 1969).

The DS α^T is determined by spaces $\mathcal{F}^T$ and $\mathcal{H}$, and the control operator W^T; thus, one can identify α^T with the triple $\{\mathcal{F}^T, \mathcal{H}, W^T\}$.

Let $\mathcal{H}_\#$ be a Hilbert space, $W_\#^T : \mathcal{F}^T \to \mathcal{H}_\#$ be an operator; the triple $\alpha_\#^T = \{\mathcal{F}^T, \mathcal{H}_\#, W_\#^T\}$ is said to be *a model* of α^T if there exists an isometric operator $U : \mathcal{H}_\# \to \mathcal{H}$ satisfying

$$U^* U = \mathbb{1}_{\mathcal{H}_\#} \qquad W^T = U W_\#^T. \qquad (5.12)$$

$\mathcal{H}_\#$ and $W_\#^T$ are called the (model) inner space and control operator, respectively; U is *a transform operator*. The sets

$$\mathcal{U}_\#^\xi := W_\#^T \mathcal{F}^{T,\xi} = U^* \mathcal{U}^\xi \qquad 0 \leqslant \xi \leqslant T$$

are called reachable; the (model) wave projectors $P_\#^\xi$ are introduced as those in $\text{clos}_{\mathcal{H}_\#} \mathcal{U}_\#^T$ onto $\text{clos}_{\mathcal{H}_\#} \mathcal{U}_\#^\xi$, so that

$$P^\xi U = U P_\#^\xi \qquad P_\perp^\xi U = U P_{\#\perp}^\xi \qquad 0 \leqslant \xi \leqslant T \qquad (5.13)$$

where $P_{\#\perp}^\xi = \mathbb{1}_{\mathcal{H}_\#} - P_\#^\xi$. Relations between the DS α^T and its model are illustrated in figure 3.

The following fact justifies the introduction of models.

Proposition 5.1. *Any model determines the visualizing operator.*

Indeed, V^T is determined by operators $\Pi^{T,\xi}$ which may be expressed in model terms:

$$\Pi^{T,\xi} = (W^T)^* P^\xi W^T = (\text{see (5.12)}) = (U W_\#^T)^* P^\xi U W_\#^T = (\text{see (5.13)}) = (W_\#^T)^* P_\#^\xi W_\#^T$$

so that (5.10) takes the form

$$V^T = Y^T \int_0^T \mathrm{d}X^{T,\xi}\, \mathrm{d}[(W_\#^T)^* P_\#^\xi W_\#^T]. \qquad (5.14)$$

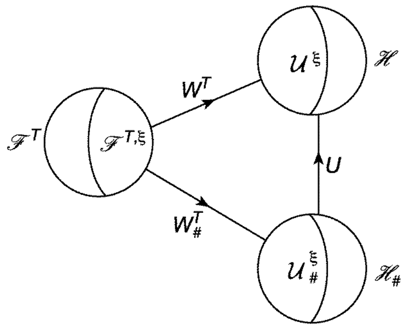

Figure 3. Model.

Relation (5.11) also may be rewritten in invariant form:

$$\widetilde{(u^f(\cdot,T))}(\gamma,\xi) = \lim_{t\to T-\xi-0}((W^T_\#)^* P^{\xi}_{\#\perp} W^T_\# f)(\gamma,t) \qquad \text{a.e. on } \Sigma^T. \quad (5.15)$$

In the remainder of section 5 we construct the concrete models corresponding to both kinds of inverse data, dynamical and spectral. The goal is to recover the visualizing operator along the scheme

$$\text{inverse data} \ \Rightarrow \ \text{model} \ \Rightarrow V^T. \quad (5.16)$$

Models were first used in the BC method in Belishev (1995) and Belishev and Ivanov (1995); later they were applied to the dynamical IP for the heat equation (Belishev 1996a).

5.7. Dynamical model

Let

$$W^T = U^T|W^T|$$

be a polar decomposition of a control operator and

$$|W^T| := [(W^T)^* W^T]^{1/2}$$

be its operator module; U^T is a canonical isometry from $\operatorname{clos}_{\mathcal{F}^T} \operatorname{Ran}|W^T|$ onto $\operatorname{clos}_{\mathcal{H}} \operatorname{Ran} W^T$ (see, e.g., Kato 1966). An obvious fact is that the triple

$$\alpha^T_{\text{din}} := \{\mathcal{F}^T, \operatorname{clos}_{\mathcal{F}^T}\operatorname{Ran}|W^T|, |W^T|\}$$

forms a model of the DS α^T, with the isometry U^T playing the role of a transform operator.

Model α^T_{din} may be constructed via a response operator. Indeed, R^{2T} determines C^T (see (3.12)), whereas

$$|W^T| = (C^T)^{1/2}.$$

Thus, denoting $\operatorname{clos}_{\mathcal{F}^T} \operatorname{Ran}(C^T)^{1/2} =: \mathcal{F}^T_{1/2}$, one can represent

$$\alpha^T_{\text{din}} = \{\mathcal{F}^T, \mathcal{F}^T_{1/2}, (C^T)^{1/2}\}.$$

The sets $\mathcal{U}^{\xi}_{1/2} := (C^T)^{1/2}\mathcal{F}^{T,\xi}$ are reachable ones in the model. Denote $\mathcal{F}^{T,\xi}_{1/2} := \operatorname{clos}_{\mathcal{F}^T} \mathcal{U}^{\xi}_{1/2}$; let $P^{\xi}_{1/2}$ be the projector in $\mathcal{F}^T_{1/2}$ onto $\mathcal{F}^{T,\xi}_{1/2}$; representation (5.14) takes the form

$$V^T = Y^T \int_0^T \mathrm{d}X^{T,\xi}\, \mathrm{d}[(C^T)^{1/2} P^{\xi}_{1/2} (C^T)^{1/2}] \quad (5.17)$$

determining V^T via C^T, and, consequently, via R^{2T}.

5.8. Spectral model

The Fourier expansion of waves (see section 2.5) gives a method to construct a model via spectral data.

Let $c = \{c_k\}_{k=1}^{\infty}$ be a (real) sequence belonging to the space ℓ_2; introduce the unitary operator $U; \ell_2 \to \mathcal{H}$,

$$Uc = \sum_{k=1}^{\infty} c_k \varphi_k.$$

The representation (2.17) induces the operator $W_{sp}^T : \mathcal{F}^T \to \ell_2$

$$W_{sp}^T f := \{c_k^f(T)\}_{k=1}^{\infty} \qquad c_k^f(T) = (f, s_k^T)_{\mathcal{F}^T}$$

the relation

$$W^T = U W_{sp}^T$$

being valid.

From the aforesaid it may be concluded that the triple

$$\alpha_{sp}^T := \{\mathcal{F}^T, \ell_2, W_{sp}^T\}$$

turns out to be a model of the DS α^T. Evidently, α_{sp}^T is determined by data $\{\lambda_k; \psi_k(\cdot)\}_{k=1}^{\infty}$.

Let P_{sp}^{ξ} be projectors in ℓ_2 onto the subspaces $\text{clos}_{\ell_2} W_{sp}^T \mathcal{F}^{T,\xi}$; in accordance with (5.14) we obtain the representation

$$V^T = Y^T \int_0^T \mathrm{d}X^{T,\xi}\, \mathrm{d}[(W_{sp}^T)^* P_{sp}^{\xi} W_{sp}^T] \tag{5.18}$$

which expresses the visualizing operator in terms of a spectral model. Note, in addition, that the adjoint operator entering in (5.18) acts by the rule

$$(W_{sp}^T)^* c = \sum_{k=1}^{\infty} c_k s_k^T$$

with the series converging in $\mathcal{F}^T$ (see (2.19)).

An important peculiarity of data $\{\lambda_k; \psi_k\}$ is that they determine V^T for *any* $T > 0$, whereas R^{2T} determines V^T for *given* T. This reflects a global character of spectral data.

6. The solving of inverse problems

We describe a way to recover part of a manifold filled by waves through the visualizing operator. Supplementing the diagram (5.16), the step $V^T \Rightarrow (\Omega^T, g)$ completes the reconstruction.

6.1. C^l_{loc}-controllability

We are going to extract information about an intrinsic geometry from wave images. Some additional properties of waves are required for this purpose. Here we present a result which strengthens property (4.3): a set of smooth waves turns out to be dense in classes of differentiable functions.

Introduce the classes of controls $C_0^N \subset \mathcal{F}^T$,

$$C_0^N := \operatorname{clos}_{C^N(\Sigma^T)} \mathcal{M}^T = \left\{ f \in C^N(\Sigma^T) \left| \left(\frac{\partial}{\partial t}\right)^j f \right|_{t=0} = 0,\ j = 0, 1, \ldots, N \right\}$$
$$N = 0, 1, \ldots$$

and denote

$$\mathcal{U}_N^T := W^T C_0^N \subset \mathcal{U}^T.$$

Lemma 6.1. *For integer N and l satisfying $N \geqslant l + 1 + [n/2]$ the relation*

$$\operatorname{clos}_{C^l} \mathcal{U}_N^T|_D = C^l(D) \qquad T > 0 \tag{6.1}$$

is valid on any compact $D \subset \operatorname{Int} \Omega^T$.

The proof can be found in Belishev and Dolgoborodov (1997). This is the result which gives the title of this section. As a corollary, we obtain the following. Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}^T$ be a C_0^N-complete system of controls, i.e.

$$\operatorname{clos}_{C^N} \operatorname{Lin}\{f_j\}_{j=1}^{\infty} = C_0^N \tag{6.2}$$

(Lin is the linear span) and $u_j = W^T f_j$ be the corresponding waves; in the conditions of lemma 6.1, relation (6.1) is equivalent to the following:

$$\operatorname{clos}_{C^l} \operatorname{Lin}\{u_j|_D\}_{j=1}^{\infty} = C^l(D). \tag{6.3}$$

Everywhere in the following we put $N = 3 + [n/2]$ so that *the waves $\{u_j\}_{j=1}^{\infty}$ form a C^2_{loc}-complete system in Ω^T.*

6.2. Wave coordinates

Property (6.3) opens the possibility of using smooth waves as coordinates on Ω^T.

Lemma 6.2. (i) *System $\{u_j\}$ separates points in Ω, i.e. for any $x', x'' \in \Omega^T$, the following conditions are equivalent:*

(a) $x' = x''$;

(b) equality $u_j(x') = u_j(x'')$ is valid for all j.

(ii) For any $x_0 \in \operatorname{Int}\Omega^T$ one can choose $u_{j_1}, \ldots, u_{j_n}$ such that the gradients $\nabla u_{j_1}(x_0), \ldots, \nabla u_{j_n}(x_0)$ form a basis in tangent space $T_{x_0}\Omega^T$.

Proof (sketch). For (ii) fix point $x_0 \in \operatorname{Int}\Omega^T$ and choose its compact vicinity $D \subset \operatorname{Int}\Omega^T$ covered by local coordinates $\eta^1, \ldots, \eta^n$ so that

$$\dim \operatorname{Lin}\{\nabla \eta^k(x_0)\}_{k=1}^n = n.$$

The C^1_{loc}-completeness permits us to approximate

$$\left\| \eta^k - \sum_{j=1}^{p} \alpha_j^k u_j \right\|_{C^1(D)} < \varepsilon \qquad k = 1, \ldots, n$$

that implies

$$\left| \nabla \eta^k(x_0) - \sum_{j=1}^{p} \alpha_j^k \nabla u_j(x_0) \right| < \varepsilon.$$

Taking ε small enough, the latter obviously leads to the equality

$$\dim \operatorname{Lin}\{\nabla\eta^k(x_0)\}_{k=1}^n = \dim \operatorname{Lin}\{\nabla u_j(x_0)\}_{j=1}^p = n$$

and, therefore, one can select subsystem $\{u_{j_k}\}_{k=1}^n$ which generates $T_{x_0}\Omega^T$.

Statement (i) may be justified along the same way using the C_{loc}-completeness of the system $\{u_j\}$. The lemma is proved. □

Property (ii) permits the use of $\{u_j\}$ as local (wave) coordinates.

Proposition 6.1. *For any $x_0 \in \operatorname{Int}\Omega^T$ one can choose a subset of waves $u_{j_1}, \ldots, u_{j_n}$ which forms a coordinate system in a vicinity of x_0.*

Another useful property of smooth waves is the following.

Proposition 6.2. (*i*) *The representation*

$$\overline{\Omega}^T = \bigcup_j \operatorname{supp} u_j \qquad T > 0 \tag{6.4}$$

is valid.

(*ii*) *The set of pairs $\{u_j, \Delta_g u_j\}_{j=1}^\infty$ determines the tensor g in Ω^T.*

Indeed, (6.4) is an evident corollary of (i), lemma 6.2. Furthermore, fixing $x_0 \in \operatorname{Int}\Omega^T$ and denoting $h_j := \Delta_g u_j$ one has the equalities in local coordinates

$$g^{kl}(x_0)\frac{\partial^2 u_j(x_0)}{\partial\eta^k\partial\eta^l} + g^k(x_0)\frac{\partial u_j(x_0)}{\partial\eta^k} = h_j(x_0) \tag{6.5}$$

(see (1.5)) which may be considered as an algebraic system to find unknown $g^{kl}(x_0)$, $g^k(x_0)$. Extending, if necessary, a number of equations $j = 1, 2, \ldots, p$ one can achieve its solvability, which simply follows from a C^2_{loc}-completeness of the system $\{u_j\}$.

The coordinate properties of waves lead to the analogous properties of their images.

Proposition 6.3. (*i*) *The representation*

$$\overline{\Theta}^T = \bigcup_j \operatorname{supp} \tilde{u}_j \qquad T > 0 \tag{6.6}$$

is valid.

(*ii*) *The pairs $\{\tilde{u}_j, \tilde{\Delta}\tilde{u}_j\}_{j=1}^\infty$ determine the tensor $\mathfrak{g}$ on Θ^T.*

Indeed, relation (6.6) follows directly from (6.4) by virtue of $\beta|_{\Theta^T} > 0$. Tensor $\mathfrak{g}$ may be found from equations $\tilde{\Delta}\tilde{u}_j = \tilde{h}_j$ which are written in sgc as follows:

$$\mathfrak{g}^{\mu\nu}\frac{\partial^2\tilde{u}_j(\gamma_0,\tau_0)}{\partial\gamma^\mu\partial\gamma^\nu} + \mathfrak{g}^\nu(\gamma_0,\tau_0)\frac{\partial\tilde{u}_j(\gamma_0,\tau_0)}{\partial\gamma^\nu} + \mathfrak{g}^0(\gamma_0,\tau_0)\tilde{u}_j(\gamma_0,\tau_0) + \frac{\partial^2\tilde{u}_j(\gamma_0,\tau_0)}{\partial\tau^2} = \tilde{h}_j(\gamma_0,\tau_0) \tag{6.7}$$

for $(\gamma_0, \tau_0) \in \Theta^T$, $j = 1, 2, \ldots$ (see (1.10)). As one can show, wave images $\{\tilde{u}_j\}_{j=1}^\infty$ form a C^2_{loc}-complete system on Θ^T that provides a solvability of (6.7) with respect to $\mathfrak{g}^{\mu\nu}$, $\mathfrak{g}^\nu$, $\mathfrak{g}^0$.

Note, in addition, that images $\{\tilde{h}_j\}$ are also determined by the visualizing operator,

$$\tilde{h}_j = \tilde{\Delta}\tilde{u}_j = (\text{see (3.21)}) = V^T\frac{\partial^2}{\partial t^2}f_j. \tag{6.8}$$

6.3. *Reconstruction of* $(\Theta^T, \mathfrak{g})$

The following result is an important step along the way from inverse data towards a manifold.

Theorem 6.1. *The visualizing operator V^T determines the manifold $(\Theta^T, \mathfrak{g})$.*

Proof. This is presented in the form of the recovering procedure.

Step 1. Choose controls $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}^T$ satisfying (6.2) (with $N = 3 + [n/2]$) and find images $\tilde{u}_j = V^T f_j$ on Σ^T. Recover pattern Θ^T by means of representation (6.6).

Step 2. To recover the tensor $\mathfrak{g}$ visualize images $\tilde{h}_j = V^T \partial^2 f_j / \partial t^2$ (see (6.8)) and find components $\mathfrak{g}^{\mu\nu}$ from equations (6.7).

The manifold is recovered; the theorem is proved. □

Manifold $(\Theta^T, \mathfrak{g})$ is an isometric copy of $(\Omega^T \setminus \omega^T, g)$ (see (v), section 1.4), and, therefore, the latter is recovered up to isometry.

To complete a reconstruction it remains for us to turn $(\Theta^T, \mathfrak{g})$ into a copy of (Ω^T, g). To do this we need, roughly speaking, to glue Θ^T along the coast θ^T and extend a metric tensor onto the glued points. An accurate realization of this plan requires us to invoke one more object described below. A reconstruction itself is postponed until section 6.5.

6.4. *Wave copy*

Whereas pattern Θ^T may be characterized as 'Ω^T in semigeodesical coordinates', the manifold introduced could be called 'Ω^T in wave coordinates'.

Let $\mathbb{R}^\infty$ be a space of real sequences $\{r_j\}$ with the metric

$$d(r', r'') = \sum_{j=1}^{\infty} 2^{-j} \frac{|r'_j - r''_j|}{1 + |r'_j - r''_j|}$$

$\{u_j\}$ being the system of smooth waves used previously. Consider the map $u : \Omega^T \to \mathbb{R}^\infty$,

$$u(x) := \{u_j(x)\}_{j=1}^{\infty}$$

and denote $\Omega_u^T := u(\Omega^T)$. The injectivity and continuity of this map follow from lemma 6.2; moreover, it is not difficult to show that u maps Ω^T onto Ω_u^T homeomorphically.

Map u transfers a Riemannian structure from Ω^T on Ω_u^T as follows.

(i) The subset $\Omega_u^T \setminus \omega_u^T$ ($\omega_u^T := u(\omega^T)$) may be covered by 'sgc' $\gamma \circ u^{-1}$, $\tau \circ u^{-1}$ and equipped with the metric tensor $g_u := \mathfrak{g}(\gamma \circ u^{-1}, \tau \circ u^{-1})$. This turns it into an isometric copy of $(\Omega^T \setminus \omega^T, g)$. To extend g_u onto the whole of Ω_u^T let us apply a trick which is useful in a future reconstruction.

(ii) Fix $m \in \omega_u^T$; let M be its (small) $\mathbb{R}^\infty$-vicinity. As proposition 6.1 guarantees, one can choose $u_{j_1} \circ u^{-1}, \ldots, u_{j_n} \circ u^{-1}$ to be local (wave) coordinates on $M \cap \Omega_u^T$. Find components of g_u in these coordinates *out of* the cut locus ω_u^T and then extend the components on m by continuity. Thus, tensor g_u is determined everywhere on Ω_u^T.

As it was constructed, manifold (Ω_u^T, g_u) appears to be isometric to (Ω^T, g). We say it is *a wave copy* of the original manifold. Relations between a manifold, its pattern, and wave copy are shown on figure 4.

The following result clarifies the role of a wave copy in reconstruction.

Lemma 6.3. *Pattern $(\Theta^T, \mathfrak{g})$ and system $\{\tilde{u}_j\}$ determine manifold (Ω_u^T, g_u).*

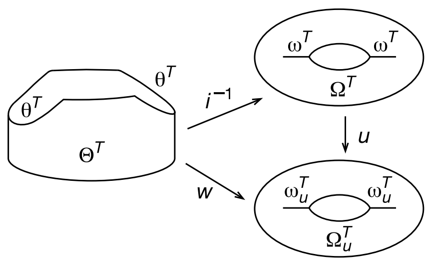

Figure 4. Wave copy.

Proof. The tensor $\mathfrak{g}$ determines the function $\beta|_{\Theta^T}$ (see section 1.3); thus, functions

$$w_j := \beta^{-1/2}\tilde{u}_j = (\text{see definition of images}) = u_j \circ i^{-1}$$

may be considered as given on Θ^T. One can extend w_j on the coast θ^T by continuity (see (iv), (v), section 1.4).

As is seen from equalities

$$w_j(\gamma,\tau) = \beta^{-1/2}(\gamma,\tau)\tilde{u}_j(\gamma,\tau) = u_j(x(\gamma,\tau)) \qquad (\gamma,\tau) \in \Theta^T$$

the map $w : \Theta^T \cup \theta^T \to \mathbb{R}^\infty$,

$$w : (\gamma,\tau) \to \{w_j(\gamma,\tau)\}_{j=1}^\infty$$

coincides with composition $u \circ i^{-1}$ which implies

$$w(\Theta^T \cup \theta^T) = \Omega_u^T.$$

To determine the tensor g_u on Ω_u^T one can repeat the steps (i) and (ii) described above, using w instead of u:

(i) introduce the sgc $\gamma \circ w^{-1}, \tau \circ w^{-1}$ on $\Omega_u^T \setminus \omega_u^T$ and tensor $g_u = \mathfrak{g}(\gamma \circ w^{-1}, \tau \circ w^{-1})$;

(ii) using local wave coordinates $w_{j_1} \circ w^{-1}, \ldots, w_{j_n} \circ w^{-1}$ on $\omega_u^T = w(\theta^T)$, extend g_u on the cut locus.

Thus, (Ω_u^T, g_u) is constructed; the lemma is proved. □

Note that map w glues points of coast θ^T: $w(\gamma',\tau) = w(\gamma'',\tau)$ iff $i^{-1}((\gamma',\tau)) = i^{-1}((\gamma'',\tau)) \in \omega^T$ (compare with (iv), section 1.4).

6.5. *The recovering of* (Ω^T, g)

To complete a reconstruction we need just to join up the results obtained above.

Theorem 6.2. (i) *The response operator* R^{2T} *determines the manifold* (Ω^T, g).

(ii) *The spectral data* $\{\lambda_k;\psi_k(\cdot)\}_{k=1}^\infty$ *determine the manifold* (Ω, g).

Proof. Either kind of data determines a model (see sections 5.7 and 5.8); models determine the visualizing operator V^T. Knowing V^T one can recover pattern (Θ^T, g) (theorem 6.1), after which the wave copy (Ω_u^T, g_u) may be found (lemma 6.3). The latter is isometric to (Ω^T, g).

The spectral data permit us to find V^T for any $T > 0$ (see the end of section 5.8). Therefore, they determine (Ω^T, g), $T > 0$ and, thus, the manifold (Ω, g) in a whole. The theorem is proved. □

So, a reconstruction is implemented in accordance with the scheme: inverse data $\Rightarrow$ model $\Rightarrow$ operator $V^T \Rightarrow$ pattern $(\Theta^T, \mathfrak{g}) \Rightarrow$ wave copy $(\Omega_u^T, g_u) \leftrightarrow (\Omega^T, g)$.

6.6. Remarks

Besides the recovering of a wave copy, there exist other ways to extract geometry from wave images.

(i) In the case of spectral reconstruction, one can visualize the images $\{\tilde{\varphi}_k\}$ of eigenfunctions and, thereafter, use them as coordinates (instead of $\{\tilde{u}_j\}$) to construct a copy of (Ω^T, g) (see Belishev and Kurylev 1992).

(ii) Let $l_{\gamma,\alpha}$ be a geodesic starting into Ω from $\gamma \in \Gamma$ in the direction $\alpha \in S_+^{n-1}$. There exists a control f which generates the wave u^f (the so-called quasiphoton, a kind of Gaussian beam) with the following remarkable property: the wave u^f is localized near $l_{\gamma,\alpha}$. Its image $\tilde{u}^f$ traces on Σ^T a curve $\tilde{\ell}_{\gamma,\alpha} := i(\ell_{\gamma,\alpha} \setminus \omega)$ and, therefore, an external observer possessing the operator V^T is able to visualize this curve. Varying γ and α the observer can recover a family $\{\tilde{\ell}_{\gamma,\alpha}\}$ which is rich enough to recover the pattern $(\Theta^T, \mathfrak{g})$, to glue it along a coast and, eventually, to reconstruct (Ω^T, g) (see Belishev and Kachalov 1992).

Later, this technique was applied to the problem with incomplete data (Kachalov and Kurylev 1993). This paper generalizes one of the results of Novikov (1988) on the more complicated case of manifolds.

(iii) The following scheme of reconstruction, in a sense, is dual to the previous one. Fix $\gamma_0 \in \Gamma$; let $f = \delta_{\gamma_0}(\gamma)\theta(t)$ be a pointwise control, u_{γ_0} be the corresponding wave. In this case the hemisphere $S_\xi[\gamma_0] = \{x \in \Omega |\, \mathrm{dist}(x, \gamma_0) = \xi\}$ necessarily belongs to $\mathrm{supp}\, u_{\gamma_0}(\cdot, \xi)$ (see Bardos and Belishev 1995, Belishev 1994). Therefore, determining the image $\tilde{u}_{\gamma_0}(\cdot, \xi)$ one can visualize on Σ^T the surface $\tilde{S}_\xi[\gamma_0] := i(S_\xi[\gamma_0] \setminus \omega)$ as a boundary of $\mathrm{supp}\, \tilde{u}_{\gamma_0}(\cdot, \xi)$. The family $\{\tilde{S}_\xi[\gamma_0]\}$; $\{\gamma_0 \in \Gamma,\ \xi \in [0, T]$ turns out to be a sufficiently informative object to determine $(\Theta^T, \mathfrak{g})$ and, further, to get (Ω^T, g). Moreover, this scheme permits us to find $g|_\Gamma$ from inverse data; thus, a reconstruction may be fulfilled without setting a metric on Γ.

Spectral reconstruction was first realized in Belishev and Kurylev (1992). Note, that the scheme used in this paper is overloaded with unnecessary details. Dynamical reconstruction was given in Belishev and Kachalov (1992). Both papers are based upon the work of Belishev (1990b).

(iv) The smoothness of a manifold is required to work with classical solutions of (2.1)–(2.3), to justify the geometrical optics (2.13) and (2.15), and to use the Holmgren–John–Tataru theorem. All of these demands may be satisfied by the C^N-smoothness with large enough finite N (see Belishev and Kachalov 1994).

(v) Note in addition that the BC method gives some results for the Kac's problem of recovering the shape of a drum. A simple generalization of the scheme (Belishev 1988) leads to the following: for a wide class of manifolds, a Riemannian compact with a border is determined by its Beltrami–Laplace operator given in *any* representation. In other words, compact (Ω, g) is a unitary invariant of Δ_g.

6.7. On recovering metrics

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain equipped with metric $\mathrm{d}s^2 = g_{kl}(x)\,\mathrm{d}x^k\,\mathrm{d}x^l$ which turns the domain into a Riemannian manifold; let R^{2T} and $\{\lambda_k; \psi_k\}$ be the inverse data of (Ω, g). Can one recover $g_{kl}(\cdot)$ in Ω via inverse data?

As it is stated, the question has a negative answer. The well known fact (see, e.g., Sylvester and Uhlmann 1991) is that any diffeomorphism $\Phi : \Omega \to \Omega$, $\Phi|_\Gamma = \mathrm{Id}$ gives another metric $g' = \Phi_*[g]$ having the same inverse data, so that it is impossible to distinguish g' from g via boundary measurements.

One way to remove this kind of non-uniqueness was proposed by Lee (see Sylvester and Uhlmann 1991). Suppose that a family of metrics produced by the group $\{\Phi\}$ contains a *unique* metric g_{extr} which minimizes the energy (Dirichlet) functional. This selected metric is determined by inverse data uniquely. To obtain g_{extr} in the framework of the BC method one can recover a wave copy (Ω_u, g_u), and then equip it with *harmonic* coordinates $\pi^1, \ldots, \pi^n : \Delta_{g_u}\pi^k = 0$. The map $\pi : \Omega_u \to \mathbb{R}^n$, $\pi(x) = \{\pi^k(x)\}_{k=1}^n$ will determine g_{extr} in Ω.

Another way is to use the pecularities of a metric. As an example, consider the case of a strictly convex surface S in $\mathbb{R}^3$ with a border lying on a plane (hatlet). The Euclidean $\mathbb{R}^3$-metric induces on S an intrinsic metric g of positive curvature. The classical result of A V Pogorelov is that g determines S up to isometry in $\mathbb{R}^3$. Therefore, having a wave copy of (S, g), one can embed it into $\mathbb{R}^3$ uniquely and recover a hatlet. The same trick works for any rigid surface with *fixed* border. This situation has multidimensional analogues.

The reasons concerning the group $\{\Phi\}$ may be applied to the case of manifolds to recover not only Laplacian but a wider class of self-adjoint operators of Schrodinger type. This extension of the BC method has been developed by Kurylev (1992, 1994a, b).

6.8. Dynamical reconstruction of vector fields

Here we would like to announce one more result of the BC method which is planned for a future publication.

Let $b = b^k\partial/\partial x^k$ be a smooth vector field on Ω; consider the problem

$$u_{tt} - \Delta_g u - bu = 0 \qquad \text{in Int } Q^T \tag{6.9}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \tag{6.10}$$

$$u|_{\Sigma^T} = f. \tag{6.11}$$

Let u^f be its solution. As a dynamical system, problem (6.9)–(6.11) is described by the same spaces and operators as the system α^T. The peculiarity of this case is that the operator $\Delta_g + b$ governing an evolution is not self-adjoint.

Assume, in addition, that the manifold (Ω, g) possesses the 'non-trapping property': any geodesic starting from any point of Ω in an arbitrary direction reaches the border Γ in a time which does not exceed T_0. This property guarantees the equality

$$\mathcal{U}^T = \mathcal{H} \qquad T > T_0/2$$

i.e. the system (6.9)–(6.11) turns out to be *exactly controllable* for large enough time (Bardos *et al* 1992, Bardos and Belishev 1995). The last fact permits us to obtain the following result.

Theorem 6.3. *The response operator R^{2T}, $T > T_0$, determines (Ω, g) and b.*

Moreover, an efficient procedure using the amplitude integral permits us to reconstruct a manifold together with a vector field on it. The proof is based upon the results of Avdonin and Belishev (1996) and the present paper.

A bounded domain in $\mathbb{R}^n$ (with the Euclidean metric) is an important example of the non-trapping manifold. Thus, the BC method is able to recover arbitrary vector fields in $\Omega \subset \mathbb{R}^n$.

Let us note in conclusion that there exist some reasons to hope for an optimal result: the hypothesis is that R^{2T} determines (Ω^T, g) and $b|_{\Omega^T}$ for *any* $T > 0$.

7. The recovering of density

Originally the BC method was proposed to recover the density of an inhomogeneous membrane. From the geometrical point of view, this case corresponds to a conformally-flat metric, and it looks naturally that the recovering procedure is based upon the relations between Cartesian and semigeodesical coordinates.

7.1. Direct and inverse problems

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary $\Gamma \in C^\infty$; ρ be a function (*density*) satisfying $\rho > 0$; $\rho,\ \rho^{-1} \in C^\infty(\overline{\Omega})$. A density induces the metric

$$\mathrm{d}s^2 = \rho|\mathrm{d}x|^2 \tag{7.1}$$

turning Ω into a Riemannian manifold, so that all of the objects introduced in section 1 are defined. We denote them as before: τ, Ω^ξ, Θ^T etc, relating them to the metric (7.1) (not the Euclidean one!).

Consider the problem

$$\rho u_{tt} - \Delta u = 0 \qquad \text{in } \Omega \times (0, T) \tag{7.2}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \tag{7.3}$$

$$u|_{\Sigma^T} = f \tag{7.4}$$

with $\Delta = \sum_{k=1}^n \partial^2/\partial(x^k)^2$. The corresponding dynamical system α^T is described by the same spaces and operators as system (2.1)–(2.3), but since the density enters into the wave equation in a special manner, some of the definitions have to be slightly corrected:

(i) the outer space of the DS α^T is $\mathcal{F}^T = L_2(\Sigma^T; \mathrm{d}\Gamma \mathrm{d}t)$, $\mathrm{d}\Gamma$ being a Lebesgue surface measure on Γ;

(ii) the inner space is $\mathcal{H} = L_2(\Omega; \rho\, \mathrm{d}x)$,

$$(u, v)_{\mathcal{H}} = \int_\Omega \mathrm{d}x\ \rho(x) u(x) v(x)$$

the subspaces $\mathcal{H}^\xi = \{y \in \mathcal{H} |\ \mathrm{supp}\, y \subset \Omega^\xi\}$, $0 \leqslant \xi \leqslant T$ corresponding to subdomain Ω^ξ filled by waves;

(iii) operators of control and observation are defined as before: $W^T f = u^f(\cdot, T)$; $O^T y = \partial_\nu v^y|_{\Sigma^T}$ (with Euclidean outward normal ν); the relation $(W^T)^* = O^T$ holds;

(iv) the response operator is $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\mathrm{Dom}\, R^T = \{f \in H^1(\Sigma^T) | f|_{t=0} = 0\}$, $R^T f = \partial_\nu u^f|_{\Sigma^T}$. By virtue of the hyperbolicity of problem (7.2)–(7.4), the operator R^{2T} is determined by values $\rho|_{\Omega^T}$ being independent on the behaviour of the density in $\Omega \setminus \Omega^T$;

(v) a spectral representation (see section 2.5) is related to the operator $L : \mathcal{H} \to \mathcal{H}$, $\mathrm{Dom}\, L = H^2(\Omega) \cap H_0^1(\Omega)$, $Ly := -\rho^{-1}\Delta y$ which is self-adjoint in $\mathcal{H}$. To find the Fourier coefficients of a wave one can use (2.17) with Lebesgue's $\mathrm{d}\Gamma$.

The dynamical IP is to recover the density ρ in Ω^T via a given operator R^{2T}; *the spectral IP* is to recover ρ in Ω via given spectral data $\{\lambda_k; \psi_k\}_{k=1}^\infty$.

7.2. Amplitude formula

Here we transform the geometrical optics relation (2.15) into the form to be convenient for the use in the IPs.

As can be shown, in the special case of the wave equation (7.2) the relation (2.15) takes the form

$$\lim_{t\to T-\xi-0} \partial_\nu v^{y_\xi}(\gamma, t) = \begin{cases} \mu(\gamma,\xi)\beta^{1/2}(\gamma,\xi)y(x(\gamma,\xi)) & (\gamma,\xi)\in\sigma_+^\xi\times\{t=T-\xi\} \\ 0 & (\gamma,\xi)\in\sigma_-^\xi\times\{t=T-\xi\} \end{cases} \tag{7.5}$$

with an additional factor

$$\mu(\gamma,\xi) := \left(\frac{\rho(x(\gamma,\xi))}{\rho(x(\gamma,0))}\right)^{(2-n)/4}$$

($n = \dim\Omega$). Considering the limit as a function of (γ, ξ), recalling the definition of images (section 1.5) and taking into account (1.8), one can rewrite (7.5) as follows,

$$\lim_{t\to T-\xi-0} \partial_\nu v^{y_\xi}(\gamma, t) = \mu(\gamma,\xi)\tilde{y}(x(\gamma,\xi)) \qquad \text{a.e. on } \Sigma^T. \tag{7.6}$$

We have

$$\partial_\nu v^{y_\xi}|_{\Sigma^T} = (\text{see section 3.2}) = O^T y_\xi = (\text{see section 4.3}) = O^T(\mathbb{1}_{\mathcal{H}} - G^\xi)y$$
$$= (\text{see (4.4)}) = O^T(G^T - G^\xi)y = (\text{see (4.8)}) = O^T(P^T - P^\xi)y$$

which leads to the relation

$$\lim_{t\to T-\xi-0} (O^T(P^T - P^\xi)y) = \mu(\gamma,\xi)\tilde{y}(x(\gamma,\xi)) \qquad \text{a.e. on } \Sigma^T \tag{7.7}$$

which is said to be *the amplitude formula* (AF). It represents an image $\tilde{y}$ as a collection of amplitudes of wave discontinuities propagating in the dual system α_*^T.

A further transformation of the AF is connected with the wave projectors entering in (7.7). They will be represented through bases consisting of waves.

The amplitude formula was first introduced in Belishev (1990b).

7.3. Wave bases

A useful corollary of the controllability of the DS α^T is that there exist bases in $\mathcal{H}^\xi$ consisting of waves.

Fix $\xi \in (0, T]$, $T < T_*$; let $\{f_j^\xi\}_{j=1}^\infty \subset \mathcal{F}^{T,\xi}$ (see (3.3)) be a complete system of controls, i.e.

$$\text{clos}_{\mathcal{F}^T} \text{Lin}\{f_j^\xi\}_{j=1}^\infty = \mathcal{F}^{T,\xi}$$

(Lin is the linear span); the relation (4.7) obviously implies

$$\text{clos}_{\mathcal{H}} \text{Lin}\{W^T f_j^\xi\}_{j=1}^\infty = \mathcal{H}^\xi.$$

The property (3.10) provides the possibility of orthogonalizing the system $\{f_j^\xi\}$ by means of the Schmidt process

$$g_1 = f_1^\xi \qquad h_1^\xi = (C^T g_1, g_1)_{\mathcal{F}^T}^{-1/2} g_1$$
$$\vdots$$
$$g_k = f_k^\xi - \sum_{j=1}^{k-1} (C^T f_k, h_j^\xi)_{\mathcal{F}^T} h_j^\xi \qquad h_k^\xi = (C^T g_k, g_k)_{\mathcal{F}^T}^{-1/2} g_k$$
$$\vdots \tag{7.8}$$

The obtained system satisfies

$$\mathrm{clos}_{\mathcal{F}^T}\{h_j^\xi\}_{j=1}^\infty = \mathcal{F}^{T,\xi} \qquad (C^T h_i^\xi, h_j^\xi)_{\mathcal{F}^T} = \delta_{ij} \tag{7.9}$$

so therefore the system of waves $\{u_j^\xi\}_{j=1}^\infty$, $u_j^\xi := W^T h_j^\xi$ forms an orthonormalized (*wave*) basis in $\mathcal{H}^\xi$:

$$\mathrm{clos}_{\mathcal{H}}\,\mathrm{Lin}\{u_j^\xi\}_{j=1}^\infty = \mathcal{H}^\xi \qquad (u_i^\xi, u_j^\xi)_{\mathcal{H}} = (\text{see (3.9), (7.9)}) = \delta_{ij}. \tag{7.10}$$

The following has to be emphasized: to carry on the C^T-orthogonalization one needs the connecting operator *only*. Therefore, an outer observer who knows R^{2T} or $\{\lambda_k, \psi_k\}$ is able to construct system $\{h_j^\xi\}$ producing a wave basis with the help of representations (3.12) and (3.19).

A wave projector may be represented through a wave basis as follows:

$$P^\xi = \sum_{j=1}^\infty (\cdot, u_j^\xi)_{\mathcal{H}}\, u_j^\xi.$$

Suppose that system $\{h_j^\xi\}$ is constructed (via inverse data) for every $\xi \in (0, T]$; then one can represent functions $O^T P^\xi y$ entering the AF in the form

$$O^T P^\xi y = \sum_{j=1}^\infty (y, u_j^\xi)_{\mathcal{H}} O^T u_j^\xi = \sum_{j=1}^\infty (y, u_j^\xi)_{\mathcal{H}} O^T W^T h_j^\xi = \sum_{j=1}^\infty (y, u_j^\xi)_{\mathcal{H}}\, C^T h_j^\xi \qquad 0 < \xi \leqslant T. \tag{7.11}$$

7.4. Images of harmonic functions

In the case of harmonic functions the coefficients (inner products) entering (7.11) can also be expressed in terms of inverse data.

Denote $\kappa^T(t) := T - t$.

Lemma 7.1. *(i) If $a \in C^1(\overline{\Omega}^T) \cap C^2(\Omega^T)$ satisfies $\Delta a = 0$ in Ω^T, the equality*

$$(a, u^f(\cdot, T))_{\mathcal{H}^T} = ((R^T)^*(\kappa^T a|_\Gamma) - \kappa^T \partial_\nu a|_\Gamma, f)_{\mathcal{F}^T} \tag{7.12}$$

is valid for any $f \in \mathcal{F}^T$.

(ii) If $a \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $\Delta a = 0$ in Ω, the equality

$$(a, u^f(\cdot, T))_{\mathcal{H}} = -\sum_{k=1}^\infty \lambda_k^{-1} (a|_\Gamma, \psi_k)_{L_2(\Gamma)} (f, s_k^T)_{\mathcal{F}^T} \tag{7.13}$$

is valid for any $f \in \mathcal{F}^T$.

Proof. (i) For $f \in \mathcal{M}^T$ one has the relations

$$\begin{aligned}(a, u^f(\cdot, T))_{\mathcal{H}^T} &= \int_\Omega \mathrm{d}x\, \rho(x) a(x) u^f(x, T) = \int_\Omega \mathrm{d}x\, \rho(x) a(x) \int_0^T \mathrm{d}t\, (T - t) u^f_{tt}(x, t) \\ &= \int_0^T \mathrm{d}t\, (T - t) \int_\Omega \mathrm{d}x\, a(x) \Delta u^f(x, t) \\ &= \int_0^T \mathrm{d}t\, (T - t) \int_\Gamma \mathrm{d}\Gamma\, [a(\gamma)\partial_\nu u^f(\gamma, t) - (\partial_\nu a)(\gamma) u^f(\gamma, t)] \\ &= \int_{\Sigma^T} \mathrm{d}\Gamma\, \mathrm{d}t \{[\kappa^T(t) a(\gamma)](R^T f)(\gamma, t) - [\kappa^T(t)\partial_\nu a(\gamma)] f(\gamma, t)\}.\end{aligned}$$

One can justify the inclusion $\kappa^T a|_\Gamma \in \mathrm{Dom}(R^T)^*$, which permits one to transform the last integral into the right-hand side of (7.12). Then the equality is extended from $\mathcal{M}^T$ on $\mathcal{F}^T$ by continuity.

(ii) Let $a = \sum_{k=1}^{\infty} a_k \varphi_k$ be the Fourier expansion with respect to the eigenbasis of operator L. Integrating by parts one can easily find the coefficients

$$a_k = -\lambda_k^{-1} \int_\Gamma \mathrm{d}\Gamma\, a(\gamma) \partial_\nu \varphi_k(\gamma). \tag{7.14}$$

Calculating the inner product through the Fourier coefficients (see (2.17) and (7.14)) one obtains (7.13). The lemma is proved. □

Blagovestchenskii was the first to discover the possibility of expressing products $(a, u^f)_{\mathcal{H}}$ and $(u^f, u^g)_{\mathcal{H}}$ via dynamical inverse data (see Belishev 1987a, Belishev and Blagovestchenskii 1992).

Let $a \in C^1(\overline{\Omega})$ be harmonic in Ω; the result of lemma 7.1 together with the amplitude formula give the possibility of recovering the function $\mu\tilde{a}$ on Σ^T through the inverse data. To do so the following procedure may be used.

Step 1. Find the operator C^T by means of (3.12) or (3.19).

Step 2. For every $\xi \in (0, T]$ construct a system $\{h_j^\xi\} \subset \mathcal{F}^{T,\xi}$ satisfying (7.9).

Step 3. Determine the function

$$O^T(P^T - P^\xi)a = \text{(see (7.11))} = \sum_{j=1}^{\infty} [(a, u_j^T)_{\mathcal{H}}\, C^T h_j^T - (a, u_j^\xi)_{\mathcal{H}}\, C^T h_j^\xi]$$

calculating the inner products by means of (7.12) or (7.13).

Step 4. Find $\mu\tilde{a}|_{\Sigma^T}$ with the help of AF (7.7).

Let $1\,(\cdot) = 1$ be a unit function in Ω; $\pi^1, \ldots, \pi^n$ are the Cartesian coordinate functions: $\pi^k(x) = x^k$, $x = \{x^1, \ldots, x^n\}$. All of them are harmonic; therefore, applying the procedure described above one can recover the functions

$$\mu\tilde{1}, \mu\tilde{\pi}^1, \ldots, \mu\tilde{\pi}^n \qquad \text{on } \Sigma^T \tag{7.15}$$

via R^{2T} or $\{\lambda_k; \psi_k\}$.

7.5. *The solving of the IPs*

To recover the density it remains for us to show that functions (7.15) determine ρ in Ω^T.

First, let us note that the function $\mu\tilde{1}$ determines a pattern on Σ^T:

$$\overline{\Theta}^T = \operatorname{supp} \mu\tilde{1}. \tag{7.16}$$

Furthermore, one can find functions

$$\frac{\mu\tilde{\pi}^1}{\mu\tilde{1}}, \ldots, \frac{\mu\tilde{\pi}^n}{\mu\tilde{1}} \qquad \text{on } \Theta^T \tag{7.17}$$

which determine a connection between sgc and Cartesian coordinates. Indeed, recalling the definition of images (section 1.5) one obtains

$$\frac{\mu\tilde{\pi}^k}{\mu\tilde{1}} = \frac{\mu(\gamma,\xi)\beta^{1/2}(\gamma,\xi)\pi^k(x(\gamma,\xi))}{\mu(\gamma,\xi)\beta^{1/2}(\gamma,\xi)1} = \pi^k(x(\gamma,\xi)).$$

So therefore, for given (γ, ξ), one can find point $x(\gamma, \xi)$ as follows:

$$x(\gamma,\xi) = \{\pi^1(x(\gamma,\xi)), \ldots, \pi^n(x(\gamma,\xi))\} = \left\{\frac{\mu\tilde{\pi}^1}{\mu\tilde{1}}(\gamma,\xi), \ldots, \frac{\mu\tilde{\pi}^n}{\mu\tilde{1}}(\gamma,\xi)\right\} \in \mathbb{R}^n. \tag{7.18}$$

In other words, functions (7.17) determine the map $i^{-1} : \Theta^T \to \Omega^T \setminus \omega^T$ transferring (γ, ξ) in $x(\gamma, \xi)$ (see section 1.4). Applying i^{-1} to a horizontal cross section of a pattern we can recover an equidistant surface in a domain:

$$i^{-1} : \{(\gamma, \tau) \in \Theta^T | \tau = \xi\} \to \Gamma^\xi \setminus \omega^T$$

(see (ii), section 1.4).

A family of surfaces $\Gamma^\xi \setminus \omega^T \subset \Omega$, $0 < \xi < T$ determines an eikonal in $\Omega^T \setminus \omega^T$ by the rule

$$\tau|_{\Gamma^\xi} = \xi \qquad 0 < \xi < T. \tag{7.19}$$

An eikonal determines the density,

$$\|\nabla_x \tau\|^2_{\mathbb{R}^n} = \rho \qquad \text{in } \Omega^T \setminus \omega^T \tag{7.20}$$

i.e. almost everywhere in Ω^T. By virtue of continuity of ρ, it may be taken as recovered in the whole of Ω^T.

The aforesaid may be summarized in the form of a procedure completing steps 1–4, section 7.4.

Step 5. Find functions $\mu\tilde{1}, \mu\tilde{\pi}^1, \ldots, \mu\tilde{\pi}^n$ on Σ^T; recover a pattern via (7.16).

Step 6. Using the correspondence $(\gamma, \xi) \to x(\gamma, \xi)$ (see (7.18)), recover surfaces $\Gamma^\xi \setminus \omega^T$, $0 < \xi < T$ in Ω and find an eikonal in $\Omega^T \setminus \omega^T$ (see (7.19)).

Step 7. Recover the density in Ω^T by means of (7.20).

In the case of spectral data the density may be recovered in Ω^T for any $T > 0$, i.e. in Ω.

The inverse problems are solved. Concerning the possibility of using the procedure for real calculations, the following should be noted. To obtain system $\{h_j^\xi\}$ by means of process (7.8) is, roughly speaking, the same as finding the operator $(C^T)^{-1}$. To invert C^T is similar to solving the BCP which is ill-posed (see (i), section 4.1). These reasons may be considered as an explanation given by the BC method for the fact of the ill-posedness of multidimensional IPs.

7.6. The recovering from part of a boundary

The BC method works in the case of inverse data given on part of a boundary. Here we describe briefly one variant of the recovering procedure proposed in Belishev (1987b).

(i) *Partial data*. Let σ be an open subset on Γ and $\mathcal{F}^T_\sigma$ be a subspace of controls acting from σ,

$$\mathcal{F}^T_\sigma := \{f \in \mathcal{F}^T | \operatorname{supp} f \subset \sigma \times [0, T]\}.$$

The operator $R^T_\sigma : \mathcal{F}^T_\sigma \to \mathcal{F}^T_\sigma$, $\operatorname{Dom} R^T = \mathcal{F}^T_\sigma \cap \operatorname{Dom} R^T$,

$$R^T_\sigma f := (R^T f)|_{\sigma \times [0,T]}$$

is said to be a partial response operator.

The set of pairs $\{\lambda_k; \psi_k|_\sigma\}_{k=1}^\infty$ is said to be the partial spectral data.

It is easy to see that either kind of partial data determines the partial connecting operator $C^T_\sigma : \mathcal{F}^T_\sigma \to \mathcal{F}^T_\sigma$, $C^T_\sigma f = C^T f|_{\sigma \times [0,T]}$ as previously:

$$\begin{aligned} C^T_\sigma &= (\text{see (3.12)}) = \frac{1}{2}(S^T)^* R^{2T}_\sigma J^{2T} S^T = (\text{see (3.19)}) \\ &= \sum_{k=1}^\infty (\cdot, s_k^T|_{\sigma \times [0,T]})_{\mathcal{F}^T} s_k^T|_{\sigma \times [0,T]}. \end{aligned} \tag{7.21}$$

The waves moving from σ fill the domains

$$\Omega_\sigma^\xi := \{x \in \Omega | \operatorname{dist}(x, \sigma) < \xi\} \qquad 0 < \xi \leqslant T$$

(distance in ρ-metric!). The dynamical IP is to recover ρ in Ω_σ^T via a given operator R_σ^{2T}. The spectral IP is to recover ρ in Ω via given $\{\lambda_k; \psi_k|_\sigma\}_{k=1}^\infty$.

(ii) *Controllability*. Let

$$\mathcal{F}_\sigma^{T,\xi} = \{f \in \mathcal{F}^T | \operatorname{supp} f \subset \sigma \times [T-\xi, T]\} = \mathcal{F}_\sigma^T \cap \mathcal{F}^{T,\xi} \qquad 0 \leqslant \xi \leqslant T$$

be a subspace of delayed controls; the corresponding reachable sets

$$\mathcal{U}_\sigma^\xi = W^T \mathcal{F}_\sigma^{T,\xi}$$

lie in the subspaces

$$\mathcal{H}_\sigma^\xi := \{y \in \mathcal{H} | \operatorname{supp} y \subset \Omega_\sigma^\xi\}$$

by virtue of hyperbolicity. Using the Holmgren–John–Tataru theorem one can generalize relation (4.7) as follows: for any $0 < \xi \leqslant T$ the equality

$$\operatorname{clos}_{\mathcal{H}} \mathcal{U}_\sigma^\xi = \mathcal{H}_\sigma^\xi$$

is valid.

As a corollary we obtain the possibility of constructing a wave basis in $\mathcal{H}_\sigma^\xi$. Let $\{f_j^\xi\}_{j=1}^\infty \subset \mathcal{F}_\sigma^{T,\xi}$ be a complete system of controls and $\{h_j^\xi\}_{j=1}^\infty$ be a system obtained from the first by means of C_σ^T-orthogonalization (see (7.8)); $u_j^\xi = W^T h_j^\xi$ are the corresponding waves. In accordance with (7.21) the system $\{u_j^\xi\}_{j=1}^\infty$ forms an orthonormalized basis in $\mathcal{H}_\sigma^\xi$.

As follows from (7.21), the system $\{h_j^\xi\}$ is determined by either kind of partial inverse data.

(iii) *Mark function*. Fix $m \in \Omega$ and introduce the function

$$\mathcal{E}_m(x) := \begin{cases} \dfrac{\partial}{\partial x^1} \ln \|x - m\|_{\mathbb{R}^2} - e_m(x) & \text{if } n = 2 \\ \dfrac{\partial}{\partial x^1} \|x - m\|_{\mathbb{R}^3}^{-1} - e_m(x) & \text{if } n = 3 \\ \|x - m\|_{\mathbb{R}^n}^{-n+2} - e_m(x) & \text{if } n > 3 \end{cases}$$

where e_m is harmonic in Ω and is chosen so that $\mathcal{E}_m$ satisfies

$$\Delta \mathcal{E}_m = 0 \qquad \text{in } \Omega \setminus \{m\} \tag{7.22}$$

$$\mathcal{E}_m|_\Gamma = 0. \tag{7.23}$$

We call $\mathcal{E}_m$ a mark function; one can easily check that it is not square integrable,

$$\|\mathcal{E}_m\|_{\mathcal{H}}^2 = \infty. \tag{7.24}$$

(iv) *The recovering of eikonal*. Fix $\xi < T$ and point $m \in \Omega_\sigma^T \setminus \overline{\Omega_\sigma^\xi}$, so that the mark function is harmonic in Ω_σ^ξ. In this case it may be expanded over a wave basis

$$\mathcal{E}_m = \sum_{j=1}^\infty (\mathcal{E}_m, u_j^\xi)_{\mathcal{H}^\xi} u_j^\xi$$

with the coefficients

$$(\mathcal{E}_m, u_j^\xi)_{\mathcal{H}^\xi} = (\text{see (7.12), (7.23)}) = -(\kappa^T \partial_\nu \mathcal{E}_m|_\Gamma, h_j^\xi)_{\mathcal{F}^T}$$

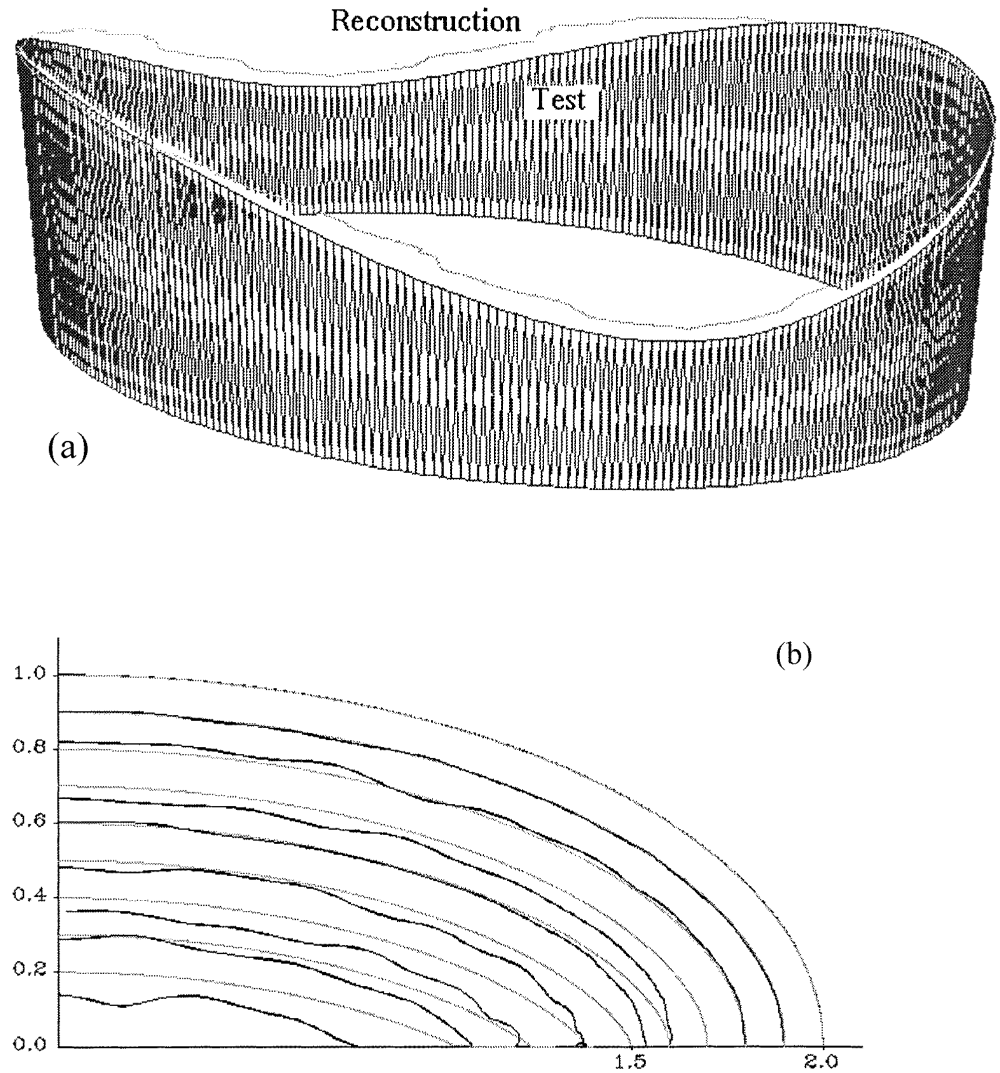

Figure 5. Spectral reconstruction in ellipse.

which implies

$$\|\mathcal{E}_m\|^2_{\mathcal{H}^\xi} = \sum_{j=1}^{\infty} (\kappa^T \partial_\nu \mathcal{E}_m|_\Gamma, h_j^\xi)^2_{\mathcal{F}^T}. \tag{7.25}$$

An important feature of this representation is that the right-hand side is determined by partial inverse data.

Let us increase ξ from zero; the value $\xi = \tau(m)$ corresponds to the moment when Ω^ξ_σ touches point m. As can be shown, one character of the touching is that the norm (7.25) tends to infinity (in accordance with (7.24)). Therefore, one can find

$$\tau(m) = \sup\{\xi > 0 \,|\, \|\mathcal{E}_m\|^2_{\mathcal{H}^\xi} < \infty\}$$

which gives a way of detecting an inclusion $m \in \Omega^T_\sigma$ and finding $\tau(m)$ via R^{2T}_σ or $\{\lambda_k; \psi_k|_\sigma\}$.

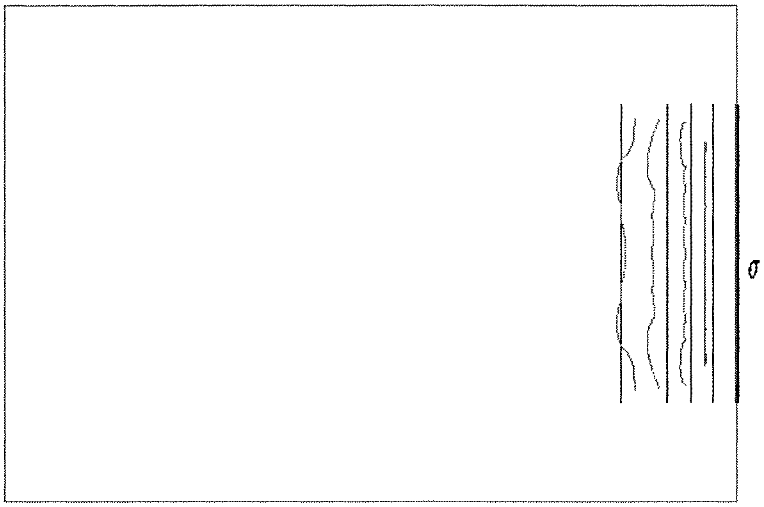

Figure 6. The recovering from part of a boundary.

(v) *The recovering of the density*. An eikonal determines the density through the Jacobi relation

$$\|\nabla_x \tau\|^2_{\mathbb{R}^n} = \rho \qquad \text{a.e. in } \Omega^T_\sigma.$$

Thus, operator R^{2T} determines $\rho|_{\Omega^T_\sigma}$; the data $\{\lambda_k; \psi_k|_\sigma\}_{k=1}^\infty$ determine $\rho|_{\Omega^T_\sigma}$ for any $T > 0$, i.e. everywhere in Ω.

The same procedure, in principle, permits one to detect and recover the unknown components of a boundary including the inner obstacles. This useful observation belongs to Ya Kurylev (private communication).

Other variants of the BC method (see Belishev 1987a, b, 1990a, b, Belishev and Kurylev 1987, 1989), including the AF, may also be adapted for partial data. They may all be generalized to the case of a Riemannian manifold.

7.7. Numerical testing

The algorithms based upon the one-dimensional variant of the BC method were tested by Belishev and Kachalov (1989), Belishev and Sheronova (1990) and, later, by He (1995).

The numerical testing in two-dimensional spectral IP was first realized by Filippov (Belishev *et al* 1994). This work was continued by Gotlib and Ivanov (Belishev *et al* 1997), who recovered a pattern of an ellipse and a family of wavefronts Γ^ξ in it via spectral data (see figure 5).

Recently Ivanov and Shirota have demonstrated that the amplitude formula permits one to reconstruct a picture of forward wavefronts Γ^ξ_σ moving from *a part* σ of a boundary. A reconstruction was implemented via spectral data in a subdomain of a rectangle covered by normal geodesics starting from σ (see figure 6). The computations were performed on an IBM PC.

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References

Avdonin S and Belishev M 1996 Boundary control and dynamical inverse problem for nonselfadjoint Sturm-Liouville operator *Contr. Cyber.* **25** 429–40

Avdonin S, Belishev M and Ivanov S 1994 The controllability in a filled domain for multidimensional wave equation with a singular boundary control *Zap. Nauchn. Semin. POMI* **210** 7–21 (Engl. transl. *J. Math. Sci.* to appear)

Avdonin S and Ivanov S 1995 *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems* (New York: Cambridge University Press)

Babich V M and Buldyrev V S 1991 *Short Wave Length Diffraction Theory (Asymptotic Methods)* (Heidelberg: Springer)

Bardos C and Belishev M 1995 The wave shaping problem. PDE and functional analysis (in memory of P Grisvard) *Progr. Nonlinear Diff. Equat. Appl.* **22** 41–59

Bardos C, Lebeau G and Rauch J 1992 Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary *SIAM J. Contr. Opt.* **30** 1024–65

Belishev M I 1987a An approach to multidimensional inverse problems for the wave equation *Dokl. Akad. Nauk SSSR* **297** 524–27 (Engl. transl. 1988 *Soviet Math. Dokl.* **36** 481–4)

—— 1987b The Gel'fand–Levitan type equations in multidimensional inverse problem for the wave equation *Zap. Nauchn. Semin. LOMI* **165** 15–20 (Engl. transl. 1990 *J. Sov. Math.* **50**)

—— 1988 On M Kac's problem of the recovering of a shape of a domain through a spectrum of the Dirichlet problem *Zap. Nauchn. Semin. LOMI* **173** 30–41 (Engl. transl. 1991 *J. Sov. Math.* **55**)

—— 1990a Wave bases in multidimensional inverse problems *Matem. Sb.* **180** 584–602 (Engl. transl. 1990 *Math. USSR Sb.* **67** 23–42)

—— 1990b Boundary control and wave fields continuation *LOMI Preprint* No P-1-90, pp 1–41 (in Russian)

—— 1994 On a justification of the Huygense Rule *Zap. Nauchn. Semin. POMI* **218** 17–24 (Engl. transl. *J. Math. Sci.* to appear)

—— 1995 The conservative model of a dissipative dynamical system *Zap. Nauchn. Semin. POMI* **230** 21–35 (Engl. transl. *J. Math. Sci.* to appear)

—— 1996a Canonical model of a dynamical system with boundary control in the inverse problem of heat conductivity *St Petersburg Math. J.* **7** 869–90

—— 1996b Boundary control and inverse problems: one-dimensional variant of the BC-method *POMI Preprint* No 10/1996, pp 1–33

Belishev M I and Blagovestchenskii A S 1992 Multidimensional analogs of the Gel'fand–Levitan–Krein equations in inverse problem for the wave equation *Ill-Posed Problems of Mathematical Physics and Analysis* (Novosibirsk: Nauka) pp 50–63 (in Russian)

Belishev M I, Blagovestchenskii A S and Ivanov S A 1997 The two-velocity dynamical system: boundary control of waves and inverse problems *Wave Motion* **25** 83–107

Belishev M I and Dolgoborodov A N 1997 Local boundary controllability in classes of smooth functions for the wave equation *POMI Preprint* No 1/1997, pp 1–9 (in Russian)

Belishev M I, Gotlib Yu V and Ivanov S A 1997 The BC-method in multidimensional spectral inverse problem: theory and numerical illustration. Control, optimization and calculus of variations, to appear

Belishev M I and Ivanov S A 1995 Boundary control and canonical realizations of two-velocity dynamical system *Zap. Nauchn. Semin. POMI* **222** 18–44 (Engl. transl. 1995 *J. Math. Sci.* to appear)

Belishev M I and Kachalov A P 1989 The boundary control methods in the spectral inverse problem for an inhomogeneous string *Zap. Nauchn. Semin. LOMI* **179** 14–22 (Engl. transl. 1991 *J. Sov. Math.* **57**)

—— 1992 Boundary control and quasiphotons in the problem of reconstruction of a Riemannian manifold via dynamical data *Zap. Nauchn. Semin. POMI* **203** 21–51 (Engl. transl. 1996 *J. Math. Sci.* **79**)

Belishev M I and Kachalov A P 1994 The operator integral in multidimensional spectral inverse problem *Zap. Nauchn. Semin. POMI* **215** 9–37 (Engl. transl. 1994 *J. Math. Sci.* to appear)

Belishev M I and Kurylev Ya V 1986 The inverse problem of the acoustical scattering in a space with a local inhomogeneousity *Zap. Nauchn. Semin. LOMI* **156** 24–34 (Engl. transl. 1990 *J. Sov. Math.* **50**)

—— 1987 Dynamical inverse problem for the multidimensional wave equation in the large *Zap. Nauchn. Semin. LOMI* **165** 21–30 (Engl. transl. 1990 *J. Sov. Math.* **50**)

—— 1989 The spectral inverse problem of the scattering of the plane waves on a halfspace with a local inhomogeneousity *Z. Vych. Mat. Mat. Fiz.* **29** 1045–56

—— 1991 Boundary control, wave field continuation and inverse problems for the wave equation *Comput. Math. Appl.* **22** 27–52

—— 1992 To the reconstruction of a Riemannian manifold via its spectral data (BC-method) *Comm. PDE* **17** 767–804

Belishev M I and Pushnitski A B 1996 On a triangular factorization of positive operators *Zap. Nauchn. Semin. POMI* **239** 45–60 (Engl. transl. *J. Math. Sci.* to appear)

Belishev M I, Ryzhov V A and Filippov V B 1994 Spectral variant of the BC method: theory and numerical testing *Dokl. RAN* **332** 414–17 (Engl. transl. 1994 *Phys. Dokl.* **39** 466–70)

Belishev M I and Sheronova T L 1990 The boundary control methods in the dynamical inverse problem for an inhomogeneous string *Zap. Nauchn. Semin. LOMI* **186** 37–50 (Engl. transl. 1995 *J. Math. Sci.* **73**)

Blagovestchenskii A S 1971 On a local approach to the solving of dynamical inverse problem for inhomogeneous string *Trudy MIAN V A Steklova* **115** 28–38

Faddeev L D 1974 The inverse scattering problem in the quantum mechanics—II. *Sovremennye Prob. Mat.* **3** 93–180

Gokhberg I Ts and Krein M G 1970 *Theory and Applications of Volterra Operators in Hilbert Space (transl. of Monographs 24)* (Providence, RI: American Mathematical Society)

Gopinath B and Sondhi M M 1971 Inversion of the telegraph equation and the synthesis of nonuniform lines *Proc. IEEE* **3** 383–92

Gromol D, Klingenberg W and Meyer W 1968 *Riemannsche Geometrie im Grossen* (Berlin: Springer)

Hartman P 1964 Geodesic parallel coordinates in the large *Am. J. Math.* **86** 705–27

He S 1995 An explicit time-domain solution for the reflection from a stratified acoustic half-space obtained by the boundary control method *Thesis* Department of Electromagnetic Theory, Royal Institute of Technology, Stockholm, TRITA-TET 95-5, December 1

Hörmander L 1992 A uniqueness theorem for second order hyperbolic differential equation *Comm. PDE* **17** 699–714

John F 1948 On linear partial differential equations with analytic coefficients. Unique continuation of data *Comm. Pure Appl. Math.* **2** 209–53

Kachalov A P and Kurylev Ya V 1993 Incomplete spectral data and reconstruction of a Riemannian manifold *J. Inv. Ill-Posed Prob.* **1** 141–53

Kalman R, Falb P and Arbib M 1969 *Topics in Mathematical System Theory* (New York: McGraw-Hill)

Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)

Kurylev Ya V 1992 On admissible groups of transformations preserving boundary spectral data in multidimensional inverse problems *Dokl. RAN* **327** 322–5

—— 1994a On the recovering of two coefficients in multidimensional isotropic inverse problem *Zap. Nauchn. Semin. POMI* **210** 158–63

—— 1994b Inverse boundary problems on Riemannian Manifolds *Contemp. Math.* **173** 181–92

Lasiecka I, Lions J-L and Triggiani R 1986 Non homogeneous boundary value problems for second order hyperbolic operators *J. Math. Pures Appl.* **65** 149–92

Lions J-L 1968 *Contrôle Optimale de Systèmes Gouvernés par des Équations aux Dérivées Partielles* (Paris: Dunod-Gauthier-Villars)

—— 1988 Exact controllability, stabilization and perturbations for distributed systems *SIAM Rev.* **30** 1–68

Nizhnik L P 1991 *Inverse Scattering Problems for the Hyperbolic Equations* (Kiev: Naukova Dumka) (in Russian)

Novikov R G 1988 A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$ *Funk. Anal. i Prilozhen.* **22** 11–22 (Engl. transl. 1988 *Func. Anal. Appl.* **22** 263–72)

Robbiano L 1991 Theoreme d'unicite adapte an controle des solutions des problemes hyperboliques *Comm. PDE* **16** 789–800

Russell D L 1971 Boundary value control theory of the higher-dimensional wave equation *SIAM J. Control* **9** 29–42

—— 1978 Controllability and stabilizability theory for linear partial differential equations *SIAM Rev.* **20** 639–739

Sondhi M M and Gopinath B 1971 Determination of vocal-tract shape from impulse response at the lips *J. Acoust. Soc. Am.* **49** 1867–73

Sylvester J and Uhlmann G 1991 Inverse problems in anisotropic media *Contemp. Math.* **122** 105–17

Tataru D 1993 Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem *Thesis* Department of Mathematics, Northwestern University

—— 1995 Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem *Comm. PDE* **20** 855–84

Wainberg B R 1982 *Asymptotic Methods in Equations of Mathematical Physics* (Moskow: Nauka) (in Russian)

Yamamoto M 1995 Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method *Inverse Problems* **11** 481–96