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Boundary Control of Axially Moving Continua: Application to a Zinc Galvanizing Line

Chang-Won Kim, Hahn Park, and Keum-Shik Hong*

Abstract: In this paper, an active vibration control of a tensioned, elastic, axially moving string is investigated. The dynamics of the translating string are described with a non-linear partial differential equation coupled with an ordinary differential equation. A right boundary control to suppress the transverse vibrations of the translating continuum is proposed. The control law is derived via the Lyapunov second method. The exponential stability of the closed-loop system is verified. The effectiveness of the proposed control law is simulated.

Keywords: Axially moving nonlinear string, boundary control, exponential stability, hyperbolic partial differential equation, Lyapunov method.

1. INTRODUCTION

Control problems of axially moving systems occur in various engineering applications: the strips in thin metal-sheet production lines, the cables, belts, and chains in power transmission lines, the magnetic tapes in recorders, band saws, and others. The dynamics of these systems can be differently modeled depending on the length, flexibility, and control objectives of the considered system. For instance, the dynamics of a moving cable of an elevator can be described by a string equation, but those of a power transmission belt in a traditional mill can be better represented by a belt equation. The difference between string and belt equations lies in whether the longitudinal elongation of the continua is considered or not.

In axially moving systems, the transverse (lateral) vibration of the moving continua often causes a serious problem in achieving good quality. It is also known that these vibrations are often caused by the eccentricity of a pulley, and/or an irregular speed of the driving motor, and/or a non-uniform material property, and/or environmental disturbances. Since the quality requirement as well as productivity in a production line is becoming stricter, these days an active or a semi-active vibration control is seriously considered.

Diverse results on the dynamics, stability, and/or active/passive controls for axially moving systems have appeared in the literature [1,3,12,20,27-29,31, 32,35-37]. Particularly, the dynamics of a band saw was modeled in [26], as an axially moving string, and its instability in relation to the moving speed and excitation frequency of the saw was investigated. By changing its damping and stiffness, a passive control strategy was reported for axially moving continua in [34]. A boundary control law that suppresses the lateral vibration of an Euler-Bernoulli beam was investigated in [25], but the beam itself was not axially moving. A boundary feedback stabilization method for a rotating body-beam system was investigated in [16]. An optimal boundary force control law that dissipates the vibration energy of an axially moving string was derived in [17]. Boundary control laws for linear and nonlinear strings were reported in [8,9], in which the dynamics of the actuator was incorporated into the control law design. An optimal control and an adaptive control of an axially moving string, respectively, were investigated in [10,11]. For a translating linear beam, the wave characteristics and an optimal boundary damping law as a function of linear velocity, linear slope, and linear force were analyzed in [19]. An adaptive vibration control for an axially moving linear beam that splits the moving part into two spans, a controlled span and an uncontrolled span was investigated in [21]. The control strategy of [21] was applied to a linear string in [22], providing experimental results. The exponential stabilization of a nonlinear beam, not axially moving, by a boundary control law was focused on in [7].

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In this paper, a vibration control method to reduce the lateral vibration of the steel strip in the continuous hot-dip zinc galvanizing line is considered, which method is depicted in Fig. 1. In order to achieve uniformity of the zinc deposit on both sides of the steel strip and to reduce the zinc consumption, the strip should pass between the air knives equidistantly. But, due to vibrations of the strip, a discrepancy between the average deposited masses on the left and right strip surfaces and a non-uniformity of the mass deposited across the strip occur.

Depending on the thickness of the strip and the distance between the two support rollers, the steel strip can be modeled in one of three ways: as a moving beam, as a moving string, or as a moving belt. In the zinc galvanizing line, the distance between the two-support rollers is quite large compared to the strip thickness and the twist motion of the strip is ignored. Therefore, modeling as a moving string is most appropriate. For the given system, in-flux and out-flux of mass occur through the boundaries, and therefore the equations of motion can be derived by applying Hamilton's principle for a system of changing mass.

The majority of research results for axially moving continua available in the literature use a linear model. The previous work of the authors [12] on this subject treated the steel strip as a linear belt, whereas in [14] a boundary control law incorporating the actuator dynamics for a linear string was investigated. In both cases, the elasticity of the string was not considered.

The contributions of this paper are the following. First, a boundary control law for a heavily flexible non-linear string is derived and its exponential stability is proved. Second, the derived boundary control law requires the use of only one sensor, which measures the angular velocity of the strip at the right end. Additionally, the damping coefficient of the actuator, having been designed, is presented.

2. EQUATIONS OF MOTION

Fig. 2 shows a schematic of the axially moving nonlinear string representing a zinc galvanizing line, which schematic will be used in deriving equations of motion and a boundary control law. The string is assumed to travel at a constant speed. The left boundary is fixed in the sense that the boundary itself does not have any vertical (transversal) movement, but it allows the material to move longitudinally. However, the right boundary permits a transversal movement of the string under a control force.

Let t be the time, x be the spatial coordinate along the longitude of motion, v be the axial speed of the string, $w(x,t)$ be the transversal displacement of the string at spatial coordinate x and time t, and L be the length of the string. Then, the absolute velocity of the string at spatial coordinate x is given by

$$\bar{v} = v\,\mathbf{i} + Dw(x,t)/Dt\,\mathbf{j}$$
$$= v\,\mathbf{i} + \{w_t(x,t) + vw_x(x,t)\}\mathbf{j}, \qquad (1)$$

where $D(\cdot)/Dt \triangleq \partial(\cdot)/\partial t + v\,\partial(\cdot)/\partial x$, and $(\,\cdot\,)_t = \partial(\,\cdot\,)/\partial t = \partial(\cdot)/\partial t$, $(\cdot)_x = \partial(\cdot)/\partial x$ denote the partial derivatives in time t and spatial coordinate x, respectively.

The kinetic energy of the axially moving string including the actuator, and the potential energy, are given by

$$T = \frac{1}{2}\int_0^L \rho A\left\{v^2 + (w_t + vw_x)^2\right\}dx + \frac{1}{2}mw_t^2(L,t),\ (2)$$
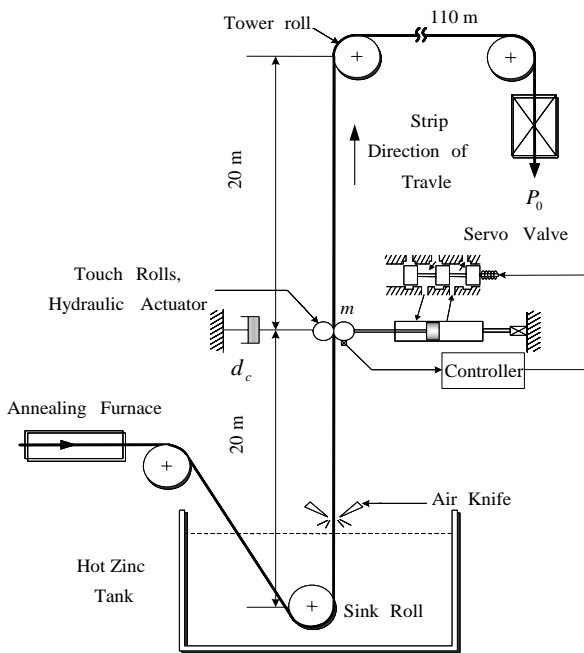

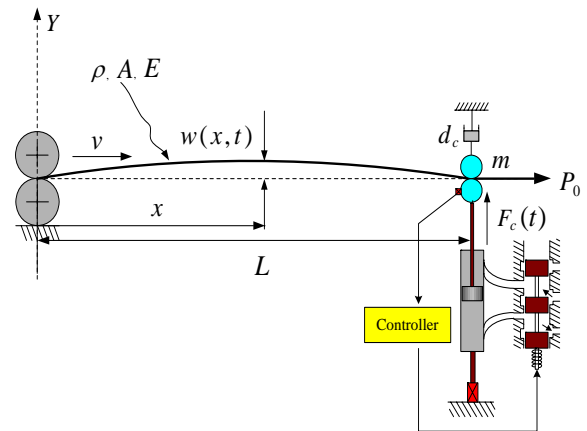Fig. 1. A translating steel strip in the zinc galvanizing line.

Fig. 2. An axially moving strip under a right boundary control force.

$$U = \int_0^L \left(P_o \varepsilon_x + \frac{EA}{2} \varepsilon_x{}^2 \right) dx \ , \qquad (3)$$

respectively, where T is the kinetic energy, U is the potential (strain) energy, ρ is the mass per unit volume (material density), A is the cross-sectional area, m is the mass of the actuator (touch roll), E is the elastic modulus of the string, and ε_x is the strain due to the tension P_0. The potential energy is proportional to the increase in string length ds when compared to the string at rest. For small slopes (see Fig. 3), the following relationship for string elongation is valid:

$$\varepsilon_x = \frac{ds - dx}{dx} = \frac{\left[dx^2 + \left(\frac{\partial w(x,t)}{\partial x} dx \right)^2 \right]^{1/2} - dx}{dx}$$

$$= \left[1 + \left(\frac{\partial w(x,t)}{\partial x} \right)^2 \right]^{1/2} - 1 \cong \frac{1}{2} \left(\frac{\partial w(x,t)}{\partial x} \right)^2 . \quad (4)$$

Therefore, (3) is rewritten as follows:

$$U = \frac{1}{2} \int_0^L \left(P_0 + \frac{EA}{4} w_x^2 \right) w_x^2 dx \ . \qquad (5)$$

Now, to derive the equations of motion, Hamilton's principle for systems with changing mass is utilized as follows [24]:

$$\delta \int_{t_1}^{t_2} \left(T - U + W_{n.c.} + W_{r.b.} \right) dt = 0 \ , \qquad (6)$$

where $W_{n.c.}$ is the non-conservative work and $W_{r.b.}$ is the virtual momentum transport at the right boundary (no variations at the left boundary). The variations of the non-conservative work and the virtual momentum transport at the right boundary are
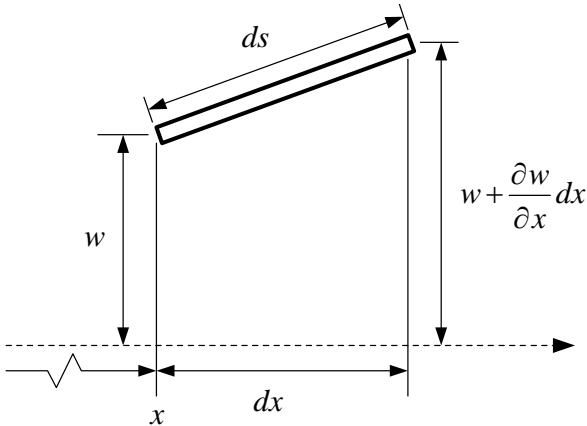

Fig. 3. Schematic of string elongation for small slopes.

$$\delta W_{n.c.} = F_c(t) \delta w(L,t) - d_c w_t(L,t) \delta w(L,t), \qquad (7)$$

$$\delta W_{r.b.} = -\rho A v \{ w_t(L,t) + v w_x(L,t) \} \delta w(L,t), \qquad (8)$$

where $F_c(t)$ is the control force, and d_c is the damping coefficient of the actuator. Now, the variations of (2) and (5), respectively, are

$$\delta T = \rho A \int_0^L (w_t + v w_x)(\delta w_t + v \delta w_x) dx$$
$$+ m w_t(L,t) \delta w_t(L,t), \qquad (9)$$

$$\delta U = P_0 \int_0^L w_x \delta w_x dx + \frac{EA}{2} \int_0^L w_x^3 \delta w_x dx \ . \qquad (10)$$

The substitution of (7)-(10) into (6) yields

$$\int_{t_1}^{t_2} \left(\delta T - \delta U + \delta W_{n.c} + \delta W_{r.b} \right) dt$$

$$= \int_{t_1}^{t_2} \int_0^L \rho A (w_t + v w_x) \delta w_t \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \int_0^L \left(\rho A v w_t + \rho A v^2 w_x \right) \delta w_x \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \int_0^L \left(-P_0 w_x - \frac{EA}{2} w_x^3 \right) \delta w_x \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \left[m w_t(L,t) \delta w_t(L,t) \delta w(L,t) \right] dt$$

$$+ \int_{t_1}^{t_2} \left\{ F_c(t) - d_c w_t(L,t) \right\} \delta w(L,t) \, dt$$

$$- \int_{t_1}^{t_2} \left\{ \rho A v w_t(L,t) + \rho A v^2 w_x(L,t) \right\} \delta w(L,t) = 0 \ . \qquad (11)$$

And the integration of (11) by parts yields

$$\int_0^L \left[(\rho A w_t + \rho A v w_x) \delta w \right]_{t_1}^{t_2} dx$$

$$- \int_{t_1}^{t_2} \int_0^L \left(\rho A w_{tt} + \rho A v w_{xt} \right) \delta w \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \left[\left(\rho A v w_t + \rho A v^2 w_x \right) \delta w \right]_0^L dt$$

$$- \int_{t_1}^{t_2} \int_0^L \left(\rho A v w_{tx} + \rho A v^2 w_{xx} \right) \delta w \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \left[\left(-P_0 w_x - \frac{EA}{2} w_x^3 \right) \delta w \right]_0^L dt$$

$$- \int_{t_1}^{t_2} \int_0^L \left(-P_0 w_{xx} - \frac{3EA}{2} w_x^2 w_{xx} \right) \delta w \, dx \, dt$$

$$+ \left[m w_t(L,t) \delta w(L,t) \right]_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \left\{ m w_{tt}(L,t) - F_c(t) + d_c w_t(L,t) \right\} \delta w(L,t) \, dt$$

$$- \int_{t_1}^{t_2} \left\{ \rho A v w_t(L,t) + \rho A v^2 w_x(L,t) \right\} \delta w(L,t) \, dt \ . \qquad (12)$$

Note that $\delta w(0,t) = 0$ because the left end is fixed

(i.e., $w(0,t) = 0$). Therefore (12) is rewritten as follows:

$$-\int_{t_1}^{t_2}\int_0^L \left(\rho A w_{tt} + 2\rho A v w_{tx}\right)\delta w\, dx\, dt$$

$$-\int_{t_1}^{t_2}\int_0^L \left\{-\left(P_0 - \rho A v^2 + \frac{3EA}{2}w_x^2\right)w_{xx}\right\}\delta w\, dx\, dt$$

$$-\int_{t_1}^{t_2}\left[m w_{tt}(L,t) - F_c(t) + d_c w_t(L,t)\right]\delta w(L,t)\, dt$$

$$-\int_{t_1}^{t_2}\left[\left\{P_0 + \frac{EA}{2}w_x^2(L,t)\right\}w_x(L,t)\right]\delta w(L,t)\, dt \;.$$

(13)

Since δw is arbitrary except for the requirement that the left end is fixed (i.e., $\delta w(0,t) = 0$), a governing equation and a boundary constraint at the right end are derived as follows:

$$\rho A w_{tt} + 2\rho A v w_{xt}$$
$$-\left(P_0 - \rho A v^2 + \frac{3EA}{2}w_x^2\right)w_{xx} = 0, \qquad (14)$$

$$F_c(t) = m w_{tt}(L,t) + d_c w_t(L,t)$$
$$+\left\{P_0 + \frac{EA}{2}w_x^2(L,t)\right\}w_x(L,t), \qquad (15)$$

where w_{tt} is the local acceleration in the transversal direction of the string, w_{xt} is the Coriolis' acceleration, and $v^2 w_{xx}$ is the centripetal acceleration.

Remark: If $3EA w_x^2 / 2 = 0$ (i.e., the system is linear), the solution of (14) can be obtained through the method of separation of variables. In this case, the natural frequency is given by $\omega_n = \frac{n\pi}{cL}(c^2 - v^2)$, $n = 1,2,3,\cdots$, where $c = \sqrt{P_0/\rho A}$ is called the wave velocity (see [18]). The natural frequency decreases as the traveling speed increases. If the traveling speed is equal to the wave velocity, the natural frequency becomes zero and a divergence of the solution occurs. In this sense, c is called the critical speed v_{cr}. Hence, the following is also assumed in this paper:

$$0 < v < v_{cr} = \sqrt{P_0/\rho A} \qquad (16)$$

if, using the parameters in Table 1, we can calculate $v_{cr} = \sqrt{P_0/\rho A}$ = 407.99 m/sec.

3. BOUNDARY CONTROL LAW

In this section, a boundary control law calculating the control force $F_c(t)$ and the range of the damping coefficient d_c, in order to suppress the transversal

Table 1. The plant parameters.

Symbols	Definitions	Values
A	cross section area	1.4×0.0045 [m^2]
L	length of the controlled part	20 [m]
P_0	tension of the strip	9,800 [kN]
m	mass of the actuator	25 [kg]
v	strip moving speed	1.8 [m/s]
ρ	mass per unit area	7,850 [kg/m^2]
d_c	damping coefficient of actuator	50 [Ns/m]
E	Elastic modulus	2×10^{11}[N/m^2]

vibration of the string, are derived. The vibrational energy of the string can be dissipated in one of two ways: one way is to apply a control action at a point in the domain $0 < x < L$, and the other is to dissipate the energy at a boundary. In this paper, it is assumed that the contact of the touch rolls with the string is firm enough and, therefore, the second view (boundary control) is explored.

A positive definite function taking the form of the total mechanical energy of the string, excluding the actuator dynamics, is first considered:

$$V_s(t) = \frac{1}{2}\int_0^L \rho A(w_t + v w_x)^2\, dx$$
$$+ \frac{1}{2}\int_0^L \left(P_0 + \frac{EA}{8}w_x^2\right)w_x^2\, dx \;, \qquad (17)$$

where the subscript s stands for string.

Lemma 1: Consider a functional $\tilde{V}$,

$$\tilde{V}(t) = V_s(t) + V_c(t), \qquad (18)$$

where the second (the script c stands for complementary) term is defined by

$$V_c(t) = \beta\rho A\int_0^L x w_x(w_t + v w_x)\, dx, \qquad (19)$$

where $\beta > 0$ is a constant. Then, (17) and (18) are equivalent; that is, there exist constants $\beta > 0$ and $0 < C_1 < 1$ satisfying

$$(1 - C_1)V_S(t) \le \tilde{V}(t) \le (1 + C_1)V_S(t). \qquad (20)$$

Proof: The existence of such β and C_1 will be proved. First, (19) becomes

$$V_c(t) = \rho A\beta\int_0^L x w_x(w_t + v w_x)\, dx$$

$$\leq \frac{\rho A \beta L}{2}\left\{\int_0^L w_x^2 dx + \int_0^L (w_t + vw_x)^2 dx\right\}$$

$$\leq \rho A \beta L\left\{\frac{1}{P_0}\cdot\frac{P_0}{2}\int_0^L w_x^2 dx + \frac{1}{2}\int_0^L (w_t + vw_x)^2 dx\right\}$$

$$+ \rho A \beta L\left\{\frac{1}{EA}\cdot\frac{EA}{8}\int_0^L w_x^4 dx\right\} \leq C_1 V_S(t), \qquad (21)$$

where

$$C_1 = \frac{\rho A \beta L}{\min(P_0, \rho A, EA)} > 0. \qquad (22)$$

Hence the following holds:

$$-C_1 V_s(t) \leq V_c(t) \leq C_1 V_s(t). \qquad (23)$$

By adding $V_s(t)$ at both sides of (23), we obtain

$$(1 - C_1)V_S(t) \leq \tilde{V}(t) \leq (1 + C_1)V_S(t). \qquad (24)$$

To have $1 - C_1 > 0$, the range of β is restricted by

$$0 < \beta < \frac{\min(P_0, \rho A, EA)}{\rho A L}. \qquad (25)$$

If verifying the range of β using the parameter values in Table 1, $0 < \beta < 1/L = 0.05$ is obtained. Note that the tension P_0 and the Young's modulus E are normally large values, and therefore, $\min\{P_0, \rho A, EA\} = \rho A$. In this paper, $\beta = 0.03$ is selected and $C_1 = 0.6$ is chosen. Lemma 1 is proved. □

Now, with Lemma 1, the following Lyapunov function candidate $V(t)$, which is basically equivalent to the total mechanical energy of the combined string and actuator, is proposed.

$$V(t) = \tilde{V}(t) + V_a(t), \qquad (26)$$

where the additional actuator-related term $V_a(t)$ is defined as

$$V_a(t) = \frac{m}{2}\{w_t(L,t) + (v + \beta L)w_x(L,t)\}^2. \qquad (27)$$

In the case of axially moving continua, the material between the two boundaries travels with time. Therefore, when we calculate the time-derivative as a Lyapunov function candidate, it should be evaluated as the time-rate of a certain control volume of the material between time t and $t + \Delta t$. In continuum mechanics, the rate of change of a certain material property (in our case, the mechanical energy) for a given control volume is the sum of the rate of change inside the control volume and the net influx through
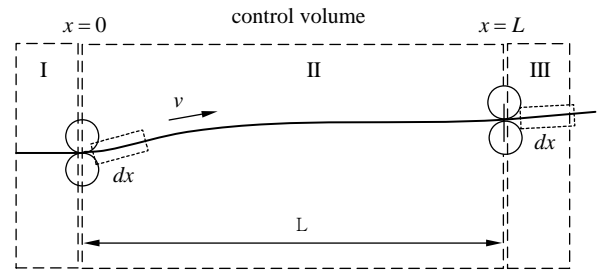

Fig. 4. The control volume with a varying right boundary.

the boundaries of the control volume. These are well established in the form of the Reynolds transport theorem [33].

To obtain the time-derivative of (26), a fixed control volume is first introduced as in Fig. 4. Volume II represents the part of the string that occupies the inner part of the control volume at an arbitrary time t, whereas I and III represent the influx and efflux of the string at $t + dt$, respectively. Using the Reynolds transport theorem, the time-derivative of (26) is given by

$$dV(t)/dt = \partial V / \partial t + v \,\partial V / \partial x \,\Big|_0^L. \qquad (28)$$

The first term on the right-hand side of (28) represents the time-rate of the equivalent energy within the control volume and the second term is the net energy flux into the control volume.

Noting that V involves three terms, V_s in (17), V_c in (19), and V_a in (27), individual terms are evaluated as follows:

$$\partial V_s(t)/\partial t = \int_0^L \rho A(w_t + vw_x)(w_{tt} + vw_{xt})dx$$

$$+ \int_0^L \left(P_0 + \frac{EA}{2}w_x^2\right)w_x w_{xt}dx$$

$$= \int_0^L \rho A(w_t + vw_x)\{(P_0 - \rho A v^2)w_{xx}\}dx$$

$$- \int_0^L \rho A v w_{xt}(w_t + vw_x)dx + \int_0^L P_0 w_x w_{xt}dx$$

$$+ \int_0^L \frac{3EA}{2}w_x^2(w_t + vw_x)dx + \int_0^L \frac{EA}{2}w_x^3 w_{xt}dx$$

$$= (P_0 - \rho A v^2)[w_t w_x]_0^L + \frac{v(P_0 - \rho A v^2)}{2}[w_x^2]_0^L$$

$$+ \frac{EA}{2}[w_x^3 w_t]_0^L - \frac{\rho A v}{2}[w_t^2]_0^L + \frac{3EAv}{8}[w_x^4],$$

$$(29)$$

$$v\,\partial V_s(t)/\partial x\Big|_0^L = v\int_0^L \rho A(w_t + vw_x)(w_{tx} + vw_{xx})dx$$

$$+ v\int_0^L P_0 w_x w_{xx}dx + \frac{EAv}{2}\int_0^L w_x^3 w_{xx}dx$$

$$= \frac{\rho A v}{2} \left[\left(w_t + v w_x \right)^2 \right]_0^L + \frac{v P_0}{2} \left[w_x^2 \right]_0^L + \frac{EAv}{8} \left[w_x^4 \right]_0^L .$$

(30)

Using (29) and (30), $dV_s(t)/dt$ becomes

$$dV_s(t)/dt = \left(P_0 - \rho A v^2 \right) \left[w_t w_x \right]_0^L$$

$$+ \frac{v \left(P_0 - \rho A v^2 \right)}{2} \left[w_x^2 \right]_0^L + \frac{EA}{2} \left[w_x^3 w_t \right]_0^L$$

$$- \frac{\rho A v}{2} \left[w_t^2 \right]_0^L + \frac{3EAv}{8} \left[w_x^4 \right]_0^L + \frac{\rho A v}{2} \left[\left(w_t + v w_x \right)^2 \right]_0^L$$

$$+ \frac{v P_0}{2} \left[w_x^2 \right]_0^L + \frac{EAv}{8} \left[w_x^4 \right]_0^L$$

$$= P_0 w_x(L,t) w_t(L,t) + v P_0 \left\{ w_x^2(L,t) - w_x^2(0,t) \right\}$$

$$+ \frac{EA}{2} w_x^3(L,t) w_t(L,t) + \frac{EAv}{2} \left\{ w_x^4(L,t) - w_x^4(0,t) \right\}.$$

(31)

For $V_c(t)$, the following are derived:

$$\partial V_c(t)/\partial t = \rho A \beta \int_0^L x w_{xt} \left(w_t + v w_x \right) dx$$

$$+ \rho A \beta \int_0^L x w_x \left(w_{tt} + v w_{xt} \right) dx , \qquad (32)$$

$$v\, \partial V_c(t)/\partial x = v \rho A \beta \int_0^L w_x \left(w_t + v w_x \right) dx$$

$$+ v \rho A \beta \int_0^L x w_{xx} \left(w_t + v w_x \right) dx$$

$$+ v \rho A \beta \int_0^L x w_x \left(w_{tx} + v w_{xx} \right) dx . \quad (33)$$

Using (32) and (33), $dV_c(t)/dt$ becomes

$$dV_c(t)/dt = v \rho A \beta \int_0^L \left(x w_{xt} w_x + x w_t w_{xx} + w_x w_t \right) dx$$

$$+ \rho A v^2 \beta \int_0^L x w_x w_{xx} dx + \rho A \beta \int_0^L x w_{xt} w_t dx$$

$$+ \beta \int_0^L x w_x \left(\rho A w_{tt} + 2 \rho A v w_{xt} + \rho A v^2 w_{xx} \right) dx$$

$$+ \rho A v^2 \beta \int_0^L w_x^2 dx .$$

(34)

To rewrite (34), the following integrations by parts are utilized:

$$\int_0^L \left(x w_{xt} w_x + x w_t w_{xx} + w_x w_t \right) dx$$

$$= \left[x w_x w_t \right]_0^L = L w_x(L,t) w_t(L,t), \qquad (35)$$

$$\int_0^L x w_x w_{xx} dx = \frac{L}{2} w_x^2(L,t) - \frac{1}{2} \int_0^L w_x^2 dx, \qquad (36)$$

$$\int_0^L x w_{xt} w_t dx = \frac{L}{2} w_t^2(L,t) - \frac{1}{2} \int_0^L w_t^2 dx. \qquad (37)$$

Also, using the governing Equation (14), the following equation is derived.

$$\beta \int_0^L x w_x \left(\rho A w_{tt} + 2 \rho A v w_{xt} + \rho A v^2 w_{xx} \right) dx$$

$$= \beta \int_0^L x w_x \left\{ \left(P_0 + \frac{3EA}{2} w_x^2 \right) w_{xx} \right\} dx .$$

(38)

Therefore, the substitution of (35)-(38) into (34) yields

$$dV_c(t)/dt = \beta \rho A L v w_x(L,t) w_t(L,t)$$

$$+ \frac{\rho A v^2 \beta L}{2} w_x^2(L,t) - \frac{\rho A v^2 \beta}{2} \int_0^L w_x^2 dx$$

$$+ \frac{\beta L P_0}{2} w_x^2(L,t) - \frac{\beta P_0}{2} \int_0^L w_x^2 dx$$

$$+ \frac{3 \beta EAL}{8} w_x^4(L,t) - \frac{3 \beta EA}{8} \int_0^L w_x^4 dx$$

$$+ \frac{\rho A \beta L}{2} w_t^2(L,t) - \frac{\rho A \beta}{2} \int_0^L w_t^2 dx$$

$$+ \rho A v^2 \beta \int_0^L w_x^2 dx .$$

(39)

Also, the time-derivative of (27) is

$$dV_a(t)/dt = m \left\{ w_t(L,t) + \left(v + \beta L \right) w_x(L,t) \right\}$$

$$\times \left\{ w_{tt}(L,t) + \left(v + \beta L \right) w_{xt}(L,t) \right\}. \qquad (40)$$

Finally, the following is the main part in this paper.

Theorem: Consider the following axially moving system

$$\rho A w_{tt} + 2 \rho A v w_{xt}$$

$$- \left(P_0 - \rho A v^2 + \frac{3EA}{2} w_x^2 \right) w_{xx} = 0 ,$$

$$w(0,t) = 0 , \qquad (41)$$

$$m w_{tt}(L,t) + d_c w_t(L,t)$$

$$+ \left\{ P_0 + \frac{EA}{2} w_x^2(L,t) \right\} w_x(L,t) = F_c(t) ,$$

$$w(x,0) = w_0(x), \, w_t(x,0) = \dot{w}_0(x) .$$

If the control force $F_c(t)$ and the damping coefficient d_c are given by

$$F_c(t) = -K w_{xt}(L,t) , \qquad (42)$$

$$\beta \rho A L / 2 < d_c < v \rho A \beta L / (v + \beta L) , \qquad (43)$$

where $K = m(v + \beta L)$ is the control gain, the closed-loop system is exponentially stable.

Proof: The substitution of (42) into (15) yields

$$m w_{tt}(L,t) = -d_c w_t(L,t) \qquad (44)$$

$$- \left\{ P_0 + \frac{EA}{2} w_x^2(L,t) \right\} w_x(L,t) - m(v + \beta L) w_{xt}(L,t) .$$

The substitution of (44) into (40) yields

$$dV_a(t)/dt = \{w_t(L,t) + (v + \beta L)w_x(L,t)\}$$
$$\times \left\{ -d_c w_t(L,t) - P_0 w_x(L,t) - \frac{EA}{2} w_x^3(L,t) \right\}. \quad (45)$$

Therefore, combining (31), (39), and (45), the time-derivative of the Lyapunov function candidate (26) is rewritten as

$$dV(t)/dt = -\frac{\beta L}{2}\left(P_0 - \rho A v^2\right)w_x^2(L,t)$$
$$-\frac{\beta EAL}{8}w_x^4(L,t) - \frac{\beta}{2}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx$$
$$-\frac{\beta\rho A}{2}\int_0^L w_t^2 dx - \frac{3\beta EA}{8}\int_0^L w_x^4 dx - v P_0 w_x^2(0,t)$$
$$-\frac{EAv}{2}w_x^4(0,t) - \left(d_c - \frac{\beta\rho AL}{2}\right)w_t^2(L,t)$$
$$+ \{\beta\rho ALv - d_c(v + \beta L)\}w_x(L,t)w_t(L,t)$$
$$\leq -\frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t) - \frac{\beta EAL}{8}w_x^4(L,t)$$
$$-\frac{\beta}{2}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx - \frac{\beta\rho A}{2}\int_0^L w_t^2 dx$$
$$-\frac{3\beta EA}{8}\int_0^L w_x^4 dx - v P_0 w_x^2(0,t) - \frac{EAv}{2}w_x^4(0,t)$$
$$-\frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right)w_t^2(L,t)$$
$$-\frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t)$$
$$-\frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right)w_t^2(L,t)$$
$$+ \{\beta\rho ALv - d_c(v + \beta L)\}w_x(L,t)w_t(L,t). \quad (46)$$

Since v is under the critical speed (see (16)), $P_0 > \rho A v^2$ is satisfied. Note that both the first and eighth terms (after the equality sign) have been split into two halves. Note also that the first eight terms in the final equation are all negative. Therefore, the negative semi-definiteness of $dV(t)/dt$ can be achieved by making the combined final three terms negative. For brevity, new notations of ϕ_1, ϕ_2, ψ are introduced as follows.

$$\phi_1 = \beta L(P_0 - \rho A v^2)/4 > 0, \quad (47)$$
$$\phi_2 = \left(d_c - \beta\rho AL/2\right)/2, \quad (48)$$
$$\psi = \{\beta\rho ALv - d_c(v + \beta L)\}/2. \quad (49)$$

From (48) and (49), if d_c satisfies the following ranges

$$d > d_c^- \triangleq \beta\rho AL/2 \quad \text{and}$$

$$d_c < d_c^+ \triangleq \beta\rho ALv/(v + \beta L), \quad (50)$$

the last three terms in (46) satisfies the following inequality:

$$-\phi_1 w_x^2 + \psi w_x w_t - \phi_2 w_t^2$$
$$\leq -\min\{\phi_1, \psi/2, \phi_2\}(w_t - w_x)^2. \quad (51)$$

Therefore, from (46) and (51), the asymptotic stability of the closed-loop system is assured.

Now, the exponential stability is demonstrated with further manipulation of the terms. (46) can be rewritten, by splitting the third term into two parts, as

$$dV(t)/dt \leq -\frac{\beta EAL}{8}w_x^4(L,t) - v P_0 w_x^2(0,t)$$
$$-\frac{EAv}{2}w_x^4(0,t) - \frac{\beta}{4}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx$$
$$-\frac{\beta}{4v^2}\left(P_0 - \rho A v^2\right)\int_0^L (v w_x^2) dx - \frac{\beta\rho A}{2}\int_0^L w_t^2 dx$$
$$-\frac{3\beta EA}{8}\int_0^L w_x^4 dx - \frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t)$$
$$-\frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right)w_t^2(L,t)$$
$$-\min\{\phi_1, \psi/2, \phi_2\}\{w_x(L,t) - w_t(L,t)\}^2$$
$$\leq -\frac{\beta EAL}{8}w_x^4(L,t) - v P_0 w_x^2(0,t) - \frac{EAv}{2}w_x^4(0,t)$$
$$-\frac{\beta\rho A}{2}\int_0^L w_t^2 dx - \frac{\beta}{4}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx$$
$$-\min\left[\frac{\beta\rho A}{2}, \frac{\beta(P_0 - \rho A v^2)}{4v^2}\right]\left[\int_0^L w_t^2 dx + \int_0^L (v w_x^2) dx\right]$$
$$-\frac{3\beta EA}{8}\int_0^L w_x^4 dx - \frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t)$$
$$-\frac{1}{2}\left(d_c - \frac{\beta\rho AL}{2}\right)w_t^2(L,t)$$
$$-\min\{\phi_1, \psi/2, \phi_2\}\{w_x(L,t) - w_t(L,t)\}^2. \quad (52)$$

By using the inequality

$$-\int_0^L w_t^2 dx - \int_0^L (v w_x^2) dx \leq -\frac{1}{2}\int_0^L (w_t + v w_x)^2 dx, \quad (53)$$

and eliminating the first four and final (negative) terms in (52), (52) can be rewritten as

$$dV(t)/dt$$
$$\leq -\min\left[\frac{\beta\rho A}{2}, \frac{\beta(P_0 - \rho A v^2)}{4v^2}\right]\left[\int_0^L (w_t + v w_x)^2 dx\right]$$

$$-\frac{\beta}{4}\left(P_0 - \rho A v^2\right)\int_0^L w_x^2 dx - \frac{3\beta EA}{8}\int_0^L w_x^4 dx$$

$$-\frac{\beta L}{4}\left(P_0 - \rho A v^2\right)w_x^2(L,t)$$

$$-\frac{1}{2}\left(d_c - \frac{\beta \rho AL}{2}\right)w_t^2(L,t)$$

$$\leq -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2}\right]$$

$$\times\left[\frac{P_0}{2}\int_0^L w_x^2 dx + \frac{EA}{8}\int_0^L w_x^4 dx + \frac{\rho A}{2}\int_0^L \left(w_t + v w_x\right)^2 dx\right]$$

$$-\frac{\beta L\left(P_0 - \rho A v^2\right)}{4(v + \beta L)^2}\cdot\frac{2}{m}\cdot\frac{m(v + \beta L)^2}{2}w_x^2(L,t)$$

$$-\frac{1}{2}\left(d_c - \frac{\beta \rho AL}{2}\right)\frac{2}{m}\cdot\frac{m}{2}w_t^2(L,t)$$

$$\leq -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2}\right]$$

$$\times\left[\frac{P_0}{2}\int_0^L w_x^2 dx + \frac{EA}{8}\int_0^L w_x^4 dx + \frac{\rho A}{2}\int_0^L \left(w_t + v w_x\right)^2 dx\right]$$

$$-\min\left[\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta \rho AL}{2}\right)\right]$$

$$\times\left[\frac{m}{2}\left\{w_t(L,t) + (v + \beta L)w_x(L,t)\right\}^2\right].$$

$$\tag{54}$$

If using (17) and (26)-(27), (54) can be expressed as

$$dV(t)/dt$$

$$\leq -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2}\right]\tilde{V}(t)$$

$$-\min\left[\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta \rho AL}{2}\right)\right]V_A(t)$$

$$\leq -\min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2},\right.$$

$$\left.\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta \rho AL}{2}\right)\right]$$

$$\times\left(\tilde{V}(t) + V_A(t)\right) = -\lambda V(t).$$

$$\tag{55}$$

(55) represents the relationship

$$V_{total}(t) \leq V_0 e^{-\lambda t},\tag{56}$$

where $V_0 = V(0)$, and λ is given by

$$\lambda = \min\left[3\beta, \frac{\beta\left(P_0 - \rho A v^2\right)}{2P_0}, \frac{\beta\left(P_0 - \rho A v^2\right)}{4\rho A v^2},\right.$$

$$\frac{\beta L\left(P_0 - \rho A v^2\right)}{2m(v + \beta L)^2}, \frac{1}{m}\left(d_c - \frac{\beta \rho AL}{2}\right)\right] > 0.\tag{57}$$

Therefore, all of the variables included in (26) converge exponentially to zero.   □

4. IMPLEMENTATION AND SIMULATION

The implementation of (42) and (43) requires two things: the feedback of $w_{xt}(L,t)$ and the satisfaction of the range $d_c^- < d_c < d_c^+$. In this paper, the satisfaction of the desired damping range is assumed, because it is related to the design problem of the actuator. If using the parameters in Table 1 with $\beta = 0.03$, the exact range is verified as follows:

$$17.67 < d_c < 27.17.\tag{58}$$

The implementation of $w_{xt}(L,t)$ can be achieved by backward differencing of $w_x(L,t)$ measured at each step.

To demonstrate the performance of the closed loop system, computer simulations using a finite difference scheme were performed. The plant parameters used for the simulations are listed in Table 1.

With $\beta = 0.03$ defined in (25), the control gain is given by

$$K = m(v + \beta L) = 15(2 + 0.03 \times 20) = 39.\tag{59}$$

For simulation purposes, let $d_c = 25$. Let the initial conditions be

$$w(x,0) = 2\sin(3\pi)\,[\text{cm}], \quad w_t(x,0) = 0\,[\text{m/s}].\tag{60}$$

Figs. 5 and 6 compare the displacement at $x = L/2$ and $x = L$, respectively, in which the used control gain is $K = 39$, the damping coefficient is $d_c = 25$, and $L = 20$ m. An immediate suppression of the vibration at $x = L/2$ is seen in Fig. 5. In Fig. 6, however, the applied control might increase the lateral vibration at the
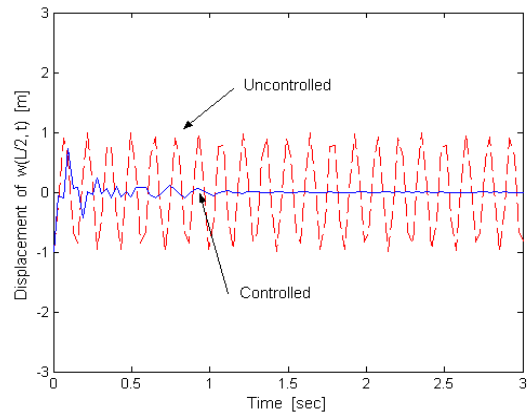

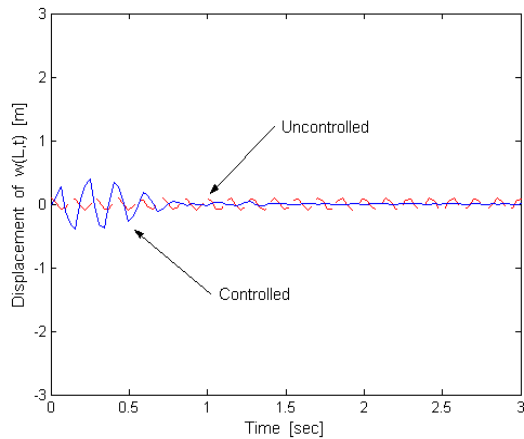Fig. 5. The transverse displacement $w(L/2,t)$: $K = 39$, $d_c = 25$, and $L=20$m.

Fig. 6. The transverse displacement $w(L,t)$: $K = 39$, $d_c = 25$, and L=20m.
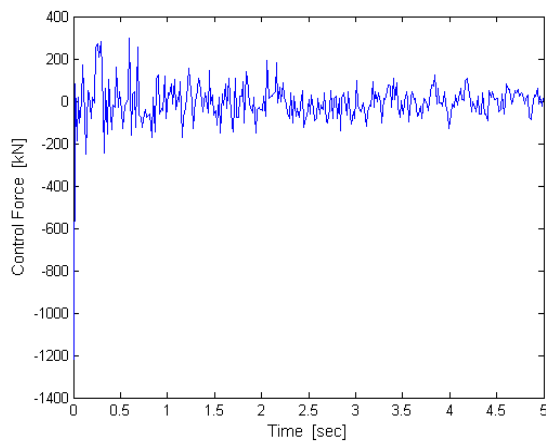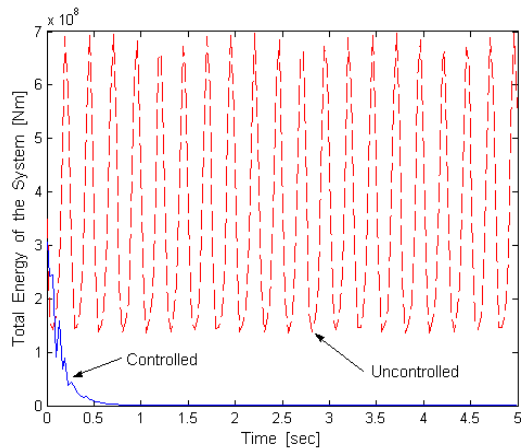

Fig. 7. The control force used in Figs. 5 and 6.

Fig. 8. Exponential decaying energy with the right boundary control.

beginning, approximately up to 0.7 seeconds, but after that, a complete suppression is seen. Fig. 7 depicts the applied control force. Finally, Fig. 8 compares the total mechanical energy, (17) with and without control, of the strip in time. It is observed that the total energy with control decays exponentially, whereas the energy without control sustains itself in time.

5. CONCLUSIONS

A transverse vibration suppression scheme in a zinc galvanizing line, in the form of the boundary control of an axially moving non-linear string, was investigated. Focused on the nonlinearity of the string, the equations of motion were derived by Hamilton's principle. The boundary control law was derived by the Lyapunov method. The implementation issues of the designed control law, also, were discussed. The exponential stability of the closed-loop system was proved. The efficiency of the designed controller was shown through simulations. Future work includes a study of the robustness of the designed controller with disturbance at the boundaries.

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