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Boundary crisis bifurcation in two parameters

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Abstract

The boundary crisis bifurcation is well known as a mechanism for destroying (or creating) a strange attractor by variation of one parameter: at the moment of the boundary crisis bifurcation the strange attractor touches its own basin of attraction. Here we follow this codimension-one bifurcation in two parameters. One might expect that this leads to a smooth curve in the two-parameter plane. Mathematically, a boundary crisis is effectively a homoclinic or heteroclinic tangency, the locus of which is a well-defined smooth curve in a two-parameter system. However, instead of a boundary crisis, the transition through this tangency curve may lead to a basin boundary metamorphosis or an interior crisis bifurcation, in which the attractor persists. This phenomenon is again well known: at the point where the type of transition changes, the boundary crisis switches to another branch of homoclinic or heteroclinic tangencies, associated with manifolds of a periodic point with a different period than before. The curve of boundary crisis bifurcations is not differentiable at such points.

In this paper we show that the curve of boundary crisis bifurcations is, in fact, not even well defined as a piecewise-smooth curve. We illustrate that there are infinitely many gaps in much the same way as the one-parameter bifurcation diagram of the attractor contains infinitely many windows where the attractor is periodic and not strange or chaotic. Throughout, we use the Hénon map to illustrate our findings.

1 Introduction

The Hénon map is the paradigm example of an invertible map that exhibits the classic period-doubling route to chaos. First introduced by Hénon [8] in 1976 it is a simplified model of the Poincaré map associated with the Lorenz equations [9], with some magnification of the Jacobian in order to show the structure. The Hénon map is simple in the sense that it is a quadratic invertible planar map with constant Jacobian. Hénon defined the map as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 + v - a\,u^2 \\ b\,u \end{pmatrix}.$$

However, we prefer the format used in [1] that is obtained via the coordinate transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\,u \\ \frac{a}{b}\,v \end{pmatrix},$$

and leads to the Hénon map in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a + b\,y - x^2 \\ x \end{pmatrix}. \tag{1}$$

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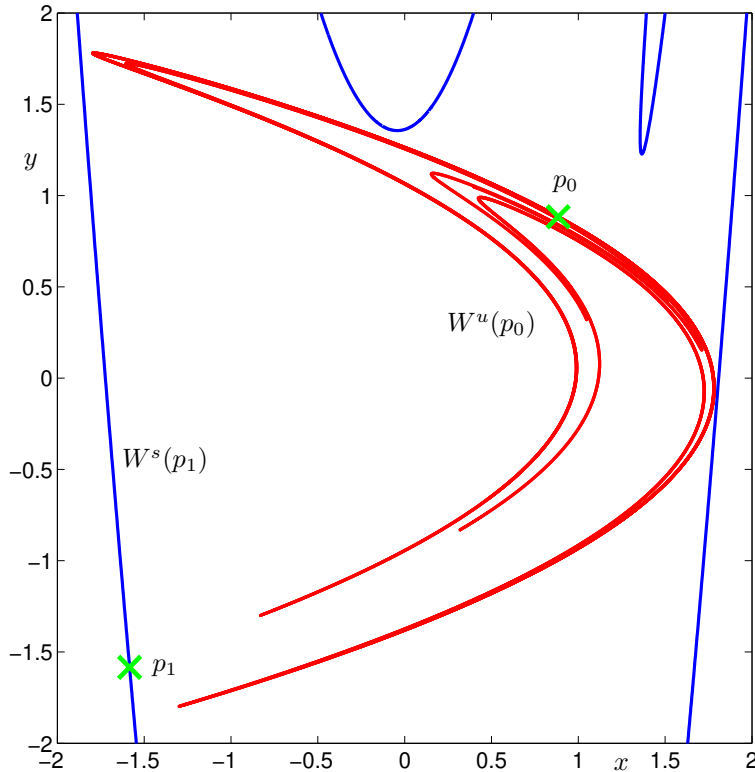
Figure 1: Fixed points and stable and unstable manifolds of the Hénon map (1) with $a = 1.4$ and $b = 0.3$. The closure of the unstable manifold $W^u(p_0)$ forms a chaotic attractor whose basin of attraction is bounded by the stable manifold $W^s(p_1)$.

We emphasize here that this coordinate transformation leaves the parameters unchanged. Hence, the two-parameter study of the boundary crisis bifurcation is the same for both maps.

Figure 1 gives an impression of the phase portrait for the classical choice of parameters, $a = 1.4$ and $b = 0.3$. For these parameter values the map has two fixed points p_0 and p_1 that are both saddles. There is one attractor, which is the chaotic attractor formed by the closure of the unstable manifold $W^u(p_0)$ of p_0. Not all initial conditions converge to the attractor; its basin is bounded by the stable manifold $W^s(p_1)$ of the other fixed point p_1 and all points outside this basin of attraction diverge off to infinity. Note that this picture can be obtained using Hénon's original coordinate system via a linear change of coordinates where the square $[-2, 2] \times [-2, 2]$ is mapped approximately to $[-1.43, 1.43] \times [-0.43, 0.43]$.

The three-dimensional bifurcation diagram in Figure 2 shows how the attractor varies with a. The pair of fixed points p_0 and p_1 are created in a saddle-node bifurcation at $a = -\frac{1}{4}(1 - b)^2 = -0.1225$. For a just larger than this value, the fixed point p_0 is stable and p_1 is a saddle. At $a = \frac{3}{4}(1 - b)^2 = 0.3675$ a period-doubling bifurcation occurs that makes p_0 a saddle as well. This is the start of a series of period-doubling bifurcations that eventually leads to the chaotic attractor shown in Figure 1. If we increase a slightly from 1.4 the attractor disappears in a boundary crisis bifurcation. As discussed in [12], the moment of the bifurcation is determined by the heteroclinic tangency of the closure of the stable manifold $W^s(p_1)$ with the closure of the unstable manifold $W^u(p_0)$. Simó reports in [12] that the tangency takes place at $a \approx 1.4269212$ and discusses the bifurcations for $b = 0.3$
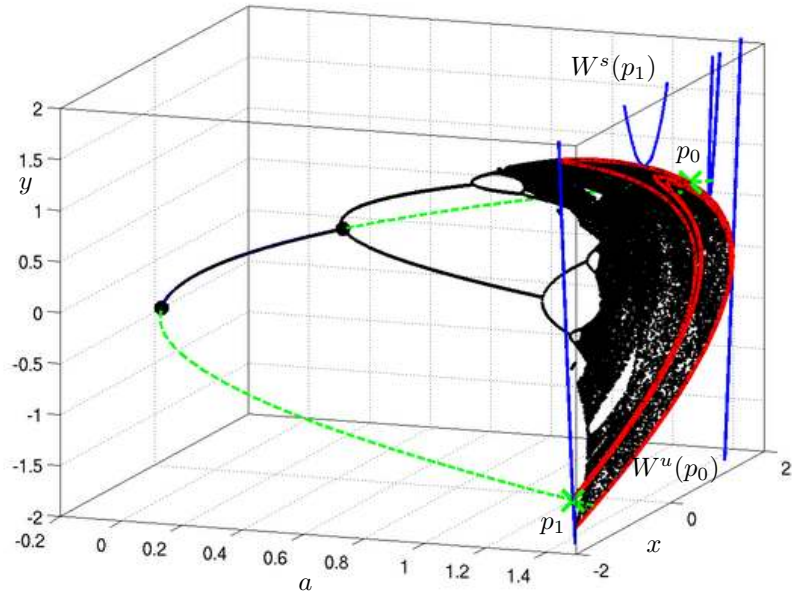
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Figure 2: The period-doubling sequence to chaos of the attractor for the Hénon map (1) with $b = 0.3$ and varying a. The branches of the fixed points p_0 and p_1 are shown also if they are unstable (dashed curves). At $a \approx 1.4269212$ a heteroclinic tangency between the closure of $W^u(p_0)$ and the stable manifolds $W^s(p_1)$ destroys the chaotic attractor.

in detail. Figure 2 illustrates the moment of the tangency by superimposing the manifolds onto the attractor.

We are interested in how the boundary crisis bifurcation is organised when the second parameter b is varied as well. Boundary crisis is a phenomenological definition that describes a topological change in the global phase portrait of a dynamical system. The actual bifurcation is a tangency between stable and unstable manifolds. However, whether such a homoclinic or heteroclinic bifurcation leads to a boundary crisis depends on other, typically global, considerations. Hence, in the above example of a boundary crisis bifurcation for $b = 0.3$, one can trace a smooth curve in the (a, b)-parameter plane that is the locus of the heteroclinic tangency between (the closures of) $W^u(p_0)$ and $W^s(p_1)$. Along this tangency curve the global manifestation of the bifurcation changes. The literature on two-parameter variation of a boundary crisis bifurcation reports that this change comes about when another curve of heteroclinic or homoclinic tangencies for a periodic orbit with a different period, transversely intersects the locus of the heteroclinic tangency between $W^u(p_0)$ and $W^s(p_1)$. Such an intersection point is called a double-crisis vertex and the locus of boundary crisis switches to the other curve. Hence, the two-parameter variation of the boundary crisis bifurcation leads to a continuous curve that is only piecewise smooth; see [3, 13] for more details.

In this paper we argue that the two-parameter variation of the boundary crisis bifurcation does not even lead to a continuous curve. In fact, we believe that there are infinitely many gaps in the "curve" in much the same ways as a one-parameter bifurcation diagram of a chaotic attractor is interspersed with infinitely many intervals where the dynamics is periodic. Namely, along the locus of the heteroclinic tangency between $W^u(p_0)$ and $W^s(p_1)$ there are intersections with saddle-node

bifurcation curves followed by a series of period-doubling bifurcation curves that form a strip in the (a, b)-parameter plane where the attractor is not chaotic and a boundary crisis bifurcation cannot take place. The intersection point of a saddle-node bifurcation curve with the locus of the heteroclinic tangency between $W^u(p_0)$ and $W^s(p_1)$ is not a switching point, but an *end point* of the locus of boundary crisis. It is an open question, but a distinct possibility, that the set of boundary crisis bifurcation points on the heteroclinic tangency curve is a Cantor set, that is, it does not contain any intervals.

This paper is organised as follows. In the next section we explain in more detail the different global manifestations of a heteroclinic or homoclinic tangency in the neighbourhood of a double-crisis vertex. Section 3 forms the core part of this paper, where we explain that the boundary crisis bifurcation in two parameters is not continuous. There are gaps along the curve where the attractor is clearly not chaotic and these gaps permeate well into the region where divergent behaviour is assumed. Section 4 discusses the consequences of this observation and relates our findings to studies of so-called strange nonchaotic attractors in quasiperiodically forced systems.

2   Two-parameter variations

The name boundary crisis bifurcation was introduced in [5]; both the boundary crisis bifurcation and the interior crisis bifurcation are discussed here. The two bifurcations are defined as *a collision between a chaotic attractor and a coexisting unstable fixed point or periodic orbit*; in a boundary crisis this causes the disappearance of the attractor, because the collision is with a point on the boundary of its basin of attraction, and in an interior crisis this causes a sudden increase (or decrease) in size of the attractor, since the colliding point lies inside the basin of attraction. The two types of crises are closely related, because they are both manifestations of a tangency between two manifolds.

A tangency between two manifolds is a structurally stable phenomenon when varying two parameters. Hence, one would expect that the boundary crisis bifurcation in the Hénon map persists if the second parameter b is varied. Such a study was done in [3, 13]. Both papers focus on the fact that the curve of boundary crises contains non-differentiable points, called *double-crisis vertices*, that are characterised by a *simultaneous* sudden change in the structure of both the attractor and its basin boundary. Figure 3 shows the double-crisis vertices that are given in [3] along a curve of boundary crisis bifurcations that we obtained by brute-force simulation. The moment of boundary crisis bifurcation was determined using bisection on a with tolerance of 10^{-6} for 21 values of $b \in [-1, 1]$. The disappearance of the attractor was decided based on 5×10^6 iterates of $50\,000$ uniformly distributed points on the line $\{0\} \times [-3, 3]$. This works well for $-0.8 \leq b \leq 0.8$. For values of b outside this interval, we used a 100×100 grid of points in the rectangle $[-1.5, 1.5] \times [-2, 2]$.

The dynamics near a double-crisis vertex is explained in detail in [13] for the vertex V_{-1}. (Note that there is an error in [13]: the b-values given for the phase portraits shown should all be multiplied with -1; compare also the discussion of V_{-1} in [3].) The essence here is to view V_{-1} as the intersection point of two crossing curves. Namely, the curve segment connecting V_{-2} with V_{-1} can be viewed as a tangency bifurcation curve involving (closures of) stable and unstable manifolds of two period-three saddles. This curve continues beyond V_{-1} where it becomes the locus of an interior crisis instead of a boundary crisis. Similarly, the curve segment between V_0 and V_{-1} is a tangency bifurcation curve involving the (closures of the) manifolds of two saddle fixed point. This curve also continues beyond V_{-1} where it becomes the locus of a so-called basin boundary metamorphosis [2, 6, 7]. In a basin
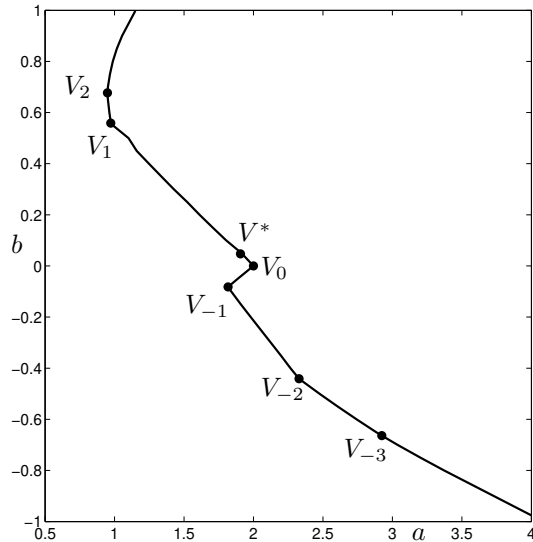
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Figure 3: A brute-force simulation produces a curve of boundary crisis bifurcations in the (a, b)-plane that contains non-differentiable points, called double-crisis vertices. For a-values to the left of this curve an attractor exists. The labelled points are the double-crisis vertices reported in [3], and the newly found double-crisis point V^*.

boundary metamorphosis, the attractor remains virtually the same, but its basin changes dramatically.

To illustrate these behaviours near a double-crisis vertex, we focus on a new double-crisis vertex $V^* \approx (1.90653, 0.04714)$ located just above V_0 in Figure 3. This vertex was not observed in previous studies.

In a small neighbourhood around V^* we distinguish four regions with topologically different phase portraits. The top right panel of Figure 4 shows these regions in the (a, b)-parameter plane. The region to the right corresponds to divergent dynamics. In the other three regions a strange attractor exists provided (a, b) is close enough to V^*. As one moves away from the bouble-crisis vertex while remaining in the same region, a (backward) period-doubling bifurcation may occur so that the attractor is no longer chaotic. In region I the strange attractor consists of three pieces. The phase portrait for $(a, b) = (1.90653, 0.05714)$ is shown in the top left panel of Figure 4. Here the parameters are so far away from V^* that the attractor is a period-three point; the attractor is not easy to see in this picture, but its coordinates are approximately: $(-0.01820, -1.42443) \mapsto (1.82481, -0.01820) \mapsto (-1.42443, 1.82481)$. All points that do not lie in the basin of attraction are coloured gray to reveal the fractal structure of the basin. Region II is illustrated with $(a, b) = (1.89653, 0.04714)$ in the bottom left panel of Figure 4. As one crosses from region I into region II a basin boundary metamorphosis takes place, which is clearly illustrated by the dramatic increase in size of the basin of attraction, and the three-piece attractor (here a period-twelve point) is relatively far away from the basin boundary. The boundary between regions I and II is the curve of heteroclinic tangencies between the manifolds of the fixed points. In region I the fixed point p_0 is not "accessible" [7] and cannot be the orbit that takes part in the boundary crisis. Indeed the solid curve that bounds region I is the locus of a heteroclinic tangency between manifolds of period-three orbits. The transition from region II to region III is also a tangency between the manifolds of these period-three orbits, but here the bifurcation causes an interior crisis instead. A representative phase portrait for region III is shown
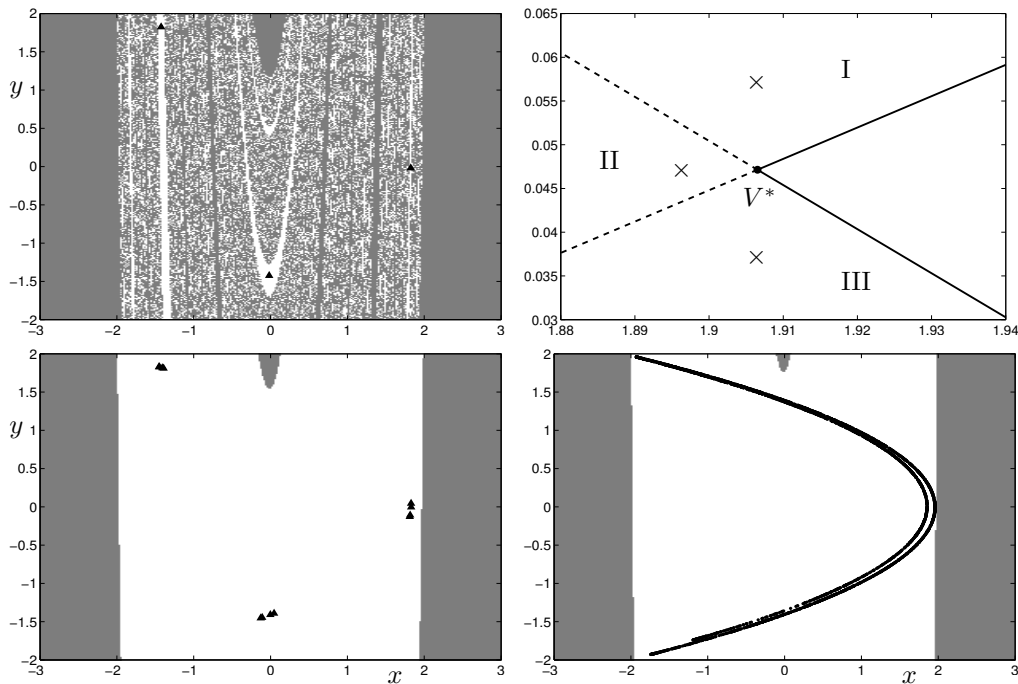
Figure 4: Dynamics of the Hénon map (1) for parameter values (marked with ×) near the double-crisis vertex $V^* \approx (a^*, b^*) = (1.90653, 0.04714)$. The parameter plane is shown in the top right picture. The phase portraits are for $(a^*, b^* + 0.01)$ in region I (top left), $(a^* - 0.01, b^*)$ in region II (bottom left) and $(a^*, b^* - 0.01)$ in region III (bottom right).

in the bottom right panel of Figure 4 where $(a, b) = (1.90653, 0.03714)$. The three distinct pieces of the attractor merged to one strange attractor which will disappear in a boundary crisis if one crosses into the divergent region.

From the point of view of a tangency between two manifolds, the double-crisis vertex V^* is the intersection of two smooth curves. One is defined by a tangency between the stable and unstable manifolds of two fixed points, and the other curve is characterised by a tangency between the manifolds of two period-three saddles.

3 Gaps in the curve of boundary crisis

The scenario near the double-crisis vertex $V^* \approx (1.90653, 0.04714)$, as described in the previous section is typical in the sense that all double-crisis vertices reported in [3] are intersections of two tangency curves involving stable and unstable manifolds of periodic orbits with two different periods. Unlike the other double-crisis vertices, V^* has a special property: consider the curve segment from V_0 to V^*. This segment is the locus of a boundary crisis bifurcation caused by a tangency between the manifolds of the two fixed points. It continues beyond V^* as a locus of basin boundary metamorphosis, but only until $E^* \approx (1.89, 0.055)$. After this point, the tangency curve again acts as the locus of boundary crisis bifurcations. More importantly, E^* is *not* a double-crisis vertex!

A detailed study of the rectangle $(a, b) \in [1.86, 1.97] \times [0.045, 0.065]$ reveals that the locus of boundary crises from V^* continues up into the divergence region of the (a, b)-parameter space. Figure 5 shows this part of the parameter space. The
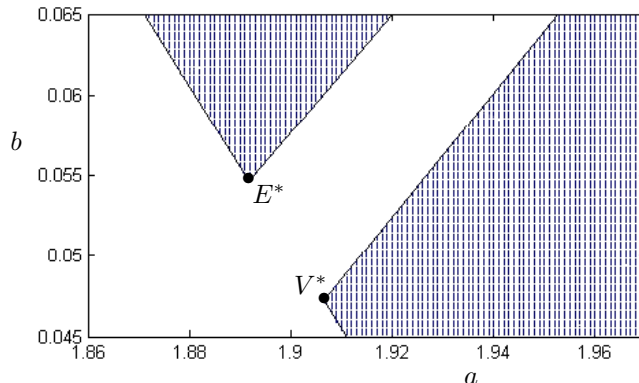
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Figure 5: A detail of the (a, b)-parameter plane showing the regions of divergent (shaded) and convergent dynamics (white) for the Hénon map (1). The attractor persists in a "corridor" that penetrates far into the divergent region. The end point E^* and the double-crisis vertex V^* mark the start of the corridor.

regions that correspond to divergent dynamics are shaded. The figure clearly shows that there is a "corridor" of parameter values for which the attractor persists. Inside the corridor, just to the left of the curve of boundary crises, the attractor is chaotic and consists of three pieces. As one moves further away from the bounding curve a series of (backward) period-doubling bifurcations takes place. The left boundary of the corridor is a locus of saddle-node bifurcations of a period-three orbit. Since the (non-chaotic) attractor disappears in a saddle-node bifurcation at this left boundary, the intersection point $E^* \approx (1.89, 0.055)$ with the heteroclinic tangency curve of the fixed points is not a double-crisis vertex. The locus of boundary crises ends at E^* and there is a distinct gap before it continues again.

We used CONTENT [4] to follow saddle-node and period-doubling bifurcation curves in two parameters. We first continued the period-doubling bifurcations of the fixed points and subsequently the period-2^k points for $k = 1, 2, 3$. These curves are shown in the top left panel of Figure 6. As expected from [3, 13], the curves lie entirely to the left of the locus of boundary crisis bifurcations. The top right panel in Figure 6 overlays the saddle-node and the first two period-doubling bifurcations for the period-three point. These curves intersect the locus of basin boundary bifurcations transversely and penetrate into the region with divergent dynamics exactly along the corridor emanating from V^*. Similarly, we continued the saddle-node and first period-doubling bifurcation curves for the period-five (bottom left) and period-seven points (bottom right). These curves intersect the locus of boundary crisis twice. This indicates that there are many more end points like E^* and corresponding double-crisis vertices V^* that lead to corridors where the attractor persists well into the region with divergent dynamics.

4   The locus of boundary crises

We studied the boundary crisis bifurcation with two varying parameters. While previous reports in the literature seem to indicate that a continuous curve of boundary crisis bifurcations exists with a finite number of non-differentiable points (double-crisis vertices), we found that there are gaps that open up to a corridor of attracting dynamics penetrating well into the region of divergent dynamics. One side of this corridor is indeed a curve of boundary crisis bifurcations, and characterised by the
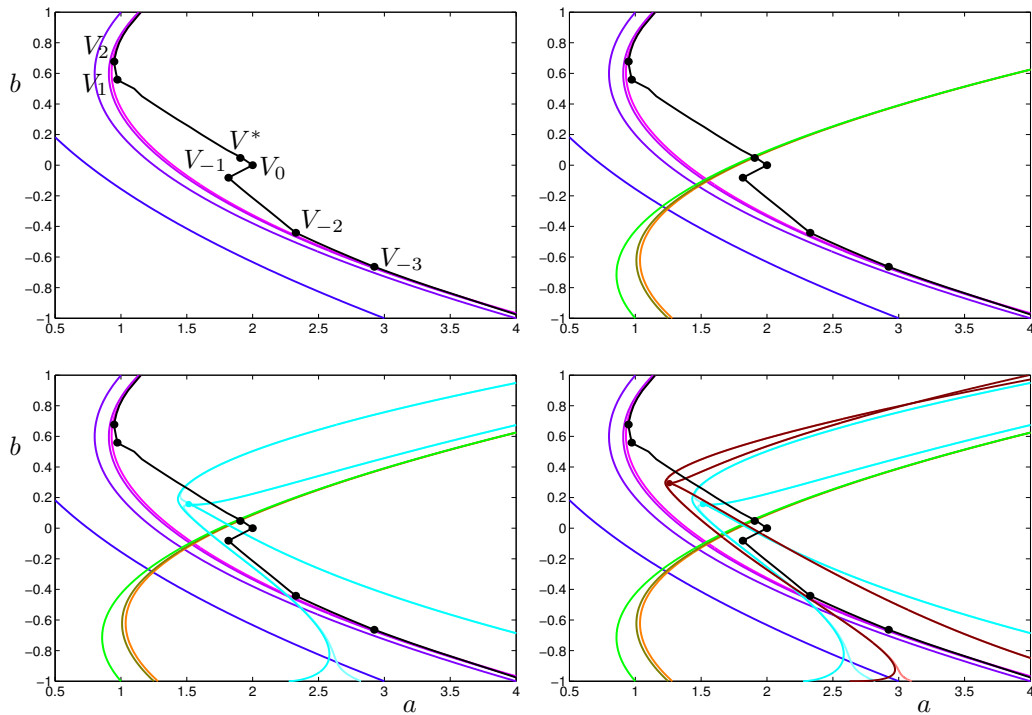
Figure 6: The locus of boundary crisis bifurcations for the Hénon map (1) is interspersed with gaps containing periodic dynamics that form corridors which penetrate far into the region with divergent dynamics. The top left panel shows the locus of boundary crisis bifurcations together with the period-doubling bifurcation curves for the period-one, period-two, period-four, and period-eight points. The other panels show in addition the saddle-node bifurcation and period-doubling bifurcation curves for the period-three point (top right), the period-five point (bottom left), and the period-seven point (bottom right), respectively.

tangency between stable and unstable manifolds of two periodic points with a fixed period. The other side of the corridor is a saddle-node bifurcation that creates the crisis orbit involved in this boundary crisis bifurcation.

As a result of our findings the locus of boundary crisis bifurcations looks very different from pictures published, for example, in [3, 13]. The segment that corresponds to a heteroclinic tangency between the (closures of) the stable and unstable manifolds of two fixed points is transversely intersected by at least five, but quite possibly infinitely many curves of saddle-node bifurcations. Each of these curves marks the onset of a period-doubling sequence to chaos starting from a periodic orbit with a particular period. The stable manifold of the periodic saddle that is created in the saddle-node bifurcation eventually becomes tangent to the unstable manifold of a saddle with the same period. The locus of this heteroclinic bifurcation is a smooth curve that corresponds to a boundary crisis. This curve also transversely intersects the heteroclinic bifurcation curve of the fixed points and the intersection point is a double-crisis vertex that has not previously been reported in the literature.

The intersection points of the saddle-node curves with the heteroclinic bifurcation curve of the fixed points are not double-crisis vertices. Here the locus of boundary crisis bifurcations ends and there is a small gap before the locus of boundary crises continues. It may well be that there are infinitely many such gaps along the curve of heteroclinic bifurcations of the stable and unstable manifolds of the fixed

points. The one-parameter bifurcation diagram of the Hénon map with $b = 0.3$ shows a period-doubling sequence to chaos, but the chaotic regime is interspersed with periodic windows where a new period-doubling scenario is initiated. When following this one-parameter variation in b, we find that these periodic windows persist right up to and even beyond the locus of boundary crisis that is defined by a heteroclinic tangency between the manifolds of the fixed points. It is known that the set of chaotic dynamics in the quadratic map is a Cantor set with positive Lebesgue measure [10]. Hence, the set of boundary crisis points along this heteroclinic bifurcation curve may well be a Cantor set.

In [11] a strange phenomenon was reported while studying the boundary crisis in two parameters for a three-dimensional quasi-periodically forced Hénon map. Here the simplest attractor is an invariant curve with quasi-periodic dynamics. As the curve undergoes a period-doubling sequence to chaos a boundary crisis bifurcation can be observed. In two parameters this bifurcation forms a curve, but the curve has a strange "bubble" where no chaotic dynamics could be traced. In [11] this phenomenon was thought to be due to the quasi-periodic forcing, but it matches very well with the results reported in this paper where no forcing is present. In fact, it may well be that the quasi-periodic forcing causes the corridor observed in the non-forced case to become a bubble, that is, closed at the "top" that penetrates into the divergent region. However, this conjecture needs further investigation.

Acknowledgements

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