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Boundary Element Method for Internal Axisymmetric Flow

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We present an accurate fast method for the computation of potential internal axisymmetric flow based on the boundary element technique. We prove that the computed velocity field asymptotically satisfies reasonable boundary conditions at infinity for various types of inlet/exit. Computation of internal axisymmetric potential flow is an essential ingredient in the three-dimensional problem of computation of velocity fields in turbomachines. We include the results of a practical application of the method to the computation of flow in turbomachines of Kaplan and Francis types.

Keywords: Boundary element method; Vortex layer; Potential axisymmetric flow; Hydraulic turbine

1 INTRODUCTION

The intrinsic three-dimensional problem for internal potential incompressible flows past objects in axisymmetric passages, such as flows past blades of turbomachines, can be solved by superposition of several flows: (a) axisymmetric flow in the passage; (b) constant whirl flow ($rv_\theta = \text{const}$) in the passage; (c) flow induced by vortex filaments on the blades; (d) flow induced by distributed sources on the boundary (including the surfaces of blades) that compensates for the nonzero normal components of velocity at the boundary of flows (a), (b), and (c). This paper is devoted to the problem of computing flow (a).

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Computation of flow (a) is also needed for quasi-three dimensional design of turbomachine blades.

In [3] the authors applied a finite element technique to axisymmetric flow. The finite element scheme was formulated using curvilinear coordinates that follow the boundary and closed form integration was used to compute the Galerkin integrals. The flow was assumed to be potential and the problem was treated as a boundary value problem for the Stokes stream function. The use of curvilinear coordinates and closed form integration was shown to provide significantly higher accuracy than comparable ordinary Cartesian finite element schemes and can be applied to viscous laminar or turbulent flows using a generalization of the Galerkin method formulated in [2].

Under the assumption that the flow is potential it is advantageous to use a boundary element method, because the boundary has codimension 1. With a comparable mesh size, boundary element methods lead to much smaller systems than other methods, such as finite element or finite difference, which involve a higher dimensional mesh covering the whole passage.

In [1] a boundary element method was formulated for internal potential axisymmetric incompressible flow in a passage with a cylindrical draft tube and applied to the simple problem of external flow past a sphere. The boundary elements and the vorticity distribution were assumed to be piecewise linear and the boundary condition that the normal velocity is zero ($v_n = 0$) was imposed at the midpoints of the elements.

In this paper we avoid the under-determinacy of the system due to boundary conditions $v_n = 0$ being imposed at midpoints of elements. This is done by imposing the boundary conditions $v_n = 0$ at the vertices of the elements. In this situation it becomes necessary to perform a more careful computation of velocity at a vertex induced by the adjoining elements. While in general, the geometry of the elements is assumed to be piecewise linear, the geometry of one or two elements adjoining a given vertex is represented by cubic splines (using Hermite interpolant polynomials). The vorticity distribution is assumed to be piecewise linear throughout.[†]

[†] In principle, to achieve greater accuracy at the cost of roughly doubling the size of the problem the vorticity distribution may be represented by Hermite splines throughout.

The above method has been applied to the practical cases of axisymmetric flow in Francis and Kaplan turbine passages with conical and cylindrical draft tubes.

2 FORMULATION OF THE PROBLEM

We consider internal fluid flow through an axisymmetric passage. We use cylindrical coordinates r,θ,z, where the z axis is the axis of symmetry of the passage. We make the following assumptions about the flow.

Hypotheses for the flow $\bar{v}$:

(i) incompressible, i.e. $\nabla \cdot \bar{v} = 0$,
(ii) potential, i.e. there exists a scalar potential φ with $\bar{v} = \nabla\varphi$,
(iii) axisymmetric, i.e. $\bar{v} = \bar{v}(r, z)$,
(iv) irrotational, i.e. the circumferential component of velocity $v_\theta = 0$.

With practical applications to hydraulic turbomachinery in mind we make the following assumptions about the geometry of the passage. The word *domain* means a connected open set.

Hypotheses for the geometry of the passage:

(v) The passage is a domain of revolution in $\mathbf{R}^3$ generated by a simply connected domain $\Omega \subseteq \{(r, z) \in R^2: r \geq 0\}$.
(vi) The boundary of the passage is the union of two disjoint C^1 surfaces of revolution. The inner surface represents the crown and the outer surface represents the bend.
(vii) The inlet and the exit are either (see Fig. 1)
  (a) *radial*, i.e. for sufficiently large r the boundary of the passage is formed by two parallel planes extending to infinity as $r \to \infty$.
  (b) or *axial*, i.e. for sufficiently large z (or $-z$) the boundary of the passage is formed by a vertical circular cylinder, two such cylinders, or a vertical cone extending to infinity as $z \to \pm\infty$. Note that the case of a cone was not considered in [1].

Governing equation Since the flow is potential and incompressible, $\nabla^2\varphi = 0$. In cylindrical coordinates, using axisymmetry we obtain

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial r^2} = 0, \qquad v_r = \frac{\partial\varphi}{\partial r}, \quad v_z = \frac{\partial\varphi}{\partial z}, \quad v_\theta = 0. \tag{1}$$

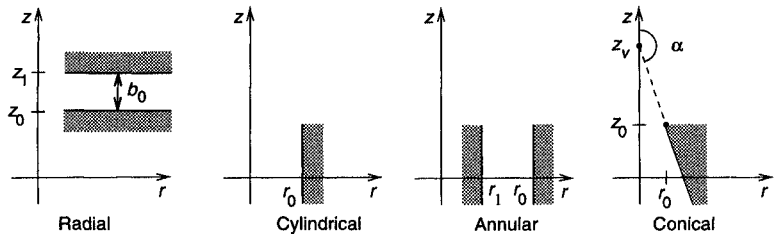

FIGURE 1 Inlet/outlet types.

Boundary conditions At the boundary of the passage we have the normal component of velocity $v_n = \bar{v} \cdot \hat{n} = 0$. At the inlet and the exit we impose the simplest possible boundary conditions. The exact form depends on the type of inlet/exit (see hypotheses for the geometry of the passage above). In the radial case we require a uniform radial flow at infinity. In the case of a cylinder or two cylinders we require a uniform vertical flow at infinity. In the case of a cone we require a uniform central flow with respect to the vertex of the cone.

In each case we can determine the required magnitude of velocity by dividing the flux q by the appropriate cross-sectional area orthogonal to the flow $\bar{v}$. In the radial case this area is a cylinder. For a cylinder each cross-section is a disk. For two cylinders each cross-section is an annulus. For a cone each cross-section is a spherical cap. The final formulas for the boundary conditions on $\bar{v}$ are given below (notation for asymptotic formulas is explained in detail in Section 3.3):

(a) Radial with distance between the planes b_0: $v_z \prec v_r \sim \pm q/(2\pi r b_0)$ as $r \to \infty$ between the planes forming the passage;
(b) Cylinder with radius r_0: $v_r \to 0$, $v_z \to \pm q/(\pi r_0^2)$ as $z \to \pm\infty$ inside the cylinder;
(c) Two cylinders with radii $0 < r_1 < r_0$: $v_r \to 0$, $v_z \to \pm q/\pi(r_0^2 - r_1^2)$ as $z \to \pm\infty$ between the cylinders;
(d) Cone with vertex $(0, z_v)$ and generatrix $r = (z - z_v)\tan\alpha$: $v_\rho \sim \pm q/(\pi \sin^2 \alpha \rho^2)$, as $\rho \to \infty$, where $\rho = |(r, z - z_v)|$ inside the cone.

3 INTEGRAL EQUATION FOR VORTICITY ON THE BOUNDARY

The boundary element method is based on the idea of using distributed sources/sinks or vorticity of *a priori* unknown intensity on the

boundary to generate potential flow. In this work, in order to easily satisfy the boundary conditions at the inlet and exit, we use distributed circular vortex filaments. In this section we reformulate the problem in terms of the distributed vortex intensity γ.

Taking into account the symmetry of the problem we assume that the distributed vortex intensity γ does not depend on θ, i.e. $\gamma = \gamma(r, z)$.

To derive the equation for γ and to specify the general boundary conditions for γ, we introduce the following notation. On the boundary let v_τ denote the tangential component of velocity inside the passage and v_τ^* the tangential component of velocity outside the passage.

3.1 Induced Velocity

The bounding surfaces are generated by C^1 curves which can be parametrized by arclength ℓ. Since the vortex intensity is axisymmetric, we can write it as a function of ℓ too. We start by determining the induced velocity due to a circular vortex filament of constant linear vortex density.

THEOREM 3.1 *The induced velocity at a test point $\bar{x}_c$ due to a circular vortex filament $z = z_0$, $r = r_0$ with intensity γ is*

$$\bar{v} = \frac{\gamma r_0}{4\pi} \int_0^{2\pi} \frac{\bar{\tau}(\theta) \times \bar{\rho}(\theta)}{|\bar{\rho}(\theta)|^3} \, d\theta,$$

where

$$\bar{\tau}(\theta) = (-\sin\theta, \cos\theta, 0), \qquad \bar{\rho}(\theta) = \bar{x}_c - (r_0 \cos\theta, r_0 \sin\theta, z_0).$$

Proof By the Biot–Savart law, the velocity at $\bar{x}_c$ induced by an infinitesimal linear vorticity $\overline{d\gamma}$ at $\bar{x}$ is

$$\overline{dv} = \frac{1}{4\pi} \frac{\overline{d\gamma} \times \bar{\rho}}{|\bar{\rho}|^3},$$

where $\bar{\rho} = \bar{x}_c - \bar{x}$. In our case $\overline{d\gamma} = \gamma \bar{\tau} r_0 \, d\theta$, where $\bar{\tau}$ is the unit vector in the direction of $\overline{d\gamma}$. Integrating around the circular filament, with $\bar{\tau}$ tangent to the circle, gives the desired result.

Now we integrate along the generatrices of the bounding surfaces of revolution to obtain the total induced velocity at a test point. For convenience we introduce an expression for the square of the distance to the test point: $R(r, z, \theta) = |\bar{\rho}|^2$.

THEOREM 3.2 *The induced velocity at a test point* $\bar{x}_c = (0, r_c, z_c)$ *due to axisymmetric distributed surface vorticity* $\gamma(\ell)$ *on the bounding surfaces of revolution is*

$$
\begin{aligned}
v_r &= \frac{1}{4\pi} \int_{\mathscr{L}} \gamma(\ell) r(\ell)(z_c - z(\ell)) \int_0^{2\pi} \frac{\sin\theta}{R(r(\ell), z(\ell), \theta)^{3/2}} \, d\theta \, d\ell, \\
v_z &= \frac{1}{4\pi} \int_{\mathscr{L}} \gamma(\ell) r(\ell) \int_0^{2\pi} \frac{r(\ell) - r_c \sin\theta}{R(r(\ell), z(\ell), \theta)^{3/2}} \, d\theta \, d\ell, \\
v_\theta &= 0,
\end{aligned}
\tag{2}
$$

where

$$
R(r, z, \theta) = r^2 + r_c^2 - 2 r_c r \sin\theta + (z_c - z)^2, \tag{3}
$$

$\mathscr{L}$ *denotes the union of curves that generate the boundary, and* ℓ *is arclength along these curves.*

Proof The direction vector to a test point $(0, r_c, z_c)$ is

$$
\begin{aligned}
\bar{\rho} &= (0, r_c, z_c) - (r(\ell)\cos\theta, r(\ell)\sin\theta, z(\ell)) \\
&= (-r(\ell)\cos\theta, r_c - r(\ell)\sin\theta, z_c - z(\ell)).
\end{aligned}
$$

Taking square of the magnitude we obtain the required expression for R. By Theorem 3.1, the induced velocity at $(0, r_c, z_c)$ due to distributed vortices on the boundary is

$$
\begin{aligned}
\bar{v} = \frac{1}{4\pi} \int_{\mathscr{L}} & \gamma(\ell) r(\ell) \\
& \times \int_0^{2\pi} \frac{(-\sin\theta, \cos\theta, 0) \times (-r(\ell)\cos\theta, r_c - r(\ell)\sin\theta, z_c - z(\ell))}{R(r(\ell), z(\ell), \theta)^{3/2}} \, d\theta \, d\ell.
\end{aligned}
$$

We take the cross product in the numerator of the integrand. Since the x coordinate of the test point is zero, by symmetry, the x component

of $\bar{v}$ is zero, so $v_0 = 0$. Furthermore, we can ignore the x component of the integrand and obtain v_r by integrating the y component.

3.2 Boundary Conditions and the Integral Equation for Vorticity

The integral equation for γ is obtained from the formulas for induced velocity and boundary conditions at the walls.

At the wall an obvious boundary condition is the absence of flow across the wall. However, using the condition $v_n = 0$ directly leads to a Fredholm integral equation of the first kind, which is unsuitable due to small diagonal entries in the matrix for the numerical solution [1].

THEOREM 3.3 *The boundary condition $v_n = 0$ is equivalent to $v_\tau = \gamma$, where v_τ is the tangential component of velocity inside the wall.*

Proof Consider the problem for potential flow outside the passage with boundary conditions: $v_\tau^* = 0$ at the passage walls, where v_τ^* is the tangential component of velocity just outside the wall; and $\bar{v} = 0$ at infinity. This problem has a unique solution $\bar{v} = 0$, i.e. the outside of the passage is a stagnation zone. In particular, $v_n = 0$ (see e.g. [1,5]).

It is well known that the discontinuity of the tangential component of velocity across a vortex sheet is precisely the vortex density, i.e. $\gamma = v_\tau - v_\tau^*$ (see e.g. [5]). To see this, integrate the velocity field along a thin closed contour, the shape of a curvilinear rectangle following the vortex sheet on opposite sides positioned orthogonally to the vortex density field (see Fig. 2). The contribution of the short sides BC and DA, transversal to the sheet, can be made arbitrarily small, since velocity is bounded. The contribution of the long sides AB and CD following the sheet is $v_\tau h - v_\tau^* h$, where h is the arclength of AB (or CD). Note that the two contributions have opposite signs, since the contour goes in opposite directions on the two sides of the sheet. By the Stokes theorem this is equal to the surface integral of the vortex

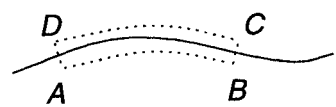

FIGURE 2 Contour of integration around a vortex sheet.

current, i.e. $v_\tau h - v_\tau^* h = \gamma h$. Taking limit as $h \to 0$ gives the desired result $v_\tau - v_\tau^* = \gamma$ and we see that the boundary condition $v_\tau^* = 0$ is equivalent to $v_\tau = \gamma$.

THEOREM 3.4 *Axisymmetric distributed surface vorticity $\gamma(\ell)$ on the bounding surfaces of revolution satisfies a Fredholm integral equation of the second kind*

$$\frac{\gamma(\ell_c)}{2} = \frac{1}{4\pi}\int_{\mathscr{L}} \gamma(\ell) r(\ell) \times \int_0^{2\pi} \frac{r'(\ell_c)(z(\ell_c) - z(\ell))\sin\theta + z'(\ell_c)(r(\ell) - r(\ell_c)\sin\theta)}{(r(\ell)^2 + r(\ell_c)^2 - 2r(\ell_c)r(\ell)\sin\theta + (z(\ell_c) - z(\ell))^2)^{3/2}} \, d\theta \, d\ell \tag{4}$$

where $\mathscr{L}$ denotes the union of curves that generate the boundary, and prime denotes differentiation with respect to arclength ℓ along these curves.

Proof At each point $(r_c, z_c) = (r(\ell_c), z(\ell_c))$ on the boundary we set $u_\tau = (v_\tau + v_\tau^*)/2$. Since by Theorem 3.3 $\gamma = v_\tau$ and $v_\tau^* = 0$, we have $u_\tau = \gamma/2$. Therefore,

$$\frac{\gamma}{2} = \frac{dr}{d\ell} v_r + \frac{dz}{d\ell} v_z. \tag{5}$$

Combining (5) and (2) gives the desired result.

3.3 *A priori* Conditions on Vorticity

In general, γ is *a priori* unknown and is obtained by solving the integral equation (4). However, in the infinite parts of the domain, namely at the inlet and exit, we impose fairly simple γ distributions and prove that the resulting velocity field asymptotically satisfies the boundary couditions at the inlet and exit. We perform the asymptotic analysis in several propositions for each of the cases of inlet/exit. The various formulas for γ are also collected in Theorem 3.14 below.

We begin with a brief explanation of asymptotic notation used in what follows. We are interested in what happens as $x \to x_0$, where $x_0 \in \mathbf{R}$ or $x_0 = \infty$.

(1) $f = \mathcal{O}(g)$ means that there exists a constant k such that for all x sufficiently close to x_0, we have $|f(x)| < k|g(x)|$.
(2) $f \sim g$ means that $f/g \to 1$.
(3) $f \prec g$ means that $f/g \to 0$.

LEMMA 3.5 *Suppose* $\gamma = \gamma_0/r$ *for* $r > \ell_0$ *on a horizontal plane* $z = z_0$. *Then as* $r_c \to \infty$, *the components of the induced velocity* v *at the test point* $(0, r_c, z_c)$ *are*

$$v_r = \frac{\gamma_0}{2}\operatorname{sgn}(z_c - z_0) r_c^{-1} + \mathcal{O}(r_c^{-2}),$$
$$v_z = \frac{\gamma_0}{2r_c} + \mathcal{O}(r_c^{-3}), \quad v_\theta = 0.$$

Proof The horizontal plane can be parametrized by $r(\ell) = \ell$ for $\ell > \ell_0$. Let $h_c = z_c - z_0$. By Theorem 3.2, with R as in (3), the components of the induced velocity field are $v_\theta = 0$,

$$v_r = \frac{\gamma_0 h_c}{4\pi} \int_0^{2\pi}\int_{\ell_0}^{\infty} \frac{\sin\theta}{R(\ell, z_0, \theta)^{3/2}}\, d\ell\, d\theta, \quad v_z = \frac{\gamma_0}{4\pi}\int_0^{2\pi}\int_{\ell_0}^{\infty} \frac{\ell - r_c \sin\theta}{R(\ell, z_0, \theta)^{3/2}}\, d\ell\, d\theta.$$

We write $R(\ell, z_0, \theta)$ (see (3)) as a polynomial $R = a + b\ell + \ell^2$ with coefficients

$$a = r_c^2 + h_c^2, \qquad b = -2r_c \sin\theta,$$

and integrate with respect to ℓ using the formulas (15) in the Appendix:

$$v_r = \frac{\gamma_0 h_c}{4\pi}\int_0^{2\pi} \sin\theta\, T_0^*\, d\theta, \qquad v_z = \frac{\gamma_0}{4\pi}\int_0^{2\pi} (T_1^* - r_c \sin\theta\, T_0^*)\, d\theta,$$

where $T_i^* = \lim_{\ell\to\infty} T_i(\ell) - T_i(\ell_0)$, $i = 0, 1$, i.e.

$$T_0^* = \frac{1}{h_c^2 + r_c^2\cos^2\theta}\left(1 - \frac{\ell_0 - r_c\sin\theta}{R(\ell_0, z_0, \theta)^{1/2}}\right),$$
$$T_1^* = \frac{1}{h_c^2 + r_c^2\cos^2\theta}\left(\frac{r_c^2 + h_c^2 - \ell_0 r_c \sin\theta}{R(\ell_0, z_0, \theta)^{1/2}} + r_c\sin\theta\right).$$

In other words,

$$v_r = \frac{\gamma_0 h_c}{4\pi}\int_0^{2\pi} \frac{\sin\theta}{h_c^2 + r_c^2\cos^2\theta}\left(1 - \frac{\ell_0 - r_c\sin\theta}{R(\ell_0, z_0, \theta)^{1/2}}\right)\mathrm{d}\theta,$$
$$v_z = \frac{\gamma_0}{4\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{R(\ell_0, z_0, \theta)^{1/2}}. \tag{6}$$

We expand $R^{-1/2}$ in a series in r_c^{-1}:

$$\begin{aligned} R(\ell_0, z_0, \theta)^{-1/2} &= r_c^{-1}[1 - 2r_c^{-1}\ell_0\sin\theta + r_c^{-2}(\ell_0^2 + h_c^2)]^{-1/2} \\ &= r_c^{-1}(1 + r_c^{-1}\ell_0\sin\theta) + \mathcal{O}(r_c^{-3}). \end{aligned}$$

Since

$$(r_c^{-1}\ell_0 - \sin\theta)(1 + r_c^{-1}\ell_0\sin\theta) = -\sin\theta + r_c^{-1}\ell_0\cos^2\theta + \mathcal{O}(r_c^{-2}),$$

$$v_r = \frac{\gamma_0 h}{4\pi}\int_0^{2\pi} \frac{\sin\theta + \sin^2\theta - r_c^{-1}\ell_0\sin\theta\cos^2\theta}{h_c^2 + r_c^2\cos^2\theta}\,\mathrm{d}\theta + \mathcal{O}(r_c^{-4}),$$
$$v_z = \frac{\gamma_0}{4\pi}\int_0^{2\pi} (r_c^{-1} + r_c^{-2}\ell_0\sin\theta)\,\mathrm{d}\theta + \mathcal{O}(r_c^{-3}).$$

Since those terms, that are odd with respect to $\theta = 0$, do not contribute to the integrals, we may rewrite

$$v_r = \frac{\gamma_0 h}{4\pi r_c^2}\int_0^{2\pi} \frac{\sin^2\theta}{r_c^{-2}h_c^2 + \cos^2\theta}\,\mathrm{d}\theta + \mathcal{O}(r_c^{-4}),$$
$$v_z = \frac{\gamma_0}{4\pi}\int_0^{2\pi} r_c^{-1}\,\mathrm{d}\theta + \mathcal{O}(r_c^{-3}).$$

The result for v_z follows immediately after integration with respect to θ. To derive the desired result for v_r, we note that $\sin^2\theta(r_c^{-2}h_c^2 + \cos^2\theta)^{-1} = -1 + (1+\eta^2)(\eta^2 + \cos^2\theta)^{-1}$, where $\eta = r_c^{-2}h_c^2$. Evaluating the needed integral (see [4, 2.562.2])

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\eta^2 + \cos^2\theta} = -\frac{2}{|\eta|(1+\eta^2)^{1/2}}\arctan\frac{|\eta|\cot\theta}{(1+\eta^2)^{1/2}}\Bigg|_0^{\pi} = \frac{2\pi}{|\eta|(1+\eta^2)^{1/2}},$$

we obtain

$$v_r = \frac{\gamma_0 h_c}{4\pi r_c^2}\left(-1 + \frac{2\pi[1+\eta^2]^{1/2}}{|\eta|}\right) + \mathcal{O}(r_c^{-4})$$
$$= \frac{\gamma_0 h_c}{2r_c}\operatorname{sgn}(h_c)[1 + r_c^{-2}h_c^2]^{1/2} + \mathcal{O}(r_c^{-2})$$

and the conclusion follows by expanding $[1 + h_c^2 r_c^{-2}]^{1/2}$ in a series in r_c^{-2}.

PROPOSITION 3.6 *Suppose* $\gamma = \gamma_0/r$ *on the horizontal plane* $z = z_0$ *and* $\gamma = -\gamma_0/r$ *on the horizontal plane* $z = z_1 = z_0 + b_0 > z_0$ *for* $r > r_0$. *Then as* $r_c \to \infty$ *the components of the induced velocity at the test point* $(0, r_c, z_c)$ *satisfy* (i) $v_\theta = 0$, (ii) $v_z = \mathcal{O}(r_c^{-3})$, *and* (iii) *if* $z_0 < z < z_1$, *then* $v_r = \gamma_0 r_c^{-1} + \mathcal{O}(r_c^{-2})$, *and if* z *is outside the interval* $[z_0, z_1]$, *then* $v_r = \mathcal{O}(r_c^{-2})$.

Proof The proof follows by the superposition principle applied to the results of Lemma 3.5 for each of the two planes. The results for v_z and v_θ are obvious, whereas

$$v_r = \frac{\gamma_0}{2}[\operatorname{sgn}(z_c - z_0) - \operatorname{sgn}(z_c - z_1)]r_c^{-1} + \mathcal{O}(r_c^{-2})$$

and we observe that if $z_0 < z < z_1$, then $\operatorname{sgn}(z_c - z_0) - \operatorname{sgn}(z_c - z_1) = 2$, while if $z \notin [z_0, z_1]$, then $\operatorname{sgn}(z_c - z_0) - \operatorname{sgn}(z_c - z_1) = 0$.

COROLLARY 3.7 *Under the assumptions of the above proposition, the flux through a cylinder of radius* r_c *between the two planes is* $2\pi\gamma_0 b_0 + \mathcal{O}(r_c^{-1})$.

PROPOSITION 3.8 *Suppose* $\gamma = \gamma_0$ *on a cylinder with radius* r_0 *for* $z \geq z_0$ *or* $z \leq z_0$. *Then as* $z_c \to \pm\infty$, *the components of the induced velocity at the test point* $(0, r_c, z_c)$ *satisfy* (i) $v_\theta = 0$, (ii) $v_r = \mathcal{O}(z_c^{-3})$, *and* (iii) *if* $r_c < r_0$, *then* $v_z = \gamma_0 + \mathcal{O}(z_c^{-2})$, *and if* $r_c > r_0$, *then* $v_z = \mathcal{O}(z_c^{-2})$.

Proof A cylindrical exit can be parametrized by $r = r_0$, $z(\ell) = \pm\ell$ for $\ell > \ell_0$, where throughout this proof we will use the $\pm$ notation to deal with the cases $z_c \to \pm\infty$. By Theorem 3.2, with R as in (3), the components of the induced velocity field are $v_\theta = 0$,

$$v_r = \frac{\gamma_0 r_0}{4\pi}\int_0^{2\pi}\int_{\ell_0}^{\infty} \frac{(z_c \mp \ell)\sin\theta}{R(r_0, \pm\ell, \theta)^{3/2}}\,d\ell\,d\theta,$$
$$v_z = \frac{\gamma_0 r_0}{4\pi}\int_0^{2\pi}\int_{\ell_0}^{\infty} \frac{r_0 - r_c\sin\theta}{R(r_0, \pm\ell, \theta)^{3/2}}\,d\ell\,d\theta.$$

We write $R(r_0 \pm \ell, \theta)$ (see (3)) as a polynomial $R = a \pm b\ell + \ell^2$ with coefficients

$$a = r_0^2 + r_c^2 - 2r_c r_0 \sin\theta + z_c^2, \quad b = -2z_c,$$

and integrate with respect to ℓ using the formulas (15) in the Appendix:

$$v_r = \frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} (z_c T_0^* \mp T_1^*) \sin\theta \, d\theta,$$
$$v_z = \frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} (r_0 - r_c \sin\theta) T_0^* \, d\theta,$$

where $T_i^* = \lim_{\ell \to \infty} T_i(\ell) - T_i(\ell_0)$, $i = 0, 1$, i.e.

$$T_0^* = \frac{1}{r_0^2 + r_c^2 - 2r_c r_0 \sin\theta} \left(\pm 1 - \frac{\pm\ell_0 - z_c}{R(r_0, \pm\ell_0, \theta)^{1/2}} \right),$$
$$T_1^* = \frac{1}{r_0^2 + r_c^2 - 2r_c r_0 \sin\theta} \left(\frac{r_0^2 + r_c^2 - 2r_c r_0 \sin\theta + z_c^2 \mp z_c \ell_0}{R(r_0, \pm\ell_0, \theta)^{1/2}} + z_c \right).$$

In other words,

$$v_r = -\frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} \frac{\sin\theta \, d\theta}{R(r_0, \pm\ell_0, \theta)^{1/2}},$$
$$v_z = \frac{\gamma_0 r_0}{4\pi} \int_0^{2\pi} \frac{r_0 - r_c \sin\theta}{r_c^2 - 2r_c r_0 \sin\theta + r_0^2} \left(\pm 1 - \frac{\pm\ell_0 - z_c}{R(r_0, \pm\ell_0, \theta)^{1/2}} \right) d\theta. \tag{7}$$

We expand $R^{-1/2}$ in a series in z_c^{-1}

$$R(r_0, \pm\ell_0, \theta)^{-1/2} = \pm z_c^{-1} [1 \mp 2\ell_0 z_c^{-1} + z_c^{-2}(r_0^2 + r_c^2 - 2r_c r_0 \sin\theta + \ell_0^2)]^{-1/2}$$
$$= \pm z_c^{-1}(1 \pm \ell_0 z_c^{-1}) + \mathcal{O}(z_c^{-3})$$

and, since $\pm z_c^{-1}(\pm\ell_0 - z_c)(1 \pm \ell_0 z_c^{-1}) = \pm(\ell_0^2 z_c^{-2} - 1)$, we obtain

$$v_r = \mp \frac{\gamma_0 r_0 (1 \pm \ell_0 z_c^{-1})}{4\pi z_c} \int_0^{2\pi} \sin\theta \, d\theta + \mathcal{O}(z_c^{-3}),$$
$$v_z = \pm \frac{\gamma_0 r_0}{2\pi} \int_0^{2\pi} \frac{r_0 - r_c \sin\theta}{r_c^2 + r_0^2 - 2r_c r_0 \sin\theta} d\theta + \mathcal{O}(z_c^{-2}).$$

By 2.551.2 [4]

$$v_z = \pm \frac{\gamma_0 r_0}{2\pi}\left[\frac{\theta}{2r_0} + \frac{1}{r_0}\arctan\frac{(r_c^2 + r_0^2)\tan(\theta/2) - 2r_c r_0}{r_0^2 - r_c^2}\right]_{-\pi}^{\pi} + \mathcal{O}(z_c^{-2}) = \pm\gamma_0 + \mathcal{O}(z_c^{-2}).$$

If $r_c < r_0$, then $v_z = \pm r_0 + \mathcal{O}(z_c^{-2})$, and if $r_c > r_0$, then $v_z = \mathcal{O}(z_c^{-2})$.

COROLLARY 3.9 *Under the assumptions of the above proposition, the flux through a horizontal plane $z = z_c$ inside the cylinder is $\pm\pi r_0^2\gamma_0 + \mathcal{O}(z_c^{-2})$.*

PROPOSITION 3.10 *Let $0 < r_1 < r_0$. Suppose that for $z \geq z_0$ or $z \leq z_0$, $\gamma = \gamma_0$ on a cylinder with radius r_0 and $\gamma = -\gamma_0$ on a cylinder with radius r_1. Then as $z_c \to \pm\infty$, the components of the induced velocity at the test point $(0, r_c, z_c)$ satisfy* (i) $v_\theta = 0$, (ii) $v_r = \mathcal{O}(z_c^{-3})$, *and* (iii) *if $r_1 < r_c < r_0$, then $v_z = \gamma_0 + \mathcal{O}(z_c^{-2})$, and if r_c is outside the interval $[r_1, r_0]$, then $v_z = \mathcal{O}(z_c^{-2})$.*

Proof This follows from Proposition 3.8 by the superposition principle.

COROLLARY 3.11 *Under the assumptions of the above proposition, the flux through a horizontal plane $z = z_c$ between the two cylinders is $\pm\pi(r_0^2 - r_1^2)\gamma_0 + \mathcal{O}(z_c^{-2})$.*

PROPOSITION 3.12 *Suppose $\gamma = \gamma_0/s^2$ on a cone with generatrix $r = (z - z_v)\tan\alpha$, where $s = |(r, z - z_v)|$. Let $s_c = |(r_c, z_c - z_v)|$. Then as $s_c \to \infty$, $v_\theta = 0$ and inside the cone the central component of $\bar{v}$ with respect to the cone vertex is*

$$v_\rho \sim \frac{\gamma_0}{2s_c^2}(\pm 1 + \cos\alpha),$$

where $+$ corresponds to the case $0 < \alpha < \pi/2$ and $-$ to the case $\pi/2 < \alpha < \pi$.

Proof A cone with angle α to the z axis can be parametrized by $z = z_v + \ell\cos\alpha$, $r = \ell\sin\alpha$. In this case $s = \ell$. Let $h_c = z_c - z_v$ and let β be the angle between the vector (r_c, h_c) and the z axis. Then $r_c = s_c\sin\beta$

and $h_c = s_c \cos\beta$. By Theorem 3.2, with R as in (3), we have $v_\theta = 0$, and

$$v_r = \frac{\gamma_0 \sin\alpha}{4\pi} \int_0^{2\pi} \int_{\ell_0}^{\infty} \frac{(h_c - \ell\cos\alpha)\sin\theta \, d\ell \, d\theta}{\ell R(\ell\sin\alpha, z_v + \ell\cos\alpha, \theta)^{3/2}},$$
$$v_z = \frac{\gamma_0 \sin\alpha}{4\pi} \int_0^{2\pi} \int_{\ell_0}^{\infty} \frac{(\ell\sin\alpha - r_c\sin\theta) \, d\ell \, d\theta}{\ell R(\ell\sin\alpha, z_v + \ell\cos\alpha, \theta)^{3/2}}.$$

We write $R(\ell\sin\alpha, z_v + \ell\cos\alpha, \theta)$ as a polynomial $R = a + b\ell + \ell^2$ with coefficients

$$a = r_c^2 + h_c^2 = s_c^2, \quad b = -2s_c\sigma, \quad \text{where } \sigma = \sin\alpha\sin\beta\sin\theta + \cos\alpha\cos\beta,$$

and integrate with respect to ℓ using the formulas (15) in the Appendix

$$v_r = \frac{\gamma_0 \sin\alpha}{4\pi} \int_0^{2\pi} (h_c T_{-1}^* - \cos\alpha \, T_0^*) \sin\theta \, d\theta,$$
$$v_z = \frac{\gamma_0 \sin\alpha}{4\pi} \int_0^{2\pi} (-r_c \sin\theta \, T_{-1}^* + \sin\alpha \, T_0^*) \, d\theta,$$

where $T_i^* = \lim_{\ell\to\infty} T_i(\ell) - T_i(\ell_0)$, $i = -1, 0$, i.e.

$$T_{-1}^* = \frac{1}{s_c^3}\left[\frac{1}{1-\sigma^2}\left(\sigma - \frac{\sigma\ell_0 - s_c(2\sigma^2 - 1)}{R_0^{1/2}}\right) + \lambda\right],$$
$$T_0^* = \frac{1}{s_c^2(1-\sigma^2)}\left[1 - \frac{\ell_0 - s_c\sigma}{R_0^{1/2}}\right],$$

where

$$R_0 = R(\ell_0\sin\alpha, z_v + \ell_0\cos\alpha, \theta) = s_c^2 - 2s_c\ell_0\sigma + \ell_0^2,$$
$$\lambda = \log\frac{s_c - \sigma\ell_0 + R_0^{1/2}}{(1-\sigma)\ell_0}.$$

In other words,

$$v_r = \frac{\gamma_0 \sin\alpha}{4\pi s_c^2} \int_0^{2\pi} \left[\frac{1}{1-\sigma^2}\left(\tau_r - \frac{\tau_r\ell_0 + s_c\mu_r}{R_0^{1/2}}\right) + \lambda\cos\beta\right] \sin\theta \, d\theta,$$
$$v_z = \frac{\gamma_0 \sin\alpha}{4\pi s_c^2} \int_0^{2\pi} \left[\frac{1}{1-\sigma^2}\left(\tau_z - \frac{\tau_z\ell_0 - s_c\mu_z}{R_0^{1/2}}\right) - \lambda\sin\beta\sin\theta\right] d\theta, \tag{8}$$

where

$$\tau_r = \sigma\cos\beta - \cos\alpha, \qquad \tau_z = \sin\alpha - \sigma\sin\beta\sin\theta,$$
$$\mu_r = \sigma\cos\alpha - \cos\beta(2\sigma^2 - 1), \qquad \mu_z = \sigma\sin\alpha - \sin\beta\sin\theta(2\sigma^2 - 1).$$

As $s_c \to \infty$, $R_0^{1/2} \sim s_c$ and

$$\lambda = \log s_c + \log\frac{2}{(1-\sigma)\ell_0} + \mathcal{O}(s_c^{-1}).$$

Therefore, since $\log s_c \sin\theta$ integrates to 0,

$$v_r \sim \frac{\gamma_0 \sin\alpha}{4\pi s_c^2}\int_0^{2\pi}\left[\frac{\tau_r - \mu_r}{1-\sigma^2} + \log\frac{2}{(1-\sigma)\ell_0}\cos\beta\right]\sin\theta\,\mathrm{d}\theta,$$
$$v_z \sim \frac{\gamma_0 \sin\alpha}{4\pi s_c^2}\int_0^{2\pi}\left[\frac{\tau_z + \mu_z}{1-\sigma^2} + \log\frac{2}{(1-\sigma)\ell_0}\sin\beta\sin\theta\right]\mathrm{d}\theta.$$

In spherical coordinates (ρ, φ, θ) centered at the vertex of the cone the velocity components $v_\theta = 0$, $v_\rho = v_r\sin\beta + v_z\cos\beta$ and $v_\varphi = -v_r\cos\beta + v_z\sin\beta$ have the following expressions:

$$v_\rho \sim \frac{\gamma_0 \sin\alpha}{4\pi s_c^2}\int_0^{2\pi}\frac{\xi_\rho}{1-\sigma^2}\,\mathrm{d}\theta,$$
$$v_\varphi \sim \frac{\gamma_0 \sin\alpha}{4\pi s_c^2}\int_0^{2\pi}\left[\frac{\xi_\varphi}{1-\sigma^2} - \log\frac{2}{(1-\sigma)\ell_0}\sin\theta\right]\mathrm{d}\theta,$$

where

$$\xi_\rho = (\tau_r - \mu_r)\sin\beta\sin\theta + (\tau_z + \mu_z)\cos\beta,$$
$$\xi_\varphi = (\mu_r - \tau_r)\cos\beta\sin\theta + (\tau_z + \mu_z)\sin\beta.$$

Since $\xi_\rho = (1+\sigma)(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\theta)$,

$$v_\rho \sim \frac{\gamma_0 \sin\alpha}{4\pi s_c^2}\int_0^{2\pi}\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\theta}{1 - \sin\alpha\sin\beta\sin\theta - \cos\alpha\cos\beta}\,\mathrm{d}\theta.$$

Using 2.551.2, 1.314.5, 1.317.1 [4], we obtain

$$v_\rho \sim \frac{\gamma_0 \sin\alpha}{2s_c^2}\left[\frac{\sin\alpha\cos\beta - \cot\alpha(1-\cos\alpha\cos\beta)}{\sqrt{(1-\cos\alpha\cos\beta)^2 - \sin^2\alpha\sin^2\beta}} + \cot\alpha\right]$$

$$= \frac{\gamma_0}{2s_c^2}\left[\frac{\cos\beta - \cos\alpha}{\sqrt{1-\cos(\alpha-\beta)}\sqrt{1-\cos(\alpha+\beta)}} + \cos\alpha\right]$$

$$= \frac{\gamma_0}{2s_c^2}(\pm 1 + \cos\alpha),$$

where $+$ corresponds to the case $0 \le \beta < \alpha < \pi/2$ and $-$ to the case $\pi/2 < \alpha < \beta \le \pi$.

COROLLARY 3.13 *Consider a spherical cap given in spherical coordinates* (ρ, φ, θ) *by* (i) $\rho = s$, $0 \le \varphi < \alpha < \pi/2$, (ii) *or* $\rho = s$, $\pi/2 < \alpha \le \varphi \le \pi$, *where* s *and* α *are constants. Under the assumptions of the above proposition, the flux through this spherical cap is asymptotically given by* $\pm\pi\gamma_0 \sin^2\alpha/s^2$, *where* $+$ *corresponds to* (i) *and* $-$ *to* (ii).

Proof By the preceding proposition the component of velocity orthogonal to the cap is $v_\rho \sim \gamma_0(\pm 1 + \cos\alpha)/(2s^2)$ and the area of the cap is $2\pi s^2(1 \mp \cos\alpha)$.

Remark In all cases, except the cone, tangential velocity outside the wall has asymptotically smaller order than inside the passage. For a cone this is not the case – induced central velocity outside the cone (with α close to 0 or π), while much smaller than inside the cone, is of the same asymptotic order. This fact inevitably leads to errors.

THEOREM 3.14 *Let* q *be constant. If* γ *satisfies the following a priori conditions for the different cases of inlet/exit*:

(a) $\gamma = \pm q/(2\pi b_0 r)$ *on two horizontal planes, a distance* b_0 *apart with opposite signs on the two planes,*
(b) $\gamma = q/(\pi r_0^2)$ *on a vertical cylinder with radius* r_0,
(c) $\gamma = \pm q/(\pi(r_0^2 - r_1^2))$ *on a vertical annular cylinder with radii* $0 < r_1 < r_0$, *with opposite signs at* r_1 *and* r_0,

(d) $\gamma = \pm q/(\pi \sin^2\alpha\rho^2)$, *where* $\rho = |(r, z - z_v)|$, *on a cone with generatrix* $r = (z - z_v)\tan\alpha$,

then the induced velocity field has flux that is asymptotic to q *and asymptotically satisfies the boundary conditions at infinity (inlet/exit) of Section* 2.

Proof This follows from the preceding results of this section.

THEOREM 3.15 *At the intersection of the inner bounding surface with the* z *axis* $\gamma = 0$.

Proof This follows from the symmetry of the problem.

4 COMPUTATIONAL APPROACH

For practical purposes the integrals in (4) are taken on parts (elements) of $\mathscr{L}$. On each element we represent $\gamma(\ell)$ by a linear spline with undetermined coefficients, take the integrals and solve the resulting system of equations for these coefficients.

Each element is taken to be linear, except when computing the induced velocity at one of its nodes. In the latter situation, we represent the element with a cubic spline. This way, though the integrals are improper in the sense that the integrands blow up, they are convergent.

The overall representation scheme is given by the Table I.

TABLE I

Type of boundary element	*Geometrical representation of the generatrix*	*Representation of vortex density* γ	*Integration*
General	linear spline	linear spline	r: "closed form" θ: numerical
Incident	cubic spline	linear spline	Split the integrand into regular and principal parts. Principal: r: "closed form" Regular: numerical
Semi-infinite	linear spline	*a priori* values	r: "closed form" θ: numerical

4.1 Boundary Elements not Incident to the Test Point

Generally each bounded element is approximated by a linear spline. If we start at a point $r(\ell_i) = r_i$, $z(\ell_i) = z_i$ this spline is of the form

$$r(\ell) = r_i + m_{ri}(\ell - \ell_i), \qquad z(\ell) = z_i + m_{zi}(\ell - \ell_i),$$

where $m_{ri}^2 + m_{zi}^2 = 1$. Letting $a_r = r_i - m_{ri}\ell_i$, $b_r = m_{ri}$, $a_z = z_i - m_{zi}\ell_i$, $b_z = m_{zi}$ we write

$$r(\ell) = a_r + b_r\ell, \qquad z(\ell) = a_z + b_z\ell.$$

The vorticity is taken to be linear as well

$$\gamma(\ell) = (1 - t)\gamma_i + t\gamma_{i+1} = \gamma_i + t(\gamma_{i+1} - \gamma_i),$$

where $t = (\ell - \ell_i)/(\ell_{i+1} - \ell_i)$ and γ_i are the nodal values of γ. We can rewrite this in the form

$$\gamma = A + B\ell, \qquad \text{where } A = \gamma_i - \frac{\ell_i(\gamma_{i+1} - \gamma_i)}{\ell_{i+1} - \ell_i}, \quad B = \frac{\gamma_{i+1} - \gamma_i}{\ell_{i+1} - \ell_i}.$$

Letting $T_n^* = T_n(\ell_{i+1}) - T_n(\ell_i)$, with a and b as given below, we obtain the following form for the integrals (2) restricted to the given element:

$$\begin{aligned} u_r &= \frac{1}{4\pi}\int_0^{2\pi}\int_{\ell_i}^{\ell_{i+1}} \frac{\bar{a}_r + \bar{b}_r\ell + \bar{c}_r\ell^2 + \bar{d}_r\ell^3}{(a + b\ell + \ell^2)^{3/2}}\,\mathrm{d}\ell\,\mathrm{d}\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi}(\bar{a}_r T_0^* + \bar{b}_r T_1^* + \bar{c}_r T_2^* + \bar{d}_r T_3^*)\,\mathrm{d}\theta, \\ u_z &= \frac{1}{4\pi}\int_0^{2\pi}\int_{\ell_i}^{\ell_{i+1}} \frac{\bar{a}_z + \bar{b}_z\ell + \bar{c}_z\ell^2 + \bar{d}_z\ell^3}{(a + b\ell + \ell^2)^{3/2}}\,\mathrm{d}\ell\,\mathrm{d}\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi}(\bar{a}_z T_0^* + \bar{b}_z T_1^* + \bar{c}_z T_2^* + \bar{d}_z T_3^*)\,\mathrm{d}\theta, \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
a &= a_r^2 + r_c^2 - 2r_c a_r \sin\theta + (z_c - a_z)^2,\\
b &= 2(a_r b_r - r_c b_r \sin\theta - (z_c - a_z)b_z),\\
\bar{a}_r &= A a_r (z_c - a_z)\sin\theta,\\
\bar{b}_r &= (B a_r (z_c - a_z) + A b_r (z_c - a_z) - A a_r b_z)\sin\theta,\\
\bar{c}_r &= (B b_r (z_c - a_z) - B a_r b_z - A b_r b_z)\sin\theta,\\
\bar{d}_r &= -B b_r b_z \sin\theta,\\
\bar{a}_z &= A a_r (a_r - r_c \sin\theta),\\
\bar{b}_z &= B a_r (a_r - r_c \sin\theta) + A b_r (a_r - r_c \sin\theta) + A a_r b_r,\\
\bar{c}_z &= B b_r (a_r - r_c \sin\theta) + B a_r b_r + A b_r^2,\\
\bar{d}_z &= B b_r^2.
\end{aligned}
$$

Note that the coefficient of ℓ^2 in the denominator of the integrands in (9) is $b_r^2 + b_z^2 = 1$.

4.2 A Pair of Boundary Elements Incident to the Test Point

Suppose the test point is at $\ell = \ell_i$ and angle $\pi/2$. Its coordinates are $(0, r_i, z_i)$, where $r_i = r(\ell_i)$ and $z_i = z(\ell_i)$. Integration over the elements $\ell_{i-1} < \ell < \ell_i$ and $\ell_i < \ell < \ell_{i+1}$ requires special consideration due to the fact that the integrand has a singularity at the test point.

As in the preceding section, we represent vortex density γ linearly. The coordinates, however, are represented by cubic splines using Hermite interpolant polynomials

$$
\begin{aligned}
H_{01}(x) &= 1 - 3x^2 + 2x^3, \qquad H_{02}(x) = 3x^2 - 2x^3,\\
H_{11}(x) &= x - 2x^2 + x^3, \qquad H_{12}(x) = -x^2 + x^3,
\end{aligned}
$$

as follows:

$$
\begin{aligned}
r(t) &= r_{j-1} H_{01}(t) + r_j H_{02}(t) + \Delta\ell_{j-1}[r'_{j-1} H_{11}(t) + r'_j H_{12}(t)],\\
z(t) &= z_{j-1} H_{01}(t) + z_j H_{02}(t) + \Delta\ell_{j-1}[z'_{j-1} H_{11}(t) + z'_j H_{12}(t)],\\
\gamma(t) &= \gamma_{j-1}(1 - t) + \gamma_j t,
\end{aligned}
$$

where

$$
t = \frac{\ell - \ell_{j-1}}{\Delta\ell_{j-1}} \text{ for } \ell_{j-1} \le \ell \le \ell_j; \quad \Delta\ell_{j-1} = \ell_j - \ell_{j-1}; \quad \text{and} \quad j = i,\ i+1.
$$

The coefficients r_j, r'_j, z_j, and z'_j are determined from the geometry of the passage via splining technique. Prime represents differentiation with respect to ℓ and the subscript j represents subsequent evaluation at ℓ_j. The coefficients γ_j are *a priori* unknown.

The contributions of the two segments surrounding the test point to the integral in (4) may be written, using symmetry with respect to θ, interchanging the order of integration, dropping the explicit dependence on ℓ, and substituting $s = \ell - \ell_i$, in the form

$$u_\tau = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\int_{-\Delta\ell_{i-1}}^{\Delta\ell_i} \frac{r'_i\gamma r(z_i - z)\sin\theta + z'_i\gamma r(r - r_i\sin\theta)}{(r^2 + r_i^2 - 2r_i r\sin\theta + (z_i - z)^2)^{3/2}}\,\mathrm{d}s\,\mathrm{d}\theta. \quad (10)$$

We use Taylor expansions, and the substitution $\delta = \pi/2 - \theta$ to analyze the behavior of the integrand near the test point

$$\begin{aligned} \sin\theta &= 1 - \tfrac{1}{2}\delta^2 + \mathcal{O}(\delta^4), \\ r &= r_0 + r'_0 s + \tfrac{1}{2}r''_0 s^2 + \tfrac{1}{6}r'''_{0\pm}s^3, \\ z &= z_0 + z'_0 s + \tfrac{1}{2}z''_0 s^2 + \tfrac{1}{6}z'''_{0\pm}s^3, \\ \gamma &= \gamma_0 + \gamma'_{0\pm}s, \end{aligned} \quad (11)$$

where prime represents differentiation with respect to s, the subscript 0 represents subsequent evaluation at $s = 0$, and $\pm$ corresponds to the cases $s < 0$ and $s > 0$.

THEOREM 4.1 *Let I be the integrand in* (10). *Then with the above notation*

$$I = \frac{r'_0\gamma r(z_0 - z)\sin\theta + z'_0\gamma r(r - r_0\sin\theta)}{(r^2 + r_0^2 - 2r_0 r\sin\theta + (z_0 - z)^2)^{3/2}} = \frac{\gamma_0 r_0 A(\varphi)}{\rho} + T(\varphi) + \mathcal{O}(\rho),$$

where $\rho\cos\varphi = r_0\delta$, $\rho\sin\varphi = s$, *and*

$$\begin{aligned} A(\varphi) &= \frac{1}{2}(z'_0 r''_0 - r'_0 z''_0)\sin^2\varphi + \frac{z'_0}{2r_0}\cos^2\varphi, \\ T(\varphi) &= \sin\varphi(\gamma'_{0\pm}r_0 + r'_0\gamma_0)A(\varphi) \\ &\quad + \gamma_0 r_0\left[\frac{r'_0 z'_0}{2r_0^2}\cos^2\varphi\sin\varphi + \frac{1}{6}(z'_0 r'''_{0\pm} - r'_0 z'''_{0\pm})\sin^3\varphi\right. \\ &\qquad \left. - \frac{3}{2}\left(\frac{r'_0}{r_0}cos^2\varphi\sin\varphi + \frac{1}{2}(r'_0 r''_0 + z'_0 z''_0)\sin^3\varphi\right)A(\varphi)\right]. \end{aligned}$$

Proof Since s is arclength, $r_0'^2 + z_0'^2 = 1$. Substituting Taylor expansions (11) into the expression $R = r^2 + r_i^2 - 2r_i r \sin\theta + (z_i - z)^2$ appearing in the denominator of I, we obtain

$$R = s^2 + r_0^2\delta^2 + r_0 r_0' s\delta^2 + s^2(\alpha_1 s^2 + \alpha_2 s + \alpha_3\delta^2) + \mathcal{O}(p^4),$$

where α_1, α_2, and α_3 are independent of s and δ. Note that $\alpha_2 = r_0' r_0'' + z_0' z_0''$. Converting to polar coordinates ρ and φ we obtain

$$R = \rho^2(1 - \alpha\rho) + \mathcal{O}(\rho^4), \quad \text{where } \alpha = \frac{r_0'}{r_0}\cos^2\varphi\sin\varphi + \alpha_2\sin^3\varphi. \tag{12}$$

Similarly, the expression in the numerator $N = r_i'(z_i - z)\sin\theta + z_i'(r - r_i\sin\theta)$ can be written as

$$\begin{aligned} N &= r_0'\left(-z_0' s - \tfrac{1}{2}z_0'' s^2 - \tfrac{1}{6}z_{0\pm}''' s^3\right)\left(1 - \tfrac{1}{2}\delta^2\right) \\ &\quad + z_0'\left(r_0' s + \tfrac{1}{2}r_0'' s^2 + \tfrac{1}{6}r_{0\pm}''' s^3 + \tfrac{1}{2}r_0\delta^2\right) + \mathcal{O}(\rho^4) \\ &= \tfrac{1}{2}(z_0' r_0'' - r_0' z_0'')s^2 + \tfrac{1}{2}z_0' r_0\delta^2 + \tfrac{1}{2}r_0' z_0' s\delta^2 \\ &\quad + \tfrac{1}{6}(z_0' r_{0\pm}''' - r_0' z_{0\pm}''')s^3 + \mathcal{O}(\rho^4). \end{aligned}$$

Therefore,

$$N = \rho^2 A(\varphi) + \rho^3 B(\varphi) + \mathcal{O}(\rho^4), \tag{13}$$

where

$$B(\varphi) = \frac{r_0' z_0'}{2r_0^2}\cos^2\varphi\sin\varphi + \frac{1}{6}(z_0' r_{0\pm}''' - r_0' z_{0\pm}''')\sin^3\varphi.$$

Combining (13) and (12) and expanding the denominator we obtain

$$I = \gamma r N R^{-3/2} = \frac{\gamma r}{\rho}\left[(A(\varphi) + \rho B(\varphi))\left(1 - \frac{3}{2}\alpha\rho\right) + \mathcal{O}(\rho^2)\right]. \tag{14}$$

Multiplying out the parentheses and (14) we obtain

$$I = \gamma r N R^{-3/2} = \gamma r\left(\frac{A(\varphi)}{\rho} + B(\varphi) - \frac{3}{2}A(\varphi)\alpha + \mathcal{O}(\rho)\right).$$

Finally, expanding γ and ρ

$$\gamma = \gamma_0 + \gamma'_{0\pm}\rho\sin\varphi + \mathcal{O}(\rho^2), \qquad r = r_0 + r'_0\rho\sin\varphi + \mathcal{O}(\rho^2),$$

and multiplying out the parentheses we obtain the required formula.

In order to integrate I accurately, we split it into its principal part and the remaining regular part

$$I = \frac{\gamma_0 r_0 A(\varphi)}{\rho} + \left(I - \frac{\gamma_0 r_0 A(\varphi)}{\rho}\right).$$

Since

$$\lim_{\rho\to 0}\left(I - \frac{\gamma_0 r_0 A(\varphi)}{\rho}\right) = T(\varphi),$$

the regular part leads to a proper integral, which is evaluated numerically.

Note that, the regular part is discontinuous at $s=0$, since r'''_{0-}, z'''_{0-}, γ'_{0-} may not equal $r'''_{0+}, z'''_{0+}, \gamma'_{0+}$. In the special case when $s=0$ ($\ell=\ell_i$) is the juncture of two arcs with different radii or an arc and a straight line segment, even r''_{0-}, z''_{0-} may not equal r''_{0+}, z''_{0+}.

THEOREM 4.2 *Let P denote the contribution to u_τ of the principal part of I.*

(a) *If ℓ_i is an interior point of a cubic spline representing the boundary generatrix, then*

$$P = \frac{\gamma_0 r_0}{2\pi}\left(-\Delta\ell_{i-1}J_1\Big|_{-\pi/2}^{-\varphi_-} + \pi r_0 J_2\Big|_{-\varphi_-}^{\varphi_+} + \Delta\ell_i J_1\Big|_{\varphi_+}^{\pi/2}\right),$$

where $\varphi_- = \arg(\pi r_0, \Delta\ell_{i-1})$, $\varphi_+ = \arg(\pi r_0, \Delta\ell_i)$, and

$$J_1(\varphi) = -\frac{1}{2}(z'_0 r''_0 - r'_0 z''_0)\cos\varphi + \frac{z'_0}{2r_0}\left(\frac{1}{2}\log\left(\frac{1-\cos\varphi}{1+\cos\varphi}\right) + \cos\varphi\right),$$

$$J_2(\varphi) = \frac{1}{2}(z'_0 r''_0 - r'_0 z''_0)\left(\frac{1}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) - \sin\varphi\right) + \frac{z'_0}{2r_0}\sin\varphi.$$

(b) *If ℓ_i is the juncture of arcs with different radii or an arc and a straight line segment, then*

$$P = \frac{\gamma_0 r_0}{2\pi}\left(-\Delta\ell_{i-1}J_1\Big|_{-\pi/2}^{-\varphi_-} + \pi r_0 J_{2-}\Big|_{-\varphi_-}^{0} + \pi r_0 J_{2+}\Big|_{0}^{\varphi_+} + \Delta\ell_i J_1\Big|_{\varphi_+}^{\pi/2}\right),$$

where $J_{2\pm}$ are constructed the same way as J_2, but using left or right second derivatives $r''_{0\pm}$ and $z''_{0\pm}$.

Proof The region of integration is a rectangle $-\Delta\ell_{i-1} \le s \le \Delta\ell_i$, $0 \le \delta \le \pi$. In order to integrate the principal part of I with respect to ρ and φ (see Theorem 4.1), we divide the region of integration into three triangles shown in the Fig. 3.

The area element in polar coordinates is $\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi$, so, using the notation of Theorem 4.1,

$$\frac{2\pi P}{\gamma_0 r_0} = \int_{-\pi/2}^{-\varphi_-}\int_0^{\rho_1(\varphi)} A(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi + \int_{-\varphi_-}^{\varphi_+}\int_0^{\rho_2(\varphi)} A(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi + \int_{\varphi_+}^{\pi/2}\int_0^{\rho_3(\varphi)} A(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi,$$

where

$$\rho_1(\varphi) = \frac{-\Delta\ell_{i-1}}{\sin\varphi}, \quad \rho_2(\varphi) = \frac{\pi r_0}{\cos\varphi}, \quad \rho_3(\varphi) = \frac{\Delta\ell_i}{\sin\varphi}.$$

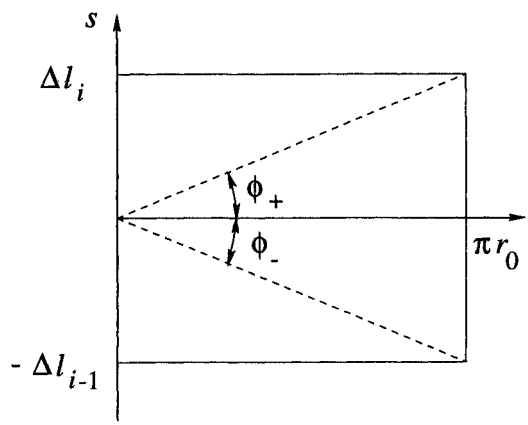

FIGURE 3 Regions of integration.

Therefore,

$$\frac{2\pi P}{\gamma_0 r_0} = -\Delta\ell_{i-1}\int_{-\pi/2}^{-\varphi_-}\frac{A(\varphi)}{\sin\varphi}\,\mathrm{d}\varphi + \pi r_0\int_{-\varphi_-}^{\varphi^+}\frac{A(\varphi)}{\cos\varphi}\,\mathrm{d}\varphi + \Delta\ell_i\int_{\varphi_+}^{\pi/2}\frac{A(\varphi)}{\sin\varphi}\,\mathrm{d}\varphi.$$

Recalling the definition of $A(\varphi)$ we see that

$$J_1'(\varphi) = \frac{A(\varphi)}{\sin\varphi} = \frac{1}{2}(z_0' r_0'' - r_0' z_0'')\sin\varphi + \frac{z_0'}{2r_0}\frac{\cos^2\varphi}{\sin\varphi},$$

$$J_2'(\varphi) = \frac{A(\varphi)}{\cos\varphi} = \frac{1}{2}(z_0' r_0'' - r_0' z_0'')\frac{\sin^2\varphi}{\cos\varphi} + \frac{z_0'}{2r_0}\cos\varphi.$$

Evaluation at the endpoints gives the desired result.

In the special case when $s=0$ is the juncture between two arcs of different radii or an arc and a straight line segment, we further split up the middle term and use right and left values of r_0'' and z_0''.

4.3 Improper Integrals at the Inlet and Exit

The inlet and the exit are of particular interest since the integrals are improper in the sense that they are taken over unbounded intervals. On these intervals we use the *a priori* values of γ, so their contribution to the integral is fixed.

Consider the integrals in (2). In each of the cases of inlet/exit we reverse the order of integration and take the inner improper integrals with respect to ℓ. Integration with respect to ℓ is in closed form using the formulas (15) given in the Appendix. Specifically we use (6) for the radial case, (7) for the cylindrical case, and (8) for the conical case.

4.4 Linear System

Since A and B are linear in γ_i and the expressions for u_r and u_z (9) are linear in A and B, we obtain a linear system of equations for the unknown nodal values γ_i.

5 APPLICATION

Below we show the streamlines and equipotentials obtained by the described method for a real passage geometry (Fig. 4). In this case the inlet is radial and the draft tube is conical. The streamlines were obtained from the velocity field data, which were in turn computed from the vorticity distribution on the boundary obtained by solving the linear system in Section 4.4 using Gaussian elimination. Equipotentials were found via an iterative technique as a family of curves orthogonal to the streamlines.

This particular example was computed using FORTRAN 77 code (not optimized) with 35 nodes per surface (70×70 linear system). We obtained the following timing:

CPU	Os	Compiler	Time
300 MHz Pentium II	Linux 2.0.30	g77 0.5.20	130.820u 0.030s 2:11.71 99.3%
296 MHz UltraSPARC-II	SunOS 5.6	g77 0.5.19	122.28u 0.59s 2:03.01 99.8%

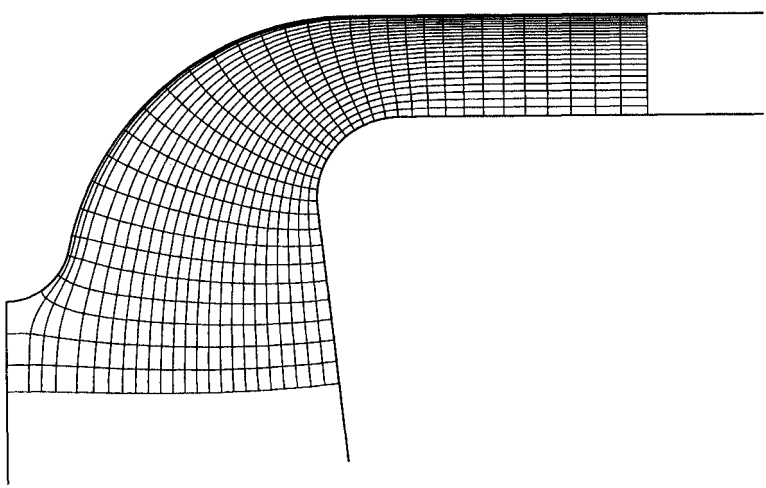

FIGURE 4 Computed streamlines and equipotentials (Butt Valley/Pacific Gas & Electric).

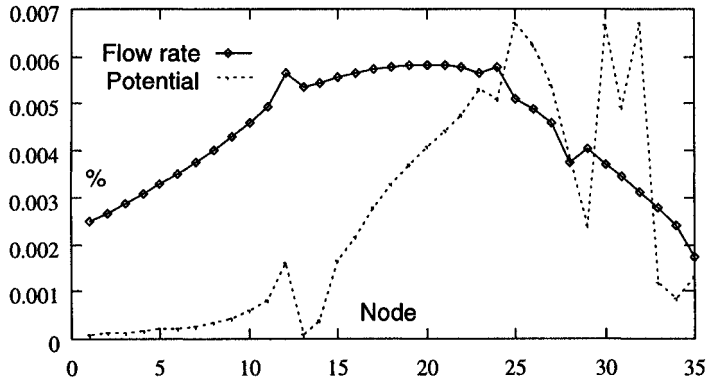

FIGURE 5 Relative errors.

The output was subjected to an internal verification procedure based on comparing computed flow rate values to the theoretical constant value across each equipotential. Also we compared the computed values of potential at the endpoints of equipotentials on the hub and the crown. Relative errors are shown in Fig. 5.

APPENDIX

The following integrals with respect to $d\ell$ are taken exactly (see e.g. 2.264.5–8, 2.269.4 in [4]). These expressions are used with various values of the coefficients a and b. Definite versions of the integrals are denoted T_n^*,

$$
\begin{aligned}
T_n &= \int \frac{x^n dx}{R^{3/2}}, \quad \text{where } R = a + bx + x^2; \\
T_0 &= \frac{2(2x+b)}{DR^{1/2}}, \quad \text{where } D = 4a - b^2; \\
T_1 &= -\frac{2(2a+bx)}{DR^{1/2}}; \\
T_2 &= -\frac{2[(2a-b^2)x - ab]}{DR^{1/2}} + \log(2R^{1/2} + 2x + b); \\
T_3 &= \frac{(a-b^2)x^2 + b(10a - 3b^2)x + a(8a - 3b^2)}{DR^{1/2}} \\
&\quad - \frac{3b}{2}\log(2R^{1/2} + 2x + b); \\
T_{-1} &= -\frac{2(bx - 2a + b^2)}{aDR^{1/2}} - a^{-3/2}\log\frac{2a + bx + 2(aR)^{1/2}}{x}, \quad (a > 0).
\end{aligned}
\tag{15}
$$

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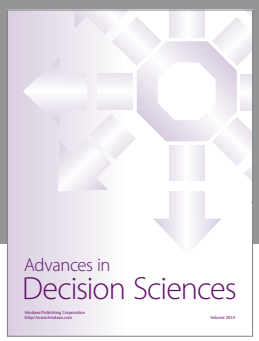

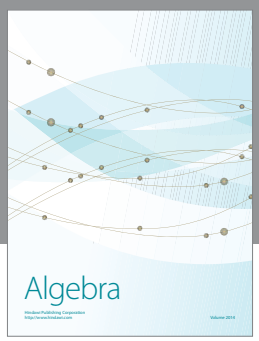

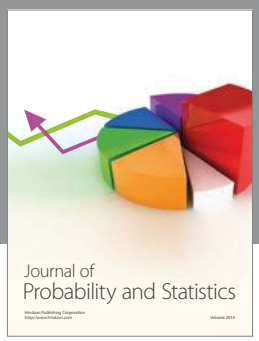

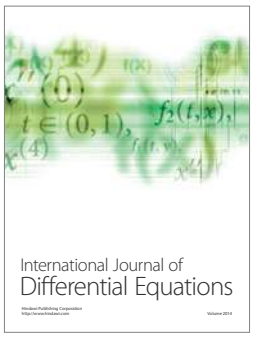

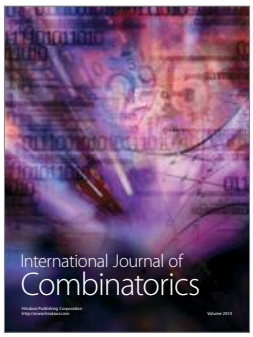

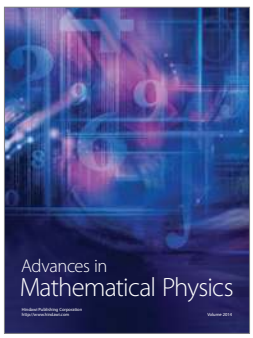

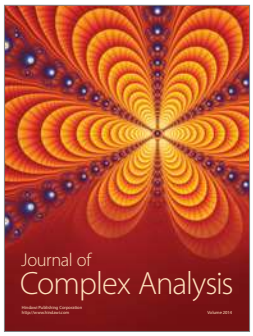

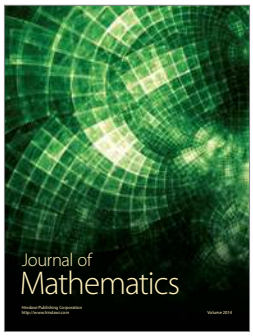

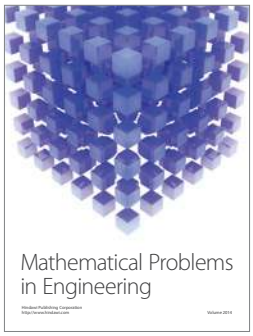

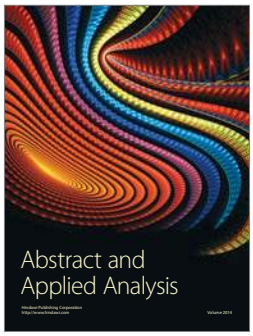

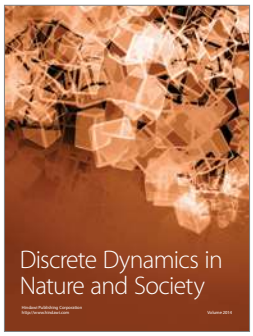

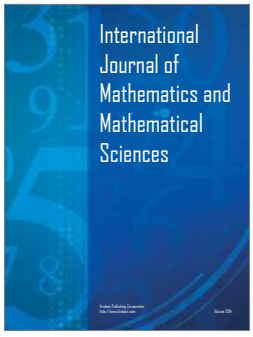

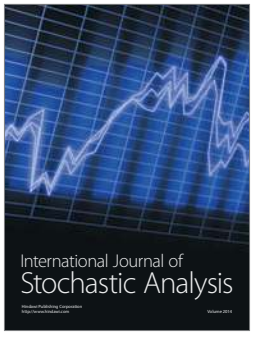

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