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# Boundary Element Analysis of Piled Rafts

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## Abstract

This work presents a Boundary Element formulation for the analysis of piled rafts in which both the plate and the soil are represented by integral equations. In this formulation the pile is represented by only one element, with three nodal points, and the shear force along the pile is approximated by a second degree polynomial. The pile-tip stress is assumed to be constant over the cross-section and a further nodal point is located there. The cap-soil interface is divided into triangular elements, with one nodal point at each corner, and the contact pressure is assumed to vary linearly across each element. The vertical displacement of each node at the cap and at the piles is represented by integral equations and a set of linear equations is obtained relating forces and displacements for all nodal points on all the interfaces and from them the nodal displacements and the overall stiffness of the system is calculated. Numerical results are presented and they correspond closely to those from other analysis.

## 1 Introduction

There have been many publications related to the analysis of piled raft foundations, in which the the supporting soil is represented by an elastic half space. The earliest formulation was proposed by Poulos & Davies[1]. In this formulation, initially load-displacement curves for the interaction of two units of piled circular-caps in which the caps are rigid were obtained and then applied to the study of rigid capped pile-groups considering the superposition of all circular cap-pile units. Although the elastic superposition is valid only when the piles are located along a circumference and submitted to the same load, these load transfer curves were applied to the analysis of generic capped pile groups. Butterfield & Banerjee[2] proposed a generic formulation for the analysis of capped pile groups however as in [1], the cap is supposed to be rigid. Only a few works were published in which the plate is supposed to be flexible. The earliest one were presented by Brown & Weisner[3] for the analysis of a long flexible footing supported on the soil and on piles. However in this formulation the footing is analyzed with the usual beam theory, with limited applications. Fatemi-Ardakani[4] presented a formulation for this analysis in which the plate is analyzed by the Boundary Element Method and the piles are represented by springs. In this formulation the stiffness of the springs are initially determined by a computer code for the analysis of pile-soil interactions problems but without taking into account the subgrade reaction between the plate and the soil resulting in a poor representation of the problem. Another problem is that as the springs transmit concentrated loads to the plate, the bending moments at the top of the piles cannot be determined by the BEM.

Hain & Lee[5] proposed a formulation for the analysis of this problem in which the plate is analyzed by the Finite Element Method and the contributions of the piles are given by load-transfer curves presented in [1]. Again the elastic superposition is imposed in piled-rafts where the requirements for it are not fulfilled.

Poulos[6] proposed a formulation for the analysis of this problem in which, as in [4], the piles are represented by springs and once more a computer code was used to evaluate its stiffness. This formulation was implemented in a finite difference method code and load-displacement curves were also used to obtain the vertical displacement of the soil at points located at the junction of the piles with the plate. Generally in all the papers mentioned before, in none of them was the full interaction between plate, pile and soil considered and the main reason for it is that its consideration would lead to a large system of equations.

The aim of this paper is to present a boundary element formulation for the analysis of piled rafts in which all the interactions are simultaneously considered. In this formulation the soil is supposed to be an elastic half space and represented by integral equations using Kelvin's fundamental solution. The plate is supposed to be thin and moreover, represented by integral equations and each pile is represented by one element with three nodal points and the shear force along it is approximated by a second degree polynomial. At the tip of the pile the stress is assumed to be constant over its cross-section and only a nodal point is located there. With this approximation, only a few unknowns are associated with each pile. Concerning the analysis of the plate-soil interaction, their interface is divided into triangular elements, with one nodal point at each corner, and the subgrade reaction is assumed to vary linearly across each element. The triangular elements have the advantage of being easily accommodated in irregular boundaries. It is assumed that the elastic half space is not disturbed by the piles. It is also assumed that the traction along the shaft of the pile is uniformly distributed around its periphery and the load points are located in its central line. Numerical results are presented and compared with those from others formulations.

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## 2 Plate-Pile-Soil Integral Equations

Figure 1 shows a generic piled-raft and the tractions applied to the cap, the soil, the pile shafts and the bases. The piles are uniform cylinders (length $L_p$, diameter $D_p$) however all piles do not necessarily have either the same length or cross section. The triangular cells on the plate-soil interface have nodes defined at their corners and the tractions on them are approximated by a linear function. Each pile is represented by one element and a second degree polynomial function is assumed to represent the tractions along its shaft. There are three nodes along the pile shaft and one at the tip of the pile and the traction in the tip is assumed to be uniformly distributed.

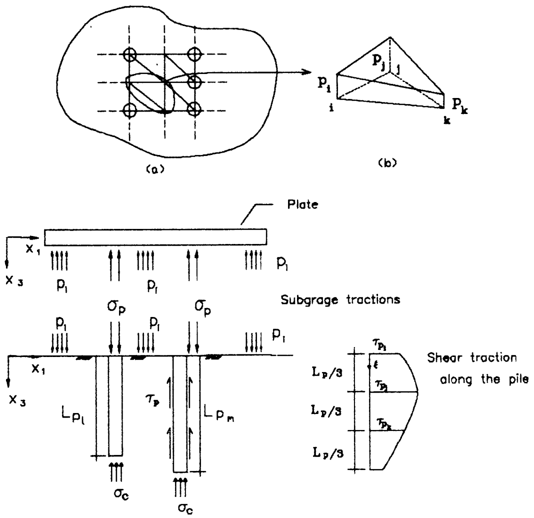


Figure 1: Plate-pile-soil interaction tractions

The tractions on the cells are approximated by a linear function in its domain and can be written as:

$$p_3 = [\Phi_n]\{p_n\} \qquad \ldots(1)$$

where $[\Phi_n]$ is the matrix of domain functions and $\{p_n\}$ the nodal tractions vector. Therefore:

$$p_3 = [\Phi_1 \;\; \Phi_2 \;\; \Phi_3][\, p_i \;\; p_j \;\; p_k]^T \qquad \ldots(2)$$

where the domain functions $\Phi_1$, $\Phi_2$ and $\Phi_3$, in rectangular coordinates, are given by:

$$\Phi_1=\frac{1}{D_t}[(x_{2j}-x_{2k})x_1+(x_{1k}-x_{2j})x_2+(x_{1j}x_{2k}-x_{1k}x_{2j})]$$

$$\Phi_2=\frac{1}{D_t}[(x_{2k}-x_{2i})x_1+(x_{1i}-x_{2k})x_2+(x_{1k}x_{2i}-x_{1i}x_{2k})] \qquad ...(3)$$

$$\Phi_3=\frac{1}{D_t}[(x_{2i}-x_{2j})x_1+(x_{1j}-x_{1i})x_2+(x_{1i}x_{2j}-x_{1j}x_{2i})]$$

with:

$$D_t=x_{1i}(x_{2j}-x_{2k})+x_{1j}(x_{2k}-x_{2i})+x_{1k}(x_{2i}-x_{2j}) \qquad ...(4)$$

The tractions along the shaft of the can be written as:

$$\tau_p=[\Phi_p]\{\tau_{pi}\} \qquad ...(5)$$

where $[\Phi_p]$ and $\{\tau_{pi}\}$ are respectively the matrix of domain functions and the vector of pile nodal tractions which can be written as:

$$\tau_p(q)=[\Phi_{p1} \quad \Phi_{p2} \quad \Phi_{p3}][\tau_{pi} \quad \tau_{pj} \quad \tau_{pk}]^T \qquad ...(6)$$

where:

$$\Phi_{p_1}(\xi)=\frac{1}{2}(9\xi^2-3\xi+2) \quad \Phi_{p_2}(\xi)=-9\xi^2+6\xi \quad \Phi_{p_3}(\xi)=\frac{1}{2}(9\xi^2-3\xi) \qquad ...(7)$$

with $\xi=\dfrac{x_3(Q)}{Lp}$

Figure 2 shows the plate with all the load applied, i.e. the external load g, the subgrade reactions p, linearly distributed in each cell, and the tension applied by the top of each pile, uniformly distributed in circular subregions.The boundary integral equation for the transverse displacement is given by:

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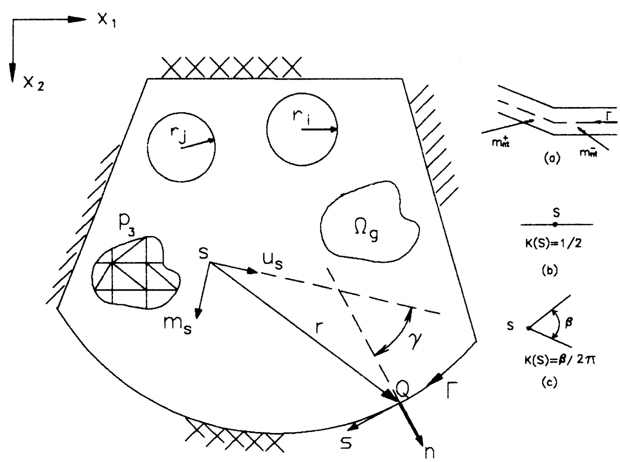


Figure 2: Plate with transverse load and pile-soil tractions

$$K(S)w(S)+\int_{\Gamma}\left[V_n^*(S,Q)\,w(Q)-m_n^*(S,Q)\frac{\partial w}{\partial n}(Q)\right]d\Gamma(Q)+\sum_{i=1}^{N_C}R_{Ci}^*(S,Q)\,w_{Ci}(Q)=$$

$$\int_{\Gamma}\left[V_n(Q)\,w^*(S,Q)-m_n(Q)\frac{\partial w^*}{\partial n}(S,Q)\right]d\Gamma(Q)+\sum_{i=1}^{N_C}R_{Ci}(Q)\,w_{Ci}^*(S,Q)+$$

$$+\int_{\Omega_g}g(q)w^*(S,q)d\Omega_g(q)+\sum_{i=1}^{Ncell}\int_{\Omega_{cell}}p_3(q)w^*(S,q)d\Omega_{cell}(q)+$$
$$+\sum_{i=1}^{Np}\int_{\Omega_T}\sigma_c(q)w^*(S,q)d\Omega_T \qquad \dots(8)$$

where $w$, $m_n$ and $V_n$ are respectively, the transverse displacement, the bending moment and the equivalent shear force along the boundary; $g(q)$, $\Omega_g$ are the transverse load and the surface where it is applied and $p_3(q)$, $\Omega_p$ and *Ncell* are the subgrade reactions, the cell's surface and the number of cells; $\sigma_T(q)$, $\Omega_T$ and $N_P$ are the traction at the top of the pile, its cross section and the number of piles. The symbol * is used here to indicate fundamental solution. In this equation Rci is the corner reaction and $K(S)=\beta/2\pi$ for a point "*S*" at a boundary corner, with an internal angle $\beta$.

From equation (8) the integral representation of the derivative of the displacement in relation to a direction $m_s$ of a system of coordinates $(m_s,u_s)$ can be derived as follows:

$$K_1(S)\frac{\partial w}{\partial m_S}(S)+K_2(S)\frac{\partial w}{\partial u_S}(S)+\int_\Gamma\left[\frac{\partial V_n^*}{\partial m_S}(S,Q)w(Q)-\frac{\partial m_n^*}{\partial m_S}(S,Q)\frac{\partial w}{\partial n}(Q)\right]d\Gamma(Q)+$$

$$+\sum_{i=1}^{N_c}\frac{\partial R_{Ci}^*}{\partial m_S}(S,Q)w_{Ci}(Q)=\int_\Gamma\left\{V_n(Q)\frac{\partial w^*}{\partial m_S}(S,Q)-m_n(Q)\frac{\partial}{\partial m_S}\left[\frac{\partial w^*}{\partial n}(S,Q)\right]\right\}d\Gamma(Q)+$$

$$+\sum_{i=1}^{N_c}R_{Ci}(Q)\frac{\partial w_{Ci}^*}{\partial m_S}(S,Q)+\int_{\Omega_g}g(q)\frac{\partial w^*}{\partial m_S}(S,q)\,d\Omega_g(q)+$$

$$+\sum_{i=1}^{Ncell}\int_{\Omega_{cell}}p_s(q)\frac{\partial w^*}{\partial m_S}(S,q)d\Omega_{cell}(q)+\sum_{i=1}^{Np}\int_{\Omega_T}\sigma_T(q)\frac{\partial w^*}{\partial m_S}(S,q)\,d\,\Omega_T(q) \qquad \ldots(9)$$

with:

$$K_1(S)=\frac{\beta}{2\pi}+\frac{\nu}{4\pi}[\sin(2\gamma)-\sin 2(\gamma+\beta)\ \ K_2(S)=\frac{\nu}{4\pi}[\cos(2\gamma)-\cos 2(\gamma+\beta)] \ldots(10)$$

The boundary of the plate is divided into linear boundary elements. In the formulation used in this paper, the displacements $w$ and $\partial w/\partial n$, and the bending moment $m_n$ along each element are approximated by a linear function, and the effective shear force, $V_n$, on the boundary is approximated by concentrated reactions applied to the element nodes (Paiva[7]). By writing the equations of the displacement and its derivatives in normal directions for all nodes at the boundary and the equations of the transverse displacement for all nodes of the cells in plate domain and performing all the integrations, the following set of linear equation can be obtained:

$$\begin{bmatrix} H^* & 0 & 0 \\ H^1 & I & 0 \\ H^2 & 0 & I \end{bmatrix} \begin{Bmatrix} w_\Gamma \\ w_{cell} \\ w_B \end{Bmatrix} = \begin{bmatrix} C^* \\ C^1 \\ C^2 \end{bmatrix} \{V_\Gamma\} + \begin{bmatrix} K & S \\ K^1 & S^2 \\ K^2 & S^2 \end{bmatrix} \begin{Bmatrix} p_\Omega \\ p_B \end{Bmatrix} + \begin{Bmatrix} G \\ G^1 \\ G^2 \end{Bmatrix} \quad \ldots(11)$$

where $\{w_\Gamma\}$, $\{w_{cell}\}$ and $\{w_p\}$ are vectors with, respectively the transverse displacement and its derivative at boundary nodes, the transverse displacement at all nodes of the cells inside the plate apart from those located in the top of the piles and the transverse displacements of the nodes located at the top of the piles.

The integral equations which relate the vertical displacement w at a generic point $S$ in the soil and the plate-pile-soil interface tractions is given by:

$$w(S) = \int_\Gamma u_{33}^*(S,Q) p_3(Q)\, d\Gamma(Q) + \sum_{i=1}^{N_p} [\int_{\Gamma_p} u_{33}^*(S,q) \tau_p(q)\, d\Gamma_p(q) +$$

$$+ \int_{\Gamma_c} u_{33}^*(S,q) \sigma_c(q)\, d\Gamma_c(q)] \quad \ldots(12)$$

where $\tau_p(q)$ is the shear force along the pile shaft, $\sigma_c(q)$ *is the* tension at the tip of the pile, $\Gamma_p$ is the lateral surface of the pile, $\Gamma_c$ is the cross section of the pile and $N_P$ is the number of piles. By writing this equation for all nodes and performing all the numerical integrations involved the following set of linear equations is obtained:

$$\begin{Bmatrix} w_{\Gamma_\Omega} \\ w_{\Gamma_b} \\ w_{\Gamma_c} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} p_\Gamma \\ \tau_{\Gamma_b} \\ \sigma_{\Gamma_c} \end{Bmatrix} \quad \ldots(13)$$

in which $\{w_{\Gamma_\Omega}\}$ is the vector of vertical displacements for all the cap-soil surface elements nodes; $\{w_{\Gamma_b}\}$ is the vector of displacements for all nodes defined along the pile shafts and $\{w_{\Gamma_c}\}$ that for the pile base traction. Similarly $\{p_\Gamma\}$, $\{\tau_{\Gamma_b}\}$ and $\{\sigma_{\Gamma_c}\}$ are vectors of the nodal values of the corresponding variables.

The axial force at a point of the pile is given by:

$$N = \int_\xi^1 \tau_P \, d\Gamma_P + \sigma_T \pi r_0^2 \quad \ldots(14)$$

and its vertical strain is given by:

$$w=-\frac{1}{E_p S}\int N\, dx_3 \qquad \text{...(15)}$$

where $E_p$ is the Modulus of Young and S is the cross secction of the pile.

From (14) and (15) an expression relating the vertical displacement of the pile due to its flexibility can be expressed as a function of its nodal values and by imposing the compatibility of the displacements of the pile and the soil at the at the nodes located at the piles, a system of equation similar to (13) is obtained, now with the flexibility of the pile included. This system of equations can also be written in its inverse form and condensed resulting in:

$$\begin{Bmatrix} p_\Gamma \\ p_B \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{Bmatrix} w_\Gamma \\ w_{cell} \\ w_B \end{Bmatrix} \qquad \text{... (16)}$$

From (11) and (16) the final system of equation for plate-pile-soil interaction analysis can finally be expressed as:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{Bmatrix} w_\Gamma \\ w_{cell} \\ w_B \end{Bmatrix} = \begin{bmatrix} C^* \\ C^1 \\ C^2 \end{bmatrix} \{V_\Gamma\} + \begin{Bmatrix} G \\ G^1 \\ G^2 \end{Bmatrix} \qquad \text{...(17)}$$

After imposing the boundary condition and solving this system of equations, displacements for all nodes are obtained. The subgrade reactions and the tractions at the top of the piles can be obtained from (13) and (14).

## 3 Examples

This formulation was successfully tested in many pile-groups with rigid and flexible caps. In figure 4 the stiffness of a two pile-group with a rigid cap, shown in figure 3, as a function of its length to diameter ratio is presented. These results show a very good concordance with those presented in [2]. In figure 5 the results of the analysis of this pile-group, submited to an uniform load and considering the flexibility of the cap are presented and they also show a very good concordance with those presented in [3] and [4].

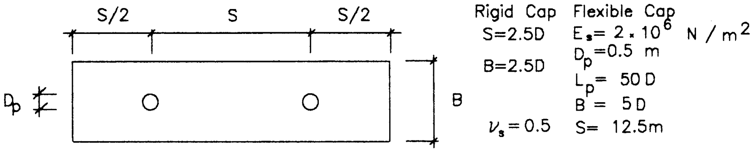


Figure 3: Rigid and flexible two piles group

For the flexible cap the results are presented as a function of $K_{st}=\dfrac{E_p I_p(1-\nu_s)}{16\pi G S^4}$, where $E_pI_p$ is the stiffness of the plate and $G_s$ is the shear modulus of the soil.

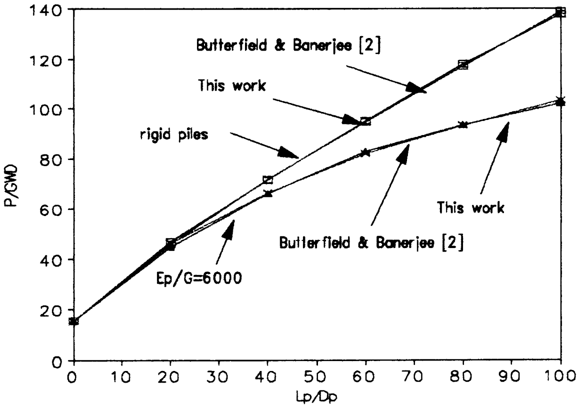


Figure 4: Stiffness of a capped two pile group

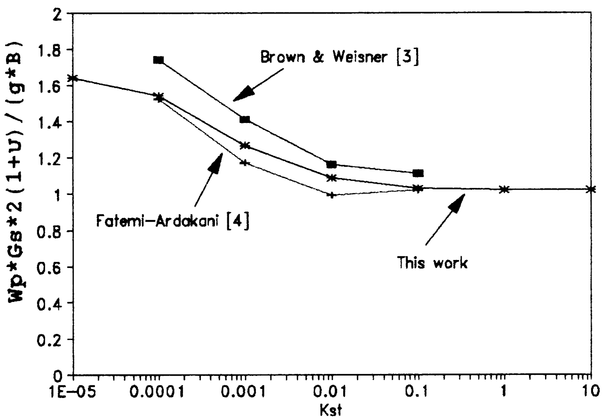


Figure 5: Displacement of the central point of the cap

# 4 Conclusion

A boundary element formulation for the analysis of flexible piled-rafts were presented in which all the compatibilities of plate, pile and soil are imposed. The contributions of the piles are transferred to the plate and in its final system of equations, only the plate unknowns at the interface plate-soil are included. The formulation was tested with rigid and flexible capped pile groups and the results show a very good concordance with those presented by other authors.

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