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BOUNDARY CONTROL OF A ONE-DIMENSIONAL, LINEAR, THERMOELASTIC ROD

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Abstract. We examine boundary control of a linear partial differential equation which describes the temperature distribution and displacement within a one-dimensional thermoelastic rod. In particular, we show that temperature or heat flux control at an endpoint is sufficient to obtain exact null-controllability. This improves earlier results for similar systems in which only partial null-controllability is obtained. We also obtain sharp regularity results for the controlled system.

Key words. linear thermoelasticity, moment problem, boundary control, regularity

AMS(MOS) subject classifications. 93B05, 80A20, 70J99

1. Introduction.

Though there is extensive literature on the topic of control and stabilization of elastic systems, relatively little has been published which includes the thermoelastic coupling. This is probably due, in part, to the relatively small affect thermoelastic damping has upon most systems of interest. However, for certain applications such as stabilization of satellite antennas, where large temperature variations are common (e.g., due to moving in and out of shadows), the need to model this coupling becomes critical. Furthermore, the recent work of Gibson et al [5], illustrates the importance of modelling even light thermoelastic damping in the design of finite-dimensional compensators.

Some notable literature on stabilization of thermoelastic systems include [13], [14], [16], [17], [18] and references therein. Very little, however, is known about the controllability structure of thermoelastic systems. In Lagnese and Lions [14], boundary control (e.g., velocity or position control on the boundary) is used to exactly control the mechanical portion of the state space. This type of controllability is called *partial exact controllability.* When this type of control is used, the thermal component of the state is ignored, and consequently if the mechanical portion is driven to rest, it will not in general remain there due to the thermal stresses which remain. The main purpose of this paper is to show that, at least for the case of a one-dimensional thermoelastic rod, exact controllability (to zero) of both mechanical and thermal components of the state space is possible by only controlling the thermal (or mechanical) component on the boundary.

A derivation of the equations of one-dimensional nonlinear thermoelasticity can be found in e.g., [21]. In the case of a homogeneous rod with uniform cross sections (see [2], [6] for the precise assumptions), the linearization of these equations can be written

$$\begin{aligned} &\frac{\partial\theta}{\partial t}(t,x)=\frac{\partial^2\theta}{\partial x^2}(t,x)-\frac{\gamma\partial^2 w}{\partial x\partial t}(t,x)\\ &\frac{\partial^2 w}{\partial t^2}(t,x)=c^2\frac{\partial^2 w}{\partial x^2}(t,x)-c^2\gamma\frac{\partial\theta}{\partial x}(t,x), \end{aligned} \tag{1.1}$$

which holds on $(t,x)\in(0,\infty)\times\Omega$ $(\Omega=(0,1))$. Here θ represents a relative temperature about the stress-free reference state $\theta=0$, and w is proportional to the displacement. The constants $\gamma>0$ and $c>0$ represent, respectively, the amount of thermal-mechanical coupling, and the small-amplitude wave speed about a constant temperature state. (See [6] for a precise definition of γ and c.) In most materials of interest, γ is several orders of magnitude smaller than 1.

The physical quantities relevant to the formulation of boundary conditions for (1.1) are the velocity v, heat flux q, stress σ and temperature θ, where the first three of these are

$$\begin{aligned} v(t,x)&=\frac{\partial w}{\partial t}(t,x)\\ q(t,x)&=-\frac{\partial\theta}{\partial x}(t,x)\\ \sigma(t,x)&=\frac{\partial w}{\partial x}(t,x)-\gamma\theta(t,x). \end{aligned}$$

In [6] it was shown that under any of the boundary conditions

$$v(t,i)=0,\qquad q(t,i)=0\qquad i=0,1; \tag{1.2}$$

$$\sigma(t,i)=0,\qquad \theta(t,i)=0\qquad i=0,1; \tag{1.3}$$

$$\sigma(t,0)=\theta(t,0)=v(t,1)=q(t,1)=0, \tag{1.4}$$

the eigenfunctions associated with (1.1) form a Riesz basis for the space of finite energy states and the corresponding eigenvalues are uniformly shifted into the left half-plane, except for possibly one or two eigenvalues located at the origin. This result is partially restated in Theorem 2.1 and will be our starting point in our examination of associated control problems.

In the case of boundary conditions (1.4), there are no eigenvalues at the origin. For this reason it will be notationally convenient to restrict our presentation to control of boundary conditions of the type (1.4), though similar results apply for control of boundary conditions of type (1.2) or (1.3).

Let $y(t) = (y_1(t), y_2(t), y_3(t))' = (w_x(t,\cdot), w_t(t,\cdot), \theta(t,\cdot))'$ represent the state of the system (1.1) at time t and let $(v_y(x), q_y(x), \sigma_y(x), \theta_y(x))$ represent the velocity, heat flux, etc., in terms of the state y. We will mainly be concerned with the following boundary control problem associated with (1.1):

$$\frac{dy}{dt} = \tau y \equiv \begin{bmatrix} 0 & D & 0 \\ c^2 D & 0 & -\gamma c^2 D \\ 0 & -\gamma D & D^2 \end{bmatrix} y \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.5}$$

$$y(0) = y^0 \quad \text{in } \Omega, \tag{1.6}$$

$$\begin{cases} \sigma_{y(t)}(0) = 0 & \theta_{y(t)}(0) = g(t) \qquad t \geq 0 \\ v_{y(t)}(1) = 0 & q_{y(t)}(1) = f(t) \qquad t \geq 0. \end{cases} \tag{1.7}$$

where $D = d/dx$, $\gamma > 0$, $c > 0$. Thus at the left end, the temperature is controlled and the stress vanishes. At the right end the heat flux is controlled while the position is fixed.

We need to define some function spaces to describe our main results. Let $\mathcal{H} = \left(L^2(\Omega)\right)^3$ with the energy inner product

$$\langle y, z \rangle = \int_0^1 y_1 \bar{z}_1 + \frac{1}{c^2} y_2 \bar{z}_2 + y_3 \bar{z}_3 \; dx,$$

and let

$$\mathcal{D}(A) = \{ y \in H^1[0,1] \times H^1[0,1] \times H^2[0,1] \mid \\ \sigma_y(0) = \theta_y(0) = v_y(1) = q_y(1) = 0 \}.$$

Now define $A : \mathcal{D}(A) \to \mathcal{H}$ by

$$Ay = \tau y \quad \forall y \in \mathcal{D}(A). \tag{1.8}$$

We denote $l^2 = \{(c_k)_{k\in\mathbb{I}} \mid \sum_{k\in\mathbb{I}} |c_k|^2 < \infty\}$, where $\mathbb{I}$ is a countable index set (usually either the integers $\mathbb{Z}$ or positive integers $\mathbb{N}$). For $\alpha \in \mathbb{R}$ define

$$S_\alpha = \left\{ \sum_{k=1}^{\infty} a_k \sin\left(k\pi - \frac{\pi}{2}\right) x \mid (a_k k^\alpha) \in l^2 \right\}, \tag{1.9}$$

$$C_\alpha = \left\{ \sum_{k=1}^{\infty} a_k \cos\left(k\pi - \frac{\pi}{2}\right) x \mid (a_k k^\alpha) \in l^2 \right\}. \tag{1.10}$$

S_α and C_α become Hilbert spaces with e.g., $\|y\|_{S_\alpha} = \|(a_k k^\alpha)\|_{l^2}$. (When $\alpha < 0$, S_α and C_α are the dual spaces to $S_{-\alpha}$ and $C_{-\alpha}$, respectively.) $C((a,b), M)$ will denote the set of functions which are continuous on the interval (a,b) with values in the space M.

Our main results are the following, together with related results given in Sections 3-5.

THEOREM 1.1. *Let $y^0 = 0$, $f \in L^2(0,\infty)$, and $g \in L^2(0,\infty)$. Then the solution to (1.5)-(1.7) belongs to $C([0,\infty), S_0 \times C_0 \times S_{-1/2})$. If additionally, $g \equiv 0$, then the solution belongs to $C([0,\infty), S_1 \times C_1 \times S_{1/2})$. These solution spaces are optimal in the sense that none of the indices $\{0, 1, 1/2, -1/2\}$ may be increased.*

THEOREM 1.2. *Assume $0 < \gamma \leq 1$ in (1.5) and $T > 2/c$.*

(i) For the boundary control problem (1.5)-(1.7), with $f \equiv 0$, given any $y^0 \in \mathcal{H}$, there exists $g \in L^2[0,T]$ such that $y \in C([0,T], S_0 \times C_0 \times S_{-1/2})$ and $y(T) = 0$.

(ii) For the boundary control problem (1.5)-(1.7), with $g \equiv 0$, given any $y^0 \in \mathcal{D}(A)$ there exists $f \in L^2[0,T]$ such that $y \in C([0,T], S_1 \times C_1 \times S_{1/2})$ and $y(T) = 0$.

In either case, T can not in general be reduced to $2/c$.

Remark 1.3. The following identifications hold.

$$S_0 = C_0 = L^2(\Omega),$$
$$S_1 = \{f \in H^1(\Omega) \mid f(0) = 0\},$$
$$C_1 = \{f \in H^1(\Omega) \mid f(1) = 0\},$$
$$S_{1/2} = [S_1, S_0]_{1/2} = \{f \in H^{1/2}(\Omega) \mid x^{-1/2} f(x) \in L^2(\Omega)\},$$
$$S_{-1/2} = S'_{1/2},$$
$$\mathcal{H} = S_0 \times C_0 \times S_0,$$
$$\mathcal{D}(A) = S_1 \times C_1 \times S_2,$$

with equivalent norms; see [15] and Section 2. (In the above, H^α denotes the usual Sobolev space of order α, $[F, G]_{1/2}$ denotes the usual interpolation space between spaces F and G, as defined in [15], and $'$ denotes duality with respect to $L^2(\Omega)$.)

In the above theorems, solutions are uniquely defined by continuous extension of the variation of constants formula; see Section 3 for details.

Proposition 5.1 gives a more general statement of Theorem 1.2 and Remark 5.2 shows that the spaces used in Theorem 1.2 are optimal in a certain sense.

The proof of Theorem 1.1 is given in Section 3 and involves an application of the Carleson measure criterion of Ho, Russell [10], and Weiss [22], which gives a sharp criterion for well-posedness of control systems. The proof of Theorem 1.2 involves reducing the control problem to a pair of coupled moment problems which are coupled through the control. A general class of such coupled moment problems are examined in Section 4 where it is shown that there are projections which decouple such moment problems into simpler ones for which known results are applicable. This leads to various controllability results, including Theorem 1.2, which are given in Section 5.

Results similar to Theorems 1.1 and 1.2 follow in the same way for boundary control systems based on the boundary conditions (1.2) or (1.3). Likewise one could also consider the case where the stress and/or velocity at an end is controlled, and similar results would follow. We mention some of these results in Section 5.

A short appendix is included which contains the proof of several technical details which will be used throughout the rest of the paper.

2. Preliminaries.

Throughout this paper, an *isomorphism* will be understood to denote a bounded, invertible operator from one Hilbert space onto another. If X is a separable Hilbert space, a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in X forms a *Riesz basis* for X if $\varphi_k = Be_k$ $(k \in \mathbb{N})$ where (e_k) is an orthonormal basis for X and B is an isomorphism. The following theorem and its corollary was proved in Hansen [6].

THEOREM 2.1. *Let A be defined by* (1.8). *The spectrum of A (and also of A^*) consists of isolated eigenvalues $(\lambda_{kj})_{k\in\mathbb{N},j\in\{1,2,3\}}$ with $\lambda_{kj} = (k\pi - \pi/2)s_{kj}$, where*

$$(s_{kj}^2 + c^2)(s_{kj} + k\pi - \pi/2) + \gamma^2 c^2 s_{kj} = 0. \tag{2.1}$$

The eigenfunctions of A (and also A^*), properly normalized, form a Riesz basis for $\mathcal{H}$.

An analysis of (2.1) in [6] shows that (λ_{kj}) can be decomposed into a real branch $(\mu_k)_{k\in\mathbb{N}}$ and a non-real branch $(\sigma_k)_{k\in\mathbb{Z}}$ with

$$\begin{cases} \mu_k = -\left(k\pi - \frac{\pi}{2}\right)^2 + O(1), & k \in \mathbb{N} \\ \sigma_k = -\frac{\gamma^2}{2} + ic\left(k\pi - \frac{\pi}{2}\right) + O(k^{-1}), & k \in \mathbb{Z}. \end{cases} \tag{2.2}$$

We let $(\psi_\lambda)_{\lambda\in\sigma(A)}$ denote the normalized eigenvectors of A^* and $(\varphi_\lambda)_{\lambda\in\sigma(A)}$ denote the biorthonormalized eigenvectors of A (each eigenvalue is counted up to its multiplicity), so that $\langle\varphi_{\lambda_k}, \psi_{\lambda_j}\rangle = \delta_{kj}$, where δ_{kj} is the Kronecker delta. There are at most a finite number of eigenvalues of multiplicity greater than one and all eigenvalues are simple if $\gamma \le 1$ (see Lemma A.1) or if $|k|$ is sufficiently large (see [6]). The form of the eigenvectors of A^* are given in the appendix.

COROLLARY 2.2. *A is the generator of a strongly continuous contraction semigroup $(\mathbb{T}_t)_{t\ge0}$ on $\mathcal{H}$ for which there exist $M > 1$ and $\beta > 0$ such that*

$$\|\mathbb{T}_t\| \le Me^{-\beta t} \qquad \forall t \ge 0.$$

Theorem 2.1 and Corollary 2.2 also hold in the case of boundary conditions (1.2) or (1.3), although the energy decay occurs the orthogonal complement of the null-space of the generator [6]. In addition, several recent papers [1], [12], [17] have shown exponential stability to hold for other sets of natural boundary conditions. These exponential stability results are important in that one can infer the existence of optimizing feedbacks for stabilization problems with quadratic cost criteria; see [5].

For any set $S \subset \mathbb{C}$ we can define an associated spectral projection $P(S) \in \mathcal{L}(\mathcal{H})$ by

$$(P(S))\,x = \frac{1}{2\pi i} \int\limits_{\Gamma} R(\lambda; A) x \, d\lambda \qquad \forall x \in \mathcal{H},$$

where $R(\lambda, A)$ is the resolvent operator of A and Γ is an appropriate contour which encloses the eigenvalues in S. There is no difficulty in defining Γ since the spectrum is discrete. In cases where Γ contains infinitely many eigenvalues, convergence for all $x \in \mathcal{H}$ is guaranteed by Theorem 2.1. Let us denote

$$P = P(\mathbb{R}), \quad \text{and} \quad Q = I - P(\mathbb{R}),$$

where I denotes the identity operator on $\mathcal{H}$. Let

$$\Lambda = P\mathcal{H}, \quad \text{and} \quad \Sigma = Q\mathcal{H}.$$

Since the projections are continuous, it follows that $\mathcal{H} = \Lambda \oplus \Sigma$.

PROPOSITION 2.3. *Let* $\mathbb{T}$ *denote the semigroup defined in* Corollary 2.2. *Then for* $t \geq 0$,

$$\mathbb{T}_t = \mathbb{S}_t P + \mathbb{G}_t Q, \tag{2.3}$$

where $\mathbb{G}$ *extends to a strongly continuous group* $(\mathbb{G}_t)_{t \in \mathbb{R}}$ *and* $\mathbb{S}$ *extends to an analytic semigroup* $(\mathbb{S}_t)_{\mathrm{Re}\ t>0}$. *The infinitesimal generators of* $\mathbb{S}$ *and* $\mathbb{G}$ *are given by the restrictions of* A, $A|_\Lambda$ *and* $A|_\Sigma$, *respectively.*

Proof. The spaces Λ and Σ are closed, $\mathbb{T}$-invariant spaces and hence the restriction of $\mathbb{T}$ to either of these spaces is a C_0 semigroup with respect to the inherited topology. For

$t \geq 0$ let $\mathbb{S}_t = \mathbb{T}_t|_\Lambda$ and $\mathbb{G}_t = \mathbb{T}_t|_\Sigma$. It follows that for $t \geq 0$, $\mathbb{T}_t = \mathbb{T}_t(P+Q) = \mathbb{S}_t P + \mathbb{G}_t Q$ hence (2.3) is valid. For any $x \in \Lambda \cap \mathcal{D}(A)$

$$Ax = \lim_{t \downarrow 0} \frac{\mathbb{T}_t x - x}{t} = \lim_{t \downarrow 0} \frac{\mathbb{S}_t x - x}{t}.$$

Thus $\mathbb{S}$ is generated by (the densely defined operator) $A|_\Lambda$ and likewise $\mathbb{G}$ is generated by $A|_\Sigma$.

It remains to show that $\mathbb{G}$ extends to a group (by $\mathbb{G}_{-t} = \mathbb{G}_t^{-1}$) and $\mathbb{S}$ has an analytic extension to $\mathrm{Re}\, t > 0$. Define $F : \mathcal{H} \to l^2$ by

$$\sum_{\lambda_k \in \sigma(A)} c_{\lambda_k} \phi_{\lambda_k} \to (c_{\lambda_k}).$$

Since (ϕ_{λ_k}) forms a Riesz basis for $\mathcal{H}$, F is an isomorphism. Define $\tilde{\mathbb{T}} = (\tilde{\mathbb{T}}_t)_{t \geq 0}$ by $\tilde{\mathbb{T}}_t = F\mathbb{T}_t F^{-1}$, and for $x \in \mathcal{H}$ define $\tilde{x}$ by $\tilde{x} = Fx$. Through this mapping, the pair $(\mathbb{T}, \mathcal{H})$ is isomorphic to $(\tilde{\mathbb{T}}, l^2)$ in the sense that $F\mathbb{T}_t x = \tilde{\mathbb{T}}_t \tilde{x}$ for any $t \geq 0$ and any $x \in \mathcal{H}$. Since any Riesz basis becomes an orthonormal basis under some equivalent inner product (see [25]), it follows that the induced topology $\|\tilde{x}\|_i = \|x\|$ is equivalent to the topology generated by the standard l^2 inner product. Thus, to show that $\mathbb{S}$ extends to an analytic semigroup $(\mathbb{S}_t)_{\mathrm{Re}\ t>0}$, it will suffice to show that $\tilde{\mathbb{T}}|_{F\Lambda}$ extends analytically to $\mathrm{Re}\ t > 0$ with respect to the standard l^2 topology. It is easily seen that $\tilde{\mathbb{T}}|_{F\Lambda}$ is a diagonal semigroup on l^2 $(= F\Lambda)$ with (diagonal) generator $(FAF^{-1})|_{F\Lambda}$. If $\mathcal{A}$ is any diagonal generator, it is easy to show that $\|R(\lambda, \mathcal{A})\|$ is inversely proportional to the distance λ is from $\sigma(\mathcal{A})$. Since $\sigma(A|_\Lambda) = \sigma((FAF^{-1})|_{F\Lambda})$ is entirely on the negative real axis, the appropriate resolvent bound ([19, p.62]) holds which shows that $\tilde{\mathbb{T}}|_{F\Lambda}$, and hence also $\mathbb{S}$, extends to an analytic semigroup in $\mathrm{Re}\ t > 0$. Likewise, since $\sigma(FAF^{-1}|_{F\Sigma})$ is contained in a vertical strip of $\mathbb{C}$, it follows from e.g., Pazy [19, p.23] that $\tilde{\mathbb{T}}|_{F\Sigma}$ extends to a group. Hence $\mathbb{G}$ also extends to a group. $\square$

It will be useful to introduce a notation for certain interpolation spaces. Since $0 \in \rho(A)$ (the resolvent set of A) and $\sigma(-A)$ is in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\ \lambda > 0\}$, for any $\alpha \in \mathbb{R}$, $(-A)^\alpha$ may be defined as in e.g., Pazy [19, p. 69]. For $\alpha > 0$, $(-A)^\alpha$ is an isomorphism from $\mathcal{D}((-A)^\alpha)$ (with graph-norm topology) to $\mathcal{H}$. For $\alpha \geq 0$, we define $\mathcal{H}_\alpha$ to be the restriction of $\mathcal{H}$ to

$\mathcal{D}((-A)^\alpha)$ with

$$\|x\|_\alpha = \|(-A)^\alpha x\|. \tag{2.4}$$

For $\alpha < 0$, we let $\mathcal{H}_\alpha$ denote the completion of $\mathcal{H}$ with respect to the norm also given by (2.4). The above spaces are explained in more detail in e.g., [8], [23]. For our problem we have that $\mathcal{D}(A) = \mathcal{D}(A^*)$. Thus it follows that $\mathcal{H}_1^* = \mathcal{H}_{-1}$ (where the duality pairing is with respect to the completion of $\langle\cdot,\cdot\rangle$). Furthermore, since the eigenfunctions of A form a Riesz basis, it can be shown that for $\alpha \in [0,1]$, $\mathcal{H}_\alpha = [\mathcal{H}_1, \mathcal{H}_0]_{1-\alpha}$, where $[\mathcal{H}_1, \mathcal{H}_0]_{1-\alpha}$ is the interpolation space defined in [15]. Using standard properties of interpolation spaces one can show $\mathcal{H}_\alpha^* = \mathcal{H}_{-\alpha}\ \forall\, \alpha \in \mathbb{R}$.

We recall a result from Weiss [23], as it applies to our problem:

Proposition 2.4. *For any $\alpha < 0$, A has a unique continuous extension to an operator on $\mathcal{H}_\alpha$ also denoted by A, which is an isomorphism from $\mathcal{H}_{\alpha+1}$ to $\mathcal{H}_\alpha$. Further if L commutes with A, i.e., if*

$$LAx = ALx \qquad \forall x \in \mathcal{H}_1 = \mathcal{D}(A),$$

then the restriction of L to $\mathcal{H}_\alpha$ ($\alpha > 0$) belongs to $\mathcal{L}(\mathcal{H}_\alpha)$. Further, L has a unique continuous extension to an operator in $\mathcal{L}(\mathcal{H}_\alpha)$ for any $\alpha < 0$.

In particular, the projections P and Q, and semigroups $\mathbb{T}$, $\mathbb{S}$ and $\mathbb{G}$ each have unique continuous extensions to $\mathcal{H}_\alpha$ (for any $\alpha < 0$). Throughout this paper we will make no notational distinction between an operator and its possible extensions. We thus define the spaces Λ_α and Σ_α by

$$\Lambda_\alpha = P\mathcal{H}_\alpha \quad \text{and} \quad \Sigma_\alpha = Q\mathcal{H}_\alpha \qquad \forall \alpha \in \mathbb{R}.$$

As a consequence of Proposition 2.4,

$$\mathcal{H}_\alpha = \Lambda_\alpha \oplus \Sigma_\alpha \qquad \forall\, \alpha \in \mathbb{R}. \tag{2.5}$$

The spaces Λ_α and Σ_α become Hilbert spaces with the norms $\|\cdot\|_{\Lambda_\alpha}$ and $\|\cdot\|_{\Sigma_\alpha}$ inherited from (2.4).

3. Regularity.

In this section we obtain via the Carleson measure criterion of Ho and Russell [10] and Weiss [22], the spaces of maximal regularity of the system (1.5)-(1.7). We begin with a discussion of the Carleson measure criterion.

Consider the control system

$$\dot{x} = \mathcal{A}x + bu(t), \tag{3.1}$$

where $x(t) \in l^2$ is the state, $u \in L^2[0,\infty)$ is the control function, $\mathcal{A}$ is assumed to be diagonal with diagonal elements ν_k which satisfy

$$\sup_{k \in \mathbb{N}} \text{Re } \nu_k = \omega_0 < 0, \tag{3.2}$$

and $b \in l^2_{-1}$, i.e., is a column vector with components b_k which satisfy

$$\sum_{k=1}^{\infty} \left| \frac{b_k}{\nu_k} \right|^2 < \infty.$$

Thus $\mathcal{A}$ generates a strongly continuous diagonal semigroup $(T_t)_{t \geq 0}$ on l^2.

For any $h > 0$ and any $\omega \in \mathbb{R}$ let

$$R(h,\omega) = \{z \in \mathbb{C} \,|\, 0 \leq \text{Re } z \leq h, \ |\text{Im } z - \omega| \leq h\}.$$

Definition 3.1. With $\mathcal{A}$, b and T as above, b satisfies the *Carleson measure criterion* for the semigroup T if there is some $M \geq 0$ such that for any $h > 0$ and any $\omega \in \mathbb{R}$,

$$\sum_{-\nu_k \in R(h,\omega)} |b_k|^2 \leq M \cdot h. \tag{3.3}$$

The Carleson measure criterion is used to determine the *admissibility* of the *input element* b in (3.1). The input element b is *admissible* for T if for some $t > 0$, the sequence $\left(\int_0^t e^{\nu_k(t-s)} b_k v(s) ds\right)_{k \in \mathbb{N}}$ lies in l^2 for all $v \in L^2[0,\infty)$. When b is admissible, for any $\tau > 0$ the operator $\Phi_\tau \colon L^2[0,\infty) \to l^2_{-1}$ defined by

$$\Phi_\tau u = \int_0^\tau T_{\tau - s} bu(s) ds \qquad \forall\, u \in L^2[0,\infty) \tag{3.4}$$

maps continuously into l^2. In this case, for any initial condition $x_0 \in l^2$ and any $u \in L^2(0,\infty)$ a unique solution of (3.1) is given by

$$x(t) = T_t x_0 + \Phi_t u, \tag{3.5}$$

with $x \in C\left([0,\infty), l^2\right)$. If b is not admissible then there exists $u \in L^2[0,\infty)$ for which the solution of (3.1) (if it can be defined at all) is not continuous in time.

Remark 3.2. It should be pointed out that the stability restriction (3.2) is unessential; we have defined the the Carleson measure criterion as it applies to stable systems. See [10] for the general definition.

Remark 3.3. In Definition 3.1, it is not necessary to verify (3.3) for every possible value of (h,ω). It is enough to consider the pairs (h_n,ω_n) for which $\nu_n = h_n + i\omega_n$. (This follows by a simple geometrical argument.)

THEOREM 3.4. (Ho and Russell, Weiss) *With b, $\mathcal{A}$ and T as above, b is admissible for T if and only if b satisfies the Carleson measure criterion for T.*

The above asserts that the control system (3.1) is well-posed on l^2 (in the above discussed sense) if and only if the sequence (b_k) satisfies (3.3).

For $\alpha \in \mathbb{R}$, we denote $l^2_\alpha = \{(c_k) \,|\, (|\nu_k|^\alpha c_k) \in l^2\}$.

Definition 3.5. Let $\alpha \in \mathbb{R}$. With b, $\mathcal{A}$ and T as above, the pair (b,T) is *well-posed on* l^2_α if for some $\tau > 0$, the operator Φ_τ defined in (3.4) maps continuously into l^2_α.

If (b,T) is well-posed on l^2_α, then we may define solutions of (3.1) by (3.5), and these solutions are continuous in time with values in l^2_α. We have the following.

COROLLARY 3.6. *Let $\alpha \in \mathbb{R}$. The pair (b,T) in (3.1) is well-posed on l^2_α if and only if $(b_k|\nu_k|^\alpha)_{k\in\mathbb{N}}$ satisfies the Carleson measure criterion for T.*

Proof. Let $\tau > 0$ and $u \in L^2(0,\infty)$. Let $(\zeta_k) = \Phi_\tau u$ as given by (3.4). If $(b_k|\nu_k|^\alpha)_{k\in\mathbb{N}}$ satisfies the Carleson measure criterion then $(\zeta_k|\nu_k|^\alpha) \in l^2$, or equivalently, $(\zeta_k) \in l^2_\alpha$. $\square$

We now return to the control problem (1.5)-(1.7). In order to apply Theorem 3.4 to our system, the input elements associated with (1.5)-(1.7) need to be identified.

Let $G : \mathbb{R}^2 \to \mathcal{H}$ denote the *Green's map* associated with (1.5)-(1.7):

$$G(u_1, u_2)' = w; \qquad \tau w = 0 \quad \text{in } \Omega,$$
$$\sigma_w(0) = 0, \quad v_w(1) = 0, \quad \theta_w(0) = u_1, \quad q_w(1) = u_2.$$

One finds that $G(u_1, u_2)' = (\gamma(-u_2 x + u_1), 0, -u_2 x + u_1)'$. If $y^0 = 0$ and $f, g \in C_0^\infty(0, \infty)$ then the (classical) solution y to (1.5)-(1.7) at time t coincides with an element of $\mathcal{H}_{-1}$ $(= \mathcal{D}(A^*)^*)$, also denoted by $y(t)$, which is given by (e.g., [3],[24])

$$y(t) = -\int_0^t A\mathbb{T}_{t-\tau} G(g, f)'(\tau)\, d\tau. \tag{3.6}$$

Since the appropriate extensions of A and $\mathbb{T}$ commute on $\mathcal{H}$,

$$\begin{aligned} y(t) &= \int_0^t \mathbb{T}_{t-s}(-AG)(g, f)'(s)\, ds \\ &\equiv \int_0^t \mathbb{T}_{t-s} B(g, f)'(s)\, ds. \end{aligned}$$

The boundary control operator B maps $\mathbb{R}^2$ into $\mathcal{H}_{-1}$ continuously and hence is a sum of two continuous functionals on $\mathcal{H}_1$. From integration by parts,

$$\begin{aligned} < B(u_1, u_2)', w > &= < -G(u_1, u_2)', A^* w > \\ &= -u_1 \bar{q}_w(0) - u_2 \bar{\theta}_w(1), \quad \forall\, w \in \mathcal{H}_1. \end{aligned}$$

Thus we define b_0, b_1 as elements of $\mathcal{H}_{-1}$ by

$$\begin{cases} \langle b_0, \bar{w} \rangle = -q_w(0) & \forall\, w \in \mathcal{H}_1, \\ \langle b_1, \bar{w} \rangle = -\theta_w(1) & \forall\, w \in \mathcal{H}_1, \end{cases} \tag{3.7}$$

so that (3.6) becomes

$$y(t) = \int_0^t \mathbb{T}_{t-s}(b_0 g(s) + b_1 f(s))\, ds \quad \text{on } \mathcal{H}_{-1}. \tag{3.8}$$

The map $(g, f)' \to y$ as given by (3.6) (or (3.8)) is bounded when considered as a map

$$(L^2(0, T))^2 \to C([0, T], \mathcal{H}_{-1}), \quad (T > 0)$$

and thus defines a generalized solution for $(g,f) \in (L^2(0,T))^2$. It follows that

$$\dot{y} = Ay + b_0 g(t) + b_1 f(t), \qquad y(0) = y^0 \in \mathcal{H} \tag{3.9}$$

has a unique continuous solution in $\mathcal{H}_{-1}$ (given by (3.8) if $y^0 = 0$) which satisfies (3.9) on $\mathcal{H}_{-2}$.

By Propositions 2.3 and 2.4 the projections P and Q continuously decompose the solutions in (3.8) by $y(t) = x(t) + z(t)$ where

$$x(t) = \int_0^t \mathbb{S}_{t-s}(Pb_0 g(s) + Pb_1 f(s))\, ds \quad \text{on } \Lambda_{-1}, \tag{3.10}$$

$$z(t) = \int_0^t \mathbb{G}_{t-s}(Qb_0 g(s) + Qb_1 f(s))\, ds \quad \text{on } \Sigma_{-1}. \tag{3.11}$$

Note that all of the results in this section which pertain to diagonal systems apply to the system (3.9) since (as in the proof of Proposition 2.3) A, $\mathbb{T}$, $\mathbb{G}$, $\mathbb{S}$, etc., can be viewed as diagonal operators on l^2 relative to the Riesz basis of eigenfunctions. Likewise an input element b may be identified with a vector in l^2_{-1} whose components are its respective Fourier coefficients. As such, one can then use the Carleson measure criterion to check well-posedness of the pairs $(b, \mathbb{T})$ on $\mathcal{H}_\alpha$.

An analysis of the admissibility of the input elements Pb_0, Pb_1, Qb_0 and Qb_1 will provide the smoothest spaces Λ_α and Σ_β in which $x(t)$ and $z(t)$ are time-continuous for all L^2 controls. This then determines the maximal regularity of the solutions $y(t)$ to the system (1.5)-(1.7). We have the following.

PROPOSITION 3.7. *In the above notation,*

(i) $(Pb_0, \mathbb{S})$ *is well-posed on* Λ_α $\forall \alpha \leq -1/4$

(ii) $(Pb_1, \mathbb{S})$ *is well-posed on* Λ_α $\forall \alpha \leq 1/4$

(iii) $(Qb_0, \mathbb{G})$ *is well-posed on* Σ_α $\forall \alpha \leq 0$

(iv) $(Qb_1, \mathbb{G})$ *is well-posed on* Σ_α $\forall \alpha \leq 1$.

Furthermore the bounds given for α *are sharp.*

Proof. We first prove (i). By (3.7), $b_0 \in \mathcal{H}_{-1}$, and hence $Pb_0 \in \Lambda_{-1}$. Therefore its series

$$\begin{aligned} Pb_0 &= \sum_{k\in\mathbb{N}} \langle Pb_0, \psi_{\mu_k}\rangle \varphi_{\mu_k} \\ &= \sum_{k\in\mathbb{N}} \langle b_0, \psi_{\mu_k}\rangle \varphi_{\mu_k} \equiv \sum_{k\in\mathbb{N}} c_k \varphi_{\mu_k} \end{aligned}$$

converges in Λ_{-1}. The coefficients (c_k) are easily computed from (3.7) and (A.5) (of the appendix). One finds that there exist positive constants m and M such that

$$mk < |c_k| < Mk \qquad \forall\, k \in \mathbb{N}. \tag{3.12}$$

For $k \in \mathbb{N}$ let $b_k = c_k/|\mu_k|^{1/4}$. The semigroup $\mathbb{S}$ can be identified with the diagonal semigroup $\tilde{\mathbb{S}} \equiv \mathrm{diag}(e^{\mu_1 t}, e^{\mu_2 t}, \dots)$ relative to the Riesz basis of eigenfunctions. Thus by Corollary 3.6, (i) holds if the sequence (b_k) satisfies the Carleson measure criterion for $\tilde{\mathbb{S}}$. Since the eigenvalues (μ_k) grow quadratically (see (2.2)), (3.12) implies that there are constants $m_1 > 0$ and $M_1 > 0$ for which

$$m_1 k < |b_k|^2 < M_1 k, \qquad \forall\, k \in \mathbb{N}.$$

It follows there are positive numbers m_2, m_3, M_2, M_3 for which

$$m_3|\mu_n| < m_2 n^2 < \sum_{k=1}^{n} |b_k|^2 < M_2 n^2 < M_3|\mu_n|, \qquad \forall\, n \in \mathbb{N}.$$

Thus if $N \in \mathbb{N}$ and $h = |\mu_N|$ we have

$$m_3 h \le \sum_{-\mu_k \in R(h,0)} |b_k|^2 \le M_3 h. \tag{3.13}$$

Thus (3.3) holds by Remark 3.3 and hence (i) holds. The first inequality in (3.13) shows that $\alpha = -1/4$ can not be increased.

The proof of (ii) is essentially the same. For (iii) and (iv), the eigenvalues lie in a vertical strip and their imaginary parts possess a uniform asymptotic separation. From this it is easy to show that (3.3) holds if and only if the sequence (b_k) in (3.3) is bounded.

The estimates (A.5) of the appendix show that Qb_0 corresponds to sequence which is bounded, and bounded away from zero. Hence (iv) follows and $\alpha = 0$ is optimal. The same estimates (A.5) also show that AQb_1 corresponds to a sequence which is bounded and bounded away from zero. Hence $(AQb_1, \mathbb{G})$ is well-posed on Σ_0. From this it follows that $(Qb_1, \mathbb{G})$ is well-posed on Σ_1 and $\alpha = 1$ is optimal. □

The next two lemmas relate the spaces Λ_α and Σ_α to the spaces S_α and C_α.

LEMMA 3.8. *Let S_α and C_α be defined by (1.9), (1.10) and assume $-1 \le \alpha \le 1$. Then*

$$\mathcal{H}_\alpha = \Lambda_\alpha \oplus \Sigma_\alpha = S_\alpha \times C_\alpha \times S_{2\alpha} \tag{3.14}$$

with equivalent norms. Furthermore,

$$\Lambda_\alpha \subset S_{2+2\alpha} \times C_{1+2\alpha} \times S_{2\alpha} \tag{3.15}$$

$$\Sigma_\alpha \subset S_\alpha \times C_\alpha \times S_{1+\alpha}. \tag{3.16}$$

Finally, the mapping $P_{12} : \Sigma_\alpha \to S_\alpha \times C_\alpha$ given by $P_{12}x = (x_1, x_2)$ and the mapping $P_3 : \Lambda_\alpha \to S_{2\alpha}$ defined by $P_3 x = x_3$ are isomorphisms.

The proof relies upon asymptotic properties of the eigenvectors and is given in the appendix.

LEMMA 3.9. *Let $|\alpha| \le 1$, $|\beta| \le 1$ and $|\beta - 2\alpha| < 1$. The following set-equivalences hold:*

$$\Lambda_\alpha + \Sigma_\beta = S_\beta \times C_\beta \times S_{2\alpha}. \tag{3.17}$$

In particular,

$$\Lambda_{-1/4} + \Sigma_0 = S_0 \times C_0 \times S_{-1/2}, \tag{3.18}$$

$$\Lambda_{1/4} + \Sigma_1 = S_1 \times C_1 \times S_{1/2}. \tag{3.19}$$

Proof. It will suffice to prove (3.18). The proof of the general case is done in the same way. If $y = x + z$ with $x \in \Lambda_{-1/4}$ and $z \in \Sigma_0$ then by Lemma 3.8 $y \in S_0 \times C_0 \times S_{-1/2}$.

Thus $\Lambda_{-1/4} + \Sigma_0 \subset S_0 \times C_0 \times S_{-1/2}$. Now let $y \equiv (y_1, y_2, y_3)' \in S_0 \times C_0 \times S_{-1/2}$. Let P_3 denote the operator defined in Lemma 3.8 and assume $\tilde{y} \equiv (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)' = P_3^{-1} y_3$. Then $P_3 \tilde{y} = \tilde{y}_3 = y_3$ and by (3.15) $\tilde{y}_1 \in S_{3/2}$ and $\tilde{y}_2 \in C_{1/2}$. Let $x = (y_1 - \tilde{y}_1, y_2 - \tilde{y}_2, 0)'$. Then $y = x + \tilde{y}$ and $x \in S_0 \times C_0 \times S_0 = \mathcal{H}_0 = \Lambda_0 + \Sigma_0$. But also $\tilde{y} \in \Lambda_{-1/4}$ hence $y \in \Lambda_{-1/4} + \Sigma_0$. This proves (3.18). $\square$

Proof of Theorem 1.1. We first examine the case where $f \equiv 0$ in (3.10), (3.11). Let $t \geq 0$. By Proposition 3.7 $x(t) \in \Lambda_{-1/4}$, $z(t) \in \Sigma_0$, and the indices $-1/4$ and 0 are optimal in that they may not be increased. Thus by Lemma 3.9 $y(t) = x(t) + z(t) \in S_0 \times C_0 \times S_{-1/2}$. Furthermore, (3.17) implies that the index $-1/2$ is optimal, and *not both* of the first two indices (0 and 0) may be increased. A sufficient condition that both the first two indices cannot be increased beyond 0 is that the operator $\mathcal{G}_T : L^2(0, \infty) \to \Sigma_0$ given by

$$\mathcal{G}_T u = \int_0^T \mathbb{G}_{T-s} Q b_0 u(s)\, ds$$

map onto a subspace of finite codimension for sufficiently large T. (Indeed, if this is so and P_{12} represents the projection operator defined in Lemma 3.8, then by Lemma 3.8, $P_{12}\mathcal{G}_T$ can not map into any of the spaces $S_\alpha \times C_\beta$ with $\alpha > 0$ or $\beta > 0$.) In the case $\gamma \leq 1$ it is shown in the proof of Proposition 5.1 that $\mathcal{G}_T$ is surjective (for large enough T). If $\gamma > 1$, the possibility of multiple eigenvalues arises, however the same proof shows that $\mathcal{G}_T$ maps onto a subspace of finite codimension. Hence the trajectories $y(t)$ are time-continuous (see Section 3) with values in $S_0 \times C_0 \times S_{-1/2}$ and (pending the proof of Proposition 5.1) each of the indices are optimal in that none may be increased.

For the case with $g \equiv 0$ in (3.10), (3.11) we have $x(t) \in \Lambda_{1/4}$ and $z(t) \in \Sigma_1$. Thus by (3.19) $y(t) \in S_1 \times C_1 \times S_{1/2}$, and as in the previous case, the indices can be shown to be optimal. $\square$

4. A moment problem of mixed parabolic-hyperbolic type.

As we will see in Section 5, the problem of controlling (3.9) from an initial state to a terminal state is equivalent to solving an associated moment problem of the form:

$$c_k = \int_0^T e^{\mu_k s} u(s) ds \qquad k \in \mathbb{N} \tag{4.1}$$

$$d_k = \int_0^T e^{\sigma_k s} u(s) ds \qquad k \in \mathbb{Z}. \tag{4.2}$$

The space of all sequences $(c_k) \cup (d_k)$ for which there exists some $u \in L^2[0,T]$ such that (4.1), (4.2) holds is called the *moment space* of (4.1)-(4.2). While the individual moment spaces of (4.1) and (4.2) are rather well understood, one can not directly use these results to infer properties of the (joint) moment space of (4.1)-(4.2). The main purpose of this section is to show that the moment space of (4.1)-(4.2) is the union of of the individual moment spaces for (4.1) and (4.2) provided T is greater than some nominal value t_c which depends upon the sequence (σ_k).

Because the results of this section pertain to a variety of sequences (σ_k), (μ_k) more general than those defined by (2.1), (2.2), we shall throughout this section consider (4.1), (4.2) with the following general assumptions on the exponents (σ_k), (μ_k).

H0. $\{(\sigma_k)\}_{k \in \mathbb{Z}} \cap \{(\mu_k)\}_{k \in \mathbb{N}} = \emptyset$.

H1. There exists $\beta \in \mathbb{C}$, $c > 0$ and $(\nu_k)_{k \in \mathbb{Z}} \in l^2$ for which (σ_k) satisfies

(i) $\sigma_k = \beta + ck\pi i + \nu_k \qquad \forall\, k \in \mathbb{Z}$,

(ii) $\sigma_k \neq \sigma_j$ unless $j = k$.

H2. There exist positive ρ, B, δ, ε and $0 \leq \theta < \frac{\pi}{2}$ for which (μ_k) satisfies

(i) $|\arg(-\mu_k)| \leq \theta \quad \forall\, k \in \mathbb{N}$,

(ii) $|\mu_k - \mu_j| \geq \delta |k^2 - j^2| \quad \forall\, k, j \in \mathbb{N}$,

(iii) $\varepsilon(\rho + Bk^2) \leq |\mu_k| \leq \rho + Bk^2 \quad \forall\, k \in \mathbb{N}$.

Assumptions (H0), (H1) and (H2) will be considered standing assumptions for all the results of this section.

Eigenvalues associated with one-dimensional hyperbolic systems often satisfy (H1), while those of one-dimensional parabolic (or "abstract parabolic") systems often satisfy

(H2). The quadratic growth and separation assumptions in (H2) parts (ii) and (iii) can be replaced by more general growth rates (see [7,Theorem 1.1]); however we will avoid this additional complication here.

It will be convenient to introduce a notation for some spaces we will need to use. For $0 \leq a < b$ let

$$\begin{aligned} W_{[a,b]} &= \text{closed span } (e^{\sigma_k t}) \quad \text{in } L^2[a,b], \\ E_{[a,b]} &= \text{closed span } (e^{-\mu_k t}) \quad \text{in } L^2[a,b]. \end{aligned}$$

With $\|\cdot\|_{[a,b]} := \|\cdot\|_{L^2[a,b]}$, $W_{[a,b]}$ and $E_{[a,b]}$ are Hilbert spaces.

Definition 4.1. Let H be a Hilbert space with closed subspaces M and N. We will say that M and N are uniformly separated if $M \cap N = \{0\}$ and their sum $M + N$ is H-closed.

Equivalently, the subspaces M and N are uniformly separated if and only if there exists $\delta > 0$ (called the minimum gap in Kato [11]) such that for any $f \in M$ and $g \in N$, each of norm 1, that $\|f - g\| \geq \delta$. See [11] for details.

The following result is the main one of this section and will allow us to decouple the moment problem (4.1), (4.2).

THEOREM 4.2. *Assume the standing hypothesis* (H0), (H1), (H2). *For each $T > 2/c$ the spaces $W_{[0,T]}$ and $E_{[0,T]}$ are uniformly separated. This does not hold for $T \leq 2/c$.*

The proof relies upon the several results which follow, and will be given later in this section.

Throughout the following we will denote

$$t_c = 2/c.$$

LEMMA 4.3. *For any $a \in \mathbb{R}$, $W_{[a,a+t_c]} = L^2[a, a+t_c]$. Furthermore, for $T \geq t_c$, $(e^{\sigma_k t})_{k \in \mathbb{Z}}$ forms a Riesz basis for each of the spaces $W_{[a,a+T]}$.*

Proof. The sequence $(\sigma_k)_{k\in\mathbb{Z}}$ lies in a vertical strip of $\mathbb{C}$, and $|\operatorname{Im} \sigma_k - ck\pi| \to 0$ as $|k| \to \infty$. This implies (see [21, p.196]) there exists N such that $(e^{s_k t})_{k\in\mathbb{Z}}$ forms a Riesz basis for $L^2(a, a+t_c)$ for any $a \in \mathbb{R}$, where $s_k = \sigma_k$ if $|k| > N$ and $s_k = \beta + ck\pi i$ if $|k| \leq N$. By [21, p.40] and [21, p.129], a Riesz basis of exponentials for $L^2(a, a+t_c)$ is stable with respect to a change of finitely many exponentials (i.e., for $|k| \leq N$, $e^{s_k t} \to e^{\sigma_k t}$). Therefore the first statement of the lemma holds and the second is true for $T = t_c$. For any $N \in \mathbb{N}$, one can choose a sequence $(e^{\tilde{s}_k t})_{k \in \mathbb{Z}}$ for which $|\operatorname{Im} \tilde{s}_k - ck\pi/N| \to 0$ as $|k \to \infty|$ and (σ_k) is a subsequence of $(\tilde{s}_k)$. As in the proof of the first statement, it follows that $(e^{\tilde{s}_k t})$ forms a Riesz basis for $L^2(a, a+Nt_c)$, for any $a \in \mathbb{R}$. Since a subset of a Riesz basis is necessarily a Riesz basis for the subspace given by its closed span, it follows that $(e^{\sigma_k t})$ forms a Riesz basis for $W_{[a,a+Nt_c]}$, for any $a \in \mathbb{R}$. Thus the second statement of the lemma is true for $T = Nt_c$ for any $N \in \mathbb{N}$. Let $t_c \leq T \leq Nt_c$. By [21, p.32], $(e^{\sigma_k t})$ forms a Riesz basis for $W_{[a,a+T]}$ if and only if there exists positive numbers m_T and M_T such that for any $n \in \mathbb{N}$ and arbitrary scalars $c_1, c_2, \ldots c_n$ one has

$$m_T \|(c_i)\|_{l^2}^2 \leq \left\| \sum_{i=1}^{n} c_i e^{\sigma_i t} \right\|_{[a,a+T]}^2 \leq M_T \|(c_i)\|_{l^2}^2. \tag{4.3}$$

Let $p_n(t) = \sum_{i=1}^{n} c_i e^{\sigma_i t}$. Since $[a, a+t_c] \subset [a, a+T] \subset [a, a+Nt_c]$ it follows that

$$\|p_n\|_{[a,a+t_c]}^2 \leq \|p_n\|_{[a,a+T]}^2 \leq \|p_n\|_{[a,a+Nt_c]}^2.$$

Furthermore, (4.3) holds if $T = t_c$ or if $T = Nt_c$. It thus follows that for arbitrary $T \in (t_c, Nt_c)$, the inequalities in (4.3) hold with $m_T = m_{t_c}$ and $M_T = M_{Nt_c}$. $\square$

The previous lemma implies that for each $f \in W_{[a,a+T]}$, with $a \in \mathbb{R}$ and $T \geq t_c$, there is a uniquely defined sequence $(c_k) \in l^2$ for which

$$f = \text{l.i.m.} \sum_{k \in \mathbb{Z}} c_k e^{\sigma_k t} \qquad t \in [a, a+T]. \tag{4.4}$$

Thus given any $f \in W_{[a,a+T]}$, we may define an extension $\tilde{f} \in L^2_{\text{loc}}(\mathbb{R})$ by

$$\tilde{f} = \text{l.i.m.} \sum_{k \in \mathbb{Z}} c_k e^{\sigma_k t} \qquad t \in \mathbb{R}. \tag{4.5}$$

LEMMA 4.4. *Let $a, b \in \mathbb{R}$ and assume $T \geq t_c$. Then the mapping $F\colon W_{[a,a+t_c]} \to W_{[b,b+T]}$ defined by*

$$Ff = \tilde{f}\Big|_{[b,b+T]}$$

is an isomorphism.

Proof. For $\alpha, \beta \in \mathbb{R}$, with $\beta \geq \alpha + t_c$, let $J_{[\alpha,\beta]} \colon W_{[\alpha,\beta]} \to l^2$ by

$$J_{[\alpha,\beta]} f = (c_k)_{k \in \mathbb{Z}},$$

where (c_k) is determined by (4.4). Lemma 4.3 implies that $J_{[\alpha,\beta]}$ is an isomorphism. Therefore $F = J_{[b,b+T]}^{-1} J_{[a,a+t_c]}$ is an isomorphism as well. □

For $a, s \in \mathbb{R}$ define $J_a(s) \in \mathcal{L}\left(L^2[a, a+t_c]\right)$ by

$$(J_a(s)f)(t) = \tilde{f}(t+s)\,\big|_{t \in [a, a+t_c]}, \tag{4.6}$$

where f and $\tilde{f}$ are given by (4.4) and (4.5). (Lemmas 4.3 and 4.4 show that J_a is well-defined.) By Lemma 4.4, for any $a, s \in \mathbb{R}$, $J_a(s)$ is an isomorphism. In fact it can be seen that $J_a = (J_a(s))_{s \in \mathbb{R}}$ forms a group. The generator of this group was characterized in Russell [20].

PROPOSITION 4.5. *For any $a \in \mathbb{R}$, $J_a = (J_a(s))_{s \in \mathbb{R}}$ is a strongly continuous group of operators on $L^2[a, a+t_c]$. J_a is generated by the derivative operator d/dt on the domain:*

$$\mathcal{D}\left(\frac{d}{dt}\right) = \{ f \in H^1[a, a+t_c] \mid f(a+t_c) - e^{\beta t_c} f(a) = \int_a^{a+t_c} q(\tau) f(\tau) d\tau \}, \tag{4.7}$$

where $q \in L^2(a, a+t_c)$ is uniquely determined by a and (σ_k), and satisfies $\|q\| \leq M_a \|(\nu_k)\|_{l^2}$ for some $M_a > 0$.

An explicit formula for q can be found in [20]. This however, will not concern us.

The next result concerns properties of the spaces $E_{[0,T]}$.

PROPOSITION 4.6. *Let $0 < \alpha < \frac{\pi}{2} - \theta$ (θ is defined in (H2)), and assume T and ν are positive. Each $f \in E_{[0,T]}$ has an analytic extension $\hat{f}$ to the the region $\Delta_\nu = \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \alpha, |\lambda| > \nu\}$. Furthermore there exist positive constants M, ω such that for any $\lambda \in \Delta_\nu$,*

$$|\hat{f}(\lambda)| \leq M e^{-\omega \rho |\lambda|} \|f\|_{[0,T]} \qquad \forall f \in E_{[0,T]}, \tag{4.8}$$

where M and ω depend only upon B, δ, ε and θ (of (H2)).

This result was proved in more generality in [7].

A key point in the above proposition is that M and ω are independent of ρ (in (H2)) and the particular sequence $(\mu_k)_{k\in\mathbb{N}}$. Thus by selectively removing a finite number of μ_k (those with the biggest real parts) from (μ_k), we can increase the ρ defined in (H2) without affecting the constants M and ω and hence obtain any desired decay rate in (4.8) for the functions generated by such a subsequence of exponentials.

We restate this key point as follows.

COROLLARY 4.7. *Let r and T be positive numbers. The space $E_{[0,T]}$ can be decomposed into the direct sum $F \oplus R$, where F is finite dimensional and all functions $f \in R$ have an analytic continuation $\hat{f}(z)$ which satisfies*

$$|\hat{f}(z)| < Me^{-r|z|}\|f\|_{[0,T]} \qquad \forall z \in \Delta_\nu, \qquad \forall f \in R. \tag{4.9}$$

Proof of Theorem 4.2. Let $\epsilon > 0$. We wish to show that the spaces $W_{[0,t_c+\varepsilon]}$ and $E_{[0,t_c+\varepsilon]}$ are uniformly separated. Let $J_{\varepsilon/2} = \big(J_{\varepsilon/2}(s)\big)_{s\in\mathbb{R}}$ denote the group defined by (4.6) and Proposition 4.5. Since groups are obviously invertible, there exist $m > 0$ and $r_0 > 0$ for which

$$\|J_{\varepsilon/2}(s)f\| > me^{-r_0 s}\|f\| \qquad \forall\, s > 0, \quad \forall\, f \in L^2(\varepsilon/2, t_c + \varepsilon/2).$$

It follows from the above and Lemma 4.4 that there exists $\tilde{m} > 0$ for which

$$\|\tilde{f}\|_{[s,s+t_c]} > \tilde{m}e^{-r_0 s}\|f\| \qquad \forall\, s > 0, \quad \forall\, f \in W_{[\varepsilon/2,t_c+\varepsilon]}, \tag{4.10}$$

where $\tilde{f}$ is defined by (4.5).

Let $r > r_0$. We may without loss of generality assume that all functions in $E_{[0,t_c+\varepsilon]}$ have analytic continuations which satisfy (4.9). (This follows since $E_{[0,t_c+\varepsilon]}$ could be decomposed as in Corollary 4.7, and since $(\mu_k)\cup(\sigma_k)$ are distinct, F is necessarily uniformly separated from $W_{[0,t_c+\varepsilon]}$.)

Now assume to the contrary, that the two spaces are not uniformly separated. Then there exists $(f_n)_{n\in\mathbb{N}} \in E_{[0,t_c+\varepsilon]}$ and $(g_n)_{n\in\mathbb{N}} \in W_{[0,t_c+\varepsilon]}$, each of norm 1 for which

$$\|f_n - g_n\|_{[0,t_c+\varepsilon]} \to 0 \qquad \text{as } n \to \infty.$$

Since each f_n has an analytic continuation $\hat{f}_n$ which satisfies the bound (4.9) (use $T = t_c + \varepsilon$ and the M determined by $\nu = \varepsilon/2$), it follows that $(\hat{f}_n)$ forms a normal family on compact subsets of $\Delta_0 = \{z \in \mathbb{C} \,|\, z \neq 0, \ |\arg z| < \alpha\}$. It follows that there exists a subsequence, still denoted $(\hat{f}_n)$, which converges uniformly on the interval $I = [\varepsilon/2, \, t_c + \varepsilon]$ to $f(t)$. Since obviously $f \in L^2(I)$, we know that $\|g_n - f\|_I \to 0$ as $n \to \infty$. Thus $f \in W_I$, and by Vitali's convergence theorem ([9], e.g.) f has an analytic continuation $\hat{f}$ to Δ_0 for which

$$|\hat{f}(t)| \leq Me^{-rt}, \qquad \forall t \geq \epsilon/2. \tag{4.11}$$

Assume for the moment that $\|f\|_I = 0$. It then follows that $\|g_n\|_I \to 0$, and consequently $\|g_n\|_{[0,\varepsilon/2]} \to 1$ as $n \to \infty$. This however is impossible by Lemma 4.4. Thus $\|f\|_I > 0$.

Let $F(s,t) = \hat{f}(s+t)$. Since $\hat{f}$ is differentiable, it follows that $\frac{\partial F}{\partial s} = \frac{\partial F}{\partial t}$ for $s+t > \varepsilon/2$. Further, $\hat{f}\,\big|_I = \tilde{f}\,\big|_I \in W_I$. Thus for all $s \in (0, \varepsilon/2)$

$$F(s, t_c + \varepsilon/2) - e^{\beta t_c} F(s, \varepsilon/2) = \int_{\varepsilon/2}^{t_c+\varepsilon/2} q(\tau) F(s,\tau)\, d\tau. \tag{4.12}$$

Morera's theorem can be used to show that the the right-hand side of (4.12) is analytic in Δ_0. Since the left-hand side of (4.12) is also analytic in this region, we conclude that (4.12) holds for all $s \in (\varepsilon/2, \infty)$. Hence right-translations of $\hat{f}$ are given by $J_{\varepsilon/2}$ (in Proposition 4.5) acting upon f, as are those of $\tilde{f}$; i.e., for $s > 0$,

$$\hat{f}\,\big|_{[s+\varepsilon/2, s+t_c+\varepsilon/2]} = J_{\varepsilon/2}(s)\left(f\,\big|_{[\varepsilon/2, \varepsilon/2+t_c]}\right) = \tilde{f}\,\big|_{[s+\varepsilon/2, s+t_c+\varepsilon/2]}.$$

It thus follows that $\tilde{f}(t) = \hat{f}(t)$ for $t > \varepsilon/2$, but this is in conflict with (4.10) and (4.11). $\square$

The following results relate Theorem 4.2 to the moment problem (4.1), (4.2).

PROPOSITION 4.8. *Let $(d_k)_{k\in\mathbb{Z}} \in l^2$. Then for any $T \geq t_c$ there is a unique $u \in W_{[0,T]}$ which solves the moment problem* (4.2). *Any $f \in L^2[0,T]$ given by $f = u + v$, with $v \in W_{[0,T]}^{\perp}$ also solves* (4.2).

Proof. This follows easily from Lemma 4.3. □

PROPOSITION 4.9. *Assume that for any $p > 0$, $(c_k)_{k\in\mathbb{N}}$ satisfies*

$$|c_k|e^{pk} \to 0 \qquad \text{as } k \to \infty. \tag{4.13}$$

Then given any $\tau > 0$ there exists a unique $u \in E_{[0,\tau]}$ which solves the moment problem (4.1). *Any $f \in L^2[0,\tau]$ given by $f = u + v$, with $v \in E_{[0,\tau]}^{\perp}$ also solves* (4.1).

Proof. From [7, Theorem 1.1] there is a $p_0 > 0$ for which the biorthonormal functions $(q_k(t))$ to $(e^{\mu_k t})$ in $E_{[0,\tau]}$ satisfy

$$\|q_k\|_{[0,\tau]} \leq Me^{p_0 k},$$

for some $M > 0$. We define $u = \sum_{k=1}^{\infty} c_k q_k$. It is easily checked that $u \in E_{[0,\tau]}$ and both u and f solve (4.1). □

Remark 4.10. The condition (4.13) can be weakened to

$$|c_k| \cdot e^{p_0 k} \to 0 \qquad \text{as } k \to \infty,$$

where a suitable p_0 can be found from [7, Theorem 1.1].

THEOREM 4.11. *Under the standing hypothesis* (H0), (H1), (H2), *given any sequence $(c_k)_{k\in\mathbb{N}}$ which satisfies* (4.13) *and any $(d_k)_{k\in\mathbb{Z}} \in l^2$, for any time $T > t_c$ there exists $u \in L^2(0,T)$ which simultaneously solves the moment problems* (4.1) *and* (4.2). *This does not hold for $T \leq t_c$.*

Proof. If $T = t_c$ the solution to (4.2) is unique (this follows from Lemma 4.3) and hence it is not in general possible to simultaneously solve (4.1) and (4.2). If $T < t_c$ then (4.2) does not necessarily have a solution and hence it is necessary that $T \geq t_c$. Thus assume $T > t_c$. By Theorem 4.2, the spaces $E \equiv E_{[0,T]}$ and $W \equiv W_{[0,T]}$ are uniformly separated. Thus $V := E + W$ is closed, and hence a Hilbert space with $\|\cdot\|_V = \|\cdot\|_{[0,T]}$. (So $V = E \oplus W$.) Let $E^\perp$, $W^\perp$ denote the orthogonal complements of E, W in V. Let P_E denote the orthogonal projection from V onto E. By a theorem in Kato [11, Ch. 4 §4], $E^\perp$ and $W^\perp$ are also uniformly separated and hence $V = E^\perp \oplus W^\perp$. From this it is easy to show that (the restriction) $P_E|_{W^\perp}$ is an isomorphism. Likewise we may define an orthogonal projection P_W for which $P_W|_{E^\perp}$ is an isomorphism. By Propositions 4.8 and 4.9 there exist $g \in W_{[0,T]}$ which solves (4.2) and $f \in E_{[0,T]}$ which solves (4.1). Let

$$u = \left(P_E|_{W^\perp}\right)^{-1} f + \left(P_W|_{E^\perp}\right)^{-1} g.$$

One easily sees that u solves both (4.1) and (4.2), and since $P_E|_{W^\perp}$ and $P_W|_{E^\perp}$ are isomorphisms, $u \in L^2[0,T]$. □

Remark 4.12. If in addition to the hypothesis of Theorem 4.11, it is known that $(d_k\sigma_k)_{k\in\mathbb{Z}} \in l^2$, then the solution u of (4.1), (4.2) may be assumed to satisfy $u(0) = 0$ and have a (distributional) derivative in L^2. This can be proved by a modification of a result in [4]. However, without any preconditions on (d_k), it can be shown that there do not in general exist smooth solutions to (4.1), (4.2) no matter how large T is.

5. Controllability.

Consider

$$\dot{y}(t) = Ay(t) + bu(T-t) \qquad 0 < t < T; \quad y(0) = y^0, \tag{5.1}$$

where A is defined in (1.8), $u \in L^2[0,T]$, b represents b_0 or b_1 in (3.7) and y^0 belongs to an appropriate space which we will specify later. If we wish to control the state to some terminal state y^T in time T, the variation of parameters formula must hold (on an appropriate space):

$$y^T - \mathbb{T}_T y^0 = \int_0^T \mathbb{T}_s bv(s)ds.$$

Using the same decomposition as in (3.10), (3.11), we must have

$$x^T - \mathbb{S}_T x^0 = \int_0^T \mathbb{S}_\tau Pbu(\tau)d\tau, \tag{5.2}$$

$$z^T - \mathbb{G}_T z^0 = \int_0^T \mathbb{G}_\tau Qbu(\tau)d\tau, \tag{5.3}$$

where $x = Py$, $z = Qy$ and likewise for x^T and z^T. In order that the solution to (5.1) exist pointwise in time, we require that (5.2) and (5.3) hold on the respective spaces in which $(\mathbb{S}, Pb)$ and $(\mathbb{G}, Qb)$ are well-posed. Thus if b represents b_0 (resp. b_1) then (5.2) should hold on $\Lambda_{-1/4}$ (resp. $\Lambda_{1/4}$) and (5.3) should hold on Σ_0 (resp. Σ_1).

When (5.2) and (5.3) are integrated against the eigenfunctions of A^* one arrives at the pair of coupled moment problems (4.1), (4.2), where (σ_k) and (μ_k) are defined by (2.1) and

$$c_k = \frac{\langle x^T, \psi_{\mu_k}\rangle - e^{\mu_k T}\langle x^0, \psi_{\mu_k}\rangle}{\langle b, \psi_{\mu_k}\rangle} \qquad d_k = \frac{\langle z^T, \psi_{\sigma_k}\rangle - e^{\sigma_k T}\langle z^0, \psi_{\sigma_k}\rangle}{\langle b, \psi_{\sigma_k}\rangle}. \tag{5.4}$$

The sequences $(\langle b, \psi_{\mu_k}\rangle)_{k\in\mathbb{N}}$ and $(\langle b, \psi_{\sigma_k}\rangle)_{k\in\mathbb{Z}}$ each consist of only nonzero terms and their asymptotic properties are given in (A.5) of the appendix.

One easily sees from (2.2) that the conditions (H0), (H1) and (H2) of the previous section are satisfied provided there are no multiple eigenvalues.

To describe the controllability of (5.1), we consider separately the problems of null-controllability and reachability. We will say that a $\mathbb{T}$-invariant space M_0 is *b-null-controllable in time T* if given any $y^0 \in M_0$ there exists $u \in L^2(0,T)$ for which (5.2), (5.3) hold (on the proper spaces) with $x^T = z^T = 0$. Likewise we will say that a $\mathbb{T}$-invariant space M_T is *b-reachable in time T* if given any $y^T \in M_T$, (5.2) and (5.3) hold with $x^0 = z^0 = 0$.

We have the following.

PROPOSITION 5.1. *Let $T > 2/c$, $\alpha \in \mathbb{R}$ and $0 < \gamma \leq 1$.*

(i) The space $\Sigma_0 + \Lambda_\alpha$ is b_0-null-controllable in time T and $\Sigma_1 + \Lambda_\alpha$ is b_1-null controllable in time T.

(ii) Let $V = \{\sum_{k\in\mathbb{N}} c_k\phi_{\mu_k} \mid (c_k)$ satisfies (4.13)$\}$. The space $\Sigma_0 + V$ is b_0-reachable in time T and $\Sigma_1 + V$ is b_1-reachable in time T.

In either case, the result does not remain true for $T \le 2/c$ or if Σ_0 or Σ_1 are replaced by larger $\mathbb{G}$-invariant spaces.

Proof. Let us first prove that $\Sigma_0 + \Lambda_\alpha$ is b_0-null-controllable in time T. (This is the case where the temperature is controlled at the left end of the rod.) Let $y^0 = x^0 + z^0$ with $x^0 \in \Lambda_\alpha$ and $z^0 \in \Sigma_0$. Since

$$z^0 = -\mathbb{G}_T^{-1} \int_0^T \mathbb{G}_\tau Q b v(\tau)\, d\tau,$$

it is necessary (since $\mathbb{G}_T$ is an isomorphism on Σ_0) that $z^0 \in \Sigma_0$ in order for (5.3) to hold on Σ_0. Thus Σ_0 can not be replaced by a larger $\mathbb{G}$-invariant space. With $x^0 \in \Lambda_\alpha$ it follows from the analyticity of $\mathbb{S}$ that $\mathbb{S}_T x^0 \in \Lambda_0$. Hence if the moment problem determined by (5.2), (5.3) has a solution then (5.2) and (5.3) will hold on the appropriate spaces $\Lambda_{-1/4}$ and Σ_0, respectively. From Lemma A.1 (in the appendix) and (2.2) one can easily see that the eigenvalues (σ_k), (μ_k) satisfy the assumptions (H0), (H1) and (H2) of Section 4. To compute (c_k) and (d_k) in (4.1), (4.2), we use (5.4) and (A.5) (in the appendix) and find there are positive numbers m and M for which

$$\text{(5.5)} \qquad m|\langle z^0, \psi_{\sigma_k}\rangle| \le |d_k| \le M|\langle z^0, \psi_{\sigma_k}\rangle|, \qquad \forall k \in \mathbb{Z},$$

$$\text{(5.6)} \qquad |c_k| \le M k^{2\alpha} e^{\mu_k T} \|x^0\|_{\Lambda_\alpha}.$$

Thus $(d_k) \in l^2$ and (c_k) satisfies (4.13). Hence by Theorem 4.11 the moment problem has a solution for $T > 2/c$ (but not in general for $T \le 2/c$). The proof that $\Lambda_\alpha + \Sigma_1$ is b_1-null-controllable is essentially the same. Thus (i) holds.

For the problem of reachability, first note that if $y^T = x^T + z^T$ with x^T and z^T as in the hypothesis, then (5.2) and (5.3) will hold on the proper spaces provided the moment problem has a solution. The moment problem which corresponds to (5.3) is easily seen to satisfy (5.5) and hence is solvable for any $(d_k) \in l^2$. Similarily, with $x^T \in V$ it is easily seen that the coefficients (c_k) satisfy (4.13). Thus (ii) holds by Theorem 4.11. ∎

More general statements can be made about the reachable space for the parabolic component (see [4]).

The proof of Theorem 1.2 now easily follows.

Proof of Theorem 1.2. Let $T > 2/c$. First consider (1.5), (1.6) with $f(t) \equiv 0$. (This is equivalent to (5.1) with $b = b_0$.) Since $y^0 \in \mathcal{H}$ certainly we have that $y^0 \in \Sigma_0 + \Lambda_{-1/4}$. By Proposition 5.1, there exists $u \in L^2(0, 2/c + \epsilon)$ for which (5.2) and (5.3) hold with $x^T = z^T = 0$. Since $x \in C([0,T], \Lambda_{-1/4})$ and $z \in C([0,T], \Sigma_0)$ it follows that by Lemma 3.9 that $y = x + z \in C([0,T], S_0 \times C_0 \times S_{-1/2})$. Part (ii) of Theorem 1.2 is proved likewise.

Remark 5.2. Theorem 1.2 is optimal in a couple of respects. In part (i), by Proposition 5.1 and Lemma 3.9, the space $\mathcal{H} = S_0 \times C_0 \times S_0$ is the largest null-controllable space of the form $S_\alpha \times C_\alpha \times S_0$. Likewise in part (ii), the space $\mathcal{D}(A) = S_1 \times C_1 \times S_2$ is the largest null-controllable space of the form $S_\alpha \times C_\alpha \times S_2$. Similar statements can be made regarding reachability. Furthermore, by Theorem 1.1, the spatial regularity of the solutions given in Theorem 1.2 is optimal in the sense described in Theorem 1.1. (This means that for general L^2-controls, no improvement in spatial regularity is possible. Of course the spatial regularity can be improved if the controls are known to be smooth. However, by Remark 4.12, there does not in general exist smooth controls unless the initial/terminal spaces are restricted.)

Remark 5.3. Proposition 5.1 implies a certain *partial exact controllability* result. Namely, for the case of temperature control ($b = b_0$), given any $y^0 \in \mathcal{H}$ and any $z^T \in L^2(\Omega) \times L^2(\Omega)$, for any $\varepsilon > 0$ it is possible to find a control $g \in L^2(0, 2/c + \varepsilon)$ which transfers y^0 to a state y^T which has z^T for its first two components. (The third component is not controlled.) More loosely stated, the mechanical components are exactly controllable on $(L^2(\Omega))^2$ in time $T = 2/c + \varepsilon$. Likewise for the case of heat flux control, the mechanical components are exactly controllable on $S_1 \times C_1$ in time T.

The above asserts that it is possible to exactly control the mechanics (position, velocity) of the rod with temperature (or heat flux) control alone. Furthermore Theorem 1.2 shows null-controllability of the whole state space (position, velocity, temperature) is possible. One could ask whether or not it is possible to drive an initial state y^0 to a terminal state of the form $y^T = (y_1^T, y_2^T, 0)'$. As the following shows, this is not in general possible without some severe restrictions.

NEGATIVE RESULT 5.4. *Let b denote b_0 or b_1. For any $n > 0$ there exists $y_n \in S_n \times C_n$ for which the state $y^T = (y_n, 0)$ is not b-reachable in any time $T > 0$.*

Sketch of proof. For $n > 0$ let $P_{12} : \mathcal{H}_n \to S_n \times C_n$ by $(y_1, y_2, y_3) \to (y_1, y_2)$ and define $M_n = P_{12}\Lambda_n$. If the space $M_n \times \{0\}$ were b_1-reachable then the corresponding moment problems must necessarily have solutions. Hence the set $\mathcal{C}$ of sequences (c_k) corresponding to PM_n should be in the moment space of (4.1). (P is the projection in (5.2).) From (5.4) and estimates in the appendix, it follows that there exists N (which depends upon n) such that if

$$\sum_{k\in\mathbb{N}} |c_k| k^N < \infty, \tag{5.7}$$

then $(c_k) \in \mathcal{C}$. Let $(q_k)_{k\in\mathbb{N}}$ denote the biorthonormal sequence to $(\exp(\mu_j t))_{k\in\mathbb{N}}$ in $E_{[0,T]}$. ($E_{[0,T]}$ was defined in Section 4.) It is known ([7]) that $\|q_k\| \geq m_1 e^{m_0 k}$ for some $m_0 > 0$, $m_1 > 0$. The solution to (4.1) is given by

$$u = \sum_{k\in\mathbb{N}} c_k q_k$$

and must converge for all (c_k) in the moment space. However there are clearly many sequences (c_k) satisfying (5.7) for which $\|c_k q_k\| \to \infty$ as $k \to \infty$. $\square$

As mentioned in the introduction, results similar to Theorems 1.1 and 1.2 apply if the stress or velocity are controlled instead of the temperature or heat flux at an endpoint. For example, consider the boundary control problem (1.5), (1.6), with the boundary conditions

$$\begin{cases} \sigma_{y(t)}(0) = g(t) & \theta_{y(t)}(0) = 0 \qquad t \geq 0 \\ v_{y(t)}(1) = f(t) & q_{y(t)}(1) = 0 \qquad t \geq 0. \end{cases} \tag{5.8}$$

This system can be shown to be equivalent to

$$\frac{dy}{dt} = Ay(t) + b_\sigma f(t) + b_v g(t), \qquad y(0) = y^0, \tag{5.9}$$

where A is defined by (1.8) and the input elements b_σ and b_v are defined by

$$\begin{cases} \langle b_\sigma, \bar{z}\rangle = -v_z(0) = -z_2(0) & \forall\, z = (z_1, z_2, z_3)' \in \mathcal{H}_1, \\ \langle b_v, \bar{z}\rangle = \ \sigma_z(1) = z_1(1) - \gamma z_3(1) & \forall\, z = (z_1, z_2, z_3)' \in \mathcal{H}_1. \end{cases}$$

Hence (5.8), (5.9) can be analyzed in the same manner as were (3.7), (3.9). In this way one can obtain the following results which we state without proof.

PROPOSITION 5.5. *Let $y^0 = 0$, $f \in L^2(0,\infty)$, and $g \in L^2(0,\infty)$. Then the solution to* (1.5), (1.6), (5.8) *belongs to $C([0,\infty), S_0 \times C_0 \times S_{1/2})$. If additionally, $g \equiv 0$, then the solution belongs to $C([0,\infty), S_0 \times C_0 \times S_1)$. These solution spaces are optimal in the sense that none of the indices $\{0, 1/2, 1\}$ may be increased.*

PROPOSITION 5.6. *Assume $0 < \gamma \le 1$ in* (1.5) *and $T > 2/c$.*

(i) For the boundary control problem (1.5), (1.6), (5.8), *with $f \equiv 0$, any $y^0 \in \mathcal{H}$ can be controlled to zero by some $g \in L^2[0,T]$. The resulting solution is in $C((0,T], S_0 \times C_0 \times S_{1/2}) \cap C([0,T], \mathcal{H})$.*

(ii) For the boundary control problem (1.5), (1.6), (5.8), *with $g \equiv 0$, any $y^0 \in \mathcal{H}$ can be controlled to zero by some $f \in L^2[0,T]$. The resulting solution is in $C((0,T], S_0 \times C_0 \times S_1) \cap C([0,T], \mathcal{H})$.*

In Propositions 5.5 and 5.6, the control time and all the spaces involved can be shown to be optimal in the same sense that those of Theorems 1.1 and 1.2 are.

Appendix.

As described in Section 2 the eigenfunctions of A^* consist of a real branch $(\mu_k)_{k\in\mathbb{N}}$ and a non-real branch $(\sigma_k)_{k\in\mathbb{Z}}$, which are determined by the characteristic equations (2.1) and satisfy the asymptotic estimates (2.2). The non-real branch consists of complex conjugate pairs for which

$$\bar{\sigma}_k = \sigma_{-k+1} \qquad \forall k \in \mathbb{N}. \tag{A.1}$$

Let $r_k = k\pi - \pi/2$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$ the associated eigenfunctions are given by (see [6])

$$\text{(A.2)} \qquad \psi_{\sigma_k} = \begin{pmatrix} \sin r_k x \\ \dfrac{\sigma_k}{r_k}\cos r_k x \\ \dfrac{-\gamma\sigma_k}{\sigma_k + r_k^2}\sin r_k x \end{pmatrix} \qquad \psi_{\mu_k} = \begin{pmatrix} \dfrac{c^2\gamma}{(\mu_k/r_k)^2 + c^2}\sin r_k x \\ \dfrac{(\mu_k/r_k)c^2\gamma}{(\mu_k/r_k)^2 + c^2}\cos r_k x \\ \sin r_k x \end{pmatrix}.$$

For $k \leq 0$, ψ_{σ_k} are given by conjugation, as in (A.1). The above eigenfunctions are not normalized (as was assumed in Section 2), but they are *almost normalized*, that is, their norms are bounded and bounded away from zero. Since all the estimates we derive here concern only the asymptotic order, all the estimates remain valid for the normalized eigenfunctions of Section 2.

For a sequence $(c_k)_{k\in\mathbb{N}}$, let us say that $c_k = \mathcal{O}(k^\alpha)$ if there are are positive numbers m and M for which $mk^\alpha \leq |c_k| \leq Mk^\alpha$. It can be seen from (2.1) and (2.2) that

$$\text{(A.3)} \qquad \psi_{\sigma_k} = \begin{pmatrix} \mathcal{O}(1)\cdot\sin r_k x \\ \mathcal{O}(1)\cdot\cos r_k x \\ \mathcal{O}(k^{-1})\sin r_k x \end{pmatrix} \qquad \psi_{\mu_k} = \begin{pmatrix} \mathcal{O}(k^{-2})\cdot\sin r_k x \\ \mathcal{O}(k^{-1})\cdot\cos r_k x \\ \mathcal{O}(1)\cdot\sin r_k x \end{pmatrix}.$$

By [6, Remark 3.3], the eigenfunctions of A likewise satisfy

$$\text{(A.4)} \qquad \phi_{\sigma_k} = \begin{pmatrix} \mathcal{O}(1)\cdot\sin r_k x \\ \mathcal{O}(1)\cdot\cos r_k x \\ \mathcal{O}(k^{-1})\sin r_k x \end{pmatrix} \qquad \phi_{\mu_k} = \begin{pmatrix} \mathcal{O}(k^{-2})\cdot\sin r_k x \\ \mathcal{O}(k^{-1})\cdot\cos r_k x \\ \mathcal{O}(1)\cdot\sin r_k x \end{pmatrix}.$$

Let b_0 and b_1 denote the input elements defined by (3.7). From (A.2) and (A.3) we have

$$\text{(A.5)} \qquad \begin{aligned} \langle b_0, \psi_{\sigma_k}\rangle &= \mathcal{O}(1) \qquad & \langle b_0, \psi_{\mu_k}\rangle &= \mathcal{O}(k) \\ \langle b_1, \psi_{\sigma_k}\rangle &= \mathcal{O}(k^{-1}) \qquad & \langle b_1, \psi_{\mu_k}\rangle &= \mathcal{O}(1). \end{aligned}$$

Proof of Lemma 3.8. The first equality in (3.14) is just (2.5). For the second, note that $\mathcal{H} = S_0 \times C_0 \times S_0$ and $\mathcal{H}_1 = S_1 \times C_1 \times S_2$. It follows from standard properties of interpolation spaces (e.g., [15]) that for $\alpha \in [0,1]$,

$$\begin{aligned} \mathcal{H}_\alpha &= [\mathcal{H}_1, \mathcal{H}]_{1-\alpha} \\ &= [S_1 \times C_1 \times S_2, S_0 \times C_0 \times S_0]_{1-\alpha} \\ &= S_\alpha \times C_\alpha \times S_{2\alpha}. \end{aligned}$$

The above also holds for $\alpha \in [-1, 0]$ by duality.

To prove (3.15), we first note from the eigenvalue estimates (2.1), for any $\alpha \in \mathbb{R}$,

$$\Lambda_\alpha = \left\{ \sum_{k=1}^{\infty} c_k \varphi_{\mu_k} \,|\, (c_k k^{2\alpha}) \in l^2 \right\}. \tag{A.6}$$

Thus if $x = (x_1, x_2, x_3)' = \sum_{k=1}^{\infty} c_k \varphi_{\mu_k} \in \Lambda_\alpha$ then by (A.4) and (A.6),

$$x_1 = \sum_{k=1}^{\infty} c_k \cdot \mathcal{O}(k^{-2}) \cdot \sin r_k x \;\; \in S_{2+2\alpha}$$

$$x_2 = \sum_{k=1}^{\infty} c_k \cdot \mathcal{O}(k^{-1}) \cdot \cos r_k x \;\; \in C_{1+2\alpha}$$

$$x_3 = \sum_{k=1}^{\infty} c_k \cdot \mathcal{O}(1) \cdot \sin r_k x \;\; \in S_{2\alpha}.$$

Hence $\Lambda_\alpha \subset S_{2+2\alpha} \times C_{1+2\alpha} \times S_{2\alpha}$. Theorem 2.1 and (A.6) imply that $(k^{-2\alpha}\phi_{\mu_k})$ forms a Riesz basis for Λ_α. Hence an equivalent norm $|\cdot|$ on Λ_α is given by $|x| = \|(c_k k^{2\alpha})\|_{l^2} = \|P_3 x\|_{S_\alpha}$. Thus P_3 is an isomorphism from Λ_α to S_α. Similar arguments show that (3.16) holds and that $P_{12}\big|_{\Sigma_\alpha}$ is an isomorphism. □

LEMMA A.1. *Let A be defined by (1.8). For $0 \le \gamma \le 1$ the spectrum of A consists entirely of simple eigenvalues.*

Proof. For $k \in \mathbb{N}$ let $r_k = k\pi - \pi/2$. First assume that for some $k \in \mathbb{N}$ the characteristic equation (2.1) has a double root, i.e., $p_k(x) \equiv (x^2 + c^2)(x + r_k) + \gamma^2 c^2 x$ can be written as

$$p_k(x) = (x + a)(x + b)^2,$$

where a and b are positive. (Any double root is clearly real, and the roots must be negative since A is dissipative.) Equating coefficients of the two polynomials leads to

$$1 + \gamma^2 = \frac{(a + 2b)(2a + b)}{ab},$$

which is impossible with $\gamma^2 \le 8$. Thus if λ is a double eigenvalue there exists distinct positive integers j, k such that $p_k(\lambda/r_k) = 0$ and $p_j(\lambda/r_j) = 0$. This can be written as

$$\lambda^3 + \lambda^2 r_k^2 + \lambda c^2 (1 + \gamma^2) r_k^2 + c^2 r_k^4 = 0 \tag{A.7}$$

$$\lambda^3 + \lambda^2 r_j^2 + \lambda c^2 (1 + \gamma^2) r_j^2 + c^2 r_j^4 = 0. \tag{A.8}$$

Let $G = 1 + \gamma^2$, $S = r_k^2 + r_j^2$ and $P = r_k^2 r_j^2$. By eliminating the λ^3 term, and respectively the constant term in (A.7), (A.8), we find (using $\lambda \neq 0$)

$$\lambda^2 + \lambda c^2 G + c^2 S = 0,$$
$$\lambda^2 S + \lambda P + c^2 GP = 0.$$

All the coefficients are positive. We again eliminate the highest order terms, and respectively the constant terms, to obtain

$$\lambda(c^2 GS - P) + c^2(S^2 - GP) = 0, \tag{A.9}$$

$$\lambda(GP - S^2) + P(c^2 G^2 - S) = 0. \tag{A.10}$$

It is easy to show that if any of the coefficients in (A.9) or (A.10) are zero, then they all are. In this case we have $GP = S^2$, but this is impossible for $\gamma^2 < 3$ since

$$S^2 = (r_k^2 + r_j^2)^2 \geq 4(r_k^2 r_j^2) = 4P. \tag{A.11}$$

We may thus assume that none of the coefficients in (A.9) or (A.10) are zero. Next we eliminate λ from (A.9), (A.10) and find

$$c^4 G^2 + c^2 S(S^2/PG - 3) + P/G = 0.$$

Since c^2 is positive, the coefficient of c^2 must be negative, and the discriminant (of the quadratic polynomial in c^2) must be positive. This leads to

$$\left(\frac{S^2}{PG} - 3\right)^2 > 4/3,$$

which is impossible by (A.11) for $\gamma^2 \leq 1$. $\square$

Remark A.2. Its worth noting that double eigenvalues are possible if γ is larger: if $c^2 = 81\pi^2/4000$ and $\gamma^2 = 91/9$ then $\lambda = (9/40)\pi^2 \exp(i2\pi/3)$ is a double eigenvalue (corresponding to $k = 1$ and $j = 2$ in the notation of (A.7), (A.8)). This shows that null-controllability does not hold for all $\gamma > 0$. Our restriction: $0 < \gamma \leq 1$ is only sufficient to insure that no double eigenvalues occur.

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