BOUNDARY EFFECT OF RICCI CURVATURE

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Abstract. On a compact Riemannian manifold with boundary, we study how Ricci curvature of the interior affects the geometry of the boundary. First we establish integral inequalities for functions defined solely on the boundary and apply them to obtain geometric inequalities involving the total mean curvature. Then we discuss related rigidity questions and prove Ricci curvature rigidity results for manifolds with boundary.

1. Introduction and statement of results

In this paper, we consider the question how the Ricci curvature of a compact manifold with boundary affects the boundary geometry of the manifold. For the scalar curvature the same question is related to the quasi-local mass problem in general relativity. Indeed, much of the formulation of the results in this paper is motivated by that in [17, 20, 10].

We begin with integral inequalities that hold for functions solely defined on the boundary. For simplicity, all manifolds and functions in this paper are assumed to be smooth.

Theorem 1.1. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with nonempty boundary Σ. Let K be a constant that is a lower bound of the Ricci curvature of g, i.e.* $\mathrm{Ric} \geq Kg$. *Let H be the mean curvature of Σ in (Ω, g) with respect to the outward normal. Suppose $H > 0$. Given any function η on Σ, define*

$$A(\eta) = \int_\Sigma \frac{\eta^2}{H} d\sigma, \quad B(\eta) = \int_\Sigma \frac{\eta \Delta_\Sigma \eta}{H} d\sigma,$$

$$C(\eta) = \int_\Sigma \left[\frac{(\Delta_\Sigma \eta)^2}{H} - \mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) \right] d\sigma,$$

where ∇_Σ, Δ_Σ are the gradient, the Laplacian on Σ respectively, $\mathbb{II}$ is the second fundamental form of Σ and $d\sigma$ is the volume form on Σ.

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Then, for each nontrivial η, either

$$\left(\frac{B(\eta)}{A(\eta)}\right)^2 \leq \frac{C(\eta)}{A(\eta)} \tag{1.1}$$

or

$$\frac{1}{2}K \leq -\frac{B(\eta)}{A(\eta)} - \sqrt{\left(\frac{B(\eta)}{A(\eta)}\right)^2 - \frac{C(\eta)}{A(\eta)}}. \tag{1.2}$$

Remark 1.1. If the term $\mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta)$ were absent in $C(\eta)$, then (1.1) would always hold by Hölder inequality.

Remark 1.2. When Ω is the closure of a bounded domain in $\mathbb{R}^3$, the functional $C(\eta)$, up to a constant multiple of $\frac{1}{8\pi}$, is the 2nd variation of the Wang-Yau quasi-local energy ([21, 22]) at $\Sigma = \partial\Omega$ in $\mathbb{R}^{3,1}$, where $\mathbb{R}^{3,1}$ is the 4-dimensional Minkowski spacetime. (See [10, 11] for details.)

Remark 1.3. If Σ has a component Σ_0 on which $\mathbb{II} > 0$, then (1.1) always fails for an η which is a non-constant eigenfunction on Σ_0 and zero elsewhere. In this case, (1.2) yields estimates on the first nonzero eigenvalue of Σ_0. (See Corollary 2.1 for details.)

The conclusion of Theorem 1.1 is easily seen to be equivalent to a statement

$$\int_\Sigma \mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma \leq \int_\Sigma \frac{1}{H} \left(\Delta_\Sigma \eta + t\eta\right)^2 d\sigma \tag{1.3}$$

for all constants $t \leq \frac{1}{2}K$. In Theorem 2.1 of Section 2, we prove a more general version of (1.3) which allows $H \geq 0$. Interpreted this way, Theorem 1.1 and its generalization (Theorem 2.1) have natural applications to the total mean curvature of the boundary.

We first state the case of nonnegative Ricci curvature.

Theorem 1.2. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with nonnegative Ricci curvature, with connected boundary Σ which has nonnegative mean curvature H. Let $X : \Sigma \to \mathbb{R}^m$ be an isometric immersion of Σ into some Euclidean space $\mathbb{R}^m$ of dimension $m \geq n$. Then*

$$\int_\Sigma H \, d\sigma \leq \int_{\Sigma'} \frac{|\vec{H}_0|^2}{H} \, d\sigma, \tag{1.4}$$

where $\vec{H}_0$ is the mean curvature vector of the immersion X, $|\vec{H}_0|$ is the length of $\vec{H}_0$, and $\Sigma' = \{x \in \Sigma \mid \vec{H}_0(x) \neq 0\}$. Moreover, if equality in (1.4) holds, then

a) $H = |\vec{H}_0|$ *identically on* Σ.
b) (Ω, g) *is flat and* $X(\Sigma)$ *lies in an* n*-dimensional plane in* $\mathbb{R}^m$.
c) (Ω, g) *is isometric to a domain in* $\mathbb{R}^n$ *if* X *is an embedding.*

Remark 1.4. In light of the Nash imbedding theorem [12], the boundary Σ always admits an isometric immersion into some Euclidean space. Therefore, Theorem 1.2 applies to any compact Riemannian manifold with nonnegative Ricci curvature, with mean convex boundary (i.e. $H \geq 0$). One may compare Theorem 1.2 with the result in [17] in which a weaker curvature condition $R \geq 0$ is assumed, where R is the scalar curvature, while a more stringent boundary condition is imposed.

Remark 1.5. If (Ω, g) has nonnegative Ricci curvature and nonempty mean convex boundary, it was shown in [7, 8] (also cf. [5]) that $\partial\Omega$ has at most two components, and $\partial\Omega$ has two components only if (Ω, g) is isometric to $N \times I$ for a connected closed manifold N and an interval I. This is why we only consider connected boundary in Theorem 1.2.

Remark 1.6. Theorem 1.2 generalizes [5, Proposition 2], which proves that b) and c) hold under a pointwise assumption $H \geq |\vec{H}_0|$. Indeed the proof in [5] can be easily adapted to prove our Theorem 1.2.

Remark 1.7. If $H > 0$ on Σ, Theorem 1.2 implies $\int_\Sigma H d\sigma \leq C$ where $C > 0$ is a constant depending only on the induced metric on Σ and a positive lower bound of H.

Next, we give an analogous result for manifolds with positive Ricci curvature.

Theorem 1.3. *Let* (Ω, g) *be an* n*-dimensional, compact Riemannian manifold with positive Ricci curvature, with connected boundary* Σ *which has nonnegative mean curvature* H*. Let* $k > 0$ *be a constant such that*

$$\mathrm{Ric} \geq (n-1)kg.$$

Suppose there exists an isometric immersion $X : \Sigma \to \mathbb{S}^m_k$*, where* $\mathbb{S}^m_k$ *is the sphere of dimension* $m \geq n$ *with constant sectional curvature* k*. Then*

$$\int_\Sigma H \, d\sigma < \int_\Sigma \frac{|\vec{H}_{\mathbb{S}}|^2 + \frac{1}{4}(n-1)^2 k}{H} \, d\sigma, \tag{1.5}$$

where $\vec{H}_{\mathbb{S}}$ *is the mean curvature vector of the immersion* X*.*

Like (1.4), (1.5) imposes constraints on the boundary mean curvature when the Ricci curvature of the interior has a positive lower bound. For instance, consider the standard hemisphere $(\mathbb{S}^n_+, g_{\mathbb{S}})$ of dimension n.

Let $\Omega \subset \mathbb{S}^n_+$ be a smooth domain with connected boundary. It follows from Theorem 1.3 that there does *not* exist a metric g on Ω satisfying $\mathrm{Ric} \geq (n-1)$, $g|_{T\partial\Omega} = g_{\mathbb{S}}|_{T\partial\Omega}$ and $H \geq \sqrt{(H_{\mathbb{S}})^2 + \frac{1}{4}(n-1)^2}$, where H and $H_{\mathbb{S}}$ are the mean curvature of $\partial\Omega$ in (Ω, g) and $(\Omega, g_{\mathbb{S}})$ respectively. This could be compared with the first step, i.e. [1, Theorem 4], in the construction of the counterexample to Min-Oo's Conjecture, in which a metric on $\mathbb{S}^n_+$ is produced so that it satisfies $R \geq n(n-1)$, but the mean curvature of $\partial\mathbb{S}^n_+$ is raised to be everywhere positive. One may also compare this with the Ricci curvature rigidity theorems in [6].

When a manifold has negative Ricci curvature somewhere, we have

Theorem 1.4. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ which has nonnegative mean curvature H. Let $k > 0$ be a constant satisfying*

$$\mathrm{Ric} \geq -(n-1)kg.$$

Suppose Σ has a component Σ_0 which admits an isometric immersion

$$X = (t, x_1, \dots, x_n) : \Sigma_0 \longrightarrow \mathbb{H}^m_{-k} \subset \mathbb{R}^{m,1},$$

where $\mathbb{R}^{m,1}$ is the $(m+1)$-dimensional Minkowski spacetime with $m \geq n$ and

$$\mathbb{H}^m_{-k} = \left\{ (y_0, y_1, \dots, y_m) \subset \mathbb{R}^{m,1} \ \middle| \ -y_0^2 + \sum_{i=1}^m y_i^2 = -\frac{1}{k}, \ y_0 > 0 \right\}.$$

Then

$$\begin{aligned} &\int_{\Sigma_0} H d\sigma + \int_{\Sigma_0} \mathbb{II}(\nabla_{\Sigma} t, \nabla_{\Sigma} t) d\sigma \\ &< \int_{\Sigma_0'} \frac{1}{H} \left\{ |\vec{H}_{\mathbb{H}}|^2 - \frac{1}{4}(n-1)^2 k + \left[\Delta_{\Sigma} t - \frac{1}{2}(n-1)kt \right]^2 \right\} d\sigma, \end{aligned} \tag{1.6}$$

where $\vec{H}_{\mathbb{H}}$ is the mean curvature vector of the immersion X into $\mathbb{H}^m_{-k}$,

$$|\vec{H}_{\mathbb{H}}|^2 - \frac{1}{4}(n-1)^2 k + \left[\Delta_{\Sigma} t - \frac{1}{2}(n-1)kt \right]^2 \geq 0 \ \text{ on } \Sigma_0,$$

and Σ_0' is the set consisting of $x \in \Sigma_0$ such that

$$|\vec{H}_{\mathbb{H}}|^2(x) - \frac{1}{4}(n-1)^2 k + \left[\Delta_{\Sigma} t - \frac{1}{2}(n-1)kt \right]^2 (x) > 0.$$

Remark 1.8. The term $\int_{\Sigma_0} \mathbb{II}(\nabla_{\Sigma} t, \nabla_{\Sigma} t) d\sigma$ in (1.6) can be dropped if either $\mathbb{II} \geq 0$ or $X(\Sigma_0) \subset \mathbb{H}^m_{-k} \cap \{t = t_0\}$ for some constant t_0. For instance, this is the case if Σ_0 can be isometrically immersed in a sphere.

The fact that (1.5) and (1.6) are strict inequalities is due to the characterization of equality case in Theorem 2.1. This leads naturally to rigidity questions in the context of Theorem 1.3 and 1.4. We have the following two related results.

Theorem 1.5. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Suppose*

- $\mathrm{Ric} \geq (n-1)\, g$
- *there exists an isometric immersion $X : \Sigma \to \mathbb{S}^m$, where $\mathbb{S}^m$ is a standard sphere of dimension $m \geq n$*
- $\mathrm{I\!I}\,(v,v) \geq |\mathrm{I\!I}_{\mathbb{S}}\,(v,v)|$*, for any $v \in T\Sigma$. Here $\mathrm{I\!I}$ is the second fundamental form of Σ in (Ω, g) and $\mathrm{I\!I}_{\mathbb{S}}$ is the vector-valued, second fundamental form of the immersion X.*

Then (Ω, g) is spherical, i.e. having constant sectional curvature 1*. Moreover if Σ is simply connected, then (Ω, g) is isometric to a domain in $\mathbb{S}^n_+$.*

Theorem 1.6. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Suppose*

- $\mathrm{Ric} \geq -(n-1)\, g$
- *there exists an isometric immersion $X : \Sigma \to \mathbb{H}^m$, where $\mathbb{H}^m$ is a hyperbolic space of dimension $m \geq n$*
- $\mathrm{I\!I}\,(v,v) \geq |\mathrm{I\!I}_{\mathbb{H}}\,(v,v)|$*, for any $v \in T\Sigma$. Here $\mathrm{I\!I}$ is the second fundamental form of Σ in (Ω, g) and $\mathrm{I\!I}_{\mathbb{H}}$ is the vector-valued, second fundamental form of the immersion X.*

Then (Ω, g) is hyperbolic, i.e. having constant sectional curvature -1*. Moreover if Σ is simply connected, then (Ω, g) isometric to a domin in $\mathbb{H}^n$.*

The rest of this paper is organized as follows. In Section 2, we prove Theorem 2.1 which implies Theorem 1.1. In Section 3, we consider applications of Theorem 2.1 to the total boundary mean curvature and prove Theorem 1.2 – 1.4. In Section 4, we discuss the related rigidity question and prove Theorem 1.5 and 1.6.

2. A geometric Poincaré type inequality

The main result of this section is the following geometric Poincaré type inequality for functions defined on the boundary of a compact Riemannian manifold.

Theorem 2.1. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with nonempty boundary Σ. Suppose*

$$\mathrm{Ric} \geq (n-1)kg \quad \textit{and} \quad H \geq 0,$$

where Ric *is the Ricci curvature of g, k is some constant, and H is the mean curvature of Σ in (Ω, g) with respect to the outward normal. Suppose H is not identically zero. Then*

$$\int_\Sigma \mathbb{II}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma \leq \int_{\Sigma \setminus \left\{ \Delta_\Sigma \eta + t\eta = 0 \right\}} \frac{1}{H} \left(\Delta_\Sigma \eta + t\eta \right)^2 d\sigma \tag{2.1}$$

for any nontrivial function η on Σ and any constant $t \leq \frac{1}{2}(n-1)k$. Here $\mathbb{II}(\cdot, \cdot)$ is the second fundamental form of Σ, ∇_Σ and Δ_Σ denote the gradient and the Laplacian on Σ respectively. Moreover, equality in (2.1) holds only if either $k > 0$, $t = 0$ and η is a constant; or $k = t = 0$ and η is the boundary value of some function u on Ω satisfying $\nabla^2 u = 0$. Here ∇^2 denotes the Hessian on (Ω, g).

Remark 2.1. The case $k = 0$, $H > 0$ and $t = 0$ in (2.1) was first proved in [11] and is related to the second variation of Wang-Yau quasi-local energy [21, 22] at a closed 2-surface in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$.

Proof. The basic tool we use is Reilly's formula ([14])

$$\begin{aligned} &\int_\Omega \left[|\nabla^2 u|^2 - (\Delta u)^2 + \mathrm{Ric}(\nabla u, \nabla u) \right] dV \\ &= \int_\Sigma \left[-\mathbb{II}(\nabla_\Sigma u, \nabla_\Sigma u) - 2(\Delta_\Sigma u) \frac{\partial u}{\partial \nu} - H \left(\frac{\partial u}{\partial \nu} \right)^2 \right] d\sigma, \end{aligned} \tag{2.2}$$

which follows from integrating the Bochner formula. Here Δ, dV denote the Laplacian, the volume form on (Ω, g) respectively; ν is the outward unit normal to Σ, and u is any function defined on Ω.

Given any nontrivial η on Σ and any constant $\lambda \leq nk$, let u be the unique solution to

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } \Omega \\ u = \eta & \text{at } \Sigma. \end{cases} \tag{2.3}$$

The fact that (2.3) has a unique solution in the case $k > 0$ follows from another theorem of Reilly ([14, Theorem 4]) which states that the first Dirichlet eigenvalue λ_1 of Δ satisfies $\lambda_1 \geq nk$ and $\lambda_1 = nk$ if and only if (Ω, g) is isometric to a hemisphere in which case $\mathbb{II}$ is identically zero. Plug this u in (2.2), using the fact

$$|\nabla^2 u|^2 = \frac{1}{n}(\Delta u)^2 + |\nabla^2 u - \frac{1}{n}(\Delta u) g|^2,$$
$$\lambda \int_\Omega u^2 dV = \int_\Omega |\nabla u|^2 dV - \int_\Sigma u \frac{\partial u}{\partial \nu} d\sigma,$$

and the assumption $\mathrm{Ric} \geq (n-1)kg$, we have

(2.4)
$$\begin{aligned}&\left(1-\frac{1}{n}\right)(nk-\lambda)\int_\Omega |\nabla u|^2 dV + \int_\Omega |\nabla^2 u + \frac{1}{n}\lambda u g|^2 dV\\ \leq &\int_\Sigma \left[-\mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) - 2\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right)\frac{\partial u}{\partial \nu} - H\left(\frac{\partial u}{\partial \nu}\right)^2\right] d\sigma.\end{aligned}$$

Given any constant $\epsilon > 0$, (2.4) implies

(2.5)
$$\begin{aligned}&\int_\Sigma \left[-\mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) + \frac{1}{H+\epsilon}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right)^2 + \epsilon\left(\frac{\partial u}{\partial \nu}\right)^2\right] d\sigma\\ \geq &\int_\Sigma \left[\frac{1}{\sqrt{H+\epsilon}}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right) + \sqrt{H+\epsilon}\frac{\partial u}{\partial \nu}\right]^2 d\sigma\\ &+ \left(1-\frac{1}{n}\right)(nk-\lambda)\int_\Omega |\nabla u|^2 dV + \int_\Omega |\nabla^2 u + \frac{1}{n}\lambda u g|^2 dV\\ \geq &\ 0.\end{aligned}$$

Define $\Sigma_{\eta,\lambda} = \left\{x \in \Sigma \mid \Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta = 0\right\}$. By Lebesgue's monotone convergence theorem, we have

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\Sigma\setminus\Sigma_{\eta,\lambda}} \frac{1}{H+\epsilon}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right)^2 d\sigma\\ &= \int_{\Sigma\setminus\Sigma_{\eta,\lambda}} \frac{1}{H}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right)^2 d\sigma.\end{aligned} \tag{2.6}$$

Therefore, it follows from (2.5) and (2.6) that

$$\int_\Sigma \mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma \leq \int_{\Sigma\setminus\Sigma_{\eta,\lambda}} \frac{1}{H}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right)^2 d\sigma, \tag{2.7}$$

which proves (2.1) by setting $t = \frac{n-1}{2n}\lambda$.

Next, suppose

$$\int_\Sigma \mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma = \int_{\Sigma\setminus\Sigma_{\eta,\lambda}} \frac{1}{H}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right)^2 d\sigma. \tag{2.8}$$

In particular, this shows

$$\frac{1}{\sqrt{H}}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right) \in L^2(\Sigma \setminus \Sigma_{\eta,\lambda}) \tag{2.9}$$

and the set $\{x \in \Sigma \setminus \Sigma_{\eta,\lambda} \mid H(x) = 0\}$ has $d\sigma$-measure zero. Hence, by (2.4) and (2.8), we have

$$\begin{aligned} &\left(1-\frac{1}{n}\right)(nk-\lambda)\int_\Omega |\nabla u|^2 dV + \int_\Omega |\nabla^2 u + \frac{1}{n}\lambda u g|^2 dV \\ \leq\ & -\int_{\Sigma\setminus\Sigma_{\eta,\lambda}} \left[\frac{1}{\sqrt{H}}\left(\Delta_\Sigma \eta + \frac{n-1}{2n}\lambda\eta\right) + \sqrt{H}\frac{\partial u}{\partial \nu}\right]^2 d\sigma \\ & - \int_{\Sigma_{\eta,\lambda}} H \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma, \end{aligned} \tag{2.10}$$

which implies

$$(nk-\lambda)|\nabla u| = 0, \quad \nabla^2 u + \frac{1}{n}\lambda u g = 0 \quad \text{on } \Omega \tag{2.11}$$

and

$$\Delta_\Sigma \eta + H\frac{\partial u}{\partial \nu} + \frac{n-1}{2n}\lambda\eta = 0 \quad \text{at } \Sigma. \tag{2.12}$$

If $\lambda < nk$, (2.11) shows u is identically a constant, therefore $\lambda = 0$, $k > 0$ and η is a constant on Σ.

If $\lambda = nk$, (2.11) shows

$$\nabla^2 u + kug = 0 \ \text{ on } \Omega, \tag{2.13}$$

which implies

$$\Delta_\Sigma \eta + H\frac{\partial u}{\partial \nu} + (n-1)k\eta = 0 \ \text{ at } \Sigma. \tag{2.14}$$

Comparing (2.14) to (2.12) with $\lambda = nk$, we have $k = \lambda = 0$. This completes the proof. □

When $H > 0$, (2.1) is simplified to

$$\int_\Sigma \mathrm{I\!I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma \leq \int_\Sigma \frac{1}{H}\left(\Delta_\Sigma \eta + t\eta\right)^2 d\sigma.$$

In this case, Theorem 2.1 is equivalent to a statement that, given any nontrivial η on Σ, the quadratic form

$$Q_\eta(t) := A(\eta)t^2 + 2B(\eta)t + C(\eta)$$

satisfies

$$Q_\eta(t) \geq 0, \quad \forall\ t \leq \frac{1}{2}(n-1)k, \tag{2.15}$$

where

$$A(\eta) = \int_\Sigma \frac{\eta^2}{H} d\sigma, \qquad B(\eta) = \int_\Sigma \frac{\eta \Delta_\Sigma \eta}{H} d\sigma, \tag{2.16}$$

$$C(\eta) = \int_\Sigma \left[\frac{(\Delta_\Sigma \eta)^2}{H} - \mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) \right] d\sigma. \tag{2.17}$$

Clearly (2.15) is equivalent to asserting that, for each fixed η, either

$$B(\eta)^2 \leq A(\eta) C(\eta) \tag{2.18}$$

or

$$\frac{1}{2}(n-1)k \leq -\frac{B(\eta)}{A(\eta)} - \sqrt{\left(\frac{B(\eta)}{A(\eta)}\right)^2 - \frac{C(\eta)}{A(\eta)}}. \tag{2.19}$$

This explains how Theorem 1.1 follows from Theorem 2.1.

Next, we apply Theorem 2.1 to eigenvalue estimates on the boundary.

Corollary 2.1. *Let (Ω, g), Σ, k, H, $\mathrm{I\!I}$ be given as in Theorem 2.1. Suppose Σ has a component Σ_0 which is convex, i.e. $\mathrm{I\!I} > 0$ on Σ_0. Let $\kappa > 0$ be a constant such that $\mathrm{I\!I} \geq \kappa \gamma$, where γ is the induced metric on Σ_0. Let λ be a positive eigenvalue of Δ_Σ on (Σ_0, γ). If $\kappa^2 + 2k > 0$, then*

$$\lambda \notin \left(\frac{1}{4}(n-1)\left[\kappa - \sqrt{\kappa^2 + 2k}\right]^2, \frac{1}{4}(n-1)\left[\kappa + \sqrt{\kappa^2 + 2k}\right]^2 \right). \tag{2.20}$$

In particular, if $k \geq 0$, the first nonzero eigenvalue $\lambda_1(\Sigma_0)$ of (Σ_0, γ) satisfies

$$\lambda_1(\Sigma_0) \geq \frac{1}{4}(n-1)\left[\kappa + \sqrt{\kappa^2 + 2k}\right]^2. \tag{2.21}$$

Proof. By defining $\eta = 0$ everywhere on $\Sigma \setminus \Sigma_0$, Theorem 2.1 implies

$$\int_{\Sigma_0} \mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma \leq \int_{\Sigma_0} \frac{1}{H} \left(\Delta_\Sigma \eta + t\eta\right)^2 d\sigma,$$

for any η defined on Σ_0 and any $t \leq \frac{1}{2}(n-1)k$. Let $A(\eta)$, $B(\eta)$ and $C(\eta)$ be given in (2.16) and (2.17) with Σ replaced by Σ_0. Suppose η is a nonzero eigenfunction, i.e. $\Delta_\Sigma \eta + \lambda \eta = 0$. Then

$$B(\eta)^2 - A(\eta)C(\eta) = \left(\int_{\Sigma_0} \frac{\eta^2}{H} d\sigma \right) \left(\int_{\Sigma_0} \mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma \right) > 0.$$

Therefore, (2.19) holds, which shows

$$\frac{1}{2}(n-1)k \leq \lambda - \left(\frac{\int_{\Sigma_0} \mathrm{I\!I}(\nabla_\Sigma \eta, \nabla_\Sigma \eta) d\sigma}{\int_{\Sigma_0} \frac{\eta^2}{H} d\sigma} \right)^{\frac{1}{2}}. \tag{2.22}$$

On the other hand,

$$\int_{\Sigma_0} \mathrm{III}(\nabla_{\Sigma}\eta, \nabla_{\Sigma}\eta) d\sigma \geq \kappa \int_{\Sigma_0} |\nabla_{\Sigma}\eta|^2 d\sigma = \kappa\lambda \int_{\Sigma_0} \eta^2 d\sigma \tag{2.23}$$

and

$$\int_{\Sigma_0} \frac{\eta^2}{H} d\sigma \leq \frac{1}{(n-1)\kappa} \int_{\Sigma_0} \eta^2 d\sigma. \tag{2.24}$$

Hence, (2.22) – (2.24) imply

$$\frac{1}{2}(n-1)k \leq \lambda - \kappa\sqrt{(n-1)\lambda}. \tag{2.25}$$

When $\kappa^2 + 2k > 0$, it follows from (2.25) that

$$\sqrt{\lambda} \notin \left(\frac{1}{2}\sqrt{n-1}\left[\kappa - \sqrt{\kappa^2+2k}\right], \frac{1}{2}\sqrt{n-1}\left[\kappa + \sqrt{\kappa^2+2k}\right] \right),$$

which completes the proof. □

Remark 2.2. Corollary 2.1 is motivated by results in [3, 23]. If $k = 0$, (2.21) reduces to $\lambda_1(\Sigma_0) \geq (n-1)\kappa^2$ which is the estimate in [23].

3. Application to total mean curvature

In this section, we recall the statement of Theorem 1.2 – 1.4 and give their proof. We begin with the case of nonnegative Ricci curvature.

Theorem 3.1. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with nonnegative Ricci curvature, with connected boundary Σ which has nonnegative mean curvature H. Let $X : \Sigma \to \mathbb{R}^m$ be an isometric immersion of Σ into some Euclidean space $\mathbb{R}^m$ of dimension $m \geq n$. Then*

$$\int_{\Sigma} H \, d\sigma \leq \int_{\Sigma'} \frac{|\vec{H}_0|^2}{H} \, d\sigma, \tag{3.1}$$

where $\vec{H}_0$ is the mean curvature vector of the immersion X and $\Sigma' \subset \Sigma$ is the set $\{\vec{H}_0(x) \neq 0\}$. Moreover, if equality in (3.1) holds, then

a) *$H = |\vec{H}_0|$ identically on Σ.*
b) *(Ω, g) is flat and $X(\Sigma)$ lies in an n-dimensional plane in $\mathbb{R}^m$.*
c) *(Ω, g) is isometric to a domain in $\mathbb{R}^n$ if X is an embedding.*

Proof. Since X is an isometric immersion, one has

$$\Delta_{\Sigma} X = \vec{H}_0. \tag{3.2}$$

At any $x \in \Sigma$, let $\{v_\alpha \mid \alpha = 1, \ldots, n-1\} \subset T_x\Sigma$ be an orthonormal frame that diagonalizes III, i.e. $\mathrm{III}(v_\alpha, v_\beta) = \delta_{\alpha\beta}\kappa_\alpha$ where $\{\kappa_1, \ldots, \kappa_{n-1}\}$

are the principal curvature of Σ in (Ω, g) at x. Let $\{e_1, \ldots, e_m\}$ denote the standard basis in $\mathbb{R}^m$ and x_i be the i-th component of X. Then

$$\begin{aligned} \sum_{i=1}^{m} \mathrm{III}(\nabla_{\Sigma} x_i, \nabla_{\Sigma} x_i) &= \sum_{i=1}^{m} \sum_{\alpha,\beta=1}^{n-1} \mathrm{III}(v_\alpha, v_\beta) \langle e_i, v_\alpha \rangle \langle e_i, v_\beta \rangle \\ &= \sum_{\alpha=1}^{n-1} \kappa_\alpha = H. \end{aligned} \tag{3.3}$$

Set $k = 0$ in Theorem 2.1 and choose $\eta = x_i$, $t = 0$ in (2.1), we have

$$\int_{\Sigma} \mathrm{III}(\nabla_{\Sigma} x_i, \nabla_{\Sigma} x_i) d\sigma \leq \int_{\Sigma} \frac{1}{H} \left(\Delta_{\Sigma} x_i \right)^2 1_{\Sigma_i'} d\sigma \tag{3.4}$$

where $1_{\Sigma_i'}$ is the characteristic function of the set $\Sigma_i' = \Sigma \setminus \{\Delta_{\Sigma} x_i = 0\}$. Summing (3.4) over i, using (3.2), (3.3) and the fact $\Sigma_i' \subset \Sigma'$, we have

$$\int_{\Sigma} H d\sigma \leq \int_{\Sigma'} \frac{1}{H} |\vec{H}_0|^2 d\sigma, \tag{3.5}$$

which proves (3.1).

Next suppose

$$\int_{\Sigma} H d\sigma = \int_{\Sigma'} \frac{1}{H} |\vec{H}_0|^2 d\sigma. \tag{3.6}$$

Then it follows from (3.4) that

$$\int_{\Sigma} \mathrm{III}(\nabla_{\Sigma} x_i, \nabla_{\Sigma} x_i) d\sigma = \int_{\Sigma} \frac{1}{H} \left(\Delta_{\Sigma} x_i \right)^2 1_{\Sigma_i'} d\sigma, \quad \forall\ i. \tag{3.7}$$

By the rigidity part of Theorem 2.1, there exist functions u_i, $1 \leq i \leq m$, such that $u_i = x_i$ at Σ and

$$\nabla^2 u_i = 0 \text{ on } \Omega. \tag{3.8}$$

Moreover, by (2.12) or (2.14), we have

$$\vec{H}_0 + H d\Phi(\nu) = 0 \text{ at } \Sigma, \tag{3.9}$$

where $\Phi : \Omega \to \mathbb{R}^m$ is a (harmonic) map defined by $\Phi = (u_1, \ldots, u_m)$, $d\Phi = (du_1, \ldots, du_m)$ is the associated tangent map, and ν is the unit outward normal to Σ in (Ω, g).

We claim

$$d\Phi(\nu)(x) \neq 0, \quad \forall\ x \in \Sigma. \tag{3.10}$$

To see this, first consider a point $y \in \Sigma'$ ($\Sigma' \neq \emptyset$ by (3.2)). At y, (3.9) implies

$$d\Phi(\nu)(y) \neq 0 \ \text{ and } \ d\Phi(\nu)(y) \perp X(\Sigma). \tag{3.11}$$

Hence, the rank of $d\Phi$ at y is n by (3.11) and the fact $\Phi|_\Sigma = X$. On the other hand, (3.8) shows du_i is parallel on Ω, $\forall\ i$. Therefore, the rank of $d\Phi$ equals n everywhere on Ω. In particular, this proves (3.10).

By (3.9) and (3.10), we now have

$$\{x \in \Sigma \mid H(x) \neq 0\} = \Sigma'. \tag{3.12}$$

Thus (3.6) becomes

$$\int_{\Sigma'} H d\sigma = \int_{\Sigma'} H|d\Phi(\nu)|^2 d\sigma \tag{3.13}$$

by (3.9). As Σ' is a nonempty open set in Σ, (3.12) and (3.13) imply

$$|d\Phi(\nu)|(z) = 1 \text{ and } d\Phi(\nu)(z) \perp X(\Sigma),\ \forall\ z \in \Sigma'. \tag{3.14}$$

It follows from (3.14), (3.9) and (3.12) that $H = |\vec{H}_0|$ identically on Σ.

The rest of the claims now follows from [5, Proposition 2]. For completeness, we include the proof. By (3.14) and the fact $\Phi|_\Sigma = X$, one knows $g = \sum_{i=1}^m du_i \otimes du_i$ at Σ'. As a result, $g = \sum_{i=1}^m du_i \otimes du_i$ on Ω as both tensors are parallel. Clearly this shows (Ω, g) is flat and Φ is an isometric immersion. Next, let v, w be any tangent vectors to Ω. (3.8) implies

$$0 = vw(\Phi) - \nabla_v w(\Phi) = \overline{\nabla}_{d\Phi(v)}(d\Phi(w)) - d\Phi(\nabla_v w), \tag{3.15}$$

where ∇ and $\overline{\nabla}$ denote the connection on (Ω, g) and $\mathbb{R}^m$ respectively. By definition, (3.15) shows $\Phi : \Omega \to \mathbb{R}^m$ is totally geodesic. Therefore, $\Phi(\Omega)$ (hence $X(\Sigma)$) lies in an n-dimensional plane in $\mathbb{R}^m$. Without losing generality, one can assume $\Phi(\Omega) \subset \mathbb{R}^n$. If $X : \Sigma \to \mathbb{R}^m$ is indeed an embedding, then $X(\Sigma) = \partial W$ where W is the closure of a bounded domain in $\mathbb{R}^n$. Since Φ is an immersion and $\Phi|_\Sigma = X$, one can show $\Phi(\Omega) \subset W$ and $\Phi : \Omega \setminus \Sigma \to W \setminus \partial W$ (by checking that $\Phi(\Omega) \setminus W$ is both open and closed in $\mathbb{R}^n \setminus W$). On the other hand, Φ being a local isometry implies $\Phi : \Omega \to W$ is a covering map. Therefore, Φ is a homeomorphism, and hence an isometry between Ω and W. □

Theorem 3.2. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with positive Ricci curvature, with connected boundary Σ which has nonnegative mean curvature H. Let $k > 0$ be a constant such that*

$$\mathrm{Ric} \geq (n-1)kg.$$

Suppose there exists an isometric immersion $X : \Sigma \to \mathbb{S}^m_k$, where $\mathbb{S}^m_k$ is the sphere of dimension $m \geq n$ with constant sectional curvature k. Then

$$\int_\Sigma H\ d\sigma < \int_\Sigma \frac{|\vec{H}_{\mathbb{S}}|^2 + \frac{1}{4}(n-1)^2 k}{H}\ d\sigma, \tag{3.16}$$

where $\vec{H}_{\mathbb{S}}$ is the mean curvature vector of the immersion X into $\mathbb{S}^m_k$.

Proof. We identify $\mathbb{S}^m_k$ with the sphere of radius $\frac{1}{\sqrt{k}}$ centered at the origin in $\mathbb{R}^{m+1}$, i.e. $\mathbb{S}^m_k = \left\{(y_1, \ldots, y_{m+1}) \subset \mathbb{R}^{m+1} \mid \sum_{i=1}^{m+1} y_i^2 = \frac{1}{k}\right\}$ and view

$$X = (x_1, \ldots, x_{m+1}) : \Sigma \longrightarrow \mathbb{S}^m_k \subset \mathbb{R}^{m+1}$$

as an isometric immersion of Σ into $\mathbb{R}^{m+1}$. Let $\vec{H}_0$ denote the mean curvature vector of $X : \Sigma \to \mathbb{R}^{m+1}$, then

$$\vec{H}_0 = \vec{H}_{\mathbb{S}} + k\langle \vec{H}_0, X\rangle X, \tag{3.17}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{m+1}$. Apply the fact

$$\Delta_\Sigma X = \vec{H}_0 \quad \text{and} \quad \langle X, X\rangle = \frac{1}{k}, \tag{3.18}$$

we have

$$\begin{aligned} 0 &= \sum_{i=1}^{m+1} \left(x_i \Delta_\Sigma x_i + |\nabla_\Sigma x_i|^2 \right) \\ &= \langle \vec{H}_0, X\rangle + (n-1). \end{aligned} \tag{3.19}$$

In Theorem 2.1, choose $\eta = x_i$ and $t = \frac{1}{2}(n-1)k > 0$ in (2.1), we have

$$\int_\Sigma \mathrm{III}(\nabla_\Sigma x_i, \nabla_\Sigma x_i) d\sigma < \int_{\Sigma'} \frac{1}{H} \left[\Delta_\Sigma x_i + \frac{1}{2}(n-1)k x_i \right]^2 1_{\Sigma'_i} d\sigma \tag{3.20}$$

where $1_{\Sigma'_i}$ is the characteristic function of the set

$$\Sigma'_i = \Sigma \setminus \left\{ \Delta_\Sigma x_i + \frac{1}{2}(n-1)k x_i = 0 \right\}.$$

Summing (3.20) over i and using (3.17) – (3.19), we have

$$\begin{aligned} \int_\Sigma H d\sigma &< \int_\Sigma \frac{1}{H} \left[|\vec{H}_0|^2 + (n-1)k\langle \vec{H}_0, X\rangle + \frac{1}{4}(n-1)^2 k^2 |X|^2 \right] d\sigma \\ &= \int_\Sigma \frac{1}{H} \left[|\vec{H}_{\mathbb{S}}|^2 + \frac{1}{4}(n-1)^2 k \right] d\sigma, \end{aligned}$$

where we have also used (3.3). This proves (3.16). □

Theorem 3.3. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ which has nonnegative mean curvature H. Let $k > 0$ be a constant satisfying*

$$\mathrm{Ric} \geq -(n-1)kg.$$

Suppose Σ has a component Σ_0 which admits an isometric immersion

$$X = (t, x_1, \ldots, x_n) : \Sigma_0 \longrightarrow \mathbb{H}^m_{-k} \subset \mathbb{R}^{m,1},$$

where $\mathbb{R}^{m,1}$ is the $(m+1)$-dimensional Minkowski spacetime with $m \geq n$ and

$$\mathbb{H}^m_{-k} = \left\{ (y_0, y_1, \ldots, y_m) \subset \mathbb{R}^{m,1} \ \Big| \ -y_0^2 + \sum_{i=1}^m y_i^2 = -\frac{1}{k}, \ y_0 > 0 \right\}.$$

Then

$$\begin{aligned} & \int_{\Sigma_0} H d\sigma + \int_{\Sigma_0} \mathrm{III}(\nabla_{\Sigma} t, \nabla_{\Sigma} t) d\sigma \\ < \ & \int_{\Sigma_0'} \frac{1}{H} \left\{ |\vec{H}_{\mathbb{H}}|^2 - \frac{1}{4}(n-1)^2 k + \left[\Delta_{\Sigma} t - \frac{1}{2}(n-1)kt \right]^2 \right\} d\sigma, \end{aligned} \tag{3.21}$$

where $\vec{H}_{\mathbb{H}}$ is the mean curvature vector of the immersion X into $\mathbb{H}^m_{-k}$,

$$|\vec{H}_{\mathbb{H}}|^2 - \frac{1}{4}(n-1)^2 k + \left[\Delta_{\Sigma} t - \frac{1}{2}(n-1)kt \right]^2 \geq 0 \quad \text{on } \Sigma_0, \tag{3.22}$$

and Σ_0' is the set consisting of all $x \in \Sigma_0$ such that

$$|\vec{H}_{\mathbb{H}}|^2(x) - \frac{1}{4}(n-1)^2 k + \left[\Delta_{\Sigma} t - \frac{1}{2}(n-1)kt \right]^2 (x) > 0.$$

Proof. Let $\vec{H}_{\mathbb{M}}$ be the mean curvature vector of $X : \Sigma_0 \to \mathbb{R}^{m,1}$, then

$$\vec{H}_{\mathbb{M}} = \vec{H}_{\mathbb{H}} - k\langle \vec{H}_{\mathbb{M}}, X \rangle X, \tag{3.23}$$

and

$$\Delta_{\Sigma} X = \vec{H}_{\mathbb{M}}, \quad \langle X, X \rangle = -\frac{1}{k}, \tag{3.24}$$

where $\langle \cdot, \cdot \rangle = -dy_0^2 + \sum_{i=1}^m dy_i^2$ is the Lorentzian product on $\mathbb{R}^{m,1}$. By (3.24), we have

$$\begin{aligned} 0 = & -(t\Delta_{\Sigma} t + |\nabla_{\Sigma} t|^2) + \sum_{i=1}^m \left(x_i \Delta_{\Sigma} x_i + |\nabla_{\Sigma} x_i|^2 \right) \\ = & \ \langle \vec{H}_{\mathbb{M}}, X \rangle + (n-1). \end{aligned} \tag{3.25}$$

At any $x \in \Sigma_0$, let $\{v_\alpha \mid \alpha = 1, \ldots, n-1\}$ be an orthonormal frame in $T_x\Sigma_0$ such that $\mathrm{III}(v_\alpha, v_\beta) = \delta_{\alpha\beta}\kappa_\alpha$ where $\{\kappa_1, \ldots, \kappa_{n-1}\}$ are the principal curvature of Σ_0 in (Ω, g) at x. We have

$$\mathrm{III}(\nabla_{\Sigma} t, \nabla_{\Sigma} t) = \sum_{\alpha=1}^{n-1} \kappa_\alpha \langle \partial_{y_0}, v_\alpha \rangle^2 \tag{3.26}$$

and

$$\begin{aligned}\sum_{i=1}^{m}\mathbb{III}(\nabla_{\Sigma}x_i,\nabla_{\Sigma}x_i) &= \sum_{\alpha=1}^{n-1}\kappa_\alpha\left(\sum_{i=1}^{m}\langle\partial_{y_i},v_\alpha\rangle^2\right)\\ &= \sum_{\alpha=1}^{n-1}\kappa_\alpha\left(1+\langle\partial_{y_0},v_\alpha\rangle^2\right).\end{aligned}\tag{3.27}$$

Therefore,

$$\int_{\Sigma_0}Hd\sigma+\int_{\Sigma_0}\mathbb{III}(\nabla_{\Sigma}t,\nabla_{\Sigma}t)d\sigma=\sum_{i=1}^{m}\int_{\Sigma_0}\mathbb{III}(\nabla_{\Sigma}x_i,\nabla_{\Sigma}x_i)d\sigma.\tag{3.28}$$

Now choose $\eta = x_i$ on Σ_0, $\eta = 0$ on $\Sigma\setminus\Sigma_0$, and $t=-\frac{1}{2}(n-1)k<0$ in Theorem 2.1, we have

$$\int_{\Sigma_0}\mathbb{III}(\nabla_{\Sigma}x_i,\nabla_{\Sigma}x_i)d\sigma < \int_{\Sigma'_{0i}}\frac{1}{H}\left[\Delta_{\Sigma}x_i-\frac{1}{2}(n-1)kx_i\right]^2 1_{\Sigma'_{0i}}d\sigma\tag{3.29}$$

where $1_{\Sigma'_{0i}}$ is the characteristic function of the set

$$\Sigma'_{0i}=\Sigma_0\setminus\left\{\Delta_{\Sigma}x_i-\frac{1}{2}(n-1)kx_i=0\right\}.$$

Direct calculation using (3.23) – (3.25) shows

$$\begin{aligned}&\sum_{i=1}^{m}\left[\Delta_{\Sigma}x_i-\frac{1}{2}(n-1)kx_i\right]^2\\ &=|\vec{H}_{\mathbb{H}}|^2-\frac{1}{4}(n-1)^2k+\left[\Delta_{\Sigma}t-\frac{1}{2}(n-1)kt\right]^2,\end{aligned}\tag{3.30}$$

which also proves (3.22). Summing (3.29) over $i=1,\dots,m$ and using (3.28) – (3.30) together with the fact $\Sigma'_{0i}\subset\Sigma'_0$, we have

$$\begin{aligned}&\int_{\Sigma_0}Hd\sigma+\int_{\Sigma_0}\mathbb{III}(\nabla_{\Sigma}t,\nabla_{\Sigma}t)d\sigma\\ &<\int_{\Sigma'_0}\frac{1}{H}\left\{|\vec{H}_{\mathbb{H}}|^2-\frac{1}{4}(n-1)^2k+\left[\Delta_{\Sigma}t-\frac{1}{2}(n-1)kt\right]^2\right\}d\sigma.\end{aligned}$$

This completes the proof. □

4. Rigidity results

Inequalities (3.16) and (3.21) are not sharp in the context of Theorem 3.2 and Theorem 3.3. In these cases, one wonders if there exist sharp integral inequalities involving H and $|\vec{H}_{\mathbb{S}}|$ (or $|\vec{H}_{\mathbb{H}}|$) which include a rigidity statement in the case of equality.

In what follows, by scaling the metric, we assume $\mathrm{Ric} \geq (n-1)g$ or $\mathrm{Ric} \geq -(n-1)g$. In the latter case, the scalar curvature R of g satisfies $R \geq -n(n-1)$. By the results in [20, 18, 9], there exists a sharp integral inequality relating H and $|\vec{H}_{\mathbb{H}}|$ if the manifold Ω is spin and the boundary Σ embeds isometrically in the hyperbolic space $\mathbb{H}^n$ as a convex hypersurface. On the other hand, the counterexample to Min-Oo's conjecture in [1] shows that even the pointwise condition $H = |\vec{H}_{\mathbb{S}}|$ is not sufficient to guarantee rigidity if one only assumes $R \geq n(n-1)$. This gives rise to the following rigidity question:

Question 4.1. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Let $D \subset \mathbb{S}^n_+$ be a bounded domain with smooth boundary ∂D, where $\mathbb{S}^n_+$ is the standard n-dimensional hemisphere. Suppose*

- $\mathrm{Ric} \geq (n-1)\, g$
- *there exists an isometry $X : \Sigma \to \partial D$*
- *$H \geq H_{\mathbb{S}} \circ X$, where H, $H_{\mathbb{S}}$ are the mean curvature of Σ, ∂D in (Ω, g), $\mathbb{S}^n_+$ respectively.*

Is (Ω, g) isometric to D in $\mathbb{S}^n_+$?

At this stage, we do not know the answer to Question 4.1. However, it was shown in [6] that Question 4.1 has an affirmative answer if the assumption $H \geq H_{\mathbb{S}} \circ X$ is replaced by a stronger assumption on the second fundamental forms.

Theorem 4.1 ([6])**.** *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Let $D \subset \mathbb{S}^n_+$ be a bounded domain with smooth boundary ∂D, where $\mathbb{S}^n_+$ is the standard n-dimensional hemisphere. Suppose*

- $\mathrm{Ric} \geq (n-1)\, g$
- *there exists an isometry $X : \Sigma \to \partial D$*
- *$\mathbb{II} \geq \mathbb{II}_{\mathbb{S}} \circ X$, where $\mathbb{II}$, $\mathbb{II}_{\mathbb{S}}$ are the second fundamental form of Σ, ∂D in (Ω, g), $\mathbb{S}^n_+$ respectively.*

Then (Ω, g) is isometric to D in $\mathbb{S}^n_+$.

In the rest of this section, we prove Theorem 1.5 and 1.6, which are analogues of Theorem 4.1 when the boundary is only isometrically immersed in a sphere or in a hyperbolic space of higher dimension.

Theorem 4.2. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Suppose*

- $\mathrm{Ric} \geq (n-1)\, g$
- *there exists an isometric immersion $X : \Sigma \to \mathbb{S}^m$, where $\mathbb{S}^m$ is a standard sphere of dimension $m \geq n$*
- $\mathbb{II}\,(v,v) \geq |\mathbb{II}_{\mathbb{S}}\,(v,v)|$, *for any $v \in T\Sigma$. Here $\mathbb{II}$ is the second fundamental form of Σ in (Ω, g) and $\mathbb{II}_{\mathbb{S}}$ is the vector-valued, second fundamental form of the immersion X.*

Then (Ω, g) is spherical, i.e. having constant sectional curvature 1.

We divide the proof of Theorem 4.2 into a few steps. First, we fix some notations. Let $\overline{\nabla}$ denote the covariant derivative on $\mathbb{S}^m$, which is identified with the unit sphere centered at the origin in $\mathbb{R}^{m+1}$. Given any $\alpha = (\alpha_1, \ldots, \alpha_{m+1}) \in \mathbb{S}^m$, let $F = F_\alpha$ be the restriction of the linear function $\langle \alpha, x\rangle = \alpha_1 x_1 + \cdots + \alpha_{m+1} x_{m+1}$ to $\mathbb{S}^m$. The gradient of F on $\mathbb{S}^m$, denoted by $\overline{\nabla} F$, is

$$\overline{\nabla} F\,(x) = \alpha - \langle \alpha, x\rangle\, x, \;\; x \in \mathbb{S}^m. \tag{4.1}$$

On Σ, define $f = F \circ X$. For simplicity, given any $p \in \Sigma$, we let $\overline{\nabla}^{\perp} F$ be the component of $\overline{\nabla} F(X(p))$ normal to $X_*\,(T_p\Sigma)$. Given $v, w \in T_p\Sigma$, recall that $\mathbb{II}_{\mathbb{S}}\,(v,w) = \left(\overline{\nabla}_{X_*(v)} X_*(w)\right)^{\perp}$ is the component of $\overline{\nabla}_{X_*(v)} X_*(w)$ normal to $X_*\,(T_p\Sigma)$. We let $\vec{H}_{\mathbb{S}}$ denote the mean curvature vector of X, which is the trace of $\mathbb{II}_{\mathbb{S}}$.

Lemma 4.1. *Along Σ, one has*

$$f^2 + |\nabla_{\Sigma} f|^2 + \left|\overline{\nabla}^{\perp} F\right|^2 = 1, \tag{4.2}$$

$$\Delta_{\Sigma} f + (n-1)\, f - \left\langle \vec{H}_{\mathbb{S}}, \overline{\nabla}^{\perp} F \right\rangle = 0, \tag{4.3}$$

$$\left\langle \overline{\nabla}_{X_*(\nabla_{\Sigma} f)} \overline{\nabla}^{\perp} F, \vec{n} \right\rangle + \left\langle \mathbb{II}_{\mathbb{S}}\,(\nabla_{\Sigma} f, \nabla_{\Sigma} f)\,, \vec{n}\right\rangle = 0. \tag{4.4}$$

Here $\vec{n}$ is any vector that is normal to $X(\Sigma)$ in $\mathbb{S}^m$.

Proof. (4.2) follows from the fact $F^2 + \left|\overline{\nabla} F\right|^2 = 1$. To show (4.3) and (4.4), we note that F on $\mathbb{S}^m$ satisfies

$$\overline{\nabla}^2 F = -F g_{\mathbb{S}}, \tag{4.5}$$

where $g_{\mathbb{S}}$ is the standard metric on $\mathbb{S}^m$. (4.5) readily implies

$$\nabla^2_{\Sigma} f(v,w) - \langle \mathbb{II}_{\mathbb{S}}(v,w), \overline{\nabla}^{\perp} F\rangle = -f\langle v, w\rangle, \quad \forall\; v, w \in T\Sigma, \tag{4.6}$$

where ∇^2_{Σ} denotes the Hessian on Σ. Taking trace of (4.6) gives (4.3). (4.5) also implies ,

$$\begin{aligned} 0 &= \overline{\nabla}^2 F\left(X_*(\nabla_{\Sigma} f), \vec{n}\right) \\ &= \left\langle \overline{\nabla}_{X_*(\nabla_{\Sigma} f)} \overline{\nabla}^{\perp} F, \vec{n} \right\rangle + \left\langle \mathbb{II}_{\mathbb{S}}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right), \vec{n} \right\rangle, \end{aligned}$$

which proves (4.4). □

The condition $\mathbb{II}(v,v) \geq |\mathbb{II}_{\mathbb{S}}(v,v)|$, $\forall\, v \in T\Sigma$, implies $H \geq |\vec{H}_{\mathbb{S}}| \geq 0$. By Reilly's theorem ([14, Theorem 4]), to prove Theorem 4.2, it suffices to assume $\lambda_1 > n$, where λ_1 is the first Dirichlet eigenvalue of (Ω, g). Under this assumption, we let u be the unique solution to

$$\left\{ \begin{array}{rcl} \Delta u + nu &=& 0 \ \text{ on } \Omega, \\ u &=& f \ \text{ at } \Sigma. \end{array} \right. \tag{4.7}$$

On (Ω, g), define

$$\phi = |\nabla u|^2 + u^2.$$

A basic fact about ϕ is that it is subharmonic, which follows from

$$\frac{1}{2}\Delta\phi = \left|\nabla^2 u + ug\right|^2 + \mathrm{Ric}(\nabla u, \nabla u) - (n-1)\,|\nabla u|^2 \geq 0. \tag{4.8}$$

On Σ, define $\chi = \dfrac{\partial u}{\partial \nu}$, where ν is the unit outward normal to Σ in (Ω, g). Then

$$\phi|_{\Sigma} = |\nabla_{\Sigma} f|^2 + \chi^2 + f^2 = 1 + \chi^2 - \left|\overline{\nabla}^{\perp} F\right|^2 \tag{4.9}$$

by (4.2) in Lemma 4.1.

Lemma 4.2. *Along Σ, the normal derivative of ϕ is given by*

$$\frac{1}{2}\frac{\partial \phi}{\partial \nu} = \langle \nabla_{\Sigma} f, \nabla_{\Sigma} \chi \rangle - \mathbb{II}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right) - H\chi^2 - \left\langle \vec{H}_{\mathbb{S}}, \overline{\nabla}^{\perp} F \right\rangle \chi.$$

Proof. Direct calculation gives

$$\begin{aligned} \frac{1}{2}\frac{\partial \phi}{\partial \nu} &= \nabla^2 u\left(\nabla u, \nu\right) + f\chi \\ &= \nabla^2 u\left(\nabla_{\Sigma} f, \nu\right) + \chi\left[\nabla^2 u\left(\nu, \nu\right) + f\right] \\ &= \langle \nabla_{\Sigma} f, \nabla_{\Sigma} \chi \rangle - \mathbb{II}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right) + \chi\left[\nabla^2 u\left(\nu, \nu\right) + f\right]. \end{aligned} \tag{4.10}$$

By (4.2) and (4.7), at Σ we have

$$\begin{aligned} -nf = \Delta u &= \Delta_{\Sigma} f + H\chi + \nabla^2 u\left(\nu, \nu\right) \\ &= -\left(n-1\right) f + \left\langle \vec{H}_{\mathbb{S}}, \overline{\nabla}^{\perp} F \right\rangle + H\chi + \nabla^2 u\left(\nu, \nu\right), \end{aligned}$$

which gives

$$\nabla^2 u\,(\nu,\nu) + f = -\left\langle \vec{H}_{\mathbb{S}}, \overline{\nabla}^{\perp} F\right\rangle - H\chi. \tag{4.11}$$

The lemma follows from (4.10) and (4.11). □

Proof of Theorem 4.2. Given any $q \in \Sigma$, we choose $\alpha = X(q) \in \mathbb{S}^m$. Then $\overline{\nabla} F(X(q)) = 0$ by (4.1). Hence,

$$\nabla_{\Sigma} f(q) = 0,\ \overline{\nabla}^{\perp} F(q) = 0 \ \text{ and } \ \phi(q) = 1 + \chi^2(q). \tag{4.12}$$

Consider $p \in \Sigma$ such that $\phi(p) = \max_{\Omega} \phi$. By (4.9) and (4.12),

$$\chi^2(p) \geq \left|\overline{\nabla}^{\perp} F\right|^2 (p). \tag{4.13}$$

Since $\langle \nabla_{\Sigma} f, \nabla_{\Sigma}\phi\rangle(p) = 0$, taking $\vec{n} = \overline{\nabla}^{\perp} F$ in (4.4), at p we have

$$\begin{aligned} \chi \left\langle \nabla_{\Sigma} f, \nabla_{\Sigma}\chi\right\rangle &= \left\langle \overline{\nabla}_{X_*(\nabla_{\Sigma} f)} \overline{\nabla}^{\perp} F, \overline{\nabla}^{\perp} F\right\rangle \\ &= -\left\langle \mathrm{I\!I\!I}_{\mathbb{S}}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right), \overline{\nabla}^{\perp} F\right\rangle. \end{aligned} \tag{4.14}$$

If $\chi(p) \neq 0$, it follows from Lemma 4.2, (4.13) and (4.14) that

$$\begin{aligned} \frac{1}{2}\frac{\partial\phi}{\partial\nu}(p) &= -\frac{1}{\chi}\left\langle \mathrm{I\!I\!I}_{\mathbb{S}}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right), \overline{\nabla}^{\perp} F\right\rangle - \mathrm{I\!I\!I}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right) \\ &\quad - \left\langle \vec{H}_{\mathbb{S}}, \overline{\nabla}^{\perp} F\right\rangle \chi - H\chi^2 \\ &\leq \left|\mathrm{I\!I\!I}_{\mathbb{S}}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right)\right| - \mathrm{I\!I\!I}\left(\nabla_{\Sigma} f, \nabla_{\Sigma} f\right) + \left(\left|\vec{H}_{\mathbb{S}}\right| - H\right)\chi^2 \\ &\leq 0. \end{aligned} \tag{4.15}$$

If $\chi(p) = 0$, then Lemma 4.2 gives

$$\frac{1}{2}\frac{\partial\phi}{\partial\nu}(p) = \langle \nabla_{\Sigma} f, \nabla_{\Sigma}\chi\rangle - \mathrm{I\!I\!I}(\nabla_{\Sigma} f, \nabla_{\Sigma} f). \tag{4.16}$$

Moreover, $\overline{\nabla}^{\perp} F(p) = 0$ by (4.13). Taking the second order derivative of ϕ along $\nabla_{\Sigma} f$ at p, we have

$$\begin{aligned} 0 &\geq \frac{1}{2}\nabla_{\Sigma} f(\nabla_{\Sigma} f(\phi))(p) \\ &= \nabla_{\Sigma} f\left(\chi\langle \nabla_{\Sigma} f, \nabla_{\Sigma}\chi\rangle - \langle \overline{\nabla}_{X_*(\nabla_{\Sigma} f)} \overline{\nabla}^{\perp} F, \overline{\nabla}^{\perp} F\rangle\right) \\ &= \langle \nabla_{\Sigma} f, \nabla_{\Sigma}\chi\rangle^2 - \left|\overline{\nabla}_{X_*(\nabla_{\Sigma} f)} \overline{\nabla}^{\perp} F\right|^2. \end{aligned} \tag{4.17}$$

We claim, at p,

$$(\overline{\nabla}_{X_*(\nabla_\Sigma f)}\overline{\nabla}^\perp F) = -\mathbb{II}_{\mathbb{S}}\left(\nabla_\Sigma f, \nabla_\Sigma f\right). \tag{4.18}$$

To see this, take any $v \in T\Sigma$, (4.5) and (4.9) imply

$$\begin{aligned} &\left\langle \overline{\nabla}_{X_*(\nabla_\Sigma f)}\overline{\nabla}^\perp F, X_*(v)\right\rangle \\ =\ & \overline{\nabla}^2 F(X_*(\nabla_\Sigma f), X_*(v)) - \left\langle \overline{\nabla}_{X_*(\nabla_\Sigma f)} X_*(\nabla_\Sigma f), X_*(v)\right\rangle \\ =\ & -fv(f) - \nabla_\Sigma^2 f(\nabla_\Sigma f, v) \\ =\ & -\frac{1}{2} v\left(f^2 + |\nabla_\Sigma f|^2\right) = -\frac{1}{2} v\left(\phi - \chi^2\right). \end{aligned} \tag{4.19}$$

Clearly, $v\left(\phi - \chi^2\right)$ varnishes at p. Hence, $(\overline{\nabla}_{X_*(\nabla_\Sigma f)}\overline{\nabla}^\perp F)(p)$ is normal to $X_*(T_p\Sigma)$. This, together with (4.4), implies (4.18). Now it follows from (4.16), (4.17) and (4.18) that

$$\frac{1}{2}\frac{\partial \phi}{\partial \nu}(p) \le |\mathbb{II}_{\mathbb{S}}\left(\nabla_\Sigma f, \nabla_\Sigma f\right)| - \mathbb{II}\left(\nabla_\Sigma f, \nabla_\Sigma f\right) \le 0.$$

By the strong maximum principle (precisely the Hopf boundary point lemma), we conclude that ϕ must be a constant. Hence, $\nabla^2 u = -ug$ by (4.8). Moreover, by (4.9) and (4.12),

$$\chi^2 - \left|\overline{\nabla}^\perp F\right|^2 = c \tag{4.20}$$

for some constant $c \ge 0$. We have the following two cases:

If $c > 0$, then $\chi^2 > |\overline{\nabla}^\perp F|^2 \ge 0$. This together with (4.15) and the fact ϕ is a constant implies $|\vec{H}_{\mathbb{S}}| = H = 0$. Since $\mathbb{II} \ge 0$, we have $\mathbb{II} = 0$ and $\mathbb{II}_{\mathbb{S}} = 0$. Thus, $X : \Sigma \to \mathbb{S}^m$ is totally geodesic. Hence $X(\Sigma)$ lies in an $(n-1)$-dimensional standard sphere $\mathbb{S}^{n-1} \subset \mathbb{S}^m$. Since $X : \Sigma \to \mathbb{S}^{n-1}$ is an isometric immersion, we have $X(\Sigma) = \mathbb{S}^{n-1}$; moreover $X : \Sigma \to \mathbb{S}^{n-1}$ is one-to-one as $\mathbb{S}^{n-1}$ is simply connected. Therefore, Σ is isometric to $\mathbb{S}^{n-1}$ and is totally geodesic in (Ω, g). By [6, Theorem 2], we conclude that (Ω, g) is isometric to a standard hemisphere $\mathbb{S}^n_+$.

If $c = 0$, then $\phi = 1$ on Σ (and hence on Ω). In this case, along Σ,

$$|\nabla u|^2 = |\nabla_\Sigma f|^2 + \chi^2 = |\nabla_\Sigma f|^2 + \left|\overline{\nabla}^\perp F\right|^2 = \left|\overline{\nabla} F\right|^2 \circ X.$$

In particular, $\nabla u\,(q) = 0$ by (4.12). We also note that $u\,(q) = f(q) = 1$ by the definition of F.

Finally, we are in a position to show (Ω, g) has constant sectional curvature 1. It suffices to assume (Ω, g) is not isometric to $\mathbb{S}^n_+$. Given any x in the interior of Ω, let $q_x \in \Sigma$ such that $dist(x, \Sigma) = dist(x, q_x)$, where $dist(\cdot, \cdot)$ denotes the distance functional on (Ω, g). Consider the function f and u constructed in the above proof by taking $q = q_x$. Since (Ω, g) is not isometric to $\mathbb{S}^n_+$, the constant c in (4.20) must be 0, hence u satisfies

$$\nabla^2 u = -ug, \ \nabla u(q_x) = 0, \quad u(q_x) = 1. \tag{4.21}$$

Let $\gamma : [0, L] \to (\Omega, g)$ be the geodesic satisfying $\gamma(0) = q_x$, $\gamma(L) = x$ and $L = dist(x, \Sigma)$. Let $\xi = \gamma'(0)$. Given any constant $l \in (0, L)$, there exists an open neighborhood W of ξ in $\mathbb{S}^{n-1}$ such that the exponential map $exp_{q_x}(\cdot, \cdot)$ is a diffeomorphism from $(0, l) \times W \subset \mathbb{R}^+ \times \mathbb{S}^{n-1}$ onto its image in (Ω, g). Now it is a standard fact that (4.21) implies

$$(\exp_{q_x})^*(g) = dr^2 + (\sin r)^2 g_{_{\mathbb{S}^{n-1}}} \tag{4.22}$$

on $(0, l) \times W$, where $g_{_{\mathbb{S}^{n-1}}}$ is the standard metric on $\mathbb{S}^{n-1}$ (cf. [6] for details). Therefore, g has constant sectional curvature 1 at $\gamma(t)$ for any $t < L$. By continuity, g has constant sectional curvature 1 at x. This completes the proof that Ω is spherical. □

As an application, we have the following rigidity result which is a spherical analogue of [5, Theorem 1].

Corollary 4.1. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Suppose*

- $\mathrm{Ric} \geq (n-1)g$
- *g has constant sectional curvature 1 at every point on Σ.*

If Σ is simply connected with nonnegative second fundamental form $\mathbb{II}$, then (Ω, g) is isometric to a domain in $\mathbb{S}^n_+$.

Proof. Let $R^\Sigma(\cdot, \cdot, \cdot, \cdot)$, ∇^Σ denote the curvature tensor, the connection on Σ respectively. By the Gauss equation and the Codazzi equation,

$$\begin{aligned} R^\Sigma(v_1, v_2, v_3, v_4) = {} & \langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle \\ & + \mathbb{II}(v_1, v_3)\,\mathbb{II}(v_2, v_4) - \mathbb{II}(v_1, v_4)\,\mathbb{II}(v_2, v_3), \\ 0 = {} & \left(\nabla^\Sigma_{v_1}\mathbb{II}\right)(v_2, v_3) - \left(\nabla^\Sigma_{v_2}\mathbb{II}\right)(v_1, v_3) \end{aligned}$$

where $v_1, \ldots v_4 \in T\Sigma$. As Σ is simply connected, the fundamental theorem of hypersurfaces (cf. [19]) implies there exists an isometric immersion $\Phi : \Sigma \to \mathbb{S}^n$ with $\mathbb{II}$ as its second fundamental form. Since $\mathbb{II} \geq 0$, by a result of Do Carmo and Warner in [4], Φ is an embedding and $\Phi(\Sigma)$ is a convex hypersurface in a hemisphere $\mathbb{S}^n_+$. Now apply Theorem 4.2, we conclude that (Ω, g) has constant sectional curvature 1 everywhere. Let D be the region bounded by $\Phi(\Sigma)$ in $\mathbb{S}^n_+$. We glue Ω and $\mathbb{S}^n \setminus D$ along the boundary via the isometric embedding Φ to get a closed manifold $(\widetilde{M}, \tilde{g})$. The fact that Σ has the same second fundamental form in (Ω, g) and $\mathbb{S}^n$ and that both (Ω, g) and $\mathbb{S}^n$ have constant sectional curvature 1 imply that $\tilde{g}$ is C^2 (indeed smooth) across Σ in $\widetilde{M}$. Hence, $(\widetilde{M}, \tilde{g})$ is a spherical space form whose volume is greater than half of $\mathbb{S}^n$ (because it contains a hemisphere). Therefore, $(\widetilde{M}, \tilde{g})$ is isometric to $\mathbb{S}^n$ and (Ω, g) is isometric to a domain in $\mathbb{S}^n_+$. □

Remark 4.1. One can also apply Theorem 4.1 in the above proof.

Theorem 1.5 now follows from Theorem 4.2 and Corollary 4.1. When $\mathrm{Ric} \geq -(n-1)$, similarly we have

Theorem 4.3. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Suppose*

- $\mathrm{Ric} \geq -(n-1)\, g$
- *there is an isometric immersion $X : \Sigma \to \mathbb{H}^m$, where $\mathbb{H}^m$ is a hyperbolic space of dimension $m \geq n$*
- $\mathbb{II}(v, v) \geq |\mathbb{II}_{\mathbb{H}}(v, v)|$, *for any $v \in T\Sigma$. Here $\mathbb{II}$ is the second fundamental form of Σ in (Ω, g) and $\mathbb{II}_{\mathbb{H}}$ is the vector-valued, second fundamental form of the immersion X.*

Then (Ω, g) is hyperbolic, i.e. having constant sectional curvature -1.

The proof is parallel to that of Theorem 4.2. Let $\langle \cdot, \cdot \rangle$ denote the dot product on $\mathbb{R}^{m,1}$ and $\overline{\nabla}$ be the connection on $\mathbb{H}^m$. Identify $\mathbb{H}^m$ with $\{x \in \mathbb{R}^{m,1} \mid \langle x, x \rangle = -1, x_0 > 0\}$. For any $\alpha \in X(\Sigma) \subset \mathbb{H}^m$, consider $F(x) = F_\alpha(x) = \langle \alpha, x \rangle$ on $\mathbb{H}^m$. Its gradient is $\overline{\nabla} F(x) = \alpha + \langle \alpha, x \rangle x$. Thus $\left|\overline{\nabla} F(x)\right|^2 = -1 + F^2$. Given any $p \in \Sigma$, let $\overline{\nabla}^{\perp} F \circ X(p)$ be the component of $\overline{\nabla} F \circ X(p)$ orthogonal to $X_*(T_p\Sigma)$. On Σ, define $f = F \circ X$. Let u be the smooth solution to

$$\begin{cases} \Delta u = nu & \text{on} \quad \Omega \\ u = f & \text{at} \quad \Sigma \end{cases}$$

and define $\chi = \frac{\partial u}{\partial \nu}$. Then $\phi := |\nabla u|^2 - u^2$ is subharmonic as seen from

$$\frac{1}{2}\Delta \phi = \left|\nabla^2 u - ug\right|^2 + \mathrm{Ric}(\nabla u, \nabla u) + (n-1)\,|\nabla u|^2 .$$

Similar to (4.9), we have

$$\phi|_\Sigma = |\nabla_{_\Sigma} f|^2 + \chi^2 - f^2 = -1 + \chi^2 - \left|\overline{\nabla}^{\perp} F\right|^2 .$$

By analyzing the normal derivative $\frac{\partial \phi}{\partial \nu}$ in the same way as in the proof of Theorem 4.2, we conclude by the strong maximum principle that ϕ is constant. Therefore, $\nabla^2 u = ug$ and $\chi^2 - \left|\overline{\nabla}^{\perp} F\right|^2$ is a nonnegative constant c along Σ. If $c > 0$, it implies $\mathbb{II}_{\mathbb{H}} = 0$, which contradicts the fact $\mathbb{H}^m$ does not contain a closed totally geodesic submanifold.

Therefore $\chi^2 - \left|\overline{\nabla}^{\perp} F\right|^2 = 0$ at Σ, which shows $\phi = -1$ on Ω. By the same argument as that of Theorem 4.2, we conclude that (Ω, g) has constant sectional curvature -1.

Corollary 4.2. *Let (Ω, g) be an n-dimensional, compact Riemannian manifold with boundary Σ. Suppose*

- $\mathrm{Ric} \geq -(n-1)g$
- *g has constant sectional curvature -1 at every point on Σ.*

If Σ is simply connected with nonnegative second fundamental form $\mathbb{II}$, then (Ω, g) is isometric to a domain in $\mathbb{H}^n$.

The proof is similar to that of Corollary 4.1. Since g has constant sectional curvature -1 along Σ and Σ is simply connected, by the Gauss and Codazzi equations and the fundamental theorem of hypersurfaces (cf. [19]), there exists an isometric immersion $\Phi : \Sigma \to \mathbb{H}^n$ with $\mathbb{II}$ as its second fundamental form. Since $\mathbb{II} \geq 0$, by the remark in Section 5 of Do Carmo and Warner [4], Φ is an embedding and $\Phi(\Sigma)$ is a convex hypersurface in $\mathbb{H}^n$. Apply Theorem 4.3 to (Ω, g) and the embedding Φ, we conclude that g has constant sectional curvature -1 everywhere on Ω. Now let D be the region bounded by $\Phi(\Sigma)$ in $\mathbb{H}^n$. We glue Ω and $\mathbb{H}^n \setminus D$ along the boundary via the isometric embedding Φ to get a complete manifold $(\widetilde{M}, \tilde{g})$. The fact Σ has the same second fundamental form in (Ω, g) and $\mathbb{H}^n$ and both (Ω, g) and $\mathbb{H}^n$ have constant sectional curvature -1 imply that $\tilde{g}$ is smooth across Σ in $\widetilde{M}$. Hence, $(\widetilde{M}, \tilde{g})$ is a complete, hyperbolic manifold which, outside a compact set, is isometric to $\mathbb{H}^n$ minus a ball. We conclude that $(\widetilde{M}, \tilde{g})$ is isometric to $\mathbb{H}^n$ and (Ω, g) is isometric to a domain in $\mathbb{H}^n$. This final claim can be seen, for instance, by the following:

Proposition 4.1. *Let (M, g) be a complete, n-dimensional Riemannian manifold with $\mathrm{Ric} \geq -(n-1)\,g$. Suppose that there exists a compact set $K \subset M$ s.t. $M \setminus K$ is isometric to $\mathbb{H}^n \setminus B$ where B is a geometric ball. Then M is isometric to $\mathbb{H}^n$.*

The Euclidean version of the result is well known (e.g. it appears as an exercise in [13] several times) . The hyperbolic case can be proved by similar methods. For lack of an exact reference, we outline a proof using Busemann functions. The main idea comes from Cai-Galloway [2]. We use the upper space model $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$. Without loss of generality we take $o = (0, \ldots, 0, 1)$. For a sequence $\epsilon_k \to 0$, let $S_k \subset M$ be the hypersurface corresponding to $x_n = \epsilon_k$ and q_k the point corresponding to $(0, \ldots, 0, 1/\epsilon_k)$. Let p_k be the point on S_k closest to

q_k and $\gamma_k : [-a_k, b_k] \to M$ be a minimizing geodesic from p_k to q_k s.t. $\gamma_k(0) \in K$ (it is easy to see that any minimizing geodesics from p_k to q_k must intersect K). Passing to a subsequence $\epsilon_k \to 0$ if necessary, we can assume that $\gamma_k(0) \to \bar{o}$ and γ_k converges to a geodesic line $\gamma : \mathbb{R} \to M$. We consider the following generalized Busemann function

$$\beta(x) = \lim_{k\to\infty} d(\bar{o}, S_k) - d(x, S_k).$$

Then we have

Claim: $\Delta\beta \geq n$ in the support sense.

The crucial fact is that S_k has constant mean curvature $H = n - 1$. The argument is the same as in Cai-Galloway [2].

We also have the standard Busemann function b associated with the ray $\gamma|_{[0,\infty)}$ defined by $b(x) = \lim_{k\to\infty} s - d(x, \gamma(s))$. It is known that $\Delta b \geq -n$ in the support sense. The rest of the proof is similar to Cai-Galloway [2] or the proof of the Cheeger-Gromoll splitting theorem. We have $\Delta(b+\beta) \geq 0$. By the triangle inequality one can show $b+\beta \leq 0$. On the other hand $b + \beta = 0$ along γ. Therefore by the strong maximum principle $b + \beta = 0$. Then $\beta = -b$ and it is a smooth function with $|\nabla beta| = 1$. By the Bochner formula one can show that $\nabla^2\beta = g - d\beta \otimes d\beta$. From this identity one can show that M is isometric to the warped product $(\mathbb{R} \times S^{n-1}, dt^2 + e^{2t}h)$, where (S, h) is a flat Riemannian manifold. It is then clear that (S, h) must be the standard $\mathbb{R}^{n-1}$. This finishes the proof of Proposition 4.1.

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