ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

WALTER LITTMAN

Boundary control theory for hyperbolic and parabolic partial differential equations with constant coefficients

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 5, nº 3 (1978), p. 567-580

<http://www.numdam.org/item?id=ASNSP_1978_4_5_3_567_0>

Boundary Control Theory for Hyperbolic and Parabolic Partial Differential Equations with Constant Coefficients (*).

WALTER LITTMAN (**)

dedicated to Hans Lewy

I. – Introduction.

Suppose we have a well-posed initial-boundary value problem for a hyperbolic or parabolic equation $Lu = 0$ in a cylindrical domain $\Omega = D \times \times [0, \infty)$, where D is a bounded domain in R^n. One basic problem of boundary control theory is that of null-controllability: given initial data in D at $t = 0$, can this data be supplemented with appropriate non-homogeneous time dependent boundary data, prescribed on the lateral boundary of Ω, such that the solution of the resulting initial boundary value problem will vanish for $t \geqslant T$? Furthermore, how small can T be chosen? For the wave equation, the heat equation and more general second order equations this problem has been treated by Russell [9], Fattorini [2], Seidman [10] and others. For additional references see these works.

The present work was to a large extent inspired by two sources, which we are happy to acknowledge: A number of very instructive conversations with L. Markus which acquainted us with the basic problems and work done in the area; secondly, a lecture given by Frank Jones in November 1973, the subject of which we shall describe shortly.

Once instance where the problem of boundary control theory has a rather straight-forward solution is the wave equation in an odd number of space

(*) Partially supported by NSF Grant MCS 76-05853 the University of Minnesota.
(**) University of Minnesota, School of Mathematics, Minneapolis, Minnesota.
Pervenuto alla Redazione il 28 Luglio 1977.

dimensions greater than one. (See Russell [9]). One simply extends the initial data outside D with support in a slightly larger set, and proceeds to solve the *pure* initial value problem with this data. Since the Riemann function for this problem has support on the surface of a (forward) cone (Huygen's principle), the solution restricted to Ω will vanish for $t > T$, where T can be determined from the geometry of D. The appropriate boundary data for the control problem is then simply read off from the solution determined as described above. Let us note that this procedure works whenever the Riemann function has an appropriate lacuna.

Frank Jones' result [7], alluded to earlier, consists in constructing a fundamental solution to the heat equation which can play the same role in the boundary control problem for the heat equation that the ordinary Riemann function plays for the (odd space dimensional) wave equation—a role for which the ordinary fundamental solution of the heat equation is not suited. Jones constructs a fundamental solution with support in $0 \leq t \leq \varepsilon$, with $\varepsilon > 0$ arbitrary.

The methods just described have several advantages. One is that the boundary control problem is solved once and for all without reference to any *specific* initial boundary value problem. Another is that one need only have *uniqueness* for the mixed problem. The existence of a solution to the control problem is a consequence of the procedure. (Uniqueness of the solution of the control problem is not asserted here and is, in general, not valid under our hypotheses.) A third advantage of the procedure is that it is not restricted to cylindrical domains.

We briefly describe the content in the remainder of the paper. Section 2 reviews some elementary facts about hyperbolic operators for $n = 1$. In section 3 we treat the boundary control problem for strictly hyperbolic equations in one space variable (which curiously enough, is not quite as immediate as the case n odd > 1). We dwell at greater length on the one dimensional case than might seem necessary mainly because the basis of the method is encountered here already. In section four we use a plane wave decomposition to treat the n dimensional hyperbolic case. Section five deals with a parabolic equation in one dimension, while section six deals with the n-dimensional case, again via a plane wave decomposition. The main theorem for the parabolic case is stated at the end of that section.

This paper emphasizes methods and qualitative results. Thus, for the sake of simplifying proofs, we assume strict hyperbolicity, although results continue to hold for general hyperbolic operators with constant coefficients with slight modifications in the proofs. We are also content with stating all results in the context of C^∞ functions, at least in the hyperbolic case.

We intend to discuss estimates in Sobolev norms (for the hyperbolic case), more general equations, variable coefficients and other aspects of the problem at another time.

The subject of this paper was presented at the Conference on Partial Differential Equations held at Oberwolfach, March 1975.

2. – Some elementary facts concerning hyperbolic equations in one space dimension.

We first treat the case of a strictly hyperbolic operator L corresponding to a homogeneous polynomial (i.e., with no lower order terms)

$$P_m(\xi, \tau) = a \prod_{i=1}^{m} (\alpha_i \xi + \tau) ,$$

where we assume that $a \neq 0$, the α's are real and, for the sake of simplicity, distinct. The differential operator $\mathcal{L} \equiv \alpha(\partial/\partial x) + \partial/\partial t$ represents a directional derivative in the direction $(\alpha, 1)$ and has two fundamental solutions, g_- and g_+, i.e., solutions to the equation $\mathcal{L}g = \delta(x)\,\delta(t)$, both being measures with support on the line $\alpha t = x$; g_+ vanishing for negative t, g_- for positive t.

Both represent signals propagating with velocity α and beginning or ending at $x = 0$, $t = 0$.

Let us assume that none of the α_i vanish, i.e., there is no zero velocity of propagation. Then we gain complete symmetry between the x and t variables. Moreover each operator $\mathcal{L}_i = \alpha_i(\partial/\partial x) + \partial/\partial t$ will have two fundamental solutions each of which can be uniquely determined by specifying support either in the upper or lower (closed) half plane—or alternately, in the right or left half plane. Convoluting the fundamental solutions for $\mathcal{L}_i$ with support in the (upper, lower, right, left) half plane we obtain, apart from a constant factor, a fundamental solution (G_+, G_-, G_R, G_L) to the operator L with support in that half plane. The *actual* support of G_+ is the sector in the upper half plane bounded by the two lines (passing through the origin) corresponding to the *algebraically* lowest and highest velocity of propagation; $\min \alpha_i$ and $\max \alpha_i$. Letting $\lambda_i = 1/\alpha_i$ we see that $G_R(G_L)$ will have support in the sector in the right (left) half plane bounded by the rays corresponding to the algebraically lowest and highest reciprocal velocity: i.e., $\min \lambda_i$ and $\max \lambda_i$. It will be of importance to us to notice that if we let $\bar{\lambda} = \max |\lambda_i|$ then the support of $G_R(G_L)$ is contained in the intersection of the set $|t/x| \leq \bar{\lambda}$ with the right (left) half plane.

Next we consider operators *with* lower order terms

$$L = P(D_x, D_t) = P_m(D_x, D_t) + Q(D_x, D_t), \qquad \left(D_x = \frac{1}{i}\frac{\partial}{\partial x}, D_t = \frac{1}{i}\frac{\partial}{\partial t}\right),$$

where $Q(\xi, \tau)$ is a polynomial with complex coefficients of degree $< m$. L has four fundamental solutions, which we again denote by $G_\pm$, G_R, G_L, having the same supports as those for $P_m(D)$. Furthermore these will vary continuously with the coefficients of the operator, as long as the speeds of propagation remain separated.

3. – The hyperbolic case, $n = 1$.

Let $L = P(D_x, D_t)$ be a strictly hyperbolic operator ($n = 1$), and let $\bar{\lambda}$ be the reciprocal of the slowest speed of propagation, i.e., $\bar{\lambda} = 1/\min|\alpha_i|$.

THEOREM 1. *Let C^∞ Cauchy data be prescribed at $t = 0$ in the closed interval $\bar{I}$ on the x axis. Then there exist C^∞ solution v to $Lv = 0$ in $\bar{I} \times [0, \infty)$, assuming this Cauchy data, vanishing for $t \geqslant T_1$, where T_1 is any number exceeding $T_0 \equiv$ diameter $\bar{I} \times \bar{\lambda}$.*

We recall that by $C^\infty(\bar{I})$ (for a closed interval $\bar{I}$) we mean functions extendable to C^∞ in a larger interval.

PROOF. Extend the initial data to a C_0^∞ function with support in a slightly larger interval I_1 contained in an ε neighborhood of I, and solve the equation $Lu = 0$ with the extended Cauchy data for all $-\infty < x < \infty$, $t \geqslant 0$. Call the solution u. Let $\varphi(t)$ be a C^∞ function equal to one for $t \leqslant \frac{1}{2}T_1 - \varepsilon$ and zero for $t \geqslant \frac{1}{2}T_1 + \varepsilon$, and set $L(u\varphi) = f$.

If m denotes the mid-point of I, let Ψ be a C^∞ function equal to one for $x \leqslant m - \varepsilon$, and zero for $x \geqslant m + \varepsilon$, and set

$$f_L = f \cdot \Psi, \qquad f_R = f \cdot (1 - \Psi)$$

and

$$U_L = f_L * G_L, \qquad U_R = f_R * G_R,$$

« $*$ » denoting involution, and G_L and G_R the fundamental solutions described in the last section. For ε sufficiently small (which we now so choose) U_L and U_R are C^∞ and each vanish in neighborhoods of the sets

$$\{t = 0, x \in I\} \qquad \text{and} \qquad \{t \geqslant T_1, x \in I\}.$$

(This is of course a consequence of the estimates of the supports of G_L and G_R of the previous section.) Hence the sum $U = U_L + U_R$ does also, and moreover satisfies $Lu = f$. Next, set $v = u\varphi - U$. We have $Lv = 0$, for $x \in \bar{I}$, v agrees with u near $t = 0$, has the originally assigned Cauchy data in $\bar{I}$, and vanishes for $t \geqslant T_1$. Hence it is a desired solution.

4. – The n dimensional hyperbolic case.

We shall accomplish the transition from one to several space dimensions by a standard device—a plane wave decomposition. To that affect, let us recall some basic facts concerning such decompositions, as described for example by Ludwig [8]. For $f \in \mathcal{S}(R^n)$ ($\mathcal{S}$ is the Schwartz space of rapidly decreasing C^∞ functions) the Radon transform is defined by

$$(Rf)(s, \omega) = \int\limits_{x\cdot\omega = s} f(x)\, dS$$

where ω is a unit vector and S represents $n-1$ dimensional surface area, while $-\infty < s < \infty$. For $g = g(s, \omega)$ define

$$(Kg)(s, \omega) = \begin{cases} \dfrac{1}{2(2\pi)^{n-1}} \left(\dfrac{1}{i}\dfrac{\partial}{\partial s}\right)^{n-1} g(s, \omega) & (n \text{ odd}), \\[2ex] \dfrac{1}{2(2\pi)^{n-1}}\, iH \left(\dfrac{1}{i}\dfrac{\partial}{\partial s}\right)^{n-1} g(s, \omega) & (n \text{ even}), \end{cases}$$

H denoting the (one dimensional) Hilbert transform. Then a plane wave decomposition

$$f(x) = \int\limits_{|\omega|=1} f_\omega(x\cdot\omega)\, d\omega$$

is accomplished by setting

$$f_\omega(s) = (KRf)(s, \omega)\,.$$

If f belongs to $\mathcal{S}(R^n)$ then $f_\omega(s)$ is infinitely differentiable in s and ω. In our applications the function will depend on another parameter, t, and part of the last statement has a more quantitative version:

$$\max_{|\alpha|\leqslant k} |D_s^\alpha D_t^\beta f_\omega(s, t)| \leqslant \text{Const} \max_{|\alpha|\leqslant k+\nu} (1 + |x|)^\nu\, |D_x^\alpha D_t^\beta f(x, t)|\,.$$

Here ν and γ are integers depending on the dimension n; k is an arbitrary non negative integer; the maximum on the left is taken over all $s \in R^1$ while the one on the right is taken over all $x \in R^n$. Both maxima may also be taken over the same closed t interval. « Const. » depends only on n.

We are now ready to treat the case of a strictly hyperbolic operator with constant coefficients in n-space dimensions. Let $L = P(D_{x_1}, \dots, D_{x_n}, D_t)$ be such an operator, and D a bounded domain in R^n. For each (spacial) direction ω (i.e., unit vector in x-space) let $\bar{\lambda}_\omega$ denote the reciprocal of the slowest speed of propagation in the ω direction. Let d_ω be the diameter of the ω-projection of D, i.e., D projected on a line parallel to the ω vector. Let $T_0 = \sup\limits_{\omega} \bar{\lambda}_\omega \cdot d_\omega$.

THEOREM 2. *Given assigned C^∞ Cauchy data at $t = 0$ on $\bar{D}$ (which can be extended as C^∞ functions in a neighborhood of $\bar{D}$); then for each $T_1 > T_0$ there exists a solution v to $Lv = 0$ in $\bar{D} \times \{t \geqslant 0\}$ assuming the prescribed Cauchy data in $\bar{D}$ and vanishing in $\bar{D} \times \{t \geqslant T_1\}$.*

(Note: $\bar{D}$ = closure of D.)

PROOF. Let us extend the Cauchy data so as to be C^∞ and have support in an ε neighborhood of D, and let us solve the resulting Cauchy problem in all of $R^n \times (t \geqslant 0)$, calling the solution u. Let T_1 be as in the statement of the theorem and let, as before,

$$L(u\varphi(t)) = f\,,$$

where $\varphi(t)$ is a C^∞ function such that

$$\varphi(t) = \begin{cases} 1 & \text{for } t \leqslant \frac{1}{2}T_1 - \varepsilon\,, \\ 0 & \text{for } t \geqslant \frac{1}{2}T_1 + \varepsilon\,. \end{cases}$$

For each unit vector ω let

$$I_\omega = \{s\colon s = x \cdot \omega, x \in \bar{D}\}$$

i.e., the ω projection of $\bar{D}$, and let m_ω denote the mid point of I_ω. Let $\Psi(s)$ be $C^\infty(R^1)$ such that

$$\Psi(s) = \begin{cases} 1 & \text{for } s \leqslant -1\,, \\ 0 & \text{for } s \geqslant 1\,, \end{cases}$$

and set $\Psi_\omega(s)=\Psi_{\omega,\varepsilon}(s)=\Psi((s-m_\omega)/\varepsilon)$. Now let

$$f(x,t)=\int_{|\omega|=1} f_\omega(s,t)\,d\omega \qquad (s=x\cdot\omega)$$

be the plane wave decomposition of f, and let

$$f_{\omega L}(s,t)=f_\omega(s,t)\Psi_\omega(s)\,,$$

$$f_{\omega R}(s,t)=f_\omega(s,t)\big(1-\Psi_\omega(s)\big)\,.$$

We note that

$$P(D_x,D_t)\,U(x\cdot\omega,t)=P_\omega(D_s,D_t)\,U(s,t)\,,$$

where $P_\omega(P_s,D_t)$ is a strictly hyperbolic operator in s,t space depending in a C^∞ way of ω, having the strict hyperbolicity maintained uniformly with respect to ω. As a matter of fact, the polynomial $P_\omega(\sigma,\tau)$ is nothing else but the polynomial $P(\xi,\tau)$ restricted to the 2-plane in $\xi-\tau$ space which contains the t-axis and the ω vector, with $\sigma=\omega\cdot\xi$.

From the case $n=1$ we know that the operator $P_\omega(D_s,D_t)=L_\omega$ has fundamental solutions

$$G_{\omega L}(s,t) \qquad \text{and} \qquad G_{\omega R}(s,t)$$

with supports contained in the sets

$$|t/s|\leqslant\bar\lambda_\omega \qquad s\leqslant 0\,,\ (s\geqslant 0)$$

(respectively), where $\bar\lambda_\omega$ is the maximal reciprocal propagation speed for L_ω. Letting

$$U_{\omega L}=G_{\omega L}*f_{\omega L}\,, \qquad U_{\omega R}=G_{\omega R}*f_{\omega R}\,,$$

(with convolutions carried out in $s-t$ space) we notice that these functions are C^∞ in s, t, ω and satisfy

$$L_\omega U_{\omega L}=f_{\omega L} \qquad (\text{same with «}R\text{»})\,,$$

hence

$$P(D_x,D_t)\,U_{\omega L}(\omega\cdot x,t)=f_{\omega L}(\omega\cdot x,t)$$

(and the same with « R »), with the supports of $U_{\omega L}$ and $U_{\omega R}$ both contained in a neighborhood S_ω^ε of the set

$$S_\omega\colon\ |t-\tfrac{1}{2}T_1|\leqslant\bar{\lambda}_\omega|x\cdot\omega-m_\omega|\,.$$

Moreover S_ω^ε approaches S_ω as $\varepsilon\to 0$. A moment's reflection shows that for ε sufficiently small (which we now so choose), the set S_ω^ε is bounded away from the sets

$$\{|x\cdot\omega-m_\omega|\leqslant\tfrac{1}{2}d_\omega,\ t=0\}\,,$$

$$\{\qquad ''\qquad\quad ,\ t=T_1\}\,,$$

and hence from the sets

$$(*)\qquad\qquad \bar{D}\times(t=0)$$

and

$$(**)\qquad\qquad \bar{D}\times(t=T_1)\,,$$

which are contained in the above sets (respectively). It follows that the supports of $U_{\omega L}$ and $U_{\omega R}$ both are bounded away from $(*)$ and $(**)$, uniformly in ω. Hence, if we define

$$U(t,x)=\int\limits_\omega\big(U_{\omega L}(x\cdot\omega,t)+U_{\omega R}(x\cdot\omega,t)\big)\,d\omega\,,$$

U will be a C^∞ solution to

$$LU=\int f_\omega\,d\omega=f$$

vanishing in neighborhood of $(*)$ and $(**)$. Defining (as in the case $n=1$)

$$v=u\varphi-U$$

we see that v satisfies all requirements of the theorem.

5. – The parabolic case $n=1$.

To motivate our procedure, we rederive Frank Jones' result for the one dimensional heat equation along somewhat different lines. We wish to imitate the procedure followed in the hyperbolic case. We begin by solving the traditional initial value problem, prescribing as initial data $\delta(x)$, thus arriving at the standard fundamental solution $G_+(x,t)$, with support in $t \geqslant 0$. Our next step would be to multiply G_+ by a cut-off function $\varphi(t)$, this time choosing $\varphi(t)$ equal to one for $|t| \leqslant \varepsilon$ and equal to zero for $|t| \geqslant 2\varepsilon$. To proceed with the analogy, we then must look for the analogue of G_L and G_R with supports in neighborhoods of the negative and positive x axis respectively. Such fundamental solutions do not exist as distributions, but they do exist formally (see Ehrenpreis [1], p. 209)

$$\sum_{i=0}^{\infty} \delta^{(i)}(t) \frac{x_+^{2i+1}}{(2i+1)!}$$

where $\delta^{(i)}$ is the i-th derivative of $\delta(t)$ and $x_+ = \max(x, 0)$. This «fundamental solution» has support in $(x \geqslant 0, t = 0)$ and may be interpreted as a functional in a space of functions satisfying a Gevrey condition of index < 2 in t. A second «fundamental solution» having support on $(x \leqslant 0, t = 0)$, may be obtained by replacing x_+ by $x_- = |x| - x_+$. Let us hasten to add that the existence of these generalized fundamental solutions is nothing but a reformulation of the observation of Holmgren [5] to the effect that the Cauchy problem for the heat equation in one space dimension can be solved if Cauchy data in an appropriate Gevrey class is prescribed on the t axis.

Digression on the spaces γ^δ. From the above discussion it becomes clear that we should choose the cut-off function $\varphi(t)$ in an appropriate Gevrey class. Instead we choose to work with a variation of the γ^δ classes. (For these spaces and their properties see Hörmander [6], p. 146). We say $\varphi(t) \in C_0^\infty\{t\colon |t| \leqslant 1\}$ belongs to $\gamma_1^\delta \equiv \gamma_1^\delta(t)$ if for every $\theta > 0$ there is a constant C_θ such that

$$|D_t^j \varphi(t)| \leqslant C_\theta \theta^j j^{\delta j}, \qquad j = 1, 2, 3, \ldots,$$

We define $\gamma_2^\delta \equiv \gamma_2^\delta(x, t)$ as the space of continuous functions $f(x, t)$ belonging for each x to $\gamma_1^\delta(t)$ with bounds uniform in x, and $\gamma_3^\delta \equiv \gamma_3^\delta(x, t)$ the space of continuous functions belonging for each x to $\gamma_1^\delta(t)$ with bounds uniform in x on bounded subsets of x-space. *We shall always pick* $1 < \delta < 2$.

We now come back to the parabolic equation

$$Lu = Au - u_t = 0,$$

where A is a one dimensional differential operator with constant coefficients of order $m > 1$. (The procedure with some modifications, also works with *variable* coefficients depending on x only). We assume that the highest order term in A is of even order and has positive coefficients. Under these conditions L has a fundamental solution G_+ with support in $t \geqslant 0$ which is analytic for $t > 0$. We now proceed with the construction of formal fundamental solutions G_L, G_R analogous to the ones exhibited for the heat equation. Denoting $\partial/\partial t$ by T, we set

$$G_R = \sum_{i=0}^{\infty} T^{(i)}\, \delta(t)\, a_R^{-i-1}(x)$$

(with G_L defined similarly), where $a_R^{-i-1}(x)$ $\big(a_L^{-i-1}(x)\big)$ is the unique solution to $A^{i+1}a = \delta(x)$ with support on $x \geqslant 0$ $(x \leqslant 0)$. A simple computation verifies that these are indeed *formal* fundamental solutions.

As a convenience *we shall from now on always choose* ε *so that* $0 < \varepsilon < \frac{1}{2}$. Having chosen the cut-off function $\varphi(t) \in \gamma_1^\delta$ with $\varphi(t) = 0$ for $|t| \geqslant 2\varepsilon$ and $= 1$ for $|t| \leqslant \varepsilon$, we choose $\Psi(x) \in C^\infty$ equal to one for $x \leqslant -1$ and zero for $x \geqslant 1$, set

$$f_L = \Psi(x) L\big(G_+ \cdot \varphi(t)\big) , \qquad f_R = \big(1 - \Psi(x)\big) L\big(G_+ \cdot \varphi(t)\big) ,$$

and notice that these functions belong to γ_3^δ, (together with all their x derivatives) and have their supports in the sets (respectively)

$$\{\varepsilon \leqslant t \leqslant 2\varepsilon,\ x \leqslant 1\} ,$$

$$\{\quad \text{''} \quad ,\ x \geqslant -1\} .$$

We would like to define

$$U_L = G_L * f_L \qquad \text{and} \qquad U_R = G_R * f_R ,$$

provided we can attribute a meaning to these expressions, and then proceed as in the hyperbolic case. Formally,

$$U_R = \sum_{i=0}^{\infty} \int_0^{x+1} \frac{\partial^i}{\partial t^i} f_R(t, x - y)\, a_R^{-i-1}(y)\, dy ,$$

$$U_L = \sum_{i=0}^{\infty} \int_0^{x-1} \frac{\partial^i}{\partial t^i} f_L\big(t, (x - y)\big)\, a_L^{-i-1}(y)\, dy .$$

For y in a bounded interval I, we have the estimate

$$|a_R^{-i-1}(y)| \leqslant K_I^i \frac{y^{mi}}{(mi)!} \qquad (y > 0)$$

(similarly for a_L and $y < 0$). For $f_{R,L}$ we have the estimate

$$\left|\frac{\partial^i}{\partial t^i} f_{R,L}\right| \leqslant C_{\theta,I}\theta^i(i)^{\delta i}, \qquad (0 < \theta)$$

with a similar estimate holding for any finite number of x derivatives of f replacing f.

An application of Stirling's formula now insures the absolute and uniform convergence of the series defining $U_{R,L}$ in any finite interval I. Furthermore, the series may be differentiated with respect to x or t any number of times without affecting the convergence, thus justifying the hitherto formal relations

$$L(G_{L,R} * f_{L,R}) = f_{L,R},$$

Setting $v = \varphi G_+ - U_R - U_L$ we see that $Lv = 0$ for $t > 0$, v agrees with G_+ for $t < \varepsilon$ and vanishes for $t \geqslant 2\varepsilon$, hence is our desired fundamental solution. Let us note that an alternative to the use of G_L and G_R would have been to apply Theorem 5.73 of Hörmander [6]. However then the construction would not generalize to variable coefficients (for $n = 1$).

6. – The parabolic case $n > 1$.

If $Lu = A(D_x)u - u_t$, we assume that L is parabolic in the sense that $\operatorname{Re} A_m(\xi) \leqslant -c|\xi|^m$, A_m being the principal part of A. Under these conditions there exists a fundamental solution $G(x, t)$ with support in $t \geqslant 0$, analytic in x and t for $t > 0$. We shall use the following

Estimates for the standard fundamental solution:

$$\operatorname*{Max}_{0<t_0\leqslant t\leqslant t_1} |D_t^j G(x, t)| \leqslant CK^j j!\, h(x),$$

where $h(x) \to 0$ faster than any negative power of $|x|$ as $|x| \to \infty$, and where C and K may depend on t_0 and t_1. $D_x^k D_t^j G$ may be estimated similarly with C replaced by C_k. To see this we examine the proof (see for ex. [3]

or [4]) of the estimate

$$|G(x, t)| \leqslant \text{const} \exp[-\alpha|x|^h] \qquad (\alpha > 0)$$

valid for $0 < t_0 \leqslant t \leqslant t_1$, and some $h > 1$. We notice that the same estimate holds for *complex* t contained in a disk

$$|t - t_2| \leqslant \delta \qquad \text{with} \qquad t_0 \leqslant t_2 \leqslant t_1 ,$$

and δ sufficiently small, depending on t_0, t_1 but not on t_2. Standard estimates for derivatives of complex analytic function then give the first estimate. The same argument can be applied to $D_x^k G$, yielding different constants.

We now proceed as before.

Using Stirling's formula we see that $f(x, t) = L(\varphi(t) G)$ satisfies an estimate

$$\underset{0 < t_0 \leqslant t \leqslant t_1}{\text{Max}} |D_t^j D_x^k(\cdot)| \leqslant C_{\theta,k} \theta^j j^{\delta j} h(x)$$

for $\theta > 0$, with $h(x) \to 0$ faster than any negative power of $|x|$.

Using the estimates for $f_\omega(s, t)$ and its derivatives stated in the first part of section 4, we see that $f_\omega(s, t)$ and each s derivative belongs to $\gamma_2^\delta(s, t)$, *uniformly in* ω. The same may be said for

$$f_{\omega L} = \Psi(s) f_\omega(s, t) \qquad \text{and} \qquad f_{\omega R} = (1 - \Psi(s)) f_\omega(s, t)$$

where $\Psi(\cdot)$ is the same C^∞ cut-off function as in the case $n = 1$.

The operator

$$L_\omega = A_\omega(D_s) - \frac{\partial}{\partial t} ,$$

defined (as in section 4) by

$$L U(\omega \cdot x, t) \equiv L_\omega U(s, t) \qquad (\omega \cdot x = s)$$

is again parabolic, uniformly with respect to ω, and varies smoothy with ω. We form

$$U_{\omega R} = G_{\omega L} * f_{\omega L} \qquad \text{and} \qquad U_{\omega R} = G_{\omega R} * f_{\omega R}$$

as in the case $n = 1$. The series involved will converge absolutely and uniformly on bounded s intervals and uniformly in ω, and the same holds for the differentiated series. We use here the fact that because of the parabolicity assumption, the operator $A_\omega(D_s)$ has its highest order coefficient bounded away from zero, uniformly in ω, from which the estimate for $a_{\omega,L}^{-i-1}(s)$ (analogous to the one in section 5) can be made *uniform in* ω.

The desired fundamental solution is then

$$v = \varphi(t)\,G - \int_\omega (U_{\omega L} + U_{\omega R})\,d\omega\,.$$

We have thus proved

THEOREM 3. *Let $L = A(D_x) - \partial/\partial t$ be a parabolic operator (as defined earlier in this section) in $R^n \times (-\infty < t < \infty)$. Then given $\varepsilon > 0$ there exists a fundamental solution with support in the strip $0 \leqslant t \leqslant \varepsilon$, which is C^∞ away from the origin.*

The following theorem and corollary were proved by Frank Jones for the heat equation [7]. As he remarks, the proof remains valid for more general parabolic equations having fundamental solutions with support in $0 \leqslant t \leqslant \varepsilon$. Applying his proof to our operator L we obtain

THEOREM 4. *Let $L = P(D_x) - \partial/\partial t$ be parabolic in the sense of this section. Let f belong to $\mathcal{D}'(R^{n+1})$ and suppose* $\operatorname{supp} f \subset R^n \times [a, b)$. *Then there exists u in $\mathcal{D}'(R^{n+1})$ such that $Lu = f$ and* $\operatorname{supp} u \subset R^n \times [a, b]$.

COROLLARY. *Let g be a continuous function on R^n and $\varepsilon > 0$. Then there exists a continuous function u on $R^n \times [0, \infty)$ such that*

$$\begin{aligned} Lu &= 0 && \textit{on } R^n \times (0, \infty)\,, \\ u(x, 0) &= g(x) && \textit{for all } x \textit{ in } R^n\,, \\ u(x, t) &= 0 && \textit{for } x \textit{ in } R^n \textit{ and } t \geqslant \varepsilon\,. \end{aligned}$$

BIBLIOGRAPHY

[1] L. EHRENPREIS, *Fourier analysis in several complex variables*, New York, N. Y. (1970).

[2] H. O. FATTORINI - D. L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Archive Rat. Mech. Anal., **4** (1971), pp. 272-292.

[3] A. FRIEDMAN, *Generalized functions and partial differential equations*, Englewood Cliffs, N. J. (1963).

[4] I. M. Gelfand - G. E. Shilov, *Generalized Functions*, vol. 3, Moscow (1958).
[5] E. Holmgren, *Sur l'équation de la propagation de la chaleur*, Ark. Mat. Astr. Physik, **4** (1908), pp. 1-4.
[6] L. Hörmander, *Linear partial differential operators*, Academic Press Inc., New York, N. Y. (1963).
[7] B. F. Jones Jr., *A fundamental solution for the heat equation which is supported in a strip*, J. Math. Analysis and Applications, **60** (1977), pp. 314-324.
[8] D. Ludwig, *The Radon transform on Euclidean spaces*, Communications in Pure and Applied Math., **19** (1966), pp. 49-81.
[9] D. L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic equations*, Studies in Applied Math., **52** (1973), pp. 189-211.
[10] T. Seidman, *Boundary observation and control for the heat equation*: Differential games and control theory, Marcel Dekker, New York, N. Y. (1974), pp. 321-351.