Aust. J. Phys., 1981, **34**, 223–41

Boundary Effects in Solution of Boltzmann's Equation for Electron Swarms

R. E. Robson

Institüt für Theoretische Physik der Universität Düsseldorf, West Germany.
Present address: Physics Department, James Cook University of North Queensland, Townsville, Qld 4811.

Abstract

The combined effect of interaction of electrons with walls, neutral molecules and an electrostatic field is considered through analytic solution of Boltzmann's equation. In the first instance, we discuss a half-range decomposition in velocity space, corresponding to electrons moving to and from the walls, which is valid for all types of electron–molecule interactions. The half-range equations are solved in the steady state for zero field and the constant mean free path model, and it is shown that the familiar full-range 'two-term' approximation equations are adequate in this case, as far as estimating bulk properties of the electrons is concerned. For the nonzero field, again in the steady state, the full-range equations are solved for the constant collision frequency model.

1. Introduction

The influence of boundaries on the diffusion of electrons and ions in a neutral gas is generally difficult to account for accurately, so that while theories relating to boundary free circumstances have flourished in recent times (Lin *et al.* 1979; Kumar *et al.* 1980), much more work remains to be done on the incorporation of wall and electrode effects, which are obviously factors to be considered in analysing experiments (Huxley and Crompton 1974). The immediate problem is not so much to account for the various types of possible wall–particle interactions but to solve the appropriate kinetic equation with even the most idealized of boundary conditions (e.g. perfect absorption). In this respect, the kinetic theory of charged particle swarms lags behind the more extensively developed transport theory for neutral gases and neutrons (Davison 1958; Case and Zweifel 1967; Williams 1971; Cercignani 1975; Zweifel 1978). One reason for this is immediately clear: In the relevant experiments, the ions and electrons are subjected to electric and/or magnetic fields, which are not merely perturbing forces, but which often drive the charged particles far from thermal equilibrium with the neutral gas molecules. No such driving force need be considered in neutron transport theory. The combined effect of interaction of charged particles with walls, fields and neutral molecules provides some very challenging analytical and computational problems indeed, and we address ourselves in this paper to a discussion of some possible means of tackling these problems, comparing with the traditional approach through the 'two-term' (or P_1) approximation of the distribution function (Huxley and Crompton 1974; Lin *et al.* 1979; Kumar *et al.* 1980). The scope of the paper is limited to the extent that it is felt that the theory would be best served at this stage by concentrating on analytic solutions of the equations and qualitative discussion, not only because many of the equations derived

here are new, but also because such numerical work as has already been done (Lowke *et al.* 1977) needs to be complemented in this way. Further restrictions to simple collision models (constant free path, constant collision frequency) have been found necessary to make the equations tractable.

Essentially, the problem is to solve the Boltzmann equation,

$$\mathscr{D}(f) \equiv (\partial_t + \boldsymbol{c}.\nabla + \boldsymbol{a}.\partial_c)f = -J(f), \tag{1}$$

for the electron velocity distribution function $f(\boldsymbol{c}, \boldsymbol{r}, t)$, given appropriate boundary conditions. Here $\boldsymbol{a} = -e\boldsymbol{E}/m$ is the acceleration suffered by an electron of charge $-e$ and mass m in the electric field $\boldsymbol{E}$, and $J(f)$ denotes the electron–neutral molecule collision term, which is linear in f. (Electron–electron interactions, which would give rise to nonlinear collision terms, are negligible in the present circumstances, where the electron number density n is extremely low in comparison with the neutral molecule number density N.) If S denotes the boundary surface and $\hat{\boldsymbol{n}}$ a unit vector normal to S and directed *into* the gas, we may write the boundary condition as (Williams 1971; Cercignani 1975)

$$f(\boldsymbol{c}, \boldsymbol{r}, t) = \int K(\boldsymbol{c}, \boldsymbol{c}')\, f(\boldsymbol{c}', \boldsymbol{r}, t)\, \mathrm{d}\boldsymbol{c}' + f^0(\boldsymbol{c}, t) \quad (\boldsymbol{c}'.\hat{\boldsymbol{n}} < 0, \boldsymbol{r} \in S, \boldsymbol{c}.\hat{\boldsymbol{n}} > 0), \tag{2}$$

where $K(\boldsymbol{c}, \boldsymbol{c}')$, the 'interaction transfer function', accounts for electron–wall interaction processes (e.g. absorption, reflection, secondary emission) and f^0 describes any additional externally-induced flux (e.g. photo- or thermionic emission). The wall acts in general as both a sink and a source of electrons in the half spaces $\boldsymbol{c}.\hat{\boldsymbol{n}} < 0$ and >0 respectively, leading to a possible *discontinuity* in velocity space at $\boldsymbol{c}.\hat{\boldsymbol{n}} = 0$. The efficacy of considering solutions of Boltzmann's equation separately in the respective half spaces has long been recognized in other boundary-value problems and the theory has been correspondingly extensively developed (Davison 1958; Case and Zweifel 1967; Williams 1971; Cercignani 1975; Zweifel 1978). We cannot take advantage of all of this previous work, since the field term in equation (1) complicates matters and the form of the collision operator is different. One important result, obtainable as a straightforward extension of theorems in standard works (see e.g. Case and Zweifel 1967, Section 2.3) is that only the distribution of velocities *incident* on the medium, i.e. $f(\boldsymbol{c})$ for $\boldsymbol{c}.\hat{\boldsymbol{n}} > 0$, need be specified in order to determine f in the medium uniquely. The complete angular distribution at the boundary is *not* required. On the other hand, the elegant method of singular eigenfunction expansions of Case and Zweifel, so useful for solving certain model problems in neutron and neutral gas transport theory, seems to be impractical here. Instead, it appears inevitable that one must resort to truncated polynomial representations of f and thus obtain only approximate solutions of equation (1).

Normally, f is approximated by the first two terms of an expansion in spherical harmonics and $J(f)$ is also approximated using the smallness of the electron–neutral molecule mass ratio m/M (Huxley and Crompton 1974; Lowke *et al.* 1977; Lin *et al.* 1979; Kumar *et al.* 1980). If, for simplicity, it is assumed that $\boldsymbol{E}$ defines an axis of symmetry (the z axis of a system of coordinates), then the expansion is simply in terms of Legendre polynomials,

$$f(\boldsymbol{c}) \approx \sum_{l=0}^{L} F_l(c) P_l(\mu), \tag{3}$$

where

$$\mu = \hat{\boldsymbol{E}}.\hat{\boldsymbol{c}}, \quad -1 \leqslant \mu \leqslant 1.$$

Equation (3) is usually referred to either as the P_L approximation or the $L+1$ term approximation. For $L = 1$, it is also sometimes (quite misleadingly) referred to as the 'diffusion approximation'. Equation (3) is a 'full-range expansion', since the same functional form is used to represent the distribution function over all angles. This type of expansion has proved successful for dealing with free-space electron transport processes; there, $L = 1$ generally gives a sufficiently good approximation to f, as long as inelastic electron–neutral collisions are relatively infrequent (Lin *et al.* 1979). In general, however, it is not possible to satisfy the boundary conditions (2) with the truncated expansion (3), and further approximations have to be made. One can form certain angular averages of equation (2), leading to relationships amongst the expansion coefficients F_l (the Marshak (1947) boundary conditions) or simply require that (2) be satisfied for certain discrete values of the angular variable. The latter approach again provides relationships amongst the F_l (the Mark (1957) boundary condition). This is discussed fully in the literature (Case and Zweifel 1967; Williams 1971) and we only note in passing that in a recent paper on electron diffusion to walls, Lowke *et al.* (1977) employed the two-term approximation plus a Mark boundary condition in their analysis, about which we shall have more to say in Section 3. Other cruder approximations have also been used, based upon macroscopic considerations (Huxley and Crompton 1974), i.e. conditions involving integrals of $f(c)$ over c, as well as the angular variables, but really apart from the work of Lowke *et al.* the question of electron–wall interactions in swarm experiments and other similar situations never seems to have been tackled with the same seriousness as, say, in neutron transport phenomena.

The equations are developed here for what might be called 'the generalized Milne problem', in which a steady stream of electrons, driven by an electrostatic field, is incident normally on an infinite plane electrode. A discussion of this somewhat idealized model can be expected to throw some light on the effects of boundaries in the Townsend–Huxley experiment (Huxley and Crompton 1974), where the current collected by the anode is measured with a view to determining the transverse diffusion coefficient. For such a situation, it would seem imperative to represent the boundary condition at the anode accurately and the half-range equations developed in Section 2 are superior to the usual full-range theory in this respect. However, the equations are difficult to solve and we have been able to do this analytically only at zero field. For a nonzero field, there is still a requirement for an analytic full-range theory and this is presented in Section 3 for a constant collision frequency model. The results are essentially what one would expect on the basis of the numerical work of Lowke *et al.* (1977).

2. Half-range Expansions and Decomposition of Boltzmann's Equation

When boundaries influence the distribution function, half-range expansions, which seem to have been first suggested by Yvon (Davison 1958, p. 171), may be necessary to properly account for the boundary-induced discontinuity. Assuming $\boldsymbol{E}$ is normal to the surface, so that it still defines an axis of symmetry, we have the 'double P_L' approximation:

$$f(c) = \sum_{l=0} f_l^+(c)\, P_l(2\mu-1), \qquad 0 < \mu < 1, \tag{4a}$$

$$= \sum_{l=0} f_l^-(c)\, P_l(2\mu+1), \qquad -1 < \mu < 0. \tag{4b}$$

With the usual orthogonality relationship for Legendre polynomials, and noting that

$$\int_0^1 P_l(2\mu-1)\, P_{l'}(2\mu-1)\, \mathrm{d}\mu = \frac{\delta_{l'l}}{2l+1} = \int_{-1}^0 P_l(2\mu+1)\, P_{l'}(2\mu+1)\, \mathrm{d}\mu, \tag{5}$$

it follows that

$$F_l(c) = (l+\tfrac{1}{2}) \int_{-1}^{+1} f(c)\, P_l(\mu)\, \mathrm{d}\mu, \tag{6}$$

$$f_l^+(c) = (2l+1) \int_0^1 f(c)\, P_l(2\mu-1)\, \mathrm{d}\mu, \tag{7a}$$

$$f_l^-(c) = (2l+1) \int_{-1}^0 f(c)\, P_l(2\mu+1)\, \mathrm{d}\mu. \tag{7b}$$

With the definitions

$$\alpha_l^\lambda = (l+\tfrac{1}{2}) \int_0^1 P_\lambda(2\mu-1)\, P_l(\mu)\, \mathrm{d}\mu, \tag{8a}$$

$$\beta_l^\lambda = (l+\tfrac{1}{2}) \int_{-1}^0 P_\lambda(2\mu+1)\, P_l(\mu)\, \mathrm{d}\mu = (-)^{l+\lambda}\alpha_l^\lambda, \tag{8b}$$

explicit formulae and properties of which are given in the Appendix, it can be shown from equations (4) and (6) that

$$F_l = \alpha_l^0\{f_0^+ + (-)^l f_0^-\} + \alpha_l^1\{f_1^+ + (-)^{l+1} f_1^-\} + \ldots, \qquad l = 0, 1, 2, \ldots. \tag{9}$$

Using values of α_l^λ given in the Appendix, it can be shown from equation (9) that

$$F_0 = \tfrac{1}{2}(f_0^+ + f_0^-), \qquad F_1 = f_1^+ + f_1^- + \tfrac{3}{4}(f_0^+ - f_0^- - f_1^+ - f_1^-),$$

$$F_{2l} = \alpha_{2l}^1(f_1^+ - f_1^-) + \ldots, \qquad F_{2l+1} = \alpha_{2l+1}^0(f_0^+ - f_0^- - f_1^+ - f_1^-) + \ldots \quad (l \geqslant 1). \tag{9'}$$

We shall be exclusively concerned with the double P_1 approximation here and shall therefore neglect $f_2^\pm, f_3^\pm, \ldots$, as indicated in equation (9). (Notice that we have gone beyond the standard P_1 or two-term approximation even at this level.) This first stage of our calculation involves decomposition of the Boltzmann equation (1) into its scalar and vector parts in the respective half ranges, and the most important consideration here is the determination of the corresponding collision terms. Rather than beginning from first principles, it is far simpler to make use of the known forms for the full-range theory, and to express the desired half-range quantities in terms of these.

If we define

$$J_\lambda^\pm = \pm \int_0^{\pm 1} J(f)\, P_\lambda(2\mu \mp 1)\, \mathrm{d}\mu, \tag{10}$$

introduce the full-range expansion (3) in the right-hand side and make use of equations (8), then

$$J_\lambda^{\pm} = \sum_l (\pm)^{l+\lambda}(l+\tfrac{1}{2})^{-1}\alpha_l^\lambda J_l(F_l), \tag{11}$$

where the J_l are operator matrix elements of the collision operator in the full-range spherical harmonic representation. By retaining leading terms in the electron–neutral molecule mass ratio $m/M \ll 1$, these latter quantities can be written for elastic collisions as (Huxley and Crompton 1974; Kumar *et al.* 1980)

$$J_0(F_0) = -(m/M)c^{-2}\partial_c\{c^3\nu_1(F_0 + [kT/mc]\partial_c F_0)\}, \tag{12a}$$

$$J_l(F_l) = \nu_l F_l, \quad l \geqslant 1, \tag{12b}$$

where

$$\nu_l(c) = Nc\,2\pi\int_0^\pi \{1-P_l(\cos\chi)\}\,\sigma(c,\chi)\sin\chi\,\mathrm{d}\chi, \quad l = 1,2,\ldots, \tag{13}$$

are collision frequencies and T is the neutral gas temperature. For isotropic scattering, the differential cross section $\sigma(c,\chi)$ is independent of scattering angle χ and then $\nu_l = \nu_1$ for all $l \geqslant 1$; hence

$$J_l = J_1 = \nu_1, \quad l \geqslant 1. \tag{14}$$

This assumption, while not critical to the development of this theory, does simplify matters considerably. It is to be emphasized that detailed expressions for the J_l are not important at this stage. Thus one could, for example, add a term accounting for inelastic collisions to J_0 (see e.g. Kumar *et al.* 1980, equation 175) without affecting the *form* of the following equations in any way.

Substitution of equation (9) into (11) gives

$$J_\lambda^+ = A'_{0\lambda}f_0^- - A'_{1\lambda}f_1^- + A_{0\lambda}f_0^+ + A_{1\lambda}f_1^+, \tag{15a}$$

$$(-)^\lambda J_\lambda^- = A_{0\lambda}f_0^- - A_{1\lambda}f_1^- + A'_{0\lambda}f_0^+ + A'_{1\lambda}f_1^+, \tag{15b}$$

where

$$A_{r\lambda} = \sum_l (l+\tfrac{1}{2})^{-1}\alpha_l^r\alpha_l^\lambda J_l, \qquad A'_{r\lambda} = \sum_l (l+\tfrac{1}{2})^{-1}(-)^l\alpha_l^r\alpha_l^\lambda J_l. \tag{16}$$

In the general case all the $A_{r\lambda}$ and $A'_{r\lambda}$ are nonzero, and the sums (16) have to be evaluated using explicit expressions for J_l. However, with the assumption of isotropic scattering, as indicated in equation (14), and with the following identities resulting from completeness of the Legendre polynomials,

$$\sum_l (l+\tfrac{1}{2})^{-1}\alpha_l^r\alpha_l^\lambda = (2\lambda+1)^{-1}\delta_{\lambda r}, \qquad \sum_l (l+\tfrac{1}{2})^{-1}(-)^l\alpha_l^r\alpha_l^\lambda = 0, \tag{17}$$

it can be shown that

$$A_{0\lambda} = \tfrac{1}{2}(J_0+J_1)\delta_{\lambda 0}, \qquad A_{1\lambda} = \tfrac{1}{3}J_1\,\delta_{\lambda 1}, \tag{18a}$$

$$A'_{0\lambda} = \tfrac{1}{2}(J_0-J_1)\delta_{\lambda 0}, \qquad A'_{1\lambda} = 0. \tag{18b}$$

Substitution of equations (18) into (15) then gives

$$J_0^{\pm} = \tfrac{1}{2}J_0(f_0^{+}+f_0^{-}) \pm \tfrac{1}{2}J_1(f_0^{+}-f_0^{-}), \qquad J_1^{\pm} = \tfrac{1}{3}J_1 f_1^{\pm}. \tag{19}$$

The left-hand side of the Boltzmann equation (1), for the infinite plane geometry shown in Fig. 1 below, is

$$\mathscr{D}(f) = \{\partial_t + c\mu\partial_z + a\mu\partial_c + ac^{-1}(1-\mu^2)\partial_\mu\}f, \tag{20}$$

where it has been assumed that there is cylindrical symmetry about the z axis, and the only spatial variation is in the z direction. Corresponding to equation (10), we define

$$\mathscr{D}_\lambda^{\pm} = \pm \int_0^{\pm 1} \mathscr{D}(f)\, P_\lambda(2\mu \mp 1)\, \mathrm{d}\mu, \tag{21}$$

as well as the operator

$$\Delta = c\partial_z + a\partial_c. \tag{22}$$

It is then straightforward to show that

$$\mathscr{D}_0^{\pm} = (\partial_t \pm \tfrac{1}{2}\Delta)f_0^{\pm} + (\tfrac{1}{6}\Delta + \tfrac{4}{3}a/c)f_1^{\pm}, \tag{23a}$$

$$\mathscr{D}_1^{\pm} = \tfrac{1}{3}\{(\partial_t \pm \tfrac{1}{2}\Delta \mp a/c)f_1^{\pm} + \tfrac{1}{2}\Delta f_0^{\pm}\}, \tag{23b}$$

and when these are combined with equations (19), we have

$$(\partial_t \pm \tfrac{1}{2}\Delta)f_0^{\pm} + (\tfrac{1}{6}\Delta + \tfrac{4}{3}a/c)f_1^{\pm} = -\tfrac{1}{2}J_0(f_0^{+}+f_0^{-}) \mp \tfrac{1}{2}J_1(f_0^{+}-f_0^{-}), \tag{24a}$$

$$(\partial_t \pm \tfrac{1}{2}\Delta \mp a/c)f_1^{\pm} + \tfrac{1}{2}\Delta f_0^{\pm} = -J_1 f_1^{\pm}. \tag{24b}$$

When explicit expressions for the collision operators, for example equations (12), are substituted in the right-hand side, these constitute four partial differential equations in the four unknowns $f_0^{\pm}$ and $f_1^{\pm}$.

Some observations about the above decomposition of the Boltzmann equation into scalar and vector parts seem in order at this stage:

(1) Our derivation constitutes what appears to be the simplest and most direct route to the desired equations starting from the general form (1) of Boltzmann's equation. On the other hand, a formal transformation theory, similar to that developed in Section 12 of Kumar *et al.* (1980), in which the (known) full-range representation of Boltzmann's equation is transformed via the matrices (8) into its representation in terms of the half-range basis sets $P_l(2\mu \mp 1)$, would also give the same result, perhaps more elegantly.

(2) The above derivation, while specially for the double P_1 approximation, can be generalized in an obvious way to higher order approximations. However, it is difficult enough to go beyond the (full-range) P_1 approximation, and seems even more so for the double P_1 approximation.

(3) When (and only when)

$$f_1^{-} = f_1^{+}, \qquad f_0^{-} + f_1^{-} = f_0^{+} - f_1^{+}, \tag{25}$$

the half-range expansions are identical with each other and with the full-range P_1 approximation, i.e. if we require

$$f = f_0^+ + f_1^+(2\mu - 1) = f_0^- + f_1^-(2\mu + 1) = F_0 + F_1\mu,$$

then

$$F_0 = f_0^+ - f_1^+ = f_0^- + f_1^-, \qquad F_1 = 2f_1^+ = 2f_1^-. \tag{26}$$

These results also follow directly from equations (9′), if we insist that $F_l = 0$ ($l \geqslant 2$). We would expect these conditions to be true far from the boundaries when there are no discontinuities, the usual boundary-free full-range theory applying under these circumstances.

(4) The above observations suggest that it may be useful to define the linear combinations

$$\Phi_0 = \tfrac{1}{2}(f_0^+ + f_0^- - f_1^+ + f_1^-), \qquad \delta\Phi_0 = f_0^+ - f_0^- - f_1^+ - f_1^-,$$

$$\Phi_1 = f_1^+ + f_1^-, \qquad \delta\Phi_1 = f_1^+ - f_1^-. \tag{27}$$

Elementary manipulations of equations (24) give

$$\begin{aligned}
\partial_t\Phi_0 + \tfrac{1}{2}\partial_t(\delta\Phi_1) + \tfrac{1}{4}\Delta(\delta\Phi_0) + \tfrac{1}{3}(\Delta + 2a/c)\Phi_1 &= -J_0(\Phi_0) - \tfrac{1}{2}J_0(\delta\Phi_1),\\
\partial_t(\delta\Phi_0) + \partial_t\Phi_1 + \Delta\Phi_0 + \tfrac{2}{3}(\Delta + 2a/c)\delta\Phi_1 &= -J_1(\delta\Phi_0) - J_1(\Phi_1),\\
\partial_t\Phi_1 + (\Delta - a/c)\delta\Phi_1 + \Delta\Phi_0 &= -J_1(\Phi_1),\\
\partial_t(\delta\Phi_1) + (\Delta - a/c)\Phi_1 + \tfrac{1}{2}\Delta(\delta\Phi_0) &= -J_1(\delta\Phi_1).
\end{aligned} \tag{28}$$

Far from the boundary, equations (25) are expected to apply, so that with (26) we may write

$$\delta\Phi_l \to 0, \qquad \Phi_l \to F_l, \qquad l = 0, 1, \tag{29}$$

as the influence of the boundary becomes weaker. The first of equations (28) then reduces to

$$\partial_t F_0 + \tfrac{1}{3}(\Delta + 2a/c)F_1 = -J_0(F_0), \tag{30}$$

while the second and third each give

$$\partial_t F_1 + \Delta F_0 = -J_1(F_1). \tag{31}$$

These are the familiar 'two-term' or P_1 approximation equations obtained by multiplying Boltzmann's equation by $P_0(\mu)$ and $P_1(\mu)$ respectively, setting $L = 1$ in equation (3) and integrating over μ from -1 to $+1$. If equation (1) is also multiplied by $P_2(\mu)$ and integrated over all μ, we have the result (the complete chain of equations is written out in equation (2.5) of Ginzburg and Gurevich 1960) in the same approximation (i.e. setting $F_2, F_3, \ldots$ equal to zero),

$$(\Delta - a/c)F_1 = 0. \tag{32}$$

This is also precisely the content of the last of equations (28) in the limit indicated in (29). Equation (32) is not a true side condition. It merely indicates that $(\Delta - a/c)F_1$ is of the same order of magnitude as the material already neglected in the $l = 2$ equation (i.e. terms in $F_2, F_3, \ldots$) and should itself be neglected if and when it appears in results based on the two-term approximation. In any truncation scheme, at least one seemingly redundant relation must result from equations of higher order than are needed to effect the approximate solution.

(5) Purely macroscopic considerations, based on particle conservation, impose a constraint on the solution of our equations. Electron number density and particle flux are given by

$$n = 2\pi \int_0^\infty \mathrm{d}c\, c^2 \int_{-1}^{+1} \mathrm{d}\mu\, f(c,\mu) = 2\pi \int_0^\infty \mathrm{d}c\, c^2 (f_0^+ + f_0^-), \tag{33}$$

$$\Gamma_z = n\langle c_z \rangle = 2\pi \int_0^\infty \mathrm{d}c\, c^3 \int_{-1}^{+1} \mathrm{d}\mu\, \mu f(c,\mu)$$

$$= \pi \int_0^\infty \mathrm{d}c\, c^3 \{f_0^+ - f_0^- + \tfrac{1}{3}(f_1^+ + f_1^-)\}, \tag{34}$$

respectively. When expressed in terms of the quantities (27), these are

$$n = 4\pi \int_0^\infty \mathrm{d}c\, c^2 (\Phi_0 + \tfrac{1}{2}\delta\Phi_1), \tag{33'}$$

$$\Gamma_z = \pi \int_0^\infty \mathrm{d}c\, c^3 (\delta\Phi_0 + \tfrac{4}{3}\Phi_1). \tag{34'}$$

The equation of continuity is

$$\partial_t n + \partial_z \Gamma_z = 0 \tag{35}$$

for the geometry of Fig. 1, where we assume for simplicity that there are no bulk sources or sinks of electrons, such as ionization or attachment phenomena, in the gas medium. On the other hand, when the first of equations (28) is multiplied by $4\pi c^2$ and integrated over all speeds c, the equation

$$\partial_t n + \partial_z \Gamma_z = 2\pi a \int_0^\infty \mathrm{d}c\, c\, \delta\Phi_0$$

results. Thus, there will be an artificial source term introduced unless the right-hand side is zero, i.e. unless $a = 0$, or $a \neq 0$ such that

$$\int_0^\infty \mathrm{d}c\, c(f_0^+ - f_0^- - f_1^+ - f_1^-) = 0. \tag{36}$$

3. Generalized Milne Problem

As shown in Fig. 1, we have an infinite half space $z > 0$ filled by a gaseous medium. There exists a source of electrons at infinity which balances the loss to the infinite plane perfectly absorbing wall at $z = 0$, so that a steady state exists. Problems of this type are usually named after Milne (Davison 1958; Case and Zweifel 1967; Williams 1971; Cercignani 1975), who considered radiation transfer in a star. The

main differences here are the electrostatic field and the form of the electron–neutral collision term $J(f)$. The wall is perfectly absorbing, so that no electrons enter the medium at $z = 0$; thus we have the following boundary condition for $f(c, \mu, z)$:

$$f(c, \mu, 0) = 0, \quad 0 \leqslant \mu \leqslant 1. \tag{37}$$

(We set $K = 0$ and $f^0 = 0$ in equation 2.) This in turn implies that the expansion coefficients $f_l^+(c, z)$ of equation (4a) satisfy

$$f_l^+(c, 0) = 0, \quad l = 0, 1, \ldots. \tag{38}$$

As we have remarked before, this condition, plus the specification of the incident electron flux at the other 'surface' at infinity, determines uniquely the solution of Boltzmann's equation.

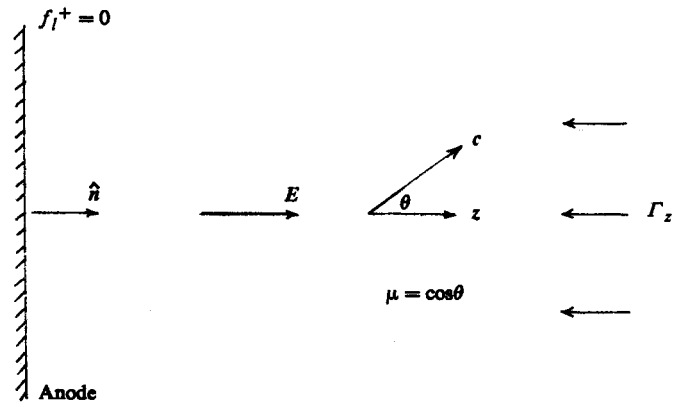

Fig. 1. Geometry for the 'generalized Milne problem', in which a steady stream of electrons is driven by an electrostatic field E from a source at infinity to a plane infinite anode at $z = 0$.

Zero Field

Setting $a = 0$ and assuming a steady state,* equations (24) become

$$\pm\tfrac{1}{2}c\partial_z f_0^{\pm} + \tfrac{1}{6}c\partial_z f_1^{\pm} = -\tfrac{1}{2}J_0(f_0^+ + f_0^-) \mp \tfrac{1}{2}\nu(f_0^+ - f_0^-), \tag{39a}$$

$$\pm\tfrac{1}{2}c\partial_z f_1^{\pm} + \tfrac{1}{2}c\partial_z f_0^{\pm} = -\nu f_1^{\pm}, \tag{39b}$$

where we have written ν for ν_1. We define the mean free path as $\lambda = c/\nu$ and write

$$f_0^+ = \tilde{A}\exp(-mc^2/2kT)z + \phi_0^+, \tag{40a}$$

$$f_0^- = \tilde{A}\exp(-mc^2/2kT)(z+\lambda) + \phi_0^-, \tag{40b}$$

$$f_1^{\pm} = -\tfrac{1}{2}\lambda\tilde{A}\exp(-mc^2/2kT) + \phi_1^{\pm}, \tag{40c}$$

* Notice that in the steady state, $\partial_t n = 0$ and the equation of continuity (35) then implies particle flux is a constant, independent of position.

where $\tilde{A}$ is a constant. The $\phi_l^{\pm}$ ($l = 0, 1$) are functions which control the behaviour as $z \to 0$, and the first terms dominate the far distance behaviour (see equations 25):

$$\lim_{z\to\infty}(f_0^+ - f_1^+) = \lim_{z\to\infty}(f_0^- + f_1^-) = \tilde{A}\exp(-mc^2/2kT)z,$$

$$\lim_{z\to\infty} f_1^- = \lim_{z\to\infty} f_1^+ = -\tfrac{1}{2}\lambda\tilde{A}\exp(-mc^2/2kT).$$

The terms linear in z in the asymptotic solution are required to provide a density gradient to drive particles from infinity to the wall. The constant $\tilde{A}$ is determined from a knowledge of either the asymptotic density (through equation 33) or from the (constant) particle flux (34).

For λ a constant, independent of c, the above equations are relatively straightforward to solve. After some lengthy algebra, we find (making use of the smallness of the mass ratio m/M to make some simplifying approximations)

$$\phi_0^+ = \tfrac{1}{2}\sum_{n=0}^{\infty}\exp(-u)L_n^{(1)}(u)\{(1+\sqrt{\tfrac{1}{3}})A_n\exp(-2\sqrt{3}\,z/\lambda) + B_n\exp(-\sqrt{(\tfrac{3}{2}n)}\,z/\lambda_\varepsilon)\}, \quad (41a)$$

$$\phi_0^- = \tfrac{1}{2}\sum_{n=0}^{\infty}\exp(-u)L_n^{(1)}(u)\{(1-\sqrt{\tfrac{1}{3}})A_n\exp(-2\sqrt{3}\,z/\lambda) + B_n\exp(-\sqrt{(\tfrac{3}{2}n)}\,z/\lambda_\varepsilon)\}, \quad (41b)$$

$$\phi_1^+ = \tfrac{1}{2}\sum_{n=0}^{\infty}\exp(-u)L_n^{(1)}(u)\left\{\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)A_n\exp(-2\sqrt{3}\,z/\lambda) + \tfrac{1}{2}\sqrt{(\tfrac{3}{2}n)}\frac{\lambda}{\lambda_\varepsilon}B_n\exp(-\sqrt{(\tfrac{3}{2}n)}\,z/\lambda_\varepsilon)\right\}, \quad (41c)$$

$$\phi_1^- = \tfrac{1}{2}\sum_{n=0}^{\infty}\exp(-u)L_n^{(1)}(u)\left\{\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)A_n\exp(-2\sqrt{3}\,z/\lambda) + \tfrac{1}{2}\sqrt{(\tfrac{3}{2}n)}\frac{\lambda}{\lambda_\varepsilon}B_n\exp(-\sqrt{(\tfrac{3}{2}n)}\,z/\lambda_\varepsilon)\right\}, \quad (41d)$$

where $L_n^{(1)}$ is a Laguerre polynomial, $u \equiv \frac{1}{2}mc^2/kT$, $\lambda_\varepsilon = (M/2m)^{\frac{1}{2}}\lambda$ is the mean free path for energy exchange, and A_n and B_n are constants to be found from application of the boundary conditions. If we assume the boundary to be perfectly absorbing, equations (38) apply, which together with (41a), (41c), (40a) and (40c) lead to

$$B_n = -(1+\sqrt{\tfrac{1}{3}})A_n, \quad A_n = 0 \quad (n \geqslant 1), \quad A_0 = -\lambda\{(\sqrt{3}-1)/(\sqrt{3}+1)\}\tilde{A}.$$

The complete solution is then

$$f_0^+(c,z) = \tilde{A}\exp(-mc^2/2kT)[z + \tfrac{1}{2}\lambda(1-\sqrt{\tfrac{1}{3}})\{1-\exp(-2\sqrt{3}\,z/\lambda)\}],$$

$$f_0^-(c,z) = \tilde{A}\exp(-mc^2/2kT)\left[z + \lambda + \tfrac{1}{2}\lambda(1-\sqrt{\tfrac{1}{3}})\left(1 - \frac{\sqrt{3}-1}{\sqrt{3}+1}\exp(-2\sqrt{3}\,z/\lambda)\right)\right],$$

$$f_1^+(c,z) = -\tfrac{1}{2}\lambda\tilde{A}\exp(-mc^2/2kT)[1-\exp(-2\sqrt{3}\,z/\lambda)],$$

$$f_1^-(c,z) = -\tfrac{1}{2}\lambda\tilde{A}\exp(-mc^2/2kT)\left[1 + \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2\exp(-2\sqrt{3}\,z/\lambda)\right], \quad (42)$$

which satisfies all the above-mentioned criteria, including constancy of particle flux. Equations (42) indicate that the asymptotic forms are attained a few mean free paths away from the wall. On the other hand, when the wall is not a perfect absorber, there will in general be another 'transition region' of thickness $\lambda_\varepsilon \gg \lambda$ before the asymptotic form is reached, as shown by the last terms in equations (41). However, these contributions are the same in each half space (i.e. identical for '+' and '−'); any discontinuity in the distribution function washes out in the inner layer of thickness $\approx\lambda$.

The full-range P_1 approximation, which has been extensively used to analyse experiments (see e.g. Robson 1976) should therefore be valid for the zero field in all but the 'boundary layer' of thickness $\approx\lambda$. The usual boundary condition $F_0 = 0$, $F_1 \neq 0$ (implying $f_0^+ = 0 = f_0^-$, $f_1^\pm \neq 0$) does not represent the true physical situation and in fact violates the fundamental assumption of the P_1 approximation that F_1 is in some sense small compared with F_0. That the method has succeeded at all probably is a reflection of the insensitivity of the bulk of the electrons to all but the coarsest details of electron–wall interactions. Thus, in experiments like that of Cavalleri (Cavalleri 1969; Huxley and Crompton 1974; Robson 1976) where *bulk* properties of the electrons in a vessel are measured, the full-range theory would appear to be quite adequate.

Nonzero Field

We were able to obtain an analytic solution for the zero-field problem by assuming λ was a constant, independent of c. When $\lambda = \lambda(c)$ further approximations are necessary, but in any case, the parameter controlling the thickness of the boundary layer might be expected to be $\approx\lambda(\bar{c})$, where $\bar{c}$ is representative of the average electron speed. Applying even a moderate electrostatic field will substantially increase $\bar{c}$, and therefore the thickness of the boundary layer (if $\lambda(c)$ increases with c), to a point where it is not negligible in comparison with the dimensions of the container, and this suggests a discussion in terms of half-range expansions. (This effect is quite different from the one resulting from the larger boundary layer of thickness $\approx\lambda_\varepsilon$ penetrating the bulk of the gas (Robson 1976).) Of course, when one is interested in effects right at the boundary surface (as in the Townsend–Huxley experiment), as well as in the bulk of the gas, the half-range theory seems highly desirable. In any case, there is sufficient motivation to seek solutions of the half-range equations for a nonzero field.

Unfortunately, equations (24) do not appear to be amenable to analytic solution, even for the simplest of model cross sections. Whether further decomposition of the $f_l^\pm$ in terms of Sonine polynomials (see e.g. Lin *et al.* (1979), equation (14), where the basis temperature expansion parameters would be different in the respective half spaces) is required or whether the numerical algorithm of Hall and Lowke (1975), used for solving the full-range equations (30) and (31), can be suitably modified, or whether some other method is needed, is not clear at this stage. Even analytic solution of the full-range equations is difficult; that some such analysis is required to complement the essentially pure numerical work of Lowke *et al.* (1977) will become clear from the discussion below. We have therefore chosen to concentrate in this paper on the full-range equations (30) and (31), in the belief that a better understanding of their solution may facilitate solution of the far more complicated half-range equations, which we leave to later work.

Assuming a steady state ($\partial_t = 0$) and a cold gas ($T = 0$), we have for equations (30) and (31)

$$\tfrac{2}{3}(a/c)F_1 + \tfrac{1}{3}\Delta F_1 = (m/M)c^{-2}\,\partial_c(c^3 \nu F_0)\,, \tag{43}$$

$$\Delta F_0 = -\nu F_1\,. \tag{44}$$

Lowke *et al.* (1977) solved these numerically for several model collision frequencies of the form $\nu \propto c^\gamma$, where γ is a constant. Their boundary condition is that the distribution function

$$F(c, z, \mu) = F_0(c, z) + F_1(c, z)\mu$$

is to vanish at $z = 0$ for backward scattering only ($\theta = 0$ and $\mu = 1$), leading to the requirement

$$F_0(c, 0) + F_1(c, 0) = 0\,. \tag{45}$$

(There is an error in the sign of their corresponding relation, but this does not affect their results in any way.) This is effectively a Mark (1957) condition, as mentioned before. They go on further to argue that equations (44) and (45) together imply the effective condition

$$F_0(c, 0) = 0\,. \tag{46}$$

Another constraint imposed upon their solution is that $\Delta F_0 = 0$ at $c = 0$ for any z and this has a significant effect on the solution, especially near the boundary. However, there is no justification for this additional requirement in the constant collision frequency model dealt with below.

As mentioned before, some approximation like equation (45) is necessary with a truncated full-range expansion, but the boundary condition (46) is somewhat more severe and indeed quite contradictory to the spirit of the P_1 approximation. One curious result is that since number density n is an integral of F_0 (see equation 47) it too vanishes at the boundary. On the other hand, $\Gamma_z = n\langle c_z\rangle$ is constant, implying that the average velocity $\langle c_z\rangle$ of electrons becomes infinite at the boundary. This is suggestive of a singularity in the distribution function.

Braglia and Lowke (1979) have compared the above method against a Monte Carlo calculation for the model $\nu \propto c^2$, with good agreement. This might suggest that the boundary conditions employed by Lowke *et al.* (1977) are substantially correct or that, for this particular model, the solution is insensitive to the boundary conditions imposed. There remain, however, some questions of physical principle to be resolved, most of which can only be answered by going to the half-range theory.

The quantities of physical interest here are the number density and mean energy of the electrons:

$$n(z) = 4\pi \int_0^\infty F_0(c, z)\, c^2\, \mathrm{d}c\,, \tag{47}$$

$$\varepsilon \equiv \langle \tfrac{1}{2}mc^2\rangle = \int_0^\infty \tfrac{1}{2}mc^2\, F_0(c, z)\, c^2\, \mathrm{d}c \Big/ \int_0^\infty F_0(c, z)\, c^2\, \mathrm{d}c\,, \tag{48}$$

whereas the particle flux,

$$\Gamma_z \equiv n\langle c_z\rangle = \frac{4\pi}{3} \int_0^\infty \mathrm{d}c\, c^3\, F_1(c, z)\,, \tag{49}$$

is a prescribed constant. The distribution function F_0^∞ far from the boundary* is obtained by solving equations (43) and (44) with the space derivatives set at zero:

$$F_0^\infty(c) = A\exp\left(-\frac{3m}{Ma^2}\int_0^c \nu^2(c)\,c\,\mathrm{d}c\right), \tag{50}$$

$$F_1^\infty(c) = -\frac{a}{\nu}\frac{\mathrm{d}}{\mathrm{d}c}F_0^\infty. \tag{51}$$

Equation (50) gives the familiar Davydov distribution. The constant A is found by normalization to either the known density at ∞ or to the particle flux (49).

We have tried a variety of methods of solving equations (43) and (44) but even for what might be considered as the simplest case, ν = constant (which Lowke *et al.* (1977) did not examine), approximation is necessary. Laplace transformation in z seems to require a knowledge of *both* F_0 and F_1 at $z = 0$ (or at least, if one function is given, the other must be determined in a self-consistent fashion from the transformed equations, a very formidable task). Separation of variables can be achieved in terms of $(c, \frac{1}{2}c^2 - az)$ instead of the pair (c, z) but this leads to an eigenvalue equation with continuous eigenvalues and otherwise offers no advantage. The most promising analytical approach seemed to be to Laplace transform the equations in the speed variable c, and this we now discuss.

The transform of a function $F(c)$ is defined by

$$[\![F]\!](s) = \int_0^\infty \exp(-\tfrac{1}{2}sc^2)F(c)\,\mathrm{d}c, \tag{52}$$

and this is simply $\mathscr{L}_s\{(2x)^{-\frac{1}{2}}F((2x)^{\frac{1}{2}})\}$. The following relations are readily established:

$$[\![c^n AF]\!] = (as+\partial_z)[\![c^{n+1}F]\!] - na[\![c^{n-1}F]\!], \tag{53a}$$

$$[\![c^{n-2}\partial_c(c^3\nu F)]\!] = s[\![c^{n+2}\nu F]\!] - (n-2)[\![c^n\nu F]\!]. \tag{53b}$$

It has been assumed that both c^nF and $c^{n+1}\nu F$ vanish as $c \to 0$.

The advantage of working with these transforms is that the physically interesting moments of the distribution function can be found quite readily from the value of the transform at $s = 0$, e.g. from equations (47)–(49),

$$n = 4\pi[\![c^2F_0]\!]_{s=0}, \quad \varepsilon = \tfrac{1}{2}m([\![c^4F_0]\!]/[\![c^2F_0]\!])_{s=0}, \quad \Gamma_z = \tfrac{4}{3}\pi[\![c^3F_1]\!]_{s=0}. \tag{54a, b, c}$$

Noting the identity

$$[\![c^{n+2}F]\!] = -2\partial_s[\![c^nF]\!], \tag{55}$$

which can be obtained directly by differentiation of an equation like (52), we can also write (54b) as

$$\varepsilon = -m(\partial_s \ln[\![c^2F_0]\!])_{s=0}. \tag{54d}$$

* Equations (50) and (51) will also describe the asymptotic behaviour in the half-range theory for, by equations (26),

$$f_1^+ = \tfrac{1}{2}F_1^\infty, \quad f_0^+ - f_1^+ = f_0^- + f_1^- = F_0^\infty, \quad \text{for } z \to \infty.$$

The transforms of c^n times equation (43) and $c^{n'}$ times (44) are respectively

$$\tfrac{1}{3}a(2-n)[\![c^{n-1}F_1]\!]+\tfrac{1}{3}(as+\partial_z)[\![c^{n+1}F_1]\!] = (m/M)(s[\![c^{n+2}vF_0]\!]+(2-n)[\![c^nvF_0]\!]), \quad (56)$$

$$-n'a[\![c^{n'-1}F_0]\!]+(as+\partial_z)[\![c^{n'+1}F_0]\!] = -[\![c^{n'}vF_1]\!]. \quad (57)$$

We shall examine the case where v is constant, and then we need consider the above equations for $n = 2$ and $n' = 3$ only:

$$\tfrac{1}{3}(as+\partial_z)[\![c^3F_1]\!] = (sm/M)v[\![c^4F_0]\!], \quad (58)$$

$$-3a[\![c^2F_0]\!]+(as+\partial_z)[\![c^4F_0]\!] = -v[\![c^3F_1]\!]. \quad (59)$$

These, together with the relation $[\![c^4F_0]\!] = -2\partial_s[\![c^2F_0]\!]$, form a closed set.* Elementary solutions with spatial dependencies $\exp(akz)$ are sought, where k is a non-negative constant. (Recall that $a = -eE/m$, so that in order to get the correct asymptotic behaviour as $z \to \infty$, we must have $k \geqslant 0$.) The complete solution is a sum over all possible such contributions and is found to be

$$[\![c^2F_0]\!](s,z) = \sum_k A_k\,\phi_k(s)\exp(akz), \quad (60)$$

where

$$\phi_k(s) = \exp\left(-\frac{3}{2}\int_0^s \frac{s+k}{(s+k)^2+s\alpha^2}\,ds\right) \quad (61)$$

$$= (1+s/\zeta^+)^{-3p(+)/2}(1+s/\zeta^-)^{-3p(-)/2}, \quad (61')$$

$$\zeta^\pm \equiv \tfrac{1}{2}\alpha^2\{1+2k/\alpha^2 \pm(1+4k/\alpha^2)^{\frac{1}{2}}\},$$

$$p(\pm) \equiv \{(1+4k/\alpha^2)^{\frac{1}{2}}\pm 1\}/2(1+4k/\alpha^2)^{\frac{1}{2}},$$

$$\alpha^2 \equiv 3mv^2/Ma^2. \quad (62)$$

The coefficients A_k are to be found from imposition of the boundary condition at $z = 0$, i.e. from

$$\sum_k A_k\,\phi_k(s) = [\![c^2F_0]\!](s,0). \quad (63)$$

Other quantities of interest are

$$[\![c^3F_1]\!](s,z) = \frac{3a}{v}s\alpha^2\sum_k \frac{A_k}{(s+k)^2+s\alpha^2}\,\phi_k(s)\exp(akz), \quad (64)$$

$$[\![c^4F_0]\!](s,z) = 3\sum_k \frac{(s+k)A_k}{(s+k)^2+s\alpha^2}\,\phi_k(s)\exp(akz). \quad (65)$$

* Closed sets of equations may be similarly derived for other model collision frequencies, and their solution represented in the form (60); the corresponding $\phi_k(s)$ is, however, far more difficult to obtain.

It is clear that the $k = 0$ term in these expressions must somehow be separated from the others. For example, splitting off the $k = 0$ term in equation (64), giving

$$[\![c^3 F_1]\!](s, z) = \frac{3a}{v} \frac{\alpha^2}{s+\alpha^2} A_0 (1 + s/\alpha^2)^{-3/2} + \frac{3as\alpha^2}{v} \sum_k{}' \frac{A_k \phi_k(s) \exp(akz)}{(s+k)^2 + s\alpha^2},$$

taking the limit as $s \to 0$ and then substituting in equation (54c) leads to the correct form for the (constant) particle flux,

$$\Gamma_z = 4\pi A_0 a/v \equiv -n_0 W, \tag{66}$$

where

$$W = |a/v| \tag{67}$$

is the drift velocity, n_0 is the number density far from the boundary, and we identify

$$A_0 = n_0/4\pi \tag{68}$$

on the basis of the discussion below.

In equation (60), the same separation

$$[\![c^2 F_0]\!](s, z) = A_0 (1 + s/\alpha^2)^{-3/2} + \sum_k{}' A_k \phi_k(s) \exp(akz) \tag{69}$$

indicates that

$$\lim_{z \to \infty} [\![c^2 F_0]\!](s, z) = A_0 (1 + s/\alpha^2)^{-3/2},$$

and hence by equation (54a)

$$n_0 \equiv \lim_{z \to \infty} n = 4\pi A_0,$$

giving (68). The first term in (69) is simply the transform of c^2 times the (normalized) Davydov distribution (50) with v a constant:

$$F_0^{\infty}(c) = n_0 (\alpha^2/2\pi)^{3/2} \exp(-\tfrac{1}{2}\alpha^2 c^2). \tag{70}$$

Although the $k = 0$ term has to be treated specially, no 'quantization' in the spectrum is evident, and the summations above must be replaced by integrals. Equation (69) is then

$$[\![c^2 F_0]\!](s, z) = \frac{n_0}{4\pi} (1 + s/\alpha^2)^{-3/2} + \int_0^{\infty} A(k) \phi_k(s) \exp(akz)\, dk, \tag{71}$$

where the lower limit has been taken as zero with the stipulation that the spectral density $A(k)$ satisfies

$$\lim_{k \to 0} A(k) = 0. \tag{72}$$

For mathematical simplicity, we use the Lowke *et al.* (1977) boundary condition (46), in spite of some misgivings in relation to its physical soundness, as mentioned before. Then, from equation (71) we get

$$\int_0^{\infty} A(k) \phi_k(s)\, dk = -\frac{n_0}{4\pi} (1 + s/\alpha^2)^{-3/2}. \tag{73}$$

This can be thought of as a transformation of the coefficients $A(k)$, through the function $\phi_k(s)$ into the expression on the right-hand side. If one knew how to invert this transformation, $A(k)$ could be found and the problem completely solved. It is certainly not a standard transformation, and the best method we could think of for inverting equation (73) was to approximate $\phi_k(s)$ by its small-s form,

$$\phi_k(s) = (1+s/k)^{-3/2} + (3\alpha^2/4k^3)s^2 + O(s^3). \tag{74}$$

The solution of equation (73) with $\phi_k(s)$ correct to first order in s is

$$A(k) = -A_0\,\delta(k-\alpha^2), \tag{75}$$

and it is a straightforward matter to show that this when substituted into (71) yields the classical result (see e.g. equation 83 below). To second order in s, the solution of (73) is approximately

$$A(k) = -A_0\{\delta(k-\alpha^2) - \tfrac{1}{5}\alpha^2 k\,\delta''(k-\alpha^2)\}. \tag{76}$$

Upon substituting in (71) and using equations (54a, b) and (68), we find for the density and mean energy

$$n(z) = n_0[1-\{1+\tfrac{1}{5}\zeta(2-\zeta)\}\exp(-\zeta)], \tag{77a}$$

$$\varepsilon(z) = \frac{3m}{2\alpha^2}\,\frac{1-(1-\tfrac{1}{5}\zeta^2)\exp(-\zeta)}{1-\{1+\tfrac{1}{5}\zeta(2-\zeta)\}\exp(-\zeta)}, \tag{77b}$$

where

$$\zeta \equiv |a|\alpha^2 z. \tag{78}$$

As $z \to \infty$, we have $n \to n_0$ and $\varepsilon \to 3m/2\alpha^2 \equiv \varepsilon_0$. At the anode, we have $n \to 0$ and $\varepsilon \to \frac{5}{2}m/\alpha^2 = \frac{5}{3}\varepsilon_0$. That the energy increases towards the anode is in accord with the findings of Lowke and collaborators and others (Lucas 1966; Sakai *et al.* 1972).

Finally, we compare these results with those obtained from the classical theory. The momentum balance equation (set $s = 0$ in equation 59) yields for the particle flux

$$\Gamma_z = -nW - \partial_z(Dn), \tag{79}$$

where W is the drift velocity (equation 67) and

$$D \equiv 2\varepsilon/3mv \tag{80}$$

is identified as the diffusion coefficient. The latter has the same z dependence as ε and therefore increases near the boundary. The drift velocity W is constant. Classically, ε is assumed to be constant, or at least, to change very slowly over whatever length scale characterizes density variations and hence equation (79) is assumed to approximate to

$$\Gamma_z = -nW - D_0\,\partial_z n, \tag{81}$$

where

$$D_0 \equiv 2\varepsilon_0/3mv = (\alpha^2 v)^{-1} \tag{82}$$

is the classical diffusion coefficient. (For the model ν constant, one need not distinguish between longitudinal and transverse coefficients (Huxley and Crompton 1974).) The solution of equation (81) subject to the boundary condition $n = 0$ at $z = 0$ is

$$n(z) = n_0\{1-\exp(-\zeta)\}. \tag{83}$$

This is quite a different form from equation (77a), even for large z far from the boundary, suggesting that the classical form is *never* adequate for this particular problem. The reason is that the mean energy has a length scale characterizing its spatial variation comparable with that of the density. (This is also clear from inspection of the results of Lowke *et al.* 1977; Braglia and Lowke 1979.) In all the usual derivations of transport coefficients (Huxley and Crompton 1974; Lin *et al.* 1979; Kumar *et al.* 1980), the assumption (either explicit or implicit) is that variations in ε are negligible in comparison with variations in n, leading to equations like (81), linearized in the gradient of n.

4. Concluding Remarks

We have developed a half-range decomposition of Boltzmann's equation (a double two-term approximation), valid for all types of scattering processes. The equations were solved analytically for the constant mean free path model and zero electric field, for the geometry of Fig. 1. There, only in the boundary layer of thickness a few free paths do the half-range equations give solutions differing from the usual two-term full-range expansion in Legendre polynomials, implying that in situations where the bulk properties of electrons are determined (as in the Cavalleri experiment) no modifications to existing theories need to be made. For nonzero fields and nonconstant free paths, the indications are, however, that boundary effects may not be so well represented through the usual theory. The half-range equations do not seem amenable to analytic solution for nonzero fields, and a numerical procedure will be required for further work with them. Even the far simpler full-range equations are difficult to solve analytically, and we had to make approximations, even for the constant collision frequency model.

This study, like the previous one by Lowke *et al.* (1977), was performed with a simple geometry. For geometries more appropriate to experiment, the task of solving Boltzmann's equation seems formidable, and here the Monte Carlo simulation technique, already used by Braglia and Lowke (1979), would seem to offer the best prospect for a more thorough understanding of experiment. Still, solutions of Boltzmann's equations are important in their own right and the half-range equations developed in Section 2 should provide a more satisfying means of dealing with boundary effects.

Acknowledgments

This work was sponsored by the Alexander von Humboldt Foundation of West Germany, to which the author expresses his gratitude. It is a pleasure to acknowledge the interest and continual encouragement of Professor K. Suchy and to thank Dr U. Weinert for helpful discussions and to acknowledge his contribution to the calculations presented in the Appendix. The author also benefitted from talks with Dr H. R. Skullerud and Dr G. L. Braglia.

References

Braglia, G. L., and Lowke, J. J. (1979). *J. Phys.* D **12**, 1831.
Case, K. M., and Zweifel, P. F. (1967). 'Linear Transport Theory' (Addison–Wesley: Reading, Mass.).
Cavalleri, G. (1969). *Phys. Rev.* **179**, 186.
Cercignani, C. (1975). 'Theory and Application of the Boltzmann Equation' (Scottish Academic: Edinburgh).
Davison, B. (1958). 'Neutron Transport Theory' (Clarendon: Oxford).
Ginzburg, V. L., and Gurevich, A. V. (1960). *Sov. Phys. Usp.* **3**, 115.
Hall, C. A., and Lowke, J. J. (1975). *J. Comput. Phys.* **19**, 297.
Huxley, L. G. H., and Crompton, R. W. (1974). 'The Diffusion and Drift of Electrons in Gases' (Wiley: New York).
Kumar, K., Skullerud, H. R., and Robson, R. E. (1980). *Aust. J. Phys.* **33**, 343.
Lin, S. L., Robson, R. E., and Mason, E. A. (1979). *J. Chem. Phys.* **71**, 3483.
Lowke, J. J., Parker, J. H., and Hall, C. A. (1977). *Phys. Rev.* A **15**, 1237.
Lucas, J. (1966). *Int. J. Electron.* **21**, 535.
Mark, C. (1957). Atomic Energy Commission of Canada Rep. Nos CRT 338, 340.
Marshak, R. E. (1947). *Phys. Rev.* **71**, 443.
Robson, R. E. (1976). *Phys. Rev.* A **13**, 1536.
Sakai, Y., Tagashira, H., and Sakamoto, S. (1972). *J. Phys.* B **5**, 1010.
Williams, M. M. R. (1971). 'Mathematical Methods in Particle Transport Theory' (Butterworths: London).
Zweifel, P. F. (1978). *Trans. Theory Stat. Phys.* **7**, 173.

Appendix

Coefficients α_l^λ of Equation (8a)

If $l < \lambda$, the $P_l(\mu)$ can be expressed as a linear combination of $P_{\lambda'}(2\mu-1)$, $\lambda' < \lambda$, and by the orthogonality property (5) of these half-range polynomials, it follows that

$$\alpha_l^\lambda = 0, \qquad l < \lambda. \tag{A1}$$

Explicit values of the first few coefficients are

$$\alpha_0^0 = \tfrac{1}{2}, \quad \alpha_1^0 = \tfrac{3}{4}, \quad \alpha_1^1 = \tfrac{1}{4}, \tag{A2}$$

and further,

$$\alpha_{2l}^0 = 0, \quad \alpha_{2l+1}^0 + \alpha_{2l+1}^1 = 0, \quad l = 1, 2, \ldots.$$

The general expression for these coefficients is

$$\alpha_l^\lambda = (l+\tfrac{1}{2}) \sum_{k(2)}^{l} (-)^{\frac{1}{2}(l-k)} \frac{(k+l-1)!!}{(l-k)!!} \frac{k!}{(k-\lambda)!(k+\lambda+1)!} \tag{A3}$$

(where the notation indicates that the summation index increases in steps of 2), and this can be proved by considering the integral

$$\int_0^1 P_\lambda(2\mu-1) P_l(\mu) \, \mathrm{d}\mu = \tfrac{1}{2} \int_{-1}^{+1} P_\lambda(z) P_l\big(\tfrac{1}{2}(z+1)\big) \, \mathrm{d}z. \tag{A4}$$

Using the standard representation,

$$P_l(x) = \sum_{k(2)}^{l} (-)^{\frac{1}{2}(l-k)} \frac{(k+l-1)!!}{(l-k)!!\,k!} x^k, \tag{A5}$$

for the Legendre polynomials as well as the inverse relation

$$x^k = \sum_{l(2)}^{k} (2l+1)\frac{k!}{(k+l+1)!!\,(k-l)!!} P_l(x), \tag{A6}$$

we obtain

$$P_l\left(\tfrac{1}{2}(z+1)\right) = \sum_{k(2)}^{l} (-)^{\frac{1}{2}(l-k)} \frac{(k+l-1)!!}{(l-k)!!\,k!\,2^k} \sum_{\tau=0}^{k} \binom{k}{\tau}$$

$$\times \sum_{\lambda(2)}^{\tau} (2\lambda+1)\frac{\tau!}{(\tau+\lambda+1)!!\,(\tau-\lambda)!!} P_\lambda(z). \tag{A7}$$

Because of the orthogonality relation for the P_l the sum over λ vanishes after insertion of equation (A7) into the integral (A4). Hence, we are left with

$$\int_0^1 P_\lambda(2\mu-1)\,P_l(\mu)\,\mathrm{d}\mu = \sum_{k(2)}^{l} (-)^{\frac{1}{2}(l-k)} \frac{(k+l-1)!!}{(l-k)!!\,2^k} s_k(\lambda), \tag{A8}$$

with

$$s_k(\lambda) = \sum_{\tau=\lambda(2)}^{k} \{(k-\tau)!\,(\lambda+\tau+1)!!\,(\tau-\lambda)!!\}^{-1}. \tag{A9}$$

(Note that (A9) vanishes unless $\lambda \leqslant \tau$, implying that (A8) vanishes unless $\lambda \leqslant l$.) A more detailed discussion of $s_k(\lambda)$ then yields (A3). For $\lambda = k$ we have

$$s_\lambda(\lambda) = \frac{1}{(2\lambda+1)!!}. \tag{A10}$$

Then we obtain from (A9) the following recurrence relation:

$$s_{k-1}(\lambda) = (k-\lambda)\,s_k(\lambda) - s_{k-1}(\lambda+1). \tag{A11}$$

Finally, it can be verified easily that the solution of the system (A11) and (A10) is

$$s_k(\lambda) = \frac{k!\,2^k}{(k-\lambda)!\,(k+\lambda+1)!}. \tag{A12}$$

Upon insertion into (A8) and using the definition (8a) of α_l^λ, we obtain (A3).

Manuscript received 22 September 1980, accepted 16 February 1981