*Journal of Elasticity* **62:** 203–216, 2001.



# Boundary Layer Solutions in Elastic Solids

YI-CHAO CHEN
*Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, U.S.A.*

K.R. RAJAGOPAL
*Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843-3123, U.S.A. E-mail: krajagopal@mengr.tamu.edu*

*Dedicated to Lewis Wheeler, on the occasion of his 60th birthday*

Received 1 December 2000; in revised form 25 April 2001

**Abstract.** Circumferential shear deformation in an annular domain is studied for a large class of incompressible isotropic elastic materials. It is demonstrated that large strains are confined in a region adjacent to a boundary, in analogy to the boundary layer phenomenon in fluid mechanics. The size of this region is quantified. An approximate solution technique for the deformation of nonlinear elastic solids, proposed by Rajagopal [7], is further studied. In this solution, akin to the boundary layer approximation in classical fluid mechanics, the full nonlinear problem is solved in a relatively small region of large strain, while the linearized problem is solved in the remaining region. Error estimates for the approximate solution are obtained.

## 1. Introduction

Whether it is the motion of fluids or solids, most of the interesting phenomena take place adjacent to solid boundaries or at interfaces. Prescribing boundary conditions and describing the observed phenomena adjacent to boundaries remain the most challenging aspect of mechanics, especially when nonlinear materials are concerned. The intense interest and effort expended in the development and study of boundary layer theory for the classical linearly viscous fluid (Schlichting [1]) notwithstanding, basic issues remain unresolved and elude our understanding of the effect of boundaries; wetting of boundaries is but one example. In the case of certain nonlinear fluids, the phenomenon of "stick-slip" is a prime example of our inadequate understanding of the effect of boundaries on the flow of such fluids. When attention is shifted to the deformation of solids, there is a lacuna with respect to the development of a boundary layer theory similar to the one that is in place for fluids, though there seems to be enough evidence to show that the development of such a theory is warranted.

By a boundary layer, we mean a narrow region adjacent to a boundary wherein the strains are large and the full nonlinear equations are assumed to hold, while exterior to this region the strains are small and the linearized equations are expected to hold. Such a theory can have relevance to a much larger class of problems involving strain localization with the nonlinear theory holding in the small region of strain localization (say due to an inclusion or defect) with the linearized theory holding outside the small region. There have been many studies where the presence of boundary layers has been established (Zhang and Rajagopal [2], Rajagopal and Tao [3], Rajagopal [4], Tao et al. [5], and Haughton [6]) within the context of a specific model. More recently, it was shown [7] within the context of a power-law neo-Hookean solid that the problem of circumferential shear deformation can be studied in the spirit of a boundary layer approach. An arbitrary thickness is assumed for the boundary layer and the nonlinear and linearized equations are solved respectively, inside and outside the boundary layer, with continuity in the displacement field and a prescribed tolerance for the difference in strain. The solution is iterated by varying the boundary layer thickness until the prescribed tolerance for the strain is met. The results compare excellently with the exact solution to the nonlinear problem in the full domain.

It is suggested that the thickness of the boundary layer can be chosen as the region in which a large portion of the strain, say 99%, is confined, i.e., at the edge of the boundary layer the strain has reduced to 1% of its maximum value. In this paper, unlike the works cited above, we do not confine ourselves to a particular constitutive function. Instead, we show that the development of boundary layers seems typical of a large class of nonlinear elastic materials, providing an even stronger case for the development of a boundary layer like theory for nonlinear elastic solids.

Here, we must point out that we do not claim that boundary layers occur for all nonlinear elastic solids, nor do we claim that boundary layers occur for all other geometries and boundary conditions than that considered here. In this paper, we concentrate on the circumferential shear deformations in annular domains because the analytic solutions exist for this problem, which allows us to carry out an explicit and quantitative study on the development of boundary layers. On the other hand, we see no reason to believe that this would be the only geometry that exhibits boundary layer phenomenon. It will be totally unsurprising if one demonstrates, perhaps through numerical analysis, that boundary layers can be identified for other problems involving more complicated geometries.

Depending on the problem under consideration, different quantities can be confined adjacent to boundaries. Traditional boundary layer theory for the linearly viscous fluid concerns the confinement of vorticity adjacent to the boundary (Schlichting [1]). In nonlinear fluids, it is possible that a variety of quantities can be confined adjacent to the boundary with the possibility of multiple deck structures with different quantities confined in the different layers. In the case of solids, it is possible to define boundary layers based on strains or stresses. The thickness of these boundary layers are not necessarily the same. For the problem considered in this paper, the thickness of the boundary layer based on strain depends on the strain level in the problem. Interestingly, however, the thickness of the boundary layer based on stress depends on the geometry of the annulus, but not on the stress or strain levels.

Error estimates are obtained for the boundary layer solution from the exact solution to the full nonlinear equations for a general class of materials. As a particular example, explicit results are given for the power-law neo-Hookean material.

## 2. Basic Equations

Consider an elastic body which in a reference configuration occupies an annular, multiply-connected domain $\Omega$. In a cylindrical polar coordinate system, $\Omega$ is denoted by

$$\Omega = \left\{(R, \Theta, Z)\colon \mathrm{R_i} \leqslant R \leqslant R_\mathrm{o},\ 0 \leqslant \Theta < 2\pi,\ -\infty < Z < \infty\right\},$$

where $R_\mathrm{i}$ and $R_\mathrm{o}$ are the inner and outer radii of the annulus, respectively.

We wish to find the deformation of the body when the inner surface is rotated through an angle $\phi$, while the outer surface is held fixed. A semi-inverse method will be used in which we consider a circumferential shear deformation of the form

$$r = R, \qquad \theta = \Theta + f(R), \qquad z = Z, \tag{1}$$

where $(r, \theta, z)$ denote the coordinates of the material particle $(R, \Theta, Z)$ after deformation, and $f \in C^1([R_\mathrm{i}, R_\mathrm{o}]; \Re)$, $\Re$ being the set of real numbers, satisfies

$$f(R_\mathrm{i}) = \phi, \qquad f(R_\mathrm{o}) = 0. \tag{2}$$

The physical components of the deformation gradient $\mathbf{F}$ for the shear deformation (1) are

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the shear strain $\gamma$ is given by

$$\gamma \equiv Rf'(R). \tag{3}$$

The physical components of the left Cauchy–Green tensor $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^\mathrm{T}$ are

$$\mathbf{B} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

The principal invariants of $\mathbf{B}$ are

$$I_1 \equiv \mathrm{tr}\mathbf{B} = 3 + \gamma^2, \qquad I_2 \equiv \frac{1}{2}\big[(\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\big(\mathbf{B}^2\big)\big] = 3 + \gamma^2, \qquad I_3 \equiv \det\mathbf{B} = 1. \tag{5}$$

The deformation (1) is isochoric.

We shall consider an elastic body that is incompressible, homogeneous, and isotropic in the reference configuration. The Cauchy stress tensor $\mathbf{T}$ for such a material takes the form (see, for example, [2])

$$\mathbf{T} = -p\mathbf{I} + g_1\mathbf{B} + g_2\mathbf{B}^2, \tag{6}$$

where $\mathbf{I}$ is the identity tensor, $g_1$ and $g_2$ are functions of two principal invariants $I_1$ and $I_2$, and $p$ is the indeterminate part of the stress due to the incompressibility constraint. We shall assume that $g_1$ and $g_2$ are of class $C^1([3,\infty)\times[3,\infty);\Re)$. Substituting (4) into (6), we find the physical components of the Cauchy stress tensor to be

$$\mathbf{T} = \begin{pmatrix} -p + g_1 + (1+\gamma^2)g_2 & \gamma g_1 + (2\gamma+\gamma^3)g_2 & 0 \\ \gamma g_1 + (2\gamma+\gamma^3)g_2 & -p + (1+\gamma^2)g_1 + (1+3\gamma^2+\gamma^4)g_2 & 0 \\ 0 & 0 & -p + g_1 + g_2 \end{pmatrix}. \tag{7}$$

Here and henceforth, the response functions $g_1$ and $g_2$ are evaluated at $(I_1, I_2) = (3+\gamma^2, 3+\gamma^2)$.

In this work, we do not require that the material be hyperelastic. If the material is hyperelastic, there exists a strain-energy function $W \in C^2([3,\infty)\times[3,\infty);\Re)$, depending on two principal invariants $I_1$ and $I_2$, such that the response functions $g_1$ and $g_2$ are given by

$$g_1 = 2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right), \qquad g_2 = -2\frac{\partial W}{\partial I_2}.$$

In the absence of body forces, the equations of equilibrium take the form

$$\mathrm{div}\mathbf{T} = 0. \tag{8}$$

For a deformation of the form (1), the components of $\mathbf{F}$ and $\mathbf{B}$ are functions of $R$ alone. We shall assume that $p$ is also a function of $R$ alone. Hence, the components of $\mathbf{T}$ are functions of $R$ as well. Substituting (7) into (8), we find that

$$\frac{\mathrm{d}}{\mathrm{d}R}\big[-p + g_1 + (1+\gamma^2)g_2\big] - \frac{\gamma^2}{R}\big[g_1 + (2+\gamma^2)g_2\big] = 0, \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}R}\big[\gamma g_1 + (2\gamma+\gamma^3)g_2\big] + \frac{2}{R}\big[\gamma g_1 + (2\gamma+\gamma^3)g_2\big] = 0. \tag{10}$$

For given response functions $g_1$ and $g_2$, one can solve, in principle, equation (10) for $\gamma$, and hence $f$. Equation (9) can be then used to find the pressure $p$.

## 3. Existence of Equilibrium Solutions

Let us define the shear stress function $\hat{\tau}$ by

$$\hat{\tau}(\gamma) \equiv \gamma g_1 + \left(2\gamma + \gamma^3\right) g_2.$$

For the deformation under consideration, the value of $\hat{\tau}$ gives the circumferential shear stress $\tau$. If the material is hyperelastic, the shear stress function is given by

$$\hat{\tau}(\gamma) = 2\gamma\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right). \tag{11}$$

Here and henceforth, the derivatives of $W$ are evaluated at $(I_1, I_2) = (3 + \gamma^2, 3 + \gamma^2)$.

Obviously, $\hat{\tau}$ is an odd function of $\gamma$. In this work, we shall assume that $\hat{\tau}$ is strictly increasing. This assumption is implied by

$$g_1 + \left(2 + 3\gamma^2\right) g_2 + 2\gamma^2\left(\frac{\partial g_1}{\partial I_1} + \frac{\partial g_1}{\partial I_2}\right) + 2\gamma^2\left(2 + \gamma^2\right)\left(\frac{\partial g_2}{\partial I_1} + \frac{\partial g_2}{\partial I_2}\right) > 0,$$

or, for a hyperelastic material, by

$$\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} + 2\gamma^2\left(\frac{\partial^2 W}{\partial I_1^2} + 2\frac{\partial^2 W}{\partial I_1 \partial I_2} + \frac{\partial^2 W}{\partial I_2^2}\right) > 0.$$

Under this assumption, function $\hat{\tau}$ can be inverted. The inverse function $\hat{\tau}^{-1} \equiv \hat{\gamma}$ is also odd and strictly increasing. Furthermore, since $g_1$ and $g_2$ are defined in an unbounded domain, function $\hat{\gamma}$ is unbounded.

PROPOSITION 1. *There exists a unique equilibrium solution $f(R)$ that satisfies the boundary conditions* (2).

*Proof.* The equation of equilibrium (10) can be rewritten as

$$\frac{\mathrm{d}\tau}{\mathrm{d}R} + \frac{2\tau}{R} = 0, \tag{12}$$

where $\tau \equiv \hat{\tau}(\gamma)$. Equation (12) admits the general solution

$$\tau = \frac{C}{R^2}, \tag{13}$$

where $C$ is a constant of integration. By (3), we can write (13) as

$$R f'(R) = \hat{\gamma}\left(\frac{C}{R^2}\right). \tag{14}$$

The equilibrium solution that satisfies the boundary condition $(2)_2$ is then given by

$$f(R) = -\int_R^{R_o} \frac{1}{\rho}\hat{\gamma}\left(\frac{C}{\rho^2}\right) \mathrm{d}\rho. \tag{15}$$

Now it suffices to show that there exists a unique $C$ such that the solution given by (15) satisfies the boundary condition $(2)_1$. Since $\hat{\gamma}$ is strictly increasing, the integral

$$-\int_{R_i}^{R_o} \frac{1}{\rho}\hat{\gamma}\left(\frac{C}{\rho^2}\right)\mathrm{d}\rho \tag{16}$$

is strictly decreasing in $C$. Moreover, since $\hat{\gamma}$ is unbounded, the integral (16) is unbounded below as $C$ increases, and unbounded above as $C$ decreases. By the continuity of the value of the integral in $C$, there exists a unique $C$ such that the integral (16) equals $\phi$. □

We note that the assumption that $\hat{\tau}(\gamma)$ is strictly increasing in $\gamma$ is made in this section only to establish the uniqueness of the solution. Such an assumption is not essentially needed for the existence of boundary layers that we shall discuss next.

## 4. Boundary Layers

In this section, we examine the variation of the shear stress and shear strain. In particular, we show that large stresses and strains are confined in a region adjacent to the inner surface. This leads to the quantitative definitions of boundary layers.

We first examine the stress distribution. Without loss of generality, we shall assume in the remainder of this paper that the constant $C$ in (13) is positive. Then, the shear stress $\tau$ is a decreasing function of the radius $R$. We further observe that the value of the stress is the greatest at the inner surface. It is anticipated that there is an annular layer adjacent to the inner surface, within which large stress variation takes place, and outside which the value of the stress is close to the value of the stress at the outer surface.

We thus introduce the following definition of stress boundary layer.

DEFINITION 1. For a given small number $\varepsilon$, the stress boundary layer is the annular region

$$\Omega_s = \left\{(R, \Theta, Z)\colon\ R_\mathrm{i} \leqslant R \leqslant R_s,\ 0 \leqslant \Theta < 2\pi,\ -\infty < Z < \infty\right\},$$

such that

$$\tau(R) < \tau_\mathrm{o} + \varepsilon(\tau_\mathrm{i} - \tau_\mathrm{o}) \quad \text{if } R > R_s \tag{17}$$

and

$$\tau(R) > \tau_\mathrm{o} + \varepsilon(\tau_\mathrm{i} - \tau_\mathrm{o}) \quad \text{if } R < R_s, \tag{18}$$

where $\tau_\mathrm{i}$ and $\tau_\mathrm{o}$ are the shear stresses at the inner and outer surfaces, respectively.

In the above definition, $R_s$ is the outer radius of the stress boundary layer, and $\varepsilon$ is the ratio of the stress variation outside the boundary layer and the total stress variation. For the concept of boundary layer to be useful, we shall take $\varepsilon$ to be a small number, for example, 0.01. A remark here is that in classical Newtonian fluid mechanics, boundary layers often result from the singular perturbation of a small parameter that is included in the original formulation of the problem. Here, we are not confined by this viewpoint, as we believe in solids (and in fluids as well) boundary layers could stem from material nonlinearity. In particular, the parameter $\varepsilon$ introduced above is not to be compared with the small parameter appearing in singular perturbation in classical fluid mechanics. Instead, $\varepsilon$ is a measure of the "smallness" of the stress (or the strain below) outside the boundary layer as opposed to that inside. It is used to capture the basic characteristics of a boundary layer.

We denote by $k$ the ratio of the outer and inner radii of the annulus, and by $h$ the relative thickness of the stress boundary layer with respect to the total thickness of the annulus:

$$k \equiv \frac{R_\mathrm{o}}{R_\mathrm{i}}, \qquad h \equiv \frac{R_s - R_\mathrm{i}}{R_\mathrm{o} - R_\mathrm{i}}.$$

By (13), (17) and (18), we have

$$\frac{C}{R_s^2} = \frac{C}{R_\mathrm{o}^2} + \varepsilon\left(\frac{C}{R_\mathrm{i}^2} - \frac{C}{R_\mathrm{o}^2}\right), \tag{19}$$

which leads to

$$h = \frac{1}{k-1}\left[\frac{k}{\sqrt{1+\varepsilon(k^2-1)}} - 1\right]. \tag{20}$$

When $\varepsilon = 0.01$ and $k = 100$, we find from (20) that

$$h \cong 0.0904,$$

that is, 99% of stress variation occurs within the boundary layer of about 9% relative thickness. Outside the boundary layer, the stress varies by only 1%.

We also note from (20) that the relative thickness $h$ of the boundary layer tends to zero as $k$ approaches $\infty$. In this case, the absolute thickness of the boundary layer is given by

$$\lim_{k\to\infty}(R_s - R_\mathrm{i}) = \lim_{k\to\infty} R_\mathrm{i} h(k-1) = R_\mathrm{i}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right).$$

Since, by (13), the stress at infinity is zero, the value of the shear stress outside the boundary layer is smaller than $\varepsilon$ times the shear stress at the inner surface.

Here, we must emphasize that we do not claim that the boundary layer as defined above is always "thin". It is obvious that the relative thickness of the boundary layer can be large for smaller values of $k$. For general boundary value problems, the existence and thickness of a boundary layer will of course depend on the geometry, boundary conditions, or the constitutive function. One of the purposes of this paper is to promote the idea that the phenomenon of boundary layer does exist for solids of certain geometry. In fact, this statement also pertains to fluids, as many studies in fluid dynamics concern the boundary layers appearing in "external domains" that extend to infinity.

We now turn to examine the variation of the shear strain $\gamma$. Evaluating (13) at $R = R_\mathrm{i}$ gives

$$\hat{\tau}(\gamma_\mathrm{i}) = \frac{C}{R_\mathrm{i}^2}, \tag{21}$$

where $\gamma_\mathrm{i}$ is the shear strain at the inner surface:

$$\gamma_\mathrm{i} \equiv R_\mathrm{i} f'(R_\mathrm{i}).$$

Since the constant $C$ is assumed to be positive, the shear stress and the shear strain are both positive. Eliminating $C$ between (13) and (21), we have

$$\frac{\hat{\tau}(\gamma_\mathrm{i})}{\hat{\tau}(\gamma)} = \frac{R^2}{R_\mathrm{i}^2}. \tag{22}$$

Equation (22) implicitly describes how the shear strain $\gamma$ depends on the radius $R$. Since $\hat{\tau}(\gamma)$ is assumed to be strictly increasing, the strain $\gamma$ decreases in $R$. Similar to the definition of the stress boundary layer, we now define the strain boundary layer.

DEFINITION 2. For a given small number $\varepsilon$, the strain boundary layer is the annular region

$$\Omega_n = \left\{(R, \Theta, Z)\colon\ R_\mathrm{i} \leqslant R \leqslant R_n,\ 0 \leqslant \Theta < 2\pi,\ -\infty < Z < \infty\right\},$$

such that

$$\gamma(R) < \gamma_\mathrm{o} + \varepsilon(\gamma_\mathrm{i} - \gamma_\mathrm{o}) \quad \text{if } R > R_n \tag{23}$$

and

$$\gamma(R) > \gamma_\mathrm{o} + \varepsilon(\gamma_\mathrm{i} - \gamma_\mathrm{o}) \quad \text{if } R < R_n, \tag{24}$$

where $\gamma_\mathrm{o}$ is the shear strain at the outer surface

$$\gamma_\mathrm{o} \equiv R_\mathrm{o} f'(R_\mathrm{o}).$$

In the above definition, $R_n$ is the outer radius of the strain boundary layer, and $\varepsilon$ is now the ratio of the strain variation outside the boundary layer and the total strain variation. Of course, for a given $\varepsilon$, Definitions 1 and 2 lead to the stress and strain boundary layers of different thicknesses in general. The following proposition facilitates comparison of the two thicknesses.

PROPOSITION 2. *If the stress function $\hat{\tau}$ is of $C^2$ and satisfies, for $\gamma \geqslant 0$,*

$$\hat{\tau}''(\gamma) \leqslant 0, \tag{25}$$

*then*

$$R_n \leqslant R_s. \tag{26}$$

*On the other hand, if*

$$\hat{\tau}''(\gamma) \geqslant 0,$$

*then*

$$R_n \geqslant R_s.$$

*Proof.* We shall show that (25) implies (26). The proof of the remaining proposition is similar. By (25) and the mean value theorem, we have

$$\hat{\tau}(\gamma_\mathrm{i}) \leqslant \hat{\tau}(\gamma_n) + (\gamma_\mathrm{i} - \gamma_n)\hat{\tau}'(\gamma_n),$$

and

$$\hat{\tau}(\gamma_\mathrm{o}) \leqslant \hat{\tau}(\gamma_n) + (\gamma_\mathrm{o} - \gamma_n)\hat{\tau}'(\gamma_n),$$

which imply

$$(\gamma_\mathrm{o} - \gamma_n)\big[\hat{\tau}(\gamma_\mathrm{i}) - \hat{\tau}(\gamma_n)\big] \geqslant (\gamma_\mathrm{i} - \gamma_n)\big[\hat{\tau}(\gamma_\mathrm{o}) - \hat{\tau}(\gamma_n)\big], \tag{27}$$

where $\gamma_n \equiv R_n f'(R_n)$. By (13), (23) and (24), inequality (27) can be written as

$$-\varepsilon\left(\frac{C}{R_\mathrm{i}^2} - \frac{C}{R_n^2}\right) \geqslant (1-\varepsilon)\left(\frac{C}{R_\mathrm{o}^2} - \frac{C}{R_n^2}\right). \tag{28}$$

The desired result then follows from (19) and (28). □

We shall refer to $\hat{\tau}'(\gamma)$ as the shear stiffness. Inequality (25) states that the shear stiffness is a non-increasing function of $\gamma$. This includes materials with a linear stress function, and materials that soften. Proposition 2 shows that for such materials, the strain boundary layer is not thicker than the stress boundary layer. In particular, the previous quantitative discussion of the stress distribution can be made appropriate for the strain distribution.

We now consider a special class of materials for which the stress function is bounded. For such materials, we define

$$\tau_m \equiv \sup_{\gamma \in \Re} \hat{\tau}(\gamma).$$

Equation (22) immediately leads to the following proposition.

PROPOSITION 3. *If the stress function $\hat{\tau}$ is bounded, then*

$$\gamma \leqslant \hat{\gamma}\left(\frac{R_\mathrm{i}^2 \tau_m}{R^2}\right). \tag{29}$$

Inequality (29) implies that the shear strain in the region outside an annulus is bounded by a number that is independent of the maximum shear strain $\gamma_\mathrm{i}$. For example, exterior to a thin annular layer of thickness $0.01R_\mathrm{i}$, the shear strain is bounded by $\hat{\gamma}(\tau_m/1.01^2)$ for arbitrarily large shear strain $\gamma_\mathrm{i}$ and shear displacement $\phi$ at the inner surface. The strain variation in this thin layer can be very large.

It is worth noting that the class of materials considered in Proposition 3 includes as a special case the classical perfectly elastic-plastic materials, with $\tau_m$ being the yield stress [10]. Indeed, when an elastic-plastic body undergoes quasi-static deformations without unloading, it behaves like a nonlinear elastic body and can be analyzed within the framework of this paper. It is well-known in classical plasticity theory that for certain geometry and boundary conditions, localized plastic zones may develop where large plastic deformations occur, while to the exterior of the plastic zones the deformation is small and the body remains elastic. A typical problem of this kind is the so-called plastic hinge [10, 11] in bending of elastic-plastic beams. When a plastic hinge develops, the beam ceases to have resistance to further increase of bending moment, and could rotate indefinitely about the plastic hinge.

As is clear from (20), the thickness of the stress boundary layer depends on $\varepsilon$ and the geometry of the annulus, but not on the stress or strain. The thickness of the strain boundary layer, on the other hand, will in general depend on the strain level. In particular, as will be shown in the following proposition, if the stress function is bounded, the thickness of the strain boundary layer tends to zero as the strain $\gamma_\mathrm{i}$ or the displacement $\phi$ at the inner surface approaches infinity.

PROPOSITION 4. *If the stress function $\hat{\tau}$ is bounded, then*

$$\lim_{|\phi|\to\infty} (R_n - R_\mathrm{i}) = 0.$$

*Proof.* We consider the case when $\phi \to -\infty$. The proof for the case when $\phi \to \infty$ is similar. By (2) and (15), we have

$$\int_{R_\mathrm{i}}^{R_\mathrm{o}} \frac{1}{\rho}\hat{\gamma}\left(\frac{C}{\rho^2}\right)\mathrm{d}\rho = -\phi. \tag{30}$$

Since $\hat{\gamma}$ in increasing, equation (30) implies

$$\frac{R_\mathrm{o} - R_\mathrm{i}}{R_\mathrm{i}}\hat{\gamma}\left(\frac{C}{R_\mathrm{i}^2}\right) > -\phi,$$

that is,

$$\gamma_\mathrm{i} > -\frac{\phi R_\mathrm{i}}{R_\mathrm{o} - R_\mathrm{i}}. \tag{31}$$

It then follows from (23) and (24) that

$$\gamma_n = \gamma_\mathrm{o} + \varepsilon(\gamma_\mathrm{i} - \gamma_\mathrm{o}) > \varepsilon\gamma_\mathrm{i} > -\frac{\varepsilon\phi R_\mathrm{i}}{R_\mathrm{o} - R_\mathrm{i}}.$$

Since the stress function $\hat{\tau}$ is strictly increasing, we must have

$$\lim_{\phi\to-\infty} \frac{C}{R_n^2} = \lim_{\phi\to-\infty} \hat{\tau}(\gamma_n) = \tau_m.$$

The same argument with (31) leads to

$$\lim_{\phi\to-\infty} \frac{C}{R_\mathrm{i}^2} = \tau_m.$$

The conclusion then follows. □

## 5. An Approximate Solution

Inspired by the boundary layer theory in fluid mechanics, Rajagopal [7] has proposed a scheme of finding approximate solutions of equilibrium in elastic solids. It exploits the fact that the strain in the region outside a boundary layer may be so small that the solutions in linear elasticity becomes a good approximation. Thus, instead of solving a full nonlinear problem in the entire domain, one only need solve the nonlinear problem in a perhaps small subdomain. This method can present considerable advantages when finding the solution numerically.

For the problem considered in the present paper, this scheme can be formulated as follows. Let $R_b$ be the outer radius of the boundary layer. The nonlinear and linearized equilibrium equations are to be solved in $[R_\mathrm{i}, R_b]$ and $[R_b, R_\mathrm{o}]$, respectively. The solutions in the two regions will be matched so that the shear displacement and the shear stress are continuous across $R_b$. The value of $R_b$ can be chosen a priori by using the criteria discussed in the previous section, or can be determined for the particular problem under consideration so that the error of the approximate solution is sufficiently small.

For $R \in [R_\mathrm{i}, R_b)$, the approximate solution is again determined by (14), with $C$ being replaced by a new constant $C_1$. Integration of equation (14) leads to

$$f^*(R) = \phi + \int_{R_\mathrm{i}}^{R} \frac{1}{\rho}\hat{\gamma}\left(\frac{C_1}{\rho^2}\right)\mathrm{d}\rho, \tag{32}$$

where the constant $C_1$ is to be determined by the continuity conditions at $R = R_b$. Here and henceforth, $*$ denotes the quantities associated with the approximate solution. The approximate solution (32) satisfies the boundary condition $(2)_1$. The corresponding shear stress is

$$\tau^*(R) = \frac{C_1}{R^2}. \tag{33}$$

For $R \in (R_b, R_o]$, the approximate solution is determined by solving the equilibrium equations for a linear stress function obtained from the original stress function. Taking $\tau = \mu\gamma$ in (13) with the new integration constant $C_1$, we find that

$$\gamma^*(R) = \frac{C_1}{\mu R^2},$$

where

$$\mu \equiv \hat{\tau}'(0), \tag{34}$$

and the constant of integration $C_1$ has been so chosen that the stress continuity condition across the interface is satisfied. In fact, the shear stress in this region is again given by (33). The corresponding shear displacement that satisfies the boundary condition $(2)_2$ is

$$f^*(R) = \frac{C_1}{2\mu}\left(\frac{1}{R_o^2} - \frac{1}{R^2}\right). \tag{35}$$

The continuity of shear displacement at $R = R_b$ requires

$$\int_{R_i}^{R_b} \frac{1}{\rho}\hat{\gamma}\left(\frac{C_1}{\rho^2}\right)\mathrm{d}\rho - \int_{R_i}^{R_o} \frac{1}{\rho}\hat{\gamma}\left(\frac{C}{\rho^2}\right)\mathrm{d}\rho = \frac{C_1}{2\mu}\left(\frac{1}{R_o^2} - \frac{1}{R_b^2}\right). \tag{36}$$

Here use has been made of $(2)_1$, (15), (32), and (35). Equation (36) implicitly determines the constant $C_1$. It is observed that when $R_b = R_o$, the solution of (36) is $C_1 = C$, corresponding to the trivial case when the nonlinear equilibrium equation is solved in the entire domain, and the approximate solution becomes exact.

We wish to estimate the error of the approximate solution. For brevity, we shall only present the error analysis for the stress. The error analysis for the strain and the displacement is similar, but more lengthy. To this end, we rewrite (36) as

$$\begin{aligned}\int_{R_i}^{R_b} \frac{1}{\rho}\hat{\gamma}\left(\frac{C_1}{\rho^2}\right)\mathrm{d}\rho - \int_{R_i}^{R_b} \frac{1}{\rho}\hat{\gamma}\left(\frac{C}{\rho^2}\right)\mathrm{d}\rho + \frac{C_1 - C}{2\mu}\left(\frac{1}{R_b^2} - \frac{1}{R_o^2}\right)\\ = \int_{R_b}^{R_o} \frac{1}{\rho}\hat{\gamma}\left(\frac{C}{\rho^2}\right)\mathrm{d}\rho - \frac{C}{2\mu}\left(\frac{1}{R_b^2} - \frac{1}{R_o^2}\right).\end{aligned} \tag{37}$$

Applying the mean value theorem to the left-hand side of (37) yields

$$\begin{aligned}(C_1 - C)\left\{\frac{1}{2C_2}\left[\hat{\gamma}\left(\frac{C_2}{R_i^2}\right) - \hat{\gamma}\left(\frac{C_2}{R_b^2}\right)\right] + \frac{1}{2\mu}\left(\frac{1}{R_b^2} - \frac{1}{R_o^2}\right)\right\}\\ = \int_{R_b}^{R_o} \frac{1}{\rho}\left[\hat{\gamma}\left(\frac{C}{\rho^2}\right) - \frac{C}{\mu\rho^2}\right]\mathrm{d}\rho,\end{aligned} \tag{38}$$

where $C_2$ is between $C$ and $C_1$. In the remainder of this paper, we shall assume that (25) holds for $\gamma \geqslant 0$. Then applying the mean-value theorem to (38), and casting bounds for appropriate terms, we find that

$$\begin{aligned} C_1 - C &= \frac{2(R_o - R_b)[\hat{\gamma}(C/R_1^2) - C/(\mu R_1^2)]}{R_1[(1/R_i^2 - 1/R_b^2)\hat{\gamma}'(\tau_2) + (1/\mu)(1/R_b^2 - 1/R_o^2)]} \\ &\leqslant \frac{2CR_i^2R_o^2(R_o - R_b)[\mu\hat{\gamma}'(\tau_1) - 1]}{R_1^3(R_o^2 - R_i^2)} \\ &\leqslant \frac{2CR_i^2R_o^2(R_o - R_b)[\mu\hat{\gamma}'(\tau_1) - 1]}{R_b^3(R_o^2 - R_i^2)}, \end{aligned} \tag{39}$$

where $R_1$, $\tau_1$ and $\tau_2$ are some intermediate values, for example, $\tau_1$ is between 0 and $C/R_b^2$.

Let $\delta$ denote the relative error of the approximate shear stress. By (13), (33), and (39), we have

$$\delta = \left|\frac{\tau^* - \tau}{\tau}\right| = \frac{C_1 - C}{C} \leqslant \frac{2R_i^2R_o^2(R_o - R_b)[\mu\hat{\gamma}'(\tau_1) - 1]}{R_b^3(R_o^2 - R_i^2)}. \tag{40}$$

It is observed from (40) that the error decreases as $R_b$ increases. In particular, $\delta$ tends to zero as $R_b$ approaches $R_o$, in which case the nonlinear equilibrium equation is solved in the entire domain, and the approximate solution becomes exact. Also note that the error becomes zero if the stress function is linear for which $\hat{\gamma}' = 1/\mu$.

## 6. An Example

In this section, we consider a power-law neo-Hookean hyperelastic material for which the strain-energy function is given by

$$W = \frac{\mu}{2}\left[\left(1 + \frac{I_1 - 3}{n}\right)^n - 1\right], \tag{41}$$

where $\mu$ and $n$ are material constants. The notation has been so chosen that (34) continues to hold. We shall assume that $0.5 \leqslant n \leqslant 1$ so that the condition (25) is satisfied. In studying the shear field near a crack tip for an anti-plane problem, Knowles [9] found that the equilibrium equations of plane-strain lose ellipticity when $n < 0.5$. When $n = 1$, the model reduces to the classical neo-Hookean model.

Substituting (5) and (41) into (11) gives

$$\begin{aligned} \hat{\tau}(\gamma) &= \mu\gamma\left(1 + \tfrac{\gamma^2}{n}\right)^{n-1}, \\ \hat{\tau}'(\gamma) &= \mu\left(1 + 2\gamma^2 - \tfrac{\gamma^2}{n}\right)\left(1 + \tfrac{\gamma^2}{n}\right)^{n-2}. \end{aligned} \tag{42}$$

By (40), (42) and the relation $\hat{\gamma}' = 1/\hat{\tau}'$, we arrive at

$$\delta \leqslant \frac{2R_{\rm i}^2 R_{\rm o}^2 (R_{\rm o} - R_b)[(1 + 2\gamma_1^2 - \gamma_1^2/n)^{-1}(1 + \gamma_1^2/n)^{2-n} - 1]}{R_b^3 (R_{\rm o}^2 - R_{\rm i}^2)}, \tag{43}$$

where $\gamma_1$ is between 0 and $\gamma(R_b)$.

As an illustrative example, we take $n = 0.75$ and $R_{\rm o} = 2R_{\rm i}$ in (43). The relative error is less than 29% if $R_b = 1.5R_{\rm i}$ and the shear strain $\gamma$ is less than 1 everywhere. Note that the error decreases rapidly in the strain. If the strain is less than 0.2, the error is less than 1.6%. For small strains, one can choose a thinner layer in which the nonlinear problem is solved. For example, if the strain is less than 0.1, then the error would be less than 2.7% even when $R_b = R_{\rm i}$, i.e., when the approximate solution in the entire domain is obtained from linear elasticity.

## References

1. H. Schlichting, *Boundary Layer Theory*. Mcgraw-Hill, Now York (1968).
2. J. Zhang and K.R. Rajagopal, Some inhomogeneous motions and deformations within the context of nonlinear elastic solids. *Internat. J. Engrg. Sci.* **30** (1992) 919–938.
3. K.R. Rajagopal and L. Tao, On an inhomogeneous deformation of a generalized neo-Hookean material. *J. Elasticity* **28** (1992) 165–184.
4. K.R. Rajagopal, Boundary layer in finite thermoelasticity. *J. Elasticity* **36** (1995) 271–301.
5. L. Tao, K.R. Rajagopal and A.S. Wineman, Circular shearing and torsion of generalized neo-Hookean materials. *IMA J. Appl. Math.* **48** (1992) 23–37.
6. D. Haughton, Boundary layer solutions for incompressible elastic cylinders. *Internat. J. Engrg. Sci.* **30** (1992) 1027–1039.
7. K.R. Rajagopal, Deformation of nonlinear elastic solids in unbounded domains. *Math. Mech. Solids* **1** (1996) 463–472.
8. C. Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics*. Springer, Berlin (1992).
9. J.K. Knowles, The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids. *Internat. J. Fracture* **13** (1997) 611–639.
10. J.M. Gere and S.P. Timoshenko, *Mechanics of Materials*, 3rd edn. PWS-KENT Publishing (1990).
11. J. Lubliner, *Plasticity Theory*. Macmillan, New York (1990).