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Boundary Harnack principle and Martin boundary for a uniform domain

Dedicated to Professor Maretsugu Yamasaki
on the occasion of his 60th birthday

By Hiroaki Aikawa

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Abstract. We establish a uniform boundary Harnack principle for a uniform domain. As applications we study the Hölder continuity of the ratios of positive harmonic functions, the Martin boundary and the Fatou theorem for a uniform domain.

1. Introduction.

There is an extensive literature on the boundary Harnack principle (abbreviated to BHP). BHP is a principle of the following type: Let D be a domain in $\boldsymbol{R}^n$ with a certain geometric property. Let V be an open set and K a compact subset of V intersecting ∂D. Then there is a positive constant $A = A(D, V, K)$ such that

$$\frac{u(x)/v(x)}{u(y)/v(y)} \le A \quad \text{for } x, y \in K \cap D, \tag{1.1}$$

whenever u and v are positive harmonic functions on D with vanishing boundary values on $V \cap \partial D$.

By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \dots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \ge 1$ such that $A^{-1} f_1 \le f_2 \le A f_1$. The constant A will be called the constant of comparison.

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Then (1.1) can read

$$\frac{u(x)}{v(x)} \approx \frac{u(y)}{v(y)} \quad \text{for } x, y \in K \cap D$$

with constant of comparison depending only on D, V and K. Let $\delta_D(x) = \operatorname{dist}(x, \partial D)$. If D is sufficiently smooth, then

$$\frac{u(x)}{u(y)} \approx \frac{\delta_D(x)}{\delta_D(y)} \quad \text{for } x, y \in K \cap D,$$

for a positive harmonic function u on D with vanishing boundary values on $V \cap \partial D$ ([**23**]). Hence (1.1) follows in this case.

BHP for nonsmooth domains is not so easy. For a Lipschitz domain BHP was obtained independently by Ancona [**4**], Dahlberg [**12**] and Wu [**24**]. Caffarelli, Fabes, Mortola and Salsa [**10**] proved BHP for positive solutions of elliptic equations in divergence form with nonsmooth coefficients on a bounded Lipschitz domain. Jerison and Kenig [**17**] introduced NTA domains and extended BHP to NTA domains. Anderson and Schoen [**5**] proved BHP for a complete manifold of negative curvature. Bañuelos, Bass and Burdzy ([**9**], [**7**] and [**8**]) employed probabilistic techniques and proved BHP for Hölder domains. The significant aspect of the work of Bañuelos, Bass and Burdzy is that they proved BHP without any exterior condition. However, BHP of Bañuelos, Bass and Burdzy is weaker than the previous BHP. It is not uniform. As was observed by Jerison and Kenig [**17**], the uniform BHP is important for further applications such as the Martin boundary, the Hölder continuity of the kernel functions, H^p and BMO spaces.

The main aim of the present paper is to establish a uniform BHP for a *uniform domain*. We say that D is a uniform domain if there exist constants A and A' such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma \subset D$ for which

$$\ell(\gamma) \le A|x_1 - x_2|, \tag{1.2}$$

$$\min\{\ell(\gamma(x_1, y)), \ell(\gamma(x_2, y))\} \le A'\delta_D(y) \quad \text{for all } y \in \gamma. \tag{1.3}$$

Here, $\ell(\gamma)$ and $\gamma(x_j, y)$ denote the length of γ and the subarc of γ connecting x_j and y, respectively (See [**14**] and [**22**]). Roughly speaking, a uniform domain is a domain satisfying only the interior conditions for an NTA domain (see [**17**]). We have

$$\text{Lipschitz} \subsetneqq \text{NTA} \subsetneqq \text{uniform} \subsetneqq \text{John}.$$

By $B(x, r)$ we denote the open ball with center at x and radius r. A simple example of a uniform domain is the unit ball minus a closed line segment, e.g.

$B(0,1)\backslash L$ with $L = \{(x_1, 0, \dots, 0) : |x_1| \le 1/2\}$ for $n \ge 3$. For another example see Proposition 1. In fact, a uniform domain enjoys only an interior condition and so it may admit irregular boundary points. Moreover, a surface ball $\partial D \cap B(\xi, r)$ may be a polar set. Hence, we always consider a generalized Dirichlet problem, i.e. boundary values have meaning outside a polar set. For simplicity, we shall say that a property holds q.e. (quasi everywhere) if it holds outside a polar set. We show the following uniform BHP.

THEOREM 1. *Let D be a uniform domain. Then there exists a constant $A_0 > 1$ depending only on D with the following property: Let $\xi \in \partial D$ and let $R > 0$ be sufficiently small. Suppose u and v are positive harmonic functions on $D \cap B(\xi, A_0 R)$, bounded on $D \cap B(\xi, A_0 R)$ and vanishing q.e. on $\partial D \cap B(\xi, A_0 R)$. Then*

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')} \quad \text{uniformly for } x, x' \in D \cap B(\xi, R)$$

where the constant of comparison depends only on D.

REMARK 1. We emphasize that the domains of positive harmonic functions u and v are localized to $D \cap B(\xi, A_0 R)$. We use localized Green functions and represent positive harmonic functions as localized Green potentials. This localization will be useful for the Hölder continuity of the ratio u/v. It enables us to avoid the deep geometric localization theorem due to Jones [**18**].

COROLLARY 1. *Let D be a uniform domain. Then the global BHP holds. That is, for an open set V and a compact subset K of V there is a positive constant $A = A(D, V, K)$ with the following property: if u and v are positive harmonic functions on D, bounded on $D \cap V$ and vanishing q.e. on $\partial D \cap V$, then* (1.1) *holds.*

Theorem 1 has several applications. With the aid of the classical technique due to Moser [**19**, Section 5], we can show the Hölder continuity of u/v at the boundary. In general, by $\operatorname{osc}_E f$ we denote $\sup_E f - \inf_E f$, the oscillation of f over E.

THEOREM 2. *Let D be a uniform domain. Then there exist $A > 0$ and $\varepsilon > 0$ depending only on D with the following property: Let $\xi \in \partial D$ and let $0 < r < R$ be sufficiently small. Suppose u and v are positive harmonic functions on $D \cap B(\xi, A_0 R)$, bounded on $D \cap B(\xi, A_0 R)$ and vanishing q.e. on $\partial D \cap B(\xi, A_0 R)$. Then*

$$\operatorname*{osc}_{D \cap B(\xi, r)} \frac{u}{v} \le A \left(\frac{r}{R}\right)^{\varepsilon} \operatorname*{osc}_{D \cap B(\xi, R)} \frac{u}{v}. \tag{1.4}$$

This result is rather surprising since the functions u and v themselves are not continuous on ∂D if D has an irregular boundary point. Combining Theorem 2 and the known interior Hölder continuity, we obtain the following Hölder continuity. See Jerison and Kenig [**17**, Theorem 7.9]

Corollary 2. *Let D be a uniform domain. Then there exist positive constants A and ε depending only on D with the following property: Let V be an open set and K a compact subset of V intersecting ∂D. If u and v are positive harmonic functions on D, bounded on $D \cap V$ and vanishing q.e. on $\partial D \cap V$, then*

$$\operatorname*{osc}_{D \cap B(x,r)} \frac{u}{v} \le A\left(\frac{r}{R}\right)^{\varepsilon} \operatorname*{osc}_{D \cap B(x,R)} \frac{u}{v} \tag{1.5}$$

for $x \in \bar{D} \cap K$ and $0 < r \le R \le \operatorname{dist}(K, V^c)$. In particular,

$$\left|\frac{u(x)/v(x)}{u(y)/v(y)} - 1\right| \le A|x-y|^{\varepsilon} \quad \text{for } x, y \in D \cap K.$$

Moreover, the ratio u/v extends to $\bar{D} \cap K$ as a Hölder continuous function.

We shall show that the Martin boundary of a bounded uniform domain is homeomorphic to the Euclidean boundary. In general by G_U we denote the Green function for the Laplacian for an open set U. For simplicity we write G for the Green function for D.

Theorem 3. *Let D be a uniform domain. For each $\xi \in \partial D$ there exists a unique minimal Martin boundary point, i.e. the limit*

$$K(x,\xi) = \lim_{D \ni y \to \xi} \frac{G(x,y)}{G(x_0,y)}$$

exists and is a minimal harmonic function on D, where x_0 is a fixed point in D. Moreover, the Martin kernel $K(x,\xi)$ is a Hölder continuous function of $\xi \in \partial D$.

Corollary 3. *The Martin boundary of a bounded uniform domain is homeomorphic to its Euclidean boundary. Each boundary point is minimal.*

The coincidence of the Martin boundary and the Euclidean boundary was given by Hunt and Wheeden [**16**] for a Lipschitz domain and by Jerison and Kenig [**17**] for an NTA domain. Our proof of Theorem 3 is different from those in [**16**] and [**17**]. They regarded the Martin kernel as the limit of the ratio of the harmonic measures. In the present setting, the harmonic measure of a surface ball may vanish. Hence we regard a kernel function as the ratio of the Green functions and we estimate them directly by using BHP. In fact, if $D = B(0,1) \setminus L$ for $n \ge 3$ as before Theorem 1, then L is polar, so that D and $B(0,1)$ have the same Green function and the same harmonic measure. It is easy to see that the Martin boundary of D is the union of the unit sphere and L. From the ratios of the harmonic measure, we cannot retrieve the Martin kernel with pole on L.

So far, we have observed that a uniform domain enjoys the same properties as an NTA domain, although it satisfies only the interior conditions. However, the harmonic measure of a uniform domain does not satisfy the doubling property. The lack of the doubling property of the harmonic measure causes strange phenomena. As an illustration let us consider the Fatou theorem. This is just on the border line; the global Fatou theorem holds and yet the local Fatou theorem does not hold. For $\xi \in \partial D$ we let $\Gamma_\alpha(\xi) = \{x \in D : |x - \xi| < (1+\alpha)\delta_D(x)\}$ where $\alpha > 0$. This is a nontangential approach region to ξ. We say that a function u defined on D has nontangential limit c at ξ if for any α, $u(x)$ restricted $\Gamma_\alpha(\xi)$ converges to c as $x \to \xi$. Let ω_D be the harmonic measure of D. We have the following global Fatou theorem.

Theorem 4. *Let D be a uniform domain. Then every positive harmonic function u on D has nontangential limits a.e. ω_D on ∂D, i.e. there is a set $E \subset \partial D$ with $\omega_D(E) = 0$ such that u has nontangential limits for $\xi \in \partial D \setminus E$.*

Let us consider a local Fatou theorem. A truncated nontangential approach region at ξ is denoted by $\Gamma_\alpha^h(\xi) = \Gamma_\alpha(\xi) \cap B(\xi, h)$. We say that a function u defined on D is nontangentially bounded from below at $\xi \in \partial D$ if there exist positive constants α, h and A such that $u(x) \geq -A$ for all $x \in \Gamma_\alpha^h(\xi)$. Let $F \subset \partial D$. We say that a function u is nontangentially bounded from below on F if u is nontangentially bounded from below at every point of F. Jerison and Kenig [**17**, Theorem 6.4] proved the following.

Theorem A. *Let D be an NTA domain. Assume that u is harmonic in D and nontangentially bounded from below on $F \subset \partial D$. Then u has nontangential limits a.e. ω_D on F.*

For a uniform domain such a local Fatou theorem does not necessarily hold.

Proposition 1. *There exist a bounded uniform domain D, a countable set $E \subset \partial D$, and a harmonic function u on D nontangentially bounded on $\partial D \setminus E$ which fails to have nontangential limits on $\partial D \setminus E$.*

Plan. Our proofs are different from the previous ones, since we assume no exterior conditions. Traditionally, BHP is proved by the Carleson estimate for positive harmonic functions vanishing on a portion of the boundary ([**11**]) and the comparison of harmonic measures of a 'box'; the Fatou theorem is proved by the maximal function with respect to the harmonic measure. In the present settings, however, the Carleson estimate for a general harmonic function is not available at first (see Remark 2 below); the harmonic measure of the domain does not have the doubling property, so that the maximal function is irrelevant. Our approach must be different. The main ingredients are as follows:

(i) Dominate the harmonic measure of the intersection of the domain and a ball by the local Green function with pole near the ball (Lemma 2).

(ii) Compare the ratios of the local Green functions with the aid of the Carleson estimate for the Green function (Lemma 3).

(iii) Represent a harmonic function as a local Green potential and use the above comparison (Proof of Theorem 1).

(iv) Use the classical Moser technique to obtain the Hölder continuity. Avoid the deep geometric localization due to Jones. (Proof of Theorems 2 and 3).

(v) Invoke the general minimal fine limit theorem and compare the minimal fine filter and nontangential filter in order to prove the Fatou theorem (Proof of Theorem 4).

The most difficult part is (i), for which we borrow the probabilistic idea of Bass and Burdzy [**9**]. The next section will prepare some technical materials for this part. In fact, Bass and Burdzy employed a deep probabilistic argument for their BHP. Their deep argument can be avoided by our (ii) and (iii).

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2. Preliminaries.

In the previous paper [**1**] (see [**2**] for other applications), we introduced the notion of *capacitary width*. Let U be an open set with Green function G_U. Define the Green capacity $\mathrm{Cap}_U(E)$ for a Borel set $E \subset U$ by

$$\mathrm{Cap}_U(E) = \sup\{\mu(E) : G_U\mu \le 1 \text{ on } U, \mu \text{ is a Borel measure supported on } E\}.$$

In the usual way $\mathrm{Cap}_U(E)$ extends to a general set $E \subset U$.

Definition. Let $0 < \eta < 1$. For $U \subset \boldsymbol{R}^n$ we define the *capacitary width* $w_\eta(U)$ by

$$w_\eta(U) = \inf\left\{ r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r)\setminus U)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} \ge \eta \text{ for all } x \in U \right\}.$$

Fundamental properties of capacitary widths are given in [**1**]. For the completeness we repeat them. We note that the constant η, $0 < \eta < 1$, is not so important. In fact, if $0 < \eta_1 < \eta_2 < 1$, then

$$w_{\eta_1}(U) \le w_{\eta_2}(U) \le A w_{\eta_1}(U) \quad \text{for any } U \subset \boldsymbol{R}^n,$$

where A depends only on the dimension n, η_1 and η_2 ([**1**, Proposition 2]). Hereafter, we fix η, $0 < \eta < 1$.

In view of the definition of a uniform domain, it is easy to see that

$$w_\eta(\{x \in D : \delta_D(x) < r\}) \le Ar \quad \text{for small } r > 0, \tag{2.1}$$

whenever D is a uniform domain.

We denote the harmonic measure of E for an open set U by $\omega_U(E)$ or by $\omega(\cdot, E, U)$. We write $C(x,r)$ and $S(x,r)$ for the closed ball and the sphere of center at x and radius r, respectively. Harmonic measures and capacitary widths are related as in the following key lemma. This lemma is implicitly proved in [**1**, Proposition 2].

Lemma 1. *There is a positive constant A_1 depending only on the dimension with the following property: if $U \neq \emptyset$ is open, $x \in U$ and $R > 0$, then*

$$\omega(x, U \cap S(x,R), U \cap B(x,R)) \le \exp\left(2 - A_1 \frac{R}{w_\eta(U)}\right). \tag{2.2}$$

Proof. For an arbitrary $\varepsilon > 0$ we have r, $w_\eta(U) \le r < w_\eta(U) + \varepsilon$ such that

$$\frac{\mathrm{Cap}_{B(y,2r)}(B(y,r) \setminus U)}{\mathrm{Cap}_{B(y,2r)}(B(y,r))} \ge \eta \quad \text{for all } y \in U. \tag{2.3}$$

For a moment we fix $y \in U$ and let $E = B(y,r) \setminus U$ and G_B the Green function for $B(y,2r)$. Let μ_E be the capacitary measure of E, i.e.

$$\mu_E \text{ is supported on } \bar{E},$$

$$\|\mu_E\| = \mathrm{Cap}_{B(y,2r)}(E),$$

$$G_B \mu_E = 1 \text{ q.e. on } E.$$

We claim

$$G_B \mu_E \ge A_2 \eta \quad \text{on } C(y,r), \tag{2.4}$$

where A_2 depends only on the dimension. To this end let ν be the capacitary measure of $B(y,r)$. Then ν is supported on $S(y,r)$ and $\|\nu\| = \mathrm{Cap}_{B(y,2r)}(B(y,r))$. By the Harnack inequality

$$G_B(\cdot, x) \approx G_B(\cdot, y) \quad \text{on } S(y, \tfrac{3}{2}r)$$

uniformly for $x \in C(y,r)$. Hence

$$G_B\mu_E(z) = \int_{\bar{E}} G_B(z,x)\,d_{\mu_E}(x) \approx G_B(z,y)\|\mu_E\|,$$

$$G_B\nu(z) = \int_{S(y,r)} G_B(z,x)\,d\nu(x) \approx G_B(z,y)\|\nu\|$$

uniformly for $z \in S(y, 3/2r)$. Since $G_B\nu \approx 1$ on $S(y, 3/2r)$, it follows from (2.3) that

$$G_B\mu_E \approx \frac{G_B\mu_E}{G_B\nu} \approx \frac{\|\mu_E\|}{\|\nu\|} = \frac{\mathrm{Cap}_{B(y,2r)}(E)}{\mathrm{Cap}_{B(y,2r)}(B(y,r))} \geq \eta$$

on $S(y, 3/2r)$. By the maximum principle (2.4) follows.

Now let us move on to the proof of (2.2). For simplicity we write Ω for $\omega(\cdot, U \cap S(x,R), U \cap B(x,R))$. Without loss of generality we may assume that $R/w_\eta(U) > 2$ and let k be the positive integer such that $2kw_\eta(U) < R \leq 2(k+1)w_\eta(U)$. Take $r > w_\eta(U)$ so close to $w_\eta(U)$ that $2kr < R$ holds. We claim

$$\sup_{U \cap C(x, R-2jr)} \Omega \leq (1 - A_2\eta)^j \tag{2.5}$$

for $j = 0, 1, \dots, k$. Since $k \approx R/w_\eta(U)$, (2.5) implies (2.2). Thus we have only to show (2.5). Let us prove (2.5) by induction. Obviously, (2.5) holds for $j = 0$. We assume that (2.5) holds for $j - 1$ and we shall prove (2.5) for $j \geq 1$. In view of the maximum principle, it is sufficient to show that

$$\sup_{U \cap S(x, R-2jr)} \Omega \leq (1 - A_2\eta)^j. \tag{2.6}$$

Let $y \in U \cap S(x, R - 2jr)$. Then $C(y, 2r) \subset C(x, R - 2(j-1)r)$, so that (2.5) for $j - 1$ implies

$$\Omega \leq (1 - A_2\eta)^{j-1} \quad \text{on } U \cap C(y, 2r).$$

Since Ω vanishes q.e. on $\partial U \cap B(x,R) \supset \partial U \cap B(y,2r)$, it follows from the maximum principle that

$$\Omega \leq (1 - A_2\eta)^{j-1}\omega(\cdot, U \cap S(y,2r), U \cap B(y,2r)) \quad \text{on } U \cap B(y,2r). \tag{2.7}$$

Let us compare $\omega(\cdot, U \cap S(y,2r), U \cap B(y,2r))$ and $1 - G_B\mu_E$, where $G_B\mu_E$ is as in (2.4). Then

$$\omega(\cdot, U \cap S(y,2r), U \cap B(y,2r)) \leq 1 - G_B\mu_E \quad \text{on } U \cap B(y,2r)$$

by the maximum principle. In particular,

$$\omega(y, U \cap S(y,2r), U \cap B(y,2r)) \leq 1 - G_B\mu_E(y) \leq 1 - A_2\eta$$

by (2.4). Substituting this to (2.7), we obtain $\Omega(y) \le (1 - A_2\eta)^j$. Hence (2.6) and so (2.5) follows. The proof is complete. □

3. Proof of Theorem 1.

The proof of Theorem 1 is based on uniform estimates of the Green function. Throughout this section we assume that D is a uniform domain and we let $\xi \in \partial D$ and $R > 0$. We shall give uniform estimates independent of ξ and R. All constants, implicit and explicit, will be independent of ξ and R, unless otherwise specified.

To facilitate the argument we introduce the quasi-hyperbolic metric $k_D(x_1, x_2)$ defined by

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta_D(x)},$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. We observe that the shortest length of a Harnack chain connecting x_1 and x_2 is comparable to $k_D(x_1, x_2)$. Hence, in view of the Harnack inequality, there is a positive constant A_3 depending only on the dimension n such that

$$\exp(-A_3 k_D(x_1, x_2)) \le \frac{h(x_1)}{h(x_2)} \le \exp(A_3 k_D(x_1, x_2))$$

for every positive harmonic function h.

For a uniform domain the following observation is important: if $\delta_D(x_1) \ge aR$, $\delta_D(x_2) \ge bR$ and $|x_1 - x_2| \le cR$, then $k_D(x_1, x_2) \le A$, where A depends only on a, b, c and D. Hence $h(x_1) \approx h(x_2)$ for any positive harmonic function h on D, where the constant of comparison depends only on a, b, c and D. Moreover observe that there is A_4, $0 < A_4 < 1$ such that

$$A_4 R \le \sup_{x \in D \cap S(\xi, R)} \delta_D(x) \le R$$

for $\xi \in \partial D$ and $R > 0$ sufficiently small, say $0 < R < 8R^*$. Let us take $\xi_R \in D \cap S(\xi, 4R)$ with $4A_4 R \le \delta_D(\xi_R) \le 4R$ for $0 < R < 2R^*$. Then, it is not so difficult to see that

$$k_D(x, \xi_R) \le A \log \frac{10R}{\delta_D(x)} \quad \text{for } x \in D \cap B(\xi, 9R), \tag{3.1}$$

where A is independent of the choice of ξ_R. In the sequel, estimates will be independent of the choice of ξ_R.

In view of the definition of a uniform domain, we find $A_5 > 9$ depending only on D such that $D \cap B(\xi, 9R)$ is included in a connected component of

$D \cap B(\xi, A_5 R)$ and

$$k_{D \cap B(\xi, A_5 R)}(x, y) \le k_D(x, y) + A \quad \text{for } x, y \in D \cap B(\xi, 9R). \tag{3.2}$$

For simplicity we let $D_R = D \cap B(\xi, (A_5 + 7)R)$ and $D'_R = D \cap B(\xi, A_5 R)$. By G_R and G'_R we denote the Green functions for D_R and D'_R, respectively. Obviously, $G_R \ge G'_R$ on $D'_R \times D'_R$. Note that D'_R needs not be connected and so $G'_R(\cdot, \xi_R)$ may vanish on some component of D'_R. However, in view of (3.1) and (3.2), $D \cap B(\xi, 9R)$ is included in the component containing ξ_R, and hence $G'_R(\cdot, \xi_R) > 0$ on $D \cap B(\xi, 9R)$.

Let us begin with a comparison of the Green function and the harmonic measure.

Lemma 2. *Let $\xi \in \partial D$ and $0 < R < 2R^*$. Then*

$$\begin{aligned} \omega(\cdot, D \cap S(\xi, 2R), D \cap B(\xi, 2R)) &\le AR^{n-2} G'_R(\cdot, \xi_R) \\ &\le AR^{n-2} G_R(\cdot, \xi_R) \quad \text{on } D \cap B(\xi, R), \end{aligned}$$

where A depends only on D.

Proof. It is sufficient to show the first inequality. We follow the idea of [**9**]. Since

$$C(\xi_R, 2^{-1}\delta_D(\xi_R)) \subset D \cap C(\xi, 6R) \backslash B(\xi, 2R) \subset D'_R \backslash B(\xi, 2R),$$

it follows from the maximum principle that

$$G'_R(\cdot, \xi_R) \le \sup_{y \in S(\xi_R, 2^{-1}\delta_D(\xi_R))} G'_R(y, \xi_R) \quad \text{on } D \cap B(\xi, 2R).$$

It is easy to see that the right hand side is comparable to R^{2-n}. Hence we can find $A_6 > 0$ depending only on D such that $A_6 R^{n-2} G'_R(\cdot, \xi_R) < 1/e$ on $D \cap B(\xi, 2R)$. Then

$$D \cap B(\xi, 2R) = \bigcup_{j \ge 0} D_j \cap B(\xi, 2R), \tag{3.3}$$

where

$$D_j = \{x \in D : \exp(-2^{j+1}) \le A_6 R^{n-2} G'_R(x, \xi_R) < \exp(-2^j)\}.$$

Let $U_j = (\bigcup_{k \ge j} D_k) \cap B(\xi, 2R)$. We claim

$$w_\eta(U_j) \le AR \exp\left(-\frac{2^j}{\lambda}\right) \tag{3.4}$$

with some $\lambda > 0$ depending only on D. Suppose $x \in U_j$. Observe from (3.1) and (3.2) that if $z \in S(\xi_R, 1/2\delta_D(\xi_R))$, then

$$k_{D'_R\setminus\{\xi_R\}}(x,z) \le k_{D'_R}(x,\xi_R) + A \le A' \log\frac{10R}{\delta_D(x)}.$$

Since $G'_R(z,\xi_R) \approx R^{2-n}$ for $z \in S(\xi_R, 1/2\delta_D(\xi_R))$, it follows from the Harnack inequality that

$$\begin{aligned}\exp(-2^j) &> A_6 R^{n-2} G'_R(x,\xi_R)\\ &\ge A R^{n-2} G'_R(z,\xi_R)\exp(-A_3 k_{D'_R\setminus\{\xi_R\}}(x,z)) \ge \left(\frac{\delta_D(x)}{10R}\right)^{\lambda}\end{aligned}$$

with $\lambda > 0$ depending only on D. Hence

$$\delta_D(x) \le 10R\exp\left(-\frac{2^j}{\lambda}\right).$$

This, together with (2.1), yields (3.4).

Now we use an inductive argument. Let $R_0 = 2R$ and

$$R_j = \left(2 - \frac{6}{\pi^2}\sum_{k=1}^{j}\frac{1}{k^2}\right)R$$

for $j \ge 1$. Then $R_j \downarrow R$ and

$$\begin{aligned}\text{(3.5)}\qquad &\sum_{j=1}^{\infty}\exp\left(2^{j+1} - \frac{R_{j-1}-R_j}{AR\exp(-2^j/\lambda)}\right)\\ &= \sum_{j=1}^{\infty}\exp\left(2^{j+1} - \frac{6}{A\pi^2}j^{-2}\exp\left(\frac{2^j}{\lambda}\right)\right) < \infty.\end{aligned}$$

We emphasize that the value of the series in (3.5) is independent of R. Let $\omega_0 = \omega(\cdot, D\cap S(\xi,2R), D\cap B(\xi,2R))$ and

$$d_j = \begin{cases}\displaystyle\sup_{x\in D_j\cap B(\xi,R_j)}\frac{\omega_0(x)}{R^{n-2}G'_R(x,\xi_R)} & \text{if } D_j\cap B(\xi,R_j)\ne\emptyset,\\ 0 & \text{if } D_j\cap B(\xi,R_j)=\emptyset.\end{cases}$$

In view of (3.3) it is sufficient to show that

$$\text{(3.6)}\qquad \sup_{j\ge 0} d_j \le A < \infty,$$

where A is independent of R.

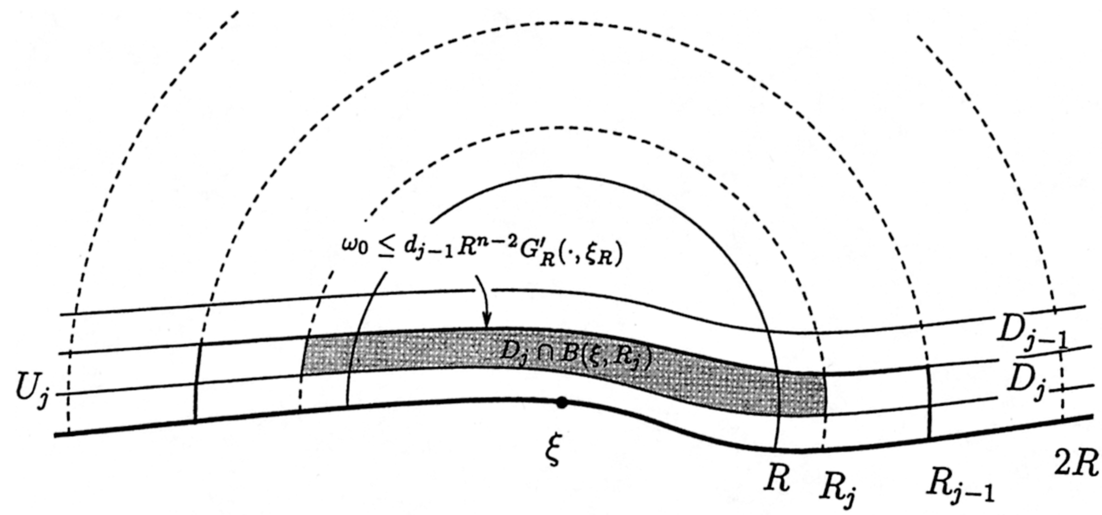

Figure 1. Maximum principle over $U_j \cap B(\xi, R_{j-1})$.

Let $j > 0$. Then the maximum principle yields that

$$\omega_0(x) \le \omega(x, U_j \cap S(\xi, R_{j-1}), U_j \cap B(\xi, R_{j-1})) + d_{j-1}R^{n-2}G'_R(x, \xi_R) \tag{3.7}$$

for $x \in U_j \cap B(\xi, R_{j-1})$. See Figure 1. If $x \in B(\xi, R_j)$, then $B(x, R_{j-1} - R_j) \cap S(\xi, R_{j-1}) = \emptyset$, so that the first term in the right hand side of (3.7) is not greater than

$$\begin{aligned} &\omega(x, U_j \cap S(x, R_{j-1} - R_j), U_j \cap B(x, R_{j-1} - R_j)) \\ &\quad \le \exp\left(2 - A_1 \frac{R_{j-1} - R_j}{w_\eta(U_j)}\right) \le \exp\left(2 - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) \end{aligned}$$

by Lemma 1 and (3.4). Moreover, $A_6 R^{n-2}G'_R(x, \xi_R) \ge \exp(-2^{j+1})$ for $x \in D_j$ by definition. Hence (3.7) becomes

$$\begin{aligned} \omega_0(x) &\le \exp\left(2 - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}R^{n-2}G'_R(x, \xi_R) \\ &\le \left(A_6 \exp\left(2 + 2^{j+1} - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}\right) R^{n-2}G'_R(x, \xi_R) \end{aligned}$$

for $x \in D_j \cap B(\xi, R_j)$. Dividing both sides by $R^{n-2}G'_R(x, \xi_R)$ and taking the supremum over $x \in D_j \cap B(\xi, R_j)$, we obtain

$$d_j \le A_6 \exp\left(2 + 2^{j+1} - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}$$

and hence

$$d_i \le A_6 \sum_{j=1}^{\infty} \exp\left(2 + 2^{j+1} - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_0 < \infty$$

by (3.5). By definition $d_0 \le A_6 e^2$. Thus (3.6) follows and the lemma is proved. □

We need an estimate for the Green function which will substitute the Carleson estimate. We have

$$G_R(x, y) \le AR^{2-n} \quad \text{uniformly for } x \in D \cap B(\xi, 2R), y \in D \cap B(\xi, 9R) \backslash B(\xi, 3R),$$

where A is independent of R, since the diameter of D_R is bounded by R up to a multiplicative constant. It is important to localize the Green function. If we replace G_R by G, then the above inequality holds for $n \ge 3$, but not for $n = 2$ in general.

Lemma 3. *Let $\xi \in \partial D$ and $0 < R < R^*$. Then*

$$\frac{G_R(x, y)}{G_R(x', y)} \approx \frac{G_R(x, y')}{G_R(x', y')} \quad \text{for } x, x' \in D \cap B(\xi, R) \text{ and } y, y' \in D \cap S(\xi, 6R)$$

with the constant of comparison depending only on D.

Proof. Let us take $x^* \in D \cap S(\xi, R)$ and $y^* \in D \cap S(\xi, 6R)$ such that $A_4 R \le \delta_D(x^*) \le R$ and $6A_4 R \le \delta_D(y^*) \le 6R$. It is sufficient to show

$$G_R(x, y) \approx \frac{G_R(x^*, y)}{G_R(x^*, y^*)} G_R(x, y^*) \tag{3.8}$$

for $x \in D \cap B(\xi, R)$ and $y \in D \cap S(\xi, 6R)$.

First we show that the left hand side of (3.8) is not less than the right hand side of (3.8) up to a multiplicative constant. To this end we fix $y \in D \cap S(\xi, 6R)$ and observe that

(i) $u(x) = G_R(x, y)$ is a positive harmonic function on $D_R \backslash \{y\}$ with vanishing q.e. on ∂D_R;

(ii) $v(x) = (G_R(x^*, y))/(G_R(x^*, y^*)) G_R(x, y^*)$ is a positive harmonic function on $D_R \backslash \{y^*\}$ with vanishing q.e. on ∂D_R.

Since $y^* \in D \cap S(\xi, 6R)$ and $6A_4 R \le \delta_D(y^*) \le 6R$, it follows that $B(y^*, 3A_4 R) \subset D \cap B(\xi, 9R) \backslash B(\xi, 3R)$.

Let us prove $u \ge Av$ on $S(y^*, A_4 R)$. Take $z \in S(y^*, A_4 R)$. Then $k_{D_R \backslash \{y^*\}}(z, x^*) \le A$, so that

$$v(z) = \frac{G_R(x^*, y)}{G_R(x^*, y^*)} G_R(z, y^*) \approx \frac{G_R(x^*, y)}{G_R(x^*, y^*)} G_R(x^*, y^*) = G_R(x^*, y) \le AR^{2-n}. \tag{3.9}$$

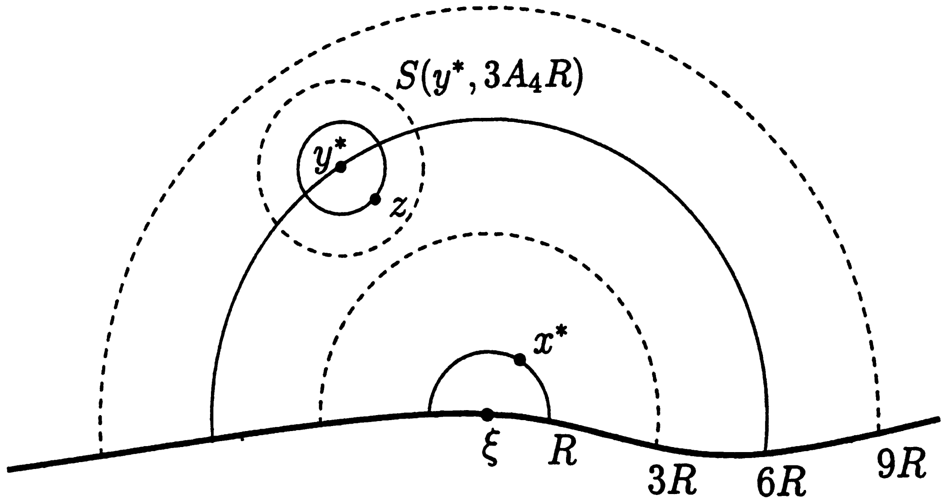

Figure 2. $k_{D_R\setminus\{y^*\}}(z, x^*) \le A$ for $z \in S(y^*, A_4R)$.

If $y \in B(y^*, 2A_4R)$, then $u(z) = G_R(z, y) \ge AR^{2-n}$, so that $u(z) \ge Av(z)$. If $y \in D\setminus B(y^*, 2A_4R)$, then

$$k_{D_R\setminus\{y\}}(z, x^*) \le k_{D_R}(z, x^*) + A \le k_D(z, x^*) + A' \le A'',$$

so that

$$v(z) \approx G_R(x^*, y) \approx G_R(z, y) = u(z)$$

by (3.9). Hence we have $u \ge Av$ on $S(y^*, A_4R)$ in any case. By the maximum principle $u \ge Av$ on $D_R\setminus B(y^*, A_4R) \supset D \cap B(\xi, R)$. Thus the left hand side of (3.8) is not less than the right hand side of (3.8) up to a multiplicative constant. See Figure 2.

For the opposite estimate of (3.8) we make use of Lemma 2. It is clear that $G_R(x, z) \le AR^{2-n} \approx G_R(x^*, y^*)$ for $x \in D \cap C(\xi, 2R)$ and $z \in D \cap B(\xi, 9R)\setminus B(\xi, 3R)$. Regarding $G_R(x, z)$ as a harmonic function of x, we obtain from the maximum principle that

$$G_R(\cdot, z) \le AG_R(x^*, y^*)\omega(\cdot, D \cap S(\xi, 2R), D \cap B(\xi, 2R)) \quad \text{on } D \cap B(\xi, 2R).$$

We obtain from Lemma 2 and the Harnack inequality that

$$\text{(3.10)} \qquad G_R(x, z) \le AG_R(x^*, y^*)R^{n-2}G_R(x, \xi_R) \le AG_R(x, y^*)$$

for $x \in D \cap B(\xi, R)$ and $z \in D \cap B(\xi, 9R)\setminus B(\xi, 3R)$. Here we have used the comparison $G_R(x^*, y^*) \approx R^{2-n}$ and $G_R(x, \xi_R) \approx G_R(x, y^*)$. Now fix $x \in D \cap B(\xi, R)$ and $y \in D \cap S(\xi, 6R)$. If $\delta_D(y) \ge 2^{-1}A_4R$, then $G_R(x, y) \approx G_R(x, y^*)$ and $G_R(x^*, y) \approx G_R(x^*, y^*)$ by the Harnack inequality, so that (3.8) follows. Hence, we may assume that $\delta_D(y) < 2^{-1}A_4R$. Then we find a point $\xi' \in \partial D$ such that

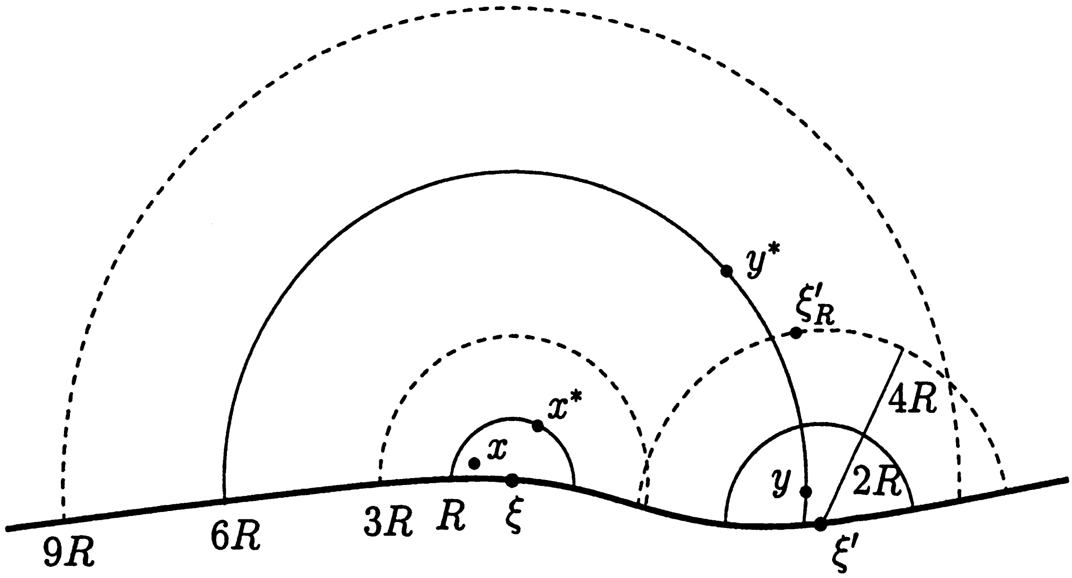

Figure 3. The case $\delta_D(y) < 2^{-1}A_4R$.

$|\xi' - y| < 2^{-1}A_4R$. Observe that $y \in D \cap B(\xi', 2^{-1}A_4R) \subset D \cap B(\xi', R)$; $5R < 6R - 2^{-1}A_4R \le |\xi - \xi'| \le 6R + 2^{-1}A_4R < 7R$ and $B(\xi', 2R) \subset B(\xi, 9R) \backslash B(\xi, 3R)$. Hence (3.10) implies $G_R(x,z) \le AG_R(x, y^*)$ for $z \in D \cap B(\xi', 2R)$, so that

$$G_R(x,y) \le AG_R(x,y^*)\omega(y, D \cap S(\xi', 2R), D \cap B(\xi', 2R)).$$

Let us invoke Lemma 2 with replacing ξ by ξ'. Since $|\xi - \xi'| \le 7R$, it follows that $D \cap B(\xi', A_5R) \subset D \cap B(\xi, (A_5 + 7)R) = D_R$. Hence

$$\begin{aligned}\omega(y, D \cap S(\xi', 2R), D \cap B(\xi', 2R)) &\le AR^{n-2}G_{D \cap B(\xi', A_5R)}(y, \xi'_R)\\ &\le AR^{n-2}G_R(y, \xi'_R) = AR^{n-2}G_R(\xi'_R, y)\end{aligned}$$

with $\xi'_R \in D \cap S(\xi', 4R)$ such that $4A_4R \le \delta_D(\xi'_R) \le 4R$. Here we have used the symmetry of the Green function. Hence

$$G_R(x,y) \le AG_R(x,y^*)R^{n-2}G_R(\xi'_R, y).$$

Observe that $|\xi'_R - y| \ge \delta_D(\xi'_R) - \delta_D(y) \ge 4A_4R - 1/2A_4R = 7/2A_4R$ and $|x^* - y| \ge \delta_D(x^*) - \delta_D(y) \ge A_4R - 1/2A_4R = 1/2A_4R$, so that $k_{D_R \backslash \{y\}}(\xi'_R, x^*) \le A$. Hence $G_R(\xi'_R, y) \approx G_R(x^*, y)$ by the Harnack inequality. See Figure 3. Since $G_R(x^*, y^*) \approx R^{2-n}$, it follows that

$$G_R(x,y) \le A\frac{G_R(x^*, y)}{G_R(x^*, y^*)}G_R(x, y^*)$$

Thus the opposite estimate of (3.8) is proved. The proof is complete. □

In order to prove Theorem 1, we represent u and v as regularized reduced functions and then as Green potentials. In general let E be a subset of D_R and let u be a positive superharmonic function on D_R. Let Φ_u^E be the family of all positive superharmonic functions v on D_R such that $v \geq u$ on E and let

$$R_u^E(x) = \inf\{v(x) : v \in \Phi_u^E\}.$$

The lower regularization $\hat{R}_u^E$ of R_u^E is called the regularized reduced function of u to E relative to D_R. It is known that $\hat{R}_u^E = u$ q.e. on E and that $\hat{R}_u^E$ is superharmonic on D_R and harmonic on $D_R \setminus \bar{E}$. For these account we refer to Helms [**15**, Chapters 7 and 8]. Here, we emphasize that the global positivity and superharmonicity of u over D_R is essential. If u were positive and superharmonic only on a neighborhood of E, then the class Φ_u^E could be empty.

Proof of Theorem 1. We prove the theorem with $A_0 = A_5 + 7$. Since u is a positive harmonic function on D_R, it follows that $\hat{R}_u^{D \cap S(\xi, 6R)}$ is a superharmonic function on D_R such that $\hat{R}_u^{D \cap S(\xi, 6R)} = u$ q.e. on $D \cap S(\xi, 6R)$ and harmonic on $D_R \setminus S(\xi, 6R)$. Moreover, $\hat{R}_u^{D \cap S(\xi, 6R)} = 0$ q.e. on ∂D_R by assumption. Hence $u = \hat{R}_u^{D \cap S(\xi, 6R)}$ on $D \cap B(\xi, 6R)$ by the maximum principle; and $\hat{R}_u^{D \cap S(\xi, 6R)}$ is a Green potential of a measure μ supported on $D \cap S(\xi, 6R)$, i.e.

$$u(x) = \int_{D \cap S(\xi, 6R)} G_R(x, y)\, d\mu(y) \quad \text{for } x \in D \cap B(\xi, 6R).$$

Let $x, x' \in D \cap B(\xi, R)$ and $y, y' \in D \cap S(\xi, 6R)$. Then

$$G_R(x, y) \approx \frac{G_R(x, y')}{G_R(x', y')} G_R(x', y)$$

by Lemma 3. Hence

$$u(x) \approx \frac{G_R(x, y')}{G_R(x', y')} \int_{D \cap S(\xi, 6R)} G_R(x', y)\, d\mu(y) = \frac{G_R(x, y')}{G_R(x', y')} u(x').$$

Therefore,

$$\frac{u(x)}{u(x')} \approx \frac{G_R(x, y')}{G_R(x', y')} \quad \text{uniformly for } y' \in D \cap S(\xi, 6R).$$

Similarly,

$$\frac{v(x)}{v(x')} \approx \frac{G_R(x, y')}{G_R(x', y')}.$$

Hence the theorem follows. □

Remark 2. The following Carleson estimate holds: If u is a positive harmonic function on $D \cap B(\xi, A_0 R)$, bounded on $D \cap B(\xi, A_0 R)$ and vanishing q.e. on $\partial D \cap B(\xi, A_0 R)$, then

$$u(x) \le A u(\xi_R) \quad \text{for } x \in D \cap B(\xi, R), \tag{3.11}$$

where $A > 0$ depends only on D. In fact, let $v = G_{D \cap B(\xi, 3A_0 R)}(\cdot, \xi_R^*)$ with $\xi_R^* \in D \cap S(\xi, 2A_0 R)$ and $2A_0 A_4 R \le \delta_D(\xi_R^*) \le 2A_0 R$. Then v is a positive harmonic function on $D \cap B(\xi, A_0 R)$, bounded on $D \cap B(\xi, A_0 R)$ and vanishing q.e. on $\partial D \cap B(\xi, A_0 R)$. Moreover, $v \le A R^{2-n} \le A v(\xi_R)$ on $D \cap B(\xi, R)$. Hence Theorem 1 yields

$$\frac{u(x)}{u(\xi_R)} \le A \frac{v(x)}{v(\xi_R)} \le A \quad \text{for } x \in D \cap B(\xi, R).$$

This proves the Carleson estimate (3.11).

The Carleson estimate for the half space was proved by Carleson [**11**] by the ingenious use of the exterior condition. The same proof applies to an NTA domain. The Carleson estimate was an important step of the proof of BHP for an NTA domain ([**17**]). For a uniform domain, however, Carleson's trick is not applicable because of the lack of the exterior condition. In the present setting, the Carleson estimate is not a tool for BHP, but one of the results of BHP.

Proof of Corollary 1. By the compactness argument we can find a small $R > 0$ and finitely many boundary points $\xi_1, \dots, \xi_k \in \partial D$ such that

$$K \cap \{x \in D : \delta_D(x) \le R/2\} \subset \bigcup_{j=1}^{k} D \cap B(\xi_j, R),$$

$$D \cap B(\xi_j, A_0 R) \subset D \cap V.$$

Fix $x_j \in D \cap B(\xi_j, R)$. Then Theorem 1 implies

$$\frac{u(x)}{v(x)} \approx \frac{u(x_j)}{v(x_j)} \quad \text{for } x \in D \cap B(\xi_j, R).$$

The usual Harnack principle yields

$$\frac{u(x_1)}{v(x_1)} \approx \cdots \approx \frac{u(x_k)}{v(x_k)} \approx \frac{u}{v} \quad \text{on } \{x \in K : \delta_D(x) \ge R/2\}.$$

Hence

$$\frac{u(x)}{v(x)} \approx \frac{u(y)}{v(y)} \quad \text{for } x, y \in K \cap D.$$

The corollary is proved. □

Remark 3. By a similar method we can show a (nonuniform) BHP for more wild domains, such as Hölder domains and John domains. Thus we have an analytic alternative proof of the results of Bañuelos, Bass and Burdzy ([**9**], [**7**] and [**8**]). Balogh and Volberg [**6**] introduced a uniformly John domain, a domain intermediate between a John domain and a uniform domain. They proved a BHP for a planar uniformly John domain with uniformly perfect boundary. Their BHP is uniform with respect to the internal metric. We can show such a BHP for a uniformly John domain without uniform perfectness of the boundary. This result will be treated elsewhere.

4. Proof of Theorems 2 and 3.

By Theorem 1 and the classical technique due to Moser [**19**, Section 5], we can show the Hölder continuity of u/v at the boundary. Let

$$M(r) = \sup_{D \cap B(\xi,r)} \frac{u}{v}, \quad m(r) = \inf_{D \cap B(\xi,r)} \frac{u}{v}.$$

Then $\operatorname{osc}_{D \cap B(\xi,r)} u/v = M(r) - m(r)$. Theorem 1 reads

$$1 \le \frac{M(r)}{m(r)} \le A_7,$$

where $A_7 > 1$ depends only on D.

Proof of Theorem 2. We have already seen in Theorem 1 that $m(r)$ and $M(r)$ are positive and finite. Observe that $M(r)v - u$ and $u - m(r)v$ are positive bounded harmonic functions on $D \cap B(\xi,r)$ with vanishing q.e. on $\partial D \cap B(\xi,r)$. Hence Theorem 1 applied to these functions and v yields

$$\sup_{D \cap B(\xi,r')} \frac{M(r)v - u}{v} \le A_7 \inf_{D \cap B(\xi,r')} \frac{M(r)v - u}{v},$$

$$\sup_{D \cap B(\xi,r')} \frac{u - m(r)v}{v} \le A_7 \inf_{D \cap B(\xi,r')} \frac{u - m(r)v}{v},$$

where $r' = r/A_0$. Hence

$$M(r) - m(r') \le A_7(M(r) - M(r')),$$

$$M(r') - m(r) \le A_7(m(r') - m(r)).$$

Adding the inequalities, we obtain

$$\operatorname*{osc}_{D \cap B(\xi,r')} \frac{u}{v} = M(r') - m(r') \le \frac{A_7 - 1}{A_7 + 1}(M(r) - m(r)) = \frac{A_7 - 1}{A_7 + 1} \operatorname*{osc}_{D \cap B(\xi,r)} \frac{u}{v}.$$

This shows (1.4) with

$$\varepsilon = \log\left(\frac{A_7+1}{A_7-1}\right)\Big/\log A_0.$$

The theorem is proved. □

REMARK 4. Dividing the both sides of (1.4) by $m(r)$, we obtain

$$\frac{M(r)}{m(r)} - 1 \le A\left(\frac{r}{R}\right)^{\varepsilon}\frac{m(R)}{m(r)}\left(\frac{M(R)}{m(R)} - 1\right) \le A\left(\frac{r}{R}\right)^{\varepsilon}\left(\frac{M(R)}{m(R)} - 1\right).$$

Since

$$\sup_{x,x'\in D\cap B(\xi,r)} \frac{u(x)}{v(x)}\Big/\frac{u(x')}{v(x')} = \frac{M(r)}{m(r)},$$

it follows that

$$\sup_{x,x'\in D\cap B(\xi,r)} \frac{u(x)}{v(x)}\Big/\frac{u(x')}{v(x')} - 1 \le A\left(\frac{r}{R}\right)^{\varepsilon}\left(\sup_{x,x'\in D\cap B(\xi,R)} \frac{u(x)}{v(x)}\Big/\frac{u(x')}{v(x')} - 1\right).$$

This is a multiplicative form of Hölder continuity.

PROOF OF COROLLARY 2. Let $x \in D$. The following interior Hölder continuity is known:

$$\operatorname*{osc}_{D\cap B(x,r)} \frac{u}{v} \le A\left(\frac{r}{R}\right)^{\varepsilon} \operatorname*{osc}_{D\cap B(x,R)} \frac{u}{v} \quad \text{for } 0 < r \le R \le \delta_D(x),$$

where A and ε depend only on the dimension. In particular,

$$\operatorname*{osc}_{D\cap B(x,r)} \frac{u}{v} \le A\left(\frac{r}{\delta_D(x)}\right)^{\varepsilon} \operatorname*{osc}_{D\cap B(x,\delta_D(x))} \frac{u}{v} \quad \text{for } 0 < r \le \delta_D(x). \tag{4.1}$$

We may assume that ε is the same as in Theorem 2, if necessary ε making smaller. The following two cases remain: $\delta_D(x) \le r \le R \le \operatorname{dist}(K, V^c)$ and $r \le \delta_D(x) \le R \le \operatorname{dist}(K, V^c)$.

CASE 1. $\delta_D(x) \le r \le R \le \operatorname{dist}(K, V^c)$. If $r \approx R$, then (1.5) is obvious. Hence we may assume that $r \le R'/4$ with $R' = R/A_0$. By definition there is $\xi \in \partial D$ with $|x - \xi| \le \delta_D(x)$. Observe that

$$B(x,r) \subset B(\xi, 2r) \subset B(\xi, R'/2) \subset B(\xi, A_0R'/2) \subset B(x,R) \subset V.$$

Hence, Theorem 2 yields

$$\operatorname*{osc}_{D\cap B(x,r)}\frac{u}{v}\le\operatorname*{osc}_{D\cap B(\xi,2r)}\frac{u}{v}\le A\left(\frac{r}{R'/2}\right)^{\varepsilon}\operatorname*{osc}_{D\cap B(\xi,R'/2)}\frac{u}{v}\le A\left(\frac{r}{R}\right)^{\varepsilon}\operatorname*{osc}_{D\cap B(x,R)}\frac{u}{v}.$$

Thus (1.5) follows.

Case 2. $r\le\delta_D(x)\le R\le\operatorname{dist}(K,V^c)$. We obtain from Case 1 with $r=\delta_D(x)$ that

$$\operatorname*{osc}_{D\cap B(x,\delta_D(x))}\frac{u}{v}\le A\left(\frac{\delta_D(x)}{R}\right)^{\varepsilon}\operatorname*{osc}_{D\cap B(x,R)}\frac{u}{v}.$$

This, together with (4.1), yields (1.5).

In the same way as in Remark 4 we obtain the multiplicative form of Hölder continuity, which readily implies the second required inequality in the corollary. The remaining is obvious. The corollary is proved. □

Let $\mathscr{H}_\xi$ be the family of all positive harmonic functions h on D vanishing q.e. on ∂D, bounded on $D\backslash B(\xi,r)$ for each $r>0$ and taking value $h(x_0)=1$. A function h in $\mathscr{H}_\xi$ is called a *kernel function* at ξ normalized at x_0.

Lemma 4. *There is a constant $A\ge1$ depending only on D such that*

$$A^{-1}\le\frac{u}{v}\le A\quad\text{for } u,v\in\mathscr{H}_\xi.$$

Proof. Let $u,v\in\mathscr{H}_\xi$ and let $r>0$. Then u and v be bounded on $D\cap B(\xi',2^{-1}r)$ for $\xi'\in\partial D\cap S(\xi,r)$. Hence Theorem 1 yields

$$\frac{u(x)}{v(x)}\approx\frac{u(x')}{v(x')}\quad\text{for } x,x'\in D\cap B(\xi',2^{-1}r/A_0),$$

where A_0 is as in Theorem 1. This, together with the Harnack inequality, shows that

$$\frac{u(x)}{v(x)}\approx\frac{u(x')}{v(x')}\quad\text{for } x,x'\in D\cap S(\xi,r),\tag{4.2}$$

where the constant of comparison is independent of r. Fix $x'\in D\cap S(\xi,r)$ for a moment. By the maximum principle we have

$$\frac{u(x)}{v(x)}\approx\frac{u(x')}{v(x')}\quad\text{for } x\in D\backslash B(\xi,r).$$

In particular,

$$A^{-1}=A^{-1}\frac{u(x_0)}{v(x_0)}\le\frac{u(x')}{v(x')}\le A.$$

Hence (4.2) becomes

$$\frac{u(x)}{v(x)} \approx 1 \quad \text{for } x \in D \backslash B(\xi, r).$$

Since $r > 0$ is arbitrary small and the constant of comparison is independent of r, the lemma follows. □

Proof of Theorem 3. Lemma 4 actually shows that $\mathscr{H}_\xi$ is a singleton. This is proved by Ancona [**4**, Lemma 6.2]. For the reader's convenience we give a short proof below. Let

$$c = \sup_{\substack{u, v \in \mathscr{H}_\xi \\ x \in D}} \frac{u(x)}{v(x)}.$$

Then $1 \le c < \infty$ by Lemma 4. It is sufficient to show that $c = 1$. Suppose to the contrary $c > 1$. Take arbitrary u, $v \in \mathscr{H}_\xi$. Then $v_1 = (cv - u)/(c-1) \in \mathscr{H}_\xi$, so that $u \le cv_1 = c(cv - u)/(c-1)$, whence $(2c-1)u \le c^2 v$ on D. This would imply

$$c = \sup_{\substack{u, v \in \mathscr{H}_\xi \\ x \in D}} \frac{u(x)}{v(x)} \le \frac{c^2}{2c-1} < c,$$

a contradiction. Thus $c = 1$ and $\mathscr{H}_\xi$ is a singleton. Moreover, the function $u \in \mathscr{H}_\xi$ is minimal. For if h is a positive harmonic function not greater than u, then $h/h(x_0) \in \mathscr{H}_\xi$, so that $h = h(x_0)u$.

Let $K(x, y) = G(x, y)/G(x_0, y)$ for $x \in D$ and $y \in D \backslash \{x_0\}$. The Martin kernel is given as the limit of $K(x, y)$ when y tends to a boundary point. If $y \to \xi \in \partial D$, then some subsequence of $\{K(\cdot, y)\}$ converges to a positive harmonic function in $\mathscr{H}_\xi$. However, since $\mathscr{H}_\xi$ is a singleton, it follows that all sequences $\{K(\cdot, y)\}$ must converge to the same positive harmonic function, the Martin kernel $K(\cdot, \xi)$ at ξ. Therefore $K(x, \cdot)$ extends continuously to $\bar{D} \backslash \{x_0\}$. The kernel function $K(\cdot, \xi)$ should be minimal. This shows the first part of the theorem.

Let us show the Hölder continuity of the kernel function. Take $\xi' \in \partial D$ and let

$$\tilde{M}(r) = \sup_{D \backslash B(\xi, r)} \frac{K(\cdot, \xi')}{K(\cdot, \xi)} = \sup_{D \cap S(\xi, r)} \frac{K(\cdot, \xi')}{K(\cdot, \xi)},$$

$$\tilde{m}(r) = \inf_{D \backslash B(\xi, r)} \frac{K(\cdot, \xi')}{K(\cdot, \xi)} = \inf_{D \cap S(\xi, r)} \frac{K(\cdot, \xi')}{K(\cdot, \xi)}$$

for $r \geq 2|\xi - \xi'|$. In the same way as in the proof of Lemma 4 we can show that

$$1 \leq \frac{\tilde{M}(r)}{\tilde{m}(r)} \leq A,$$

where $A > 1$ depends only on D. Then the same argument as in the proof of Theorem 2 shows that

$$\operatorname*{osc}_{D\setminus B(\xi,R)} \frac{K(\cdot,\xi')}{K(\cdot,\xi)} = \tilde{M}(R) - \tilde{m}(R) \leq A\left(\frac{r}{R}\right)^{\varepsilon}(\tilde{M}(r) - \tilde{m}(r)) = A\left(\frac{r}{R}\right)^{\varepsilon} \operatorname*{osc}_{D\setminus B(\xi,r)} \frac{K(\cdot,\xi')}{K(\cdot,\xi)}$$

for $R > r$. Also we have a multiplicative form of Hölder continuity,

$$\sup_{x,x' \in D\setminus B(\xi,R)} \frac{K(x,\xi')}{K(x,\xi)} \bigg/ \frac{K(x',\xi')}{K(x',\xi)} - 1 \leq A\left(\frac{r}{R}\right)^{\varepsilon}\left(\sup_{x,x' \in D\setminus B(\xi,r)} \frac{K(x,\xi')}{K(x,\xi)} \bigg/ \frac{K(x',\xi')}{K(x',\xi)} - 1\right).$$

Letting $r = 2|\xi - \xi'|$, we obtain that

$$\operatorname*{osc}_{D\setminus B(\xi,R)} \frac{K(\cdot,\xi')}{K(\cdot,\xi)} \leq A\left(\frac{|\xi - \xi'|}{R}\right)^{\varepsilon},$$

$$\sup_{x,x' \in D\setminus B(\xi,R)} \frac{K(x,\xi')}{K(x,\xi)} \bigg/ \frac{K(x',\xi')}{K(x',\xi)} - 1 \leq A\left(\frac{|\xi - \xi'|}{R}\right)^{\varepsilon}$$

for $R \geq 2|\xi - \xi'|$. Moreover, letting $x' = x_0$ and observing $K(x_0,\xi')/K(x_0,\xi) = 1$, we obtain

$$\left|\frac{K(x,\xi')}{K(x,\xi)} - 1\right| \leq A\left(\frac{|\xi - \xi'|}{R}\right)^{\varepsilon}$$

for $x \in D\setminus B(\xi,R)$ with $R \geq 2|\xi - \xi'|$. This is the form of Hölder continuity given by Jerison and Kenig [**17**, Theorem 7.1]. The theorem is proved. □

5. Proof of Theorem 4.

Jerison and Kenig [**17**] proved the Fatou theorem for an NTA domain by using the maximal function argument. Since the harmonic measure of a uniform domain need not satisfy the doubling property, the maximal function argument is not applicable in our case. Instead we shall invoke the minimal fine limit theorem and compare the minimal fine filter and nontangential filter. Without loss of generality we may assume that D is bounded. Then the Martin boundary of D is the Euclidean boundary ∂D and every boundary point is minimal (Corollary 3). For every nonnegative harmonic function h on D there is a unique measure μ_h on ∂D such that $h = K\mu_h = \int_{\partial D} K(\cdot,\xi)\, d\mu_h(\xi)$. In this section a regularized reduced function $\hat{R}_u^E$ is taken with respect to D. For simplicity we write $K_\xi = K(\cdot, \xi)$. A set $E \subset D$ is said to be minimally thin at $\xi \in \partial D$ if the regularized reduced function $\hat{R}^E_{K_\xi}$ is a Green potential. Our proof of Theorem 4 is based on the minimal fine limit theorem (see [**3**, II, Appendix], [**13**] and [**20**]).

Theorem B. *Let $h = K\mu_h$ and $H = K\mu_H$ be positive harmonic functions on D. Then, H/h has minimal fine limit $d\mu_H/d\mu_h$ for μ_h almost every boundary point ξ. That is, there is a set E minimally thin at ξ such that*

$$\lim_{\substack{x \to \xi \\ x \in D \setminus E}} \frac{H(x)}{h(x)} = \frac{d\mu_H}{d\mu_h}(\xi).$$

Theorem 4 will follow from Theorem B and the following lemma. Let $\xi \in \partial D$. We say that $\{x_j\}$ is a nontangential sequence converging to ξ if there is $\alpha > 0$ such that $x_j \in \Gamma_\alpha(\xi)$ for all large j. For $0 < a < 1$ we consider the union $\mathscr{B} = \bigcup_{j=1}^\infty B(x_j, a\delta_D(x_j))$. This is a nontangential $\mathscr{B}$-set introduced by Hunt and Wheeden [**16**].

Lemma 5. *Let $\mathscr{B} = \bigcup_{j=1}^\infty B(x_j, a\delta_D(x_j))$ be a nontangential $\mathscr{B}$-set at $\xi \in \partial D$ as above. Then $\mathscr{B}$ is not minimally thin at ξ.*

Proof. The proof is similar to [**16**]. However, we avoid the harmonic measure of D, since it does not satisfy the doubling property. Without loss of generality we may assume that $\mathscr{B} \subset D \cap B(\xi, R)$ and $x_0 \in D \setminus B(\xi, AR)$ for small $R > 0$ and sufficiently large $A > 0$. Let $a < a' < a'' < 1$. By the Harnack inequality $K_\xi \approx K_\xi(x_j)$ on $B(x_j, a\delta_D(x_j))$ with constant of comparison independent of j. Hence the regularized reduced function of K_ξ to $B(x_j, a\delta_D(x_j))$ with respect to $B(x_j, a''\delta_D(x_j))$ is comparable to $K_\xi(x_j)$ on $S(x_j, a'\delta_D(x_j))$ and so is $\hat{R}^{B(x_j, a\delta_D(x_j))}_{K_\xi}$. Therefore, the Harnack inequality and the boundary Harnack principle (Theorem 1) yield

$$\hat{R}^{B(x_j, a\delta_D(x_j))}_{K_\xi} \approx K_\xi \quad \text{on } D \cap S(\xi, (1+a)|x_j - \xi|),$$

since $\hat{R}^{B(x_j, a\delta_D(x_j))}_{K_\xi}$ is a positive harmonic function on $D \setminus C(x_j, a\delta_D(x_j))$ vanishing q.e. on ∂D. By the maximum principle

$$A^{-1} = A^{-1} K_\xi(x_0) \le \hat{R}^{B(x_j, a\delta_D(x_j))}_{K_\xi}(x_0) \le A. \tag{5.1}$$

Now let $\mathscr{B}_k = \bigcup_{j=k}^\infty B(x_j, a\delta_D(x_j))$ and $h_k = \hat{R}^{\mathscr{B}_k}_{K_\xi}$. Then $h_k(x_0) \ge \hat{R}^{B(x_k, a\delta_D(x_k))}_{K_\xi}(x_0) \ge A > 0$ by (5.1). Hence h_k reduces to a positive harmonic function h on D with $h(x_0) \ge A$. Observe that $h/h(x_0)$ is a kernel function at ξ. Hence $h/h(x_0) = K_\xi$ by the proof of Theorem 3. It is known the balayage operation is idempotent ([**13**, Section VI.3 (h)]), so that

$$\hat{R}^{\mathscr{B}}_{h_k} = \hat{R}^{\mathscr{B}}_{\hat{R}^{\mathscr{B}_k}_{K_\xi}} = \hat{R}^{\mathscr{B}_k}_{K_\xi} = h_k.$$

Letting $k \to \infty$, we obtain $\hat{R}_h^{\mathscr{B}} = h$ and hence $\hat{R}_{K_\xi}^{\mathscr{B}} = K_\xi$. Thus $\mathscr{B}$ is not minimally thin at ξ. □

PROOF OF THEOREM 4. Observe that the harmonic measure ω_D of D is given by $d\omega_D^x(\xi) = K_\xi(x)\, d\mu_1(\xi)$, where μ_1 is the representing measure of the constant function 1. Hence ω_D and μ_1 are mutually absolutely continuous. It follows from Theorem B that u has minimal fine limit at $\xi \in \partial D$ a.e. ω_D. Take such $\xi \in \partial D$. Let us prove that u has nontangential limit at ξ. If u failed to have nontangential limit at ξ, then there would exist nontangential sequences $\{x_j\}$ and $\{x_j'\}$ converging to ξ such that $\limsup_{j\to\infty} u(x_j) < \liminf_{j\to\infty} u(x_j')$. By the Harnack inequality

$$\limsup_{\substack{x\to\xi \\ x\in\mathscr{B}}} u(x) < \liminf_{\substack{x\to\xi \\ x\in\mathscr{B}'}} u(x),$$

where $\mathscr{B} = \bigcup_{j=1}^\infty B(x_j, a\delta_D(x_j))$ and $\mathscr{B}' = \bigcup_{j=1}^\infty B(x_j', a\delta_D(x_j'))$ with sufficiently small $a > 0$. Lemma 5 says that $\mathscr{B}$ and $\mathscr{B}'$ are not minimally thin at ξ. This would imply that u fails to have minimal fine limit at ξ, a contradiction. □

REMARK 5. The above proof says that if $h = K\mu_h$ and $H = K\mu_H$ are positive harmonic functions on D, then, H/h has nontangential limit $d\mu_H/d\mu_h$ for μ_h almost every boundary point.

PROOF OF PROPOSITION 1. The construction of D and E is easy. Consider the unit ball $B(0,1)$ with Whitney decomposition $\bigcup_{j=1}^\infty Q_j$ (see [**21**, Chapter VI]). Let y_j be the center of Q_j and let $D = B(0,1)\setminus\bigcup_{j=1}^\infty \{y_j\}$ and $E = \bigcup_{j=1}^\infty \{y_j\}$. We can easily observe that D is a uniform domain. In fact, suppose $x_1, x_2 \in D$. If these points lie in the same Whitney cube, then it is easy to find a curve γ connecting x_1 and x_2 satisfying (1.2) and (1.3). If these points lie in two different Whitney cubes, say Q_1 and Q_2, then the boundaries of Q_1 and Q_2 can be joined by a curve γ satisfying (1.2) and (1.3); and then each x_j can be connected with the boundary of Q_j with an appropriate curve. See Figure 4.

Now let us construct a harmonic function u on D. By differentiating the fundamental harmonic function $j+2-n$ times, we obtain a harmonic function $H_j(x)$ on $\boldsymbol{R}^n\setminus\{0\}$ homogeneous of degree $-j$ for $j \geq n-1$. Let r_j be the distance from y_j to the boundary of Q_j and let $M_j(\beta) = \sup_{S(0,\beta r_j)}|H_j(x)|$ for $0 < \beta < 1/2$. By the maximum principle $|H_j(x - y_j)| \leq M_j(\beta)$ for $x \in \boldsymbol{R}^n\setminus C(y_j, \beta r_j)$. Let

$$u(x) = \sum_{j=n-1}^\infty \frac{H_j(x-y_j)}{j^2 M_j(\beta)}.$$

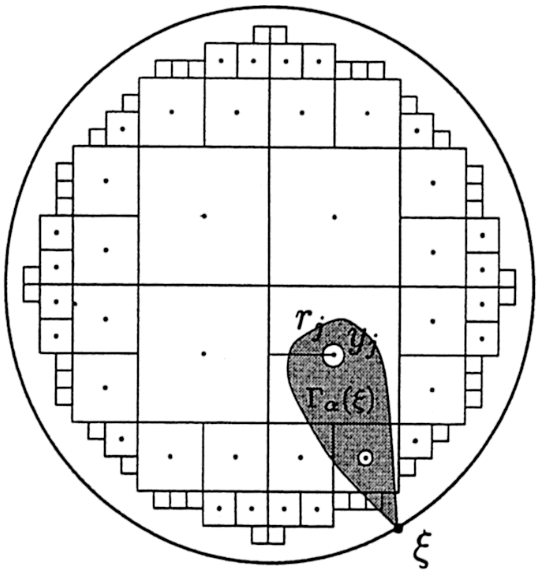

Figure 4. $D = B(0,1)\setminus\bigcup_{j=1}^{\infty}\{y_j\}$ and $\Gamma_\alpha(\xi)$.

Then u is a harmonic function on D such that

$$(5.2)\qquad |u(x)| \le \sum_{j=n-1}^{\infty}\frac{|H_j(x-y_j)|}{j^2 M_j(\beta)} \le \sum_{j=1}^{\infty}\frac{1}{j^2} < \infty \quad \text{on } D\setminus\bigcup_{j=1}^{\infty} C(y_j,\beta r_j).$$

Let $\alpha = \beta^{-1} - 2 > 0$. If $x \in \Gamma_\alpha(\xi)$ for $\xi \in S(0,1)$, then

$$r_j < |y_j - \xi| \le |x-y_j| + |x-\xi| < |x-y_j| + (1+\alpha)\delta_D(x)$$

$$\le (2+\alpha)|x-y_j| = \beta^{-1}|x-y_j|,$$

whence $x \notin C(y_j,\beta r_j)$. Hence (5.2) means that u is nontangentially bounded on $S(0,1) = \partial D\setminus E$. Obviously, $\bigcup_{j=1}^{\infty}\{y_j\}$ is of harmonic measure 0. Hence u is nontangentially bounded a.e. ω_D on ∂D. On the other hand we have

$$\sup_{S(y_i,\beta r_i/2)}\frac{|H_i(x-y_i)|}{M_i(\beta)} = \frac{M_i(\beta/2)}{M_i(\beta)} = 2^i.$$

Hence

$$\sup_{S(y_i,\beta r_i/2)}|u(x)| \ge \frac{2^i}{i^2} - \sum_{j\ne i}\frac{1}{j^2} \to \infty$$

as $i \to \infty$. For every $\xi \in S(0,1)$ we find a subsequence $\{y_{j_i}\}$ nontangentially converging to ξ with respect to $B(0,1)$. Then $\bigcup_i S(y_{j_i},\beta r_{j_i}/2)$ is nontangential at ξ with respect to D. Hence u fails to have nontangential limit at every $\xi \in S(0,1) = \partial D \setminus E$. The proposition follows. □

Remark 6. Let $0 < \rho_j < r_j/2$ be sufficiently small. Then $B(0,1) \setminus \bigcup_{j=1}^{\infty} C(y_j, \rho_j)$ is a bounded regular uniform domain for which the local Fatou theorem does not hold.

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Note added in proof. After the submission of the final form of the paper, the author was aware that F. Ferrari, (J. Fourier Anal. Appl. **4** (1998), 447–461) gave an analytic proof of a boundary Harnack principle for a Hölder domain. His result is not scale invariant.

Hiroaki Aikawa
Department of Mathematics
Shimane University
Matsue 690-8504
Japan
E-mail: haikawa@math.shimane-u.ac.jp