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BOUNDARY FEEDBACK STABILIZATION OF AN UNSTABLE HEAT EQUATION*

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Abstract. In this paper we study the problem of boundary feedback stabilization for the unstable heat equation

$$u_t(x,t) = u_{xx}(x,t) + a(x)u(x,t).$$

This equation can be viewed as a model of a heat conducting rod in which not only is the heat being diffused (mathematically due to the diffusive term u_{xx}) but also the destabilizing heat is generating (mathematically due to the term au with $a > 0$). We show that for *any* given continuously differentiable function a and *any* given positive constant λ we can *explicitly* construct a boundary feedback control law such that the solution of the equation with the control law converges to zero exponentially at the rate of λ. This is a continuation of the recent work of Boskovic, Krstic, and Liu [*IEEE Trans. Automat. Control*, 46 (2001), pp. 2022–2028] and Balogh and Krstic [*European J. Control*, 8 (2002), pp. 165–176].

Key words. heat equation, boundary control, stabilization

AMS subject classifications. 35K05, 93D15

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1. Introduction.

In this paper we continue the study of boundary feedback control of an unstable heat equation

$$u_t(x,t) = u_{xx}(x,t) + \mu u(x,t) \quad \text{in } (0,1) \times (0,\infty).$$

Hereafter, the subscripts denote the derivatives. This equation can be viewed as a model of a heat conducting rod in which not only is the heat being diffused (mathematically due to the term u_{xx}) but also the destabilizing heat is generating (mathematically due to the term μu with $\mu > 0$). This feedback control problem was recently addressed by Boskovic, Krstic, and Liu in [5], and it was shown that the unstable rod can be exponentially stabilized by a boundary feedback control law if the constant $\mu < 3\pi^2/4$; that is, the destabilizing heat generation is not very big. More recently, Balogh and Krstic [3, 4] removed the condition $\mu < 3\pi^2/4$ and replaced μ by an arbitrarily large function $a(x)$:

$$u_t(x,t) = u_{xx}(x,t) + a(x)u(x,t) \quad \text{in } (0,1) \times (0,\infty). \tag{1.1}$$

They used a backstepping method for the finite difference semidiscretized approximation of the above equation to derive a Dirichlet boundary feedback control law that makes the closed-loop system stable with an arbitrary prescribed stability margin. They showed that the integral kernel in the control law is bounded. However, some problems like the smoothness of the kernel and Neumann boundary control (usually more difficult than the Dirichlet one) were left open. Using a different method, we completely solve these problems by solving a partial differential equation of the kernel

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with strange boundary conditions (see (2.1) below). This strange boundary value problem has stood open since the work of [5] was started in 1998. We also derive Neumann boundary feedback control laws which seemingly cannot be achieved in [4]. From the proof of Lemma 2.2 below it can been seen that the feedback law is constructed explicitly and can be calculated numerically via a scheme of successive approximation. This makes its implementation possible in real problems.

The problem of boundary feedback control that we address here is not new. Some of the results on feedback stabilization of parabolic equations include the work of Amamm [2], Burns and Rubio [6], Burns, Rubio, and King [7], Day [8], Lasiecka and Triggiani [10, 11, 12, 13], and Triggiani [15]. For a detailed review of these references, we refer to [4] and [5]. In comparison with the existing literature, the novelty of the paper is the explicit construction of the feedback laws and the complete solving of the strange boundary value problem mentioned above.

The paper is organized as follows. Section 2 is devoted to the stabilization of unstable Dirichlet boundary value problems and section 3 to the stabilization of unstable Neumann boundary value problems. We raise an open problem in section 4.

2. Dirichlet boundary conditions. In what follows, we denote by $H^s(0,1)$ the usual Sobolev space (see, e.g., [1, 14]) for any $s \in \mathbf{R}$. For $s \geq 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$. We denote by $\|\cdot\|$ the norm of $L^2(0,1)$. $C^n[0,1]$ denotes the space of all n times continuously differentiable functions on $[0,1]$.

It is well known that the Dirichlet boundary value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + a(x)u(x,t) & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = 0 & \text{in } (0,\infty) \end{cases}$$

is unstable if a is positive and large. To design a boundary feedback law to stabilize it for any function $a \in C^1[0,1]$, we consider the problem

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (a(y)+\lambda)k(x,y), & 0 \leq y \leq x \leq 1, \\ k(x,0) = 0, & 0 \leq x \leq 1, \\ k_x(x,x) + k_y(x,x) + \frac{d}{dx}(k(x,x)) = a(x) + \lambda, & 0 \leq x \leq 1, \end{cases} \tag{2.1}$$

where λ is any constant. From the proof of Lemma 2.4 below we will see why we want to consider this problem. For the moment, let us assume this problem has a unique solution k for $a \in C^1[0,1]$. (This will be proved in Lemma 2.2 below.) Using the solution k, we then obtain Dirichlet boundary feedback law

$$u(1,t) = -\int_0^1 k(1,y)u(y,t)dy \qquad \text{in } (0,\infty) \tag{2.2}$$

and Neumann boundary feedback law

$$u_x(1,t) = -k(1,1)u(1,t) - \int_0^1 k_x(1,y)u(y,t)dy \qquad \text{in } (0,\infty). \tag{2.3}$$

With one of the boundary feedback laws, the system

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + a(x)u(x,t) & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 0 & \text{in } (0,\infty), \\ u(x,0) = u^0(x) & \text{in } (0,1) \end{cases} \tag{2.4}$$

is exponentially stable. In this controlled system, the left-hand end of a rod is insulated while the temperature or the heat flux at the other end is adjusted according to the measurement of k-weighted averaged temperature over the whole rod. Physically, if the destabilizing heat is generating inside the rod, then we cool the right end of the rod so that it is not overheated. To state this result, we introduce the compatible conditions for the initial data:

$$u^0(0) = 0, \quad u^0(1) = -\int_0^1 k(1,y)u^0(y)dy, \tag{2.5}$$

$$u^0(0) = 0, \quad u^0_x(1) = -k(1,1)u^0(1) - \int_0^1 k_x(1,y)u^0(y)dy. \tag{2.6}$$

THEOREM 2.1. *Assume that* $\lambda > 0$ *is any positive constant and* $a \in C^1[0,1]$ *is any function. For arbitrary initial data* $u^0(x) \in H^1(0,1)$ *with compatible condition* (2.5) *or* (2.6), *equation* (2.4) *with either* (2.2) *or* (2.3) *has a unique solution that satisfies*

$$\|u(t)\|_{H^1} \le M\|u^0\|_{H^1} e^{-\lambda t} \quad \forall t > 0, \tag{2.7}$$

where M *is a positive constant independent of* u^0.

The idea of proving the theorem is to carefully construct a transformation

$$w(x,t) = u(x,t) + \int_0^x k(x,y)u(y,t)dy$$

to convert the system (2.4) with either (2.2) or (2.3) into the exponentially stable system

$$\begin{cases} w_t = w_{xx} - \lambda w & \text{in } (0,1)\times(0,\infty), \\ w(0,t) = w(1,t) = 0 & \text{in } (0,\infty), \\ w(x,0) = w^0(x) & \text{in } (0,1) \end{cases} \tag{2.8}$$

or

$$\begin{cases} w_t = w_{xx} - \lambda w & \text{in } (0,1)\times(0,\infty), \\ w(0,t) = w_x(1,t) = 0 & \text{in } (0,\infty), \\ w(x,0) = w^0(x) & \text{in } (0,1), \end{cases} \tag{2.9}$$

where $w^0(x) = u^0(x) + \int_0^x k(x,y)u^0(y)dy$. This will be achieved in the following lemmas.

LEMMA 2.2. *Suppose that* $a \in C^1[0,1]$. *Then problem* (2.1) *has a unique solution which is twice continuously differentiable in* $0 \le y \le x \le 1$.

Proof. Using the variable changes

$$\xi = x + y, \quad \eta = x - y$$

and denoting

$$G(\xi,\eta) = k(x,y) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right),$$

problem (2.1) is transformed to

$$\begin{cases} G_{\xi\eta}(\xi,\eta) = \frac{1}{4}\left(a\left(\frac{\xi-\eta}{2}\right) + \lambda\right)G(\xi,\eta), & 0 \le \eta \le \xi \le 2, \\ G(\xi,\xi) = 0, & 0 \le \xi \le 2, \\ \frac{\partial}{\partial\xi}(G(\xi,0)) = \frac{1}{4}\left(a\left(\frac{\xi}{2}\right) + \lambda\right), & 0 \le \xi \le 2, \end{cases} \tag{2.10}$$

which is equivalent to the following integral equation:

$$G(\xi,\eta) = \frac{1}{4}\int_\eta^\xi \left(a\left(\frac{\tau}{2}\right)+\lambda\right)d\tau + \frac{1}{4}\int_\eta^\xi\int_0^\eta \left(a\left(\frac{\tau-s}{2}\right)+\lambda\right)G(\tau,s)\,ds\,d\tau. \tag{2.11}$$

By the method of successive approximations we can show that this equation has a unique continuous solution. In fact, set

$$\begin{aligned} G_0(\xi,\eta) &= \frac{1}{4}\int_\eta^\xi \left(a\left(\frac{\tau}{2}\right)+\lambda\right)d\tau,\\ G_n(\xi,\eta) &= \frac{1}{4}\int_\eta^\xi\int_0^\eta \left(a\left(\frac{\tau-s}{2}\right)+\lambda\right)G_{n-1}(\tau,s)\,ds\,d\tau \end{aligned}$$

and denote $M = \sup_{0\le x\le 1}|a(x)+\lambda|$. Then one can readily show that

$$\begin{aligned} |G_0(\xi,\eta)| &\le \frac{1}{4}M(\xi-\eta)\le M,\\ |G_1(\xi,\eta)| &\le M^2\xi\eta,\\ |G_2(\xi,\eta)| &\le \frac{M^3}{(2!)^2}\xi^2\eta^2, \end{aligned}$$

and, by induction,

$$|G_n(\xi,\eta)| \le \frac{M^{n+1}}{(n!)^2}\xi^n\eta^n.$$

These estimates show that the series

$$G(\xi,\eta) = \sum_{n=0}^\infty G_n(\xi,\eta)$$

converges absolutely and uniformly in $0\le\eta\le\xi\le 2$, and then its sum is a continuous solution of (2.11). Moreover, it follows from (2.11) that G is twice continuously differentiable because $a\in C^1[0,1]$. Indeed, differentiating (2.11) with respect to ξ gives

$$\frac{\partial G(\xi,\eta)}{\partial\xi} = \frac{1}{4}\left(a\left(\frac{\xi}{2}\right)+\lambda\right) + \frac{1}{4}\int_0^\eta \left(a\left(\frac{\xi-s}{2}\right)+\lambda\right)G(\xi,s)\,ds,$$

which implies that $\frac{\partial G(\xi,\eta)}{\partial\xi}$ is continuous since $G(\xi,\eta)$ is continuous. By analogy, we can show that other derivatives of G are continuous. $\square$

Remark 2.3. The proof of Lemma 2.2 provides a numeric computation scheme of successive approximation to compute the kernel function k in our feedback laws (2.2) and (2.3). This makes the feedback laws (2.2) and (2.3) implementable in real problems.

LEMMA 2.4. *Let $k(x,y)$ be the solution of problem* (2.1) *and define the linear bounded operator $K : H^i(0,1)\to H^i(0,1)$ $(i=0,1,2)$ by*

$$w(x) = (Ku)(x) = u(x) + \int_0^x k(x,y)u(y)\,dy \quad \textit{for } u\in H^i(0,1). \tag{2.12}$$

Then

1. *K has a linear bounded inverse $K^{-1} : H^i(0,1) \to H^i(0,1)$ ($i = 0, 1, 2$), and*
2. *K converts the system (2.2) and (2.4) and the system (2.3) and (2.4) into (2.8) and (2.9), respectively.*

Proof. To prove that (2.12) has a bounded inverse, we set

$$v(x) = \int_0^x k(x,y)u(y)dy$$

and then

$$w(x) = u(x) + v(x).$$

Hence we have

$$\begin{aligned} v(x) &= \int_0^x k(x,y)[w(y) - v(y)]dy \\ &= \int_0^x k(x,y)w(y)dy - \int_0^x k(x,y)v(y)dy. \end{aligned} \tag{2.13}$$

To show that this equation has a unique continuous solution, we set

$$\begin{aligned} v_0(x) &= \int_0^x k(x,y)w(y)dy, \\ v_n(x) &= -\int_0^x k(x,y)v_{n-1}(y)dy \end{aligned}$$

and denote $M = \sup_{0 \le y \le x \le 1} |k(x,y)|$. Then

$$\begin{aligned} |v_0(x)| &\le M\|w\|, \\ |v_1(x)| &\le M^2\|w\|x, \\ |v_2(x)| &\le \frac{M^3\|w\|}{2!}x^2, \end{aligned}$$

and, by induction,

$$|v_n(x)| \le \frac{M^{n+1}\|w\|}{n!}x^n.$$

These estimates show that the series

$$v(x) = \sum_{n=0}^{\infty} v_n(x)$$

converges absolutely and uniformly in $0 \le x \le 1$ and that its sum is a continuous solution of (2.13). Moreover, there exists a constant $C > 0$ such that

$$\|v\| \le C\|w\|. \tag{2.14}$$

This implies that there exists a bounded linear operator $\Phi : L^2(0,1) \to L^2(0,1)$ such that

$$v(x) = (\Phi w)(x)$$

and then

$$u(x) = w(x) - v(x) = ((I - \Phi)w)(x) = (K^{-1}w)(x). \tag{2.15}$$

It is clear that $K^{-1} : L^2(0,1) \to L^2(0,1)$ is bounded. To show that $K^{-1} : H^1(0,1) \to H^1(0,1)$ is bounded, we take the derivative in (2.13) and obtain

$$v_x(x) = k(x,x)w(x) + \int_0^x k_x(x,y)w(y)dy - k(x,x)v(x) - \int_0^x k_x(x,y)v(y)dy,$$

which, combined with (2.14), implies that there exists constant $C > 0$ such that

$$\|v_x\| \leq C\|w\|$$

and then by (2.15)

$$\|u\|_{H^1} \leq \|w\|_{H^1} + \|v\|_{H^1} \leq C\|w\|_{H^1}.$$

By analogy, we can show that $K^{-1} : H^2(0,1) \to H^2(0,1)$ is bounded.

To prove that the transformation (2.12) converts the system (2.2) and (2.4) and the system (2.3) and (2.4) into (2.8) and (2.9), respectively, we compute as follows:

$$\begin{aligned} w_t(x,t) &= u_t(x,t) + \int_0^x k(x,y)u_t(y,t)dy \\ &= u_t(x,t) + \int_0^x k(x,y)[u_{yy}(y,t) + a(y)u(y,t)]dy \\ &= u_t(x,t) + k(x,x)u_x(x,t) - k(x,0)u_x(0,t) \\ &\quad - k_y(x,x)u(x,t) + k_y(x,0)u(0,t) \\ &\quad + \int_0^x [k_{yy}(x,y)u(y,t) + k(x,y)a(y)u(y,t)]dy, \end{aligned} \tag{2.16}$$

$$w_x(x,t) = u_x(x,t) + k(x,x)u(x,t) + \int_0^x k_x(x,y)u(y,t)dy, \tag{2.17}$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) + \frac{d}{dx}(k(x,x))u(x,t) + k(x,x)u_x(x,t) \\ &\quad + k_x(x,x)u(x,t) + \int_0^x k_{xx}(x,y)u(y,t)dy. \end{aligned} \tag{2.18}$$

It then follows from (2.1) and (2.4) that

(2.19)

$$\begin{aligned} w_t - w_{xx} + \lambda w = {} & u_t(x,t) + k(x,x)u_x(x,t) - k(x,0)u_x(0,t) \\ & -k_y(x,x)u(x,t) + k_y(x,0)u(0,t) \\ & + \int_0^x [k_{yy}(x,y)u(y,t) + k(x,y)a(y)u(y,t)]dy \\ & -u_{xx}(x,t) - \frac{d}{dx}(k(x,x))u(x,t) - k(x,x,t)u_x(x,t) \\ & -k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy \\ & +\lambda u(x,t) + \lambda \int_0^x k(x,y)u(y,t)dy \end{aligned}$$

$$
\begin{aligned}
&= \left(a(x) - k_x(x,x) - k_y(x,x) - \frac{d}{dx}(k(x,x)) + \lambda \right) u(x,t) \\
&\quad + k_y(x,0)u(0,t) - k(x,0)u_x(0,t) \\
&\quad + \int_0^x \left[k_{yy}(x,y) - k_{xx}(x,y,t) + (a(y)+\lambda)k(x,y,t) \right] u(y,t)dy \\
&= 0.
\end{aligned}
$$

By the boundary condition of (2.4), we deduce that $w(0,t) = 0$. Using feedback law (2.2) or (2.3), we obtain

$$w(1,t) = u(1,t) + \int_0^1 k(1,y)u(y,t)dy = 0$$

or

$$w_x(1,t) = u_x(1,t) + k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)dy = 0. \qquad \square$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first note that problem (2.4) with either (2.2) or (2.3) is well posed since, by Lemma 2.4, they can be transformed to the problem (2.8) or (2.9) via the isomorphism defined by (2.12), and the problem (2.8) or (2.9) is well posed (see, e.g., [9, Chap. IV]). Moreover, there exists a positive constant $C > 0$ such that

$$
\begin{aligned}
\|u(t)\|_{H^1} &\le C\|w(t)\|_{H^1}, \\
\|w^0\|_{H^1} &\le C\|u^0\|_{H^1}.
\end{aligned}
$$

Therefore, it is sufficient to prove (2.7) for the solution w of (2.8) or (2.9). We do so only for problem (2.8) since the situation for problem (2.9) is similar.

We define the energy

$$E(t) = \frac{1}{2}\int_0^1 w(x,t)^2 dx.$$

Multiplying the first equation of (2.8) by w and integrating from 0 to 1 by parts we get

$$
\begin{aligned}
\dot{E}(t) &= w_x w\Big|_0^1 - \int_0^1 w_x(x,t)^2 dx - \lambda \int_0^1 w(x,t)^2 dx \\
&= -\int_0^1 w_x(x,t)^2 dx - \lambda \int_0^1 w(x,t)^2 dx \\
&\le -2\lambda E(t),
\end{aligned}
$$

which implies

$$E(t) \le E(0)e^{-2\lambda t} \quad \text{for } t \ge 0.$$

Set

$$V(t) = \int_0^1 w_x(x,t)^2 dx.$$

Multiplying the first equation of (2.8) by w_{xx} and integrating from 0 to 1 by parts we obtain

$$\begin{aligned}\dot{V}(t) &= -2\int_0^1 w_{xx}^2 dx + 2\lambda \int_0^1 w w_{xx} dx \\ &= -2\int_0^1 w_{xx}^2 dx - 2\lambda \int_0^1 w_x^2 dx \\ &\le -2\lambda V(t),\end{aligned}$$

which implies that

$$V(t) \le V(0)e^{-2\lambda t}.$$

This shows that (2.7) holds. □

3. Neumann boundary conditions. To stabilize the Neumann boundary value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + a(x)u(x,t) & \text{in } (0,1)\times(0,\infty), \\ u_x(0,t) = u_x(1,t) = 0 & \text{in } (0,\infty), \end{cases}$$

we consider the problem

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (a(y)+\lambda)k(x,y), & 0 \le y \le x \le 1, \\ k_y(x,0) = 0, & 0 \le x \le 1, \\ k_x(x,x) + k_y(x,x) + \frac{d}{dx}(k(x,x)) = a(x) + \lambda, & 0 \le x \le 1, \\ k(0,0) = 0, \end{cases} \tag{3.1}$$

where λ is any constant. Using the solution k, we then obtain Dirichlet boundary feedback law

$$u(1,t) = -\int_0^1 k(1,y)u(y,t)dy \qquad \text{in } (0,\infty) \tag{3.2}$$

and Neumann boundary feedback law

$$u_x(1,t) = -k(1,1)u(1,t) - \int_0^1 k_x(1,y)u(y,t)dy \qquad \text{in } (0,\infty). \tag{3.3}$$

With one of the boundary feedback laws, the system

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + a(x)u(x,t) & \text{in } (0,1)\times(0,\infty), \\ u_x(0,t) = 0 & \text{in } (0,\infty), \\ u(x,0) = u^0(x) & \text{in } (0,1) \end{cases} \tag{3.4}$$

is exponentially stable. To state this result, we introduce the compatible conditions for the initial data

$$u_x^0(0) = 0, \quad u^0(1) = -\int_0^1 k(1,y)u^0(y)dy, \tag{3.5}$$

$$u_x^0(0) = 0, \quad u_x^0(1) = -k(1,1)u^0(1) - \int_0^1 k_x(1,y)u^0(y)dy. \tag{3.6}$$

THEOREM 3.1. *Assume that $\lambda > 0$ is any positive constant and $a \in C^1[0,1]$ is any function. For arbitrary initial data $u^0(x) \in H^1(0,1)$ with the compatible condition* (3.5) *or* (3.6), *equation* (3.4) *with either* (3.2) *or* (3.3) *has a unique solution that satisfies*

$$\|u(t)\|_{H^1} \leq M\|u^0\|_{H^1}e^{-\lambda t},$$

where M is a positive constant independent of u^0.

Proof. The proof is the same as that of Theorem 2.1. The only thing we need to do is to show that problem (3.1) has a unique solution. This is given in Lemma 3.2 below. □

LEMMA 3.2. *Suppose that $a \in C^1[0,1]$. Then problem* (3.1) *has a unique solution which is twice continuously differentiable in $0 \leq y \leq x \leq 1$.*

Proof. Using the variable changes

$$\xi = x + y, \quad \eta = x - y$$

and denoting

$$G(\xi,\eta) = k(x,y) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right),$$

problem (3.1) is transformed into

$$\begin{cases} G_{\xi\eta}(\xi,\eta) = \frac{1}{4}\left(a\left(\frac{\xi-\eta}{2}\right)+\lambda\right)G(\xi,\eta), & 0 \leq \eta \leq \xi \leq 2, \\ G_\xi(\xi,\xi) = G_\eta(\xi,\xi), & 0 \leq \xi \leq 2, \\ \frac{\partial}{\partial\xi}(G(\xi,0)) = \frac{1}{4}\left(a\left(\frac{\xi}{2}\right)+\lambda\right), & 0 \leq \xi \leq 2, \\ G(0,0) = 0. \end{cases} \tag{3.7}$$

Integrating the first equation of (3.7) with respect to η from 0 to ξ gives

$$\begin{aligned} G_\xi(\xi,\xi) &= G_\xi(\xi,0) + \frac{1}{4}\int_0^\xi \left(a\left(\frac{\xi-s}{2}\right)+\lambda\right)G(\xi,s)ds \\ &= \frac{1}{4}\left(a\left(\frac{\xi}{2}\right)+\lambda\right) + \frac{1}{4}\int_0^\xi \left(a\left(\frac{\xi-s}{2}\right)+\lambda\right)G(\xi,s)ds. \end{aligned}$$

It then follows from the second equation of (3.7) that

$$\begin{aligned} \frac{d}{d\xi}[G(\xi,\xi)] &= G_\xi(\xi,\xi) + G_\eta(\xi,\xi) \\ &= 2G_\xi(\xi,\xi) \\ &= \frac{1}{2}\left(a\left(\frac{\xi}{2}\right)+\lambda\right) + \frac{1}{2}\int_0^\xi \left(a\left(\frac{\xi-s}{2}\right)+\lambda\right)G(\xi,s)ds. \end{aligned}$$

Integrating from 0 to ξ and using the fourth equation of (3.7) gives

$$G(\xi,\xi) = \frac{1}{2}\int_0^\xi \left(a\left(\frac{\tau}{2}\right)+\lambda\right)d\tau + \frac{1}{2}\int_0^\xi\int_0^\tau \left(a\left(\frac{\tau-s}{2}\right)+\lambda\right)G(\tau,s)dsd\tau. \tag{3.8}$$

Integrating twice the first equation of (3.7) first with respect to η from 0 to η and second with respect to ξ from η to ξ and using (3.8), we obtain the following integral equation:

$$
\begin{aligned}
(3.9)\quad G(\xi,\eta) = {} & \frac{1}{2}\int_0^\eta \left(a\left(\frac{\tau}{2}\right)+\lambda\right)d\tau + \frac{1}{2}\int_0^\eta\int_0^\tau \left(a\left(\frac{\tau-s}{2}\right)+\lambda\right)G(\tau,s)dsd\tau \\
& + \frac{1}{4}\int_\eta^\xi \left(a\left(\frac{\tau}{2}\right)+\lambda\right)d\tau + \frac{1}{4}\int_\eta^\xi\int_0^\eta \left(a\left(\frac{\tau-s}{2}\right)+\lambda\right)G(\tau,s)dsd\tau.
\end{aligned}
$$

As in the proof of Lemma 2.2, by the method of successive approximations we can show that this equation has a unique continuous solution. Moreover, it follows from (3.9) that G is twice continuously differentiable because $a \in C^1[0,1]$. □

Similar to Lemma 2.4, we have the following lemma.

LEMMA 3.3. *Let $k(x,y)$ be the solution of problem* (3.1) *and define the linear bounded operator $K : H^i(0,1) \to H^i(0,1)$ $(i = 0,1,2)$ by*

$$w(x) = (Ku)(x) = u(x) + \int_0^x k(x,y)u(y)dy \qquad \text{for } u \in H^i(0,1).$$

Then

1. *K has a linear bounded inverse $K^{-1} : H^i(0,1) \to H^i(0,1)$ $(i = 0,1,2)$, and*
2. *K converts the system* (3.2) *and* (3.4) *and the system* (3.3) *and* (3.4) *into*

$$\begin{cases} w_t = w_{xx} - \lambda w & \text{in } (0,1)\times(0,\infty), \\ w_x(0,t) = w(1,t) = 0 & \text{in } (0,\infty), \\ w(x,0) = w^0(x) & \text{in } (0,1) \end{cases}$$

or

$$\begin{cases} w_t = w_{xx} - \lambda w & \text{in } (0,1)\times(0,\infty), \\ w_x(0,t) = w_x(1,t) = 0 & \text{in } (0,\infty), \\ w(x,0) = w^0(x) & \text{in } (0,1), \end{cases}$$

respectively, where $w^0(x) = u^0(x) + \int_0^x k(x,y)u^0(y)dy$.

4. Remarks. An interesting problem is to stabilize the problem

$$u_t(x,t) = u_{xx}(x,t) + a(x,t)u(x,t),$$

where the function a depends on t. To address the problem, it can been seen from the computations in (2.16)–(2.19) that we have to consider the problem

$$\begin{cases} k_{xx}(x,y,t) - k_{yy}(x,y,t) - k_t(x,y,t) = (a(y,t)+\lambda)k(x,y,t), & 0 \le y \le x \le 1, \\ k_y(x,0,t) = 0, & 0 \le x \le 1, \\ k_x(x,x,t) + k_y(x,x,t) + \frac{\partial}{\partial x}(k(x,x,t)) = a(x,t)+\lambda, & 0 \le x \le 1, \end{cases}$$

where λ is any constant. But we do not know if this problem has a solution. Once we can show that this problem has a solution, all the results in sections 2 and 3 hold immediately.

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REFERENCES

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] H. Amamm, *Feedback stabilization of linear and semilinear parabolic systems*, in Semigroup Theory and Applications, P. Clement, S. Invernizzi, E. Mitidieri, and I. I. Vrabie, eds., Lecture Notes in Pure and Appl. Math. 116, Marcel Dekker, New York, 1989, pp. 21–57.
[3] A. Balogh and M. Krstic, *Infinite-step backstepping for a heat equation-like PDE with arbitrarily many unstable eigenvalues*, in Proceedings of the 2001 American Control Conference, Vol. 3, IEEE Press, Piscataway, NJ, 2001, pp. 2480–2485.
[4] A. Balogh and M. Krstic, *Infinite dimensional backstepping—style feedback transformations for a heat equation with an arbitrary level of instability*, European J. Control, 8 (2002), pp. 165–176.
[5] D. M. Boskovic, M. Krstic, and W. J. Liu, *Boundary control of an unstable heat equation via measurement of domain-averaged temperature*, IEEE Trans. Automat. Control, 46 (2001), pp. 2022–2028.
[6] J. A. Burns and D. Rubio, *A distributed parameter control approach to sensor location for optimal feedback control of thermal processes*, in Proceedings of the 36th IEEE Conference on Decision and Control, Vol. 3, IEEE Press, Piscataway, NJ, 1998, pp. 2243–2247.
[7] J. A. Burns, D. Rubio, and B. B. King, *Regularity of feedback operators for boundary control of thermal processes*, in Proceedings of the First International Conference on Nonlinear Problems in Aviation and Aerospace, Daytona Beach, FL, 1996.
[8] W. A. Day, *A decreasing property of solutions of parabolic equations with applications to thermoelasticity*, Quart. Appl. Math., 40 (1982/83), pp. 468–475.
[9] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, Transl. Math. Monogr. 23, AMS, Providence, RI, 1968.
[10] I. Lasiecka and R. Triggiani, *Stabilization of Neumann boundary feedback of parabolic equations: The case of trace in the feedback loop*, Appl. Math. Optim., 10 (1983), pp. 307–350.
[11] I. Lasiecka and R. Triggiani, *Stabilization and structural assignment of Dirichlet boundary feedback parabolic equations*, SIAM J. Control Optim., 21 (1983), pp. 766–803.
[12] I. Lasiecka and R. Triggiani, *The regulator problem for parabolic equations with Dirichlet boundary control.* I. *Riccati's feedback synthesis and regularity of optimal solution*, Appl. Math. Optim., 16 (1987), pp. 147–168.
[13] I. Lasiecka and R. Triggiani, *The regulator problem for parabolic equations with Dirichlet boundary control.* II. *Galerkin approximation*, Appl. Math. Optim., 16 (1987), pp. 187–216.
[14] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
[15] R. Triggiani, *Boundary feedback stabilization of parabolic equations*, Appl. Math. Optim., 6 (1980), pp. 201–220.