Boundary layers for cellular flows at high Péclet numbers

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Abstract

We analyze the behavior of solutions of steady advection-diffusion problems in bounded domains with prescribed Dirichlet data when the Péclet number Pe $\gg 1$ is large. We show that the solution converges to a constant in each flow cell outside a boundary layer of width $O(\varepsilon^{1/2})$, $\varepsilon = \mathrm{Pe}^{-1}$ around the flow separatrices. We construct an ε-dependent approximate "water-pipe problem" purely inside the boundary layer that provides a good approximation of the solution of the full problem but has ε-independent computational cost. We also define an asymptotic problem on the graph of streamline separatirces, and show that solution of the water-pipe problem itself may be approximated by an asymptotic, ε-independent problem on this graph. Finally, we show that the Dirichlet-to-Neumann map of the water-pipe problem approximates the Dirichlet-to-Neumann map of the separatrix problem with an error independent of the flow outside the boundary layers.

Contents

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1 Introduction

1.1 The advection-diffusion problem

We consider the steady advection-diffusion problem

$$\varepsilon\Delta\phi^\varepsilon - u\cdot\nabla\phi^\varepsilon = 0 \tag{1.1}$$

in a simply connected bounded domain $\Omega\subset\mathbb{R}^2$. The flow u is incompressible: $\nabla\cdot u = 0$, and non-penetrating through the boundary of Ω: $u\cdot n = 0$ at $\partial\Omega$ (see Figure 1.1). The small parameter $\varepsilon = \mathrm{Pe}^{-1} \ll 1$ is the inverse of the Péclet number, where $\mathrm{Pe} = fracLU\nu$ with L a legth scale typical of the size of the domain, U a velocity scale characteristic of u and ν the diffusivity. Equation (1.1) is supplemented with Dirichlet boundary data:

$$\phi^\varepsilon(\mathbf{x}) = T_0(\mathbf{x}),\ \mathbf{x}\in\partial\Omega. \tag{1.2}$$

The problem of the qualitative behavior of solutions of (1.1)-(1.2) has been studied in various areas where passive scalar advection arises, such as oceanography, meteorology and elsewhere. One of the most interesting effects is the non-trivial coupling of diffusion and strong advection at high Péclet number. Numerical and physical evidence [4, 16, 17, 18] suggests the following qualitative structure of the solution ϕ^ε inside each flow cell. There exists a boundary layer of the width $O(\sqrt{\varepsilon})$ along the separatrices between different flow cells $\mathcal{C}_j$. Outside this layer the solution is approximately equal to a constant K_j in each cell $\mathcal{C}_j$ (see Figure 1.2 for a sample numerical solution). The total dissipation rate is small asymptotically, with scaling

$$\varepsilon\int_\Omega |\nabla\phi^\varepsilon(\mathbf{x})|^2 d\mathbf{x} \sim O(\varepsilon^{1/2}).$$

The large Péclet number advection-diffusion problem arises also for the effective diffusivity in the periodic homogenization of cellular flows. The effective diffusivity in homogenization of cellular flows is given by [2, 5]

$$D_{ij}^\varepsilon = \varepsilon\int_\Omega \nabla\chi_i^\varepsilon(\mathbf{x})\cdot\nabla\chi_j^\varepsilon(\mathbf{x})d\mathbf{x}$$

where $\chi^\varepsilon(\mathbf{x})$ is the mean-zero periodic solution of the cell problem

$$\varepsilon\Delta\chi_j^\varepsilon + u\cdot\nabla\chi_j^\varepsilon = u_j.$$

This cell problem may be reduced to (1.1) with appropriate boundary conditions by representing $\chi_j = x_j + \phi_j$. Most of the mathematical studies [5, 6, 10, 11] of the advection-diffusion problem have been devoted to the problem of bounds on the effective diffusivity as $\varepsilon- > 0$. In the presence of additional symmetries the tensor $D_{ij}^\varepsilon = D^\varepsilon\delta_{ij}$. Using boundary layer analysis, Childress [4], and later [16, 17, 18], showed that the effective diffusivity scales as

$$D^\varepsilon \sim D^*\sqrt{\varepsilon}, \tag{1.3}$$

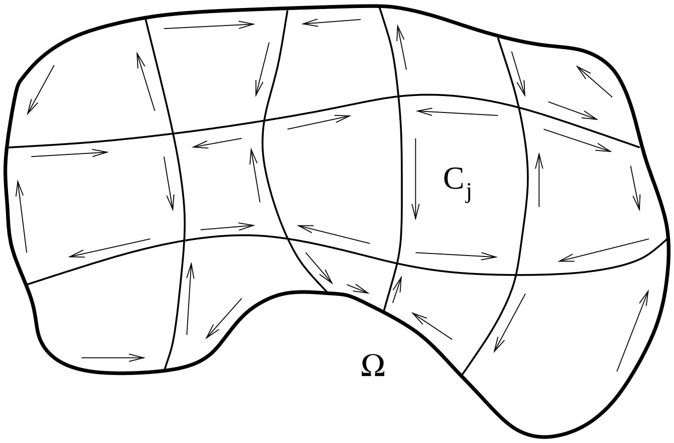

FIGURE 1.1. The domain Ω is partitioned by flow separatrices into cells $\mathcal{C}_j$.

in the special case of symmetric square cells. This asymptotic behavior was also obtained in [5, 6] using optimal bounds from variational principles. It has been recently extended to general non-square periodic cells in [11] using probabilistic techniques that have their origin in [8]. Furthermore, uniform bounds of the type

$$C_1\sqrt{\varepsilon} \le D^{\varepsilon} \le C_2\sqrt{\varepsilon}, \tag{1.4}$$

for the effective diffusivity in the periodic case have been proved in [10], generalizing the asymptotic result of [5] to finite $\varepsilon > 0$.

We recall that the case when the flow has no separatrices has been considered previously in [7, 8] including the effect of flows on the reaction-diffusion equations. The general problem (1.1)-(1.2) has been recenty analyzed in [1] in the context of the possibility of passive scalar energy cascade in a turbulent flow. In particular, the upper bound in (1.4) has been shown to be hold.

1.2 Outline of the paper

The purpose of this paper is to consider the general problem (1.1) with a large but finite Péclet number and to establish rigorously and quantitatively the above mentioned properties of the solution of the advection-diffusion problem for a small but finite $\varepsilon < 1$, without any assumptions of periodicity or symmetry for the flow. The main objective is to show that the oscillation of the solution inside each cell at distances larger than $O(\sqrt{\varepsilon})$ from the separatrices is small, and that close to separatrices the solution may be approximated by an asymptotic problem on the graph of the separatrices.

We first prove the upper bound on the dissipation rate, as in the second bound in (1.4), in the general case. The proof uses a slight modification

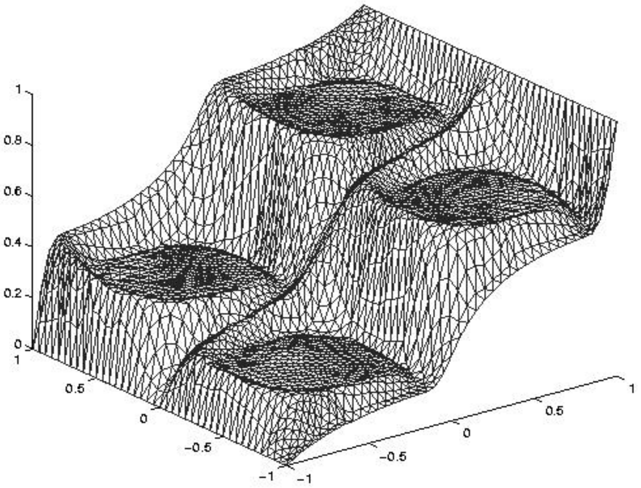

FIGURE 1.2. The temperature distribution for periodic cellular flows computed in MATLAB. $u = \nabla^{\perp}H$, $H = \sin(\pi x)\sin(\pi y)$; four cells, $\mathrm{Pe} = 20$.

of the technique of [10]. Next, we establish convergence of the solution to a constant K_j^{ε} inside each cell $\mathcal{C}_j$ at a distance $N\sqrt{\varepsilon}$ from the separatrices and obtain bounds on the rate of convergence as $N \to \infty$. The proof of this fact uses some of the ideas of integration and averaging along streamlines developed in [12] for getting bounds on the speed of a reaction-diffusion front in a cellular flow. The fact that solution is nearly constant at a distance $O(\sqrt{\varepsilon})$ away from the boundary, when the boundary data is non-constant, implies the lower bound on the dissipation rate in (1.4).

These results do not, however, allow us to conclude that the interior constants in each cell have a limit as $\varepsilon \to 0$. This requires an additional asymptotic analysis in the boundary layers. As a first step in this direction, we show that the full problem (1.1) may be restricted to an ε-dependent "water-pipe" problem inside a boundary layer of width $N\sqrt{\varepsilon}$ around the separatrices with an error decreasing as $N \to \infty$. The "water-pipe" problem has a computational cost independent of $\varepsilon \ll 1$ and provides an effective numerical tool to solve the problem at a high Péclet number.

The solution of the water-pipe problem itself is then shown to be well approximated by yet another asymptotic ε-independent problem. The latter is a many-cell generalization of a single cell problem introduced by Childress in [4], in the periodic case, and is closely related to the limit Markov chain constructed in the periodic case in [11]. In particular this allows us to show that the interior constants K_j^{ε} have a limit as $\varepsilon \to 0$ and we identify this limit in terms of the solution of the extended Childress problem. It also allows us to show that for any given boundary data $T_0(\mathbf{x})$ that is different from a constant, and any cellular flow u, there exists a positive finite limit of the scaled dissipation rate

$$\sqrt{\varepsilon}\int_\Omega |\nabla\phi^\varepsilon(\mathbf{x})|^2 d\mathbf{x} \to D^* > 0,$$

as $\varepsilon \to 0$. Finally, by means of variational principles similar to those in [3, 5, 6, 13, 14] we show that for any ε the dissipation rate can be determined from the solution of the water-pipe problem with an error independent of the flow away from the separatrices.

We note that all our results are directly applicable if homogeneous Neumann boundary conditions are prescribed on a part of the boundary, while non-uniform Dirichlet boundary conditions are prescribed on the rest of $\partial\Omega$. The generalization to that case is straightforward.

1.3 The main results

We recall that the flow u is assumed to be incompressible, thus a stream function $H(x,y)$ exists so that $u = \nabla^\perp H = (H_y, -H_x)$. Furthermore, since we assume that the normal component of u at the boundary $\partial\Omega$ vanishes, $\partial\Omega$ has to be contained in a level set of H: $\partial\Omega \subseteq \{H = H_0\}$. Therefore, either Ω is bounded by a closed streamline of the flow u or by a collection of separatrices of u that connect a finite number of singular points of H lying on the level set $\{H = H_0\}$. The latter case is of most interest to us. We will assume without loss of generality that the critical value $H_0 = 0$ and that all the critical points of H are non-degenerate. Then, the set Ω is a union of a finite number of flow cells $\mathcal{C}_j$ bounded by separatrices of u, as in Figure 1.1. We will also assume throughout the paper that the boundary data

$$T_0(\mathbf{x}) \neq \text{const},\ \mathbf{x} \in \partial\Omega$$

is sufficiently smooth but is not constant, to avoid the trivial case of the dentically constant solution. The streamlines of the flow (level sets of the stream function) are assumed to be sufficiently regular.

The main results of this paper address three issues: bounds on the dissipation rate, convergence to constants inside each cell, and approximation of the solution by an asymptotic problem on the separatrix graph, and by the ε-dependent problem inside the "water-pipe system" of boundary layers. We summarize them below although not in the order they appear in the main body of the paper. The asymptotic problem on the graph is somewhat lengthy to describe so we do not present the main result in this section. This is done in Theorem 6.1 in Section 6. However, we do mention some of the corollaries of this result below.

Bounds for the dissipation rate

Our first result provides general bounds on the dissipation rate.

Theorem 1.1. *Let us assume that* $\partial\Omega$ *is a piecewise smooth curve and the boundary data* T_0 *in (1.2) is sufficiently smooth. Then there exists a constant* $C>0$ *so that*

$$\frac{1}{C\sqrt{\varepsilon}} \le \int_\Omega |\nabla\phi(\mathbf{x})|^2 d\mathbf{x} \le \frac{C}{\sqrt{\varepsilon}}. \tag{1.5}$$

Moreover, for any given boundary data $T_0(\mathbf{x}) \neq const$ *and flow* u *there exists a positive finite limit*

$$\lim_{\varepsilon\to 0} D^\varepsilon(T_0)/\sqrt{\varepsilon} = D^*(T_0) > 0, \tag{1.6}$$

where the dissipation rate $D^\varepsilon(T_0) = \varepsilon \int_\Omega |\nabla\phi^\varepsilon(\mathbf{x})|^2 d\mathbf{x}$.

Here and below we denote by C various constants $C = C(u, T_0, \Omega)$ that may depend on the geometry of the streamlines of u, various norms of the boundary data T_0 and the domain Ω but nothing else, unless explicitly specified. In particular they are independent of the Péclet number. The proofs of various statements in Theorem 1.1 are closely related to other results in the paper. The proof of the upper bound above is self-contained. It is proved after Theorem 2.1 in Section 2. The lower bound on the dissipation rate, on the other hand, is a corollary of the fact that solution converges to a constant inside each cell. Hence, the proof of the lower bound in (1.5) is presented only after the latter claim is proved, in Proposition 3.3 in Section 3. Finally, existence of the limit (1.6) is shown only after the asymptotic boundary layer theory is developed. This statement is proved in Theorem 7.1.

Convergence to a constant inside flow cells

Convergence of the solution to a constant inside is quantified as follows. Let $\mathcal{D}(h) = \{\mathbf{x} : \ |H(\mathbf{x})| \ge h\}$, $h > 0$ be a domain strictly inside the flow cells, at distance $O(h)$ away from the separatrices.

Theorem 1.2. *There exist constants* K_j^ε *so that inside each cell* $\mathcal{C}_j$

$$\sup_{\mathbf{x}\in\mathcal{D}(N\sqrt{\varepsilon})} \left|\phi^\varepsilon(\mathbf{x}) - K_j^\varepsilon\right| \le \frac{C}{N^{3/2}}. \tag{1.7}$$

Moreover, the constants K_j^ε *converge as* $\varepsilon \to 0$ *to certain constants* K^j.

The proof of the first part of this theorem is contained in Section 3 in Theorem 3.1. Convergence of K_j^ε to their limit values and identification of the limit follow from the approximation of ϕ^ε by the solution of the Childress problem: see Theorem 6.1 in Section 6.

Approximation by the water-pipe problem

Before we introduce the asymptotic Childress problem we consider an intermediate approximation of the full advection-diffusion problem by an ε-dependent "water-pipe" problem in the system of boundary layers around the separatrices. The water-pipe problem consists of the advection-diffusion equation (1.1) in the narrow domain

$$\Omega_N^\varepsilon = \Omega \backslash \mathcal{D}(N\sqrt{\varepsilon}) = \{\mathbf{x} \in \Omega : \ |H(\mathbf{x})| \leq N\sqrt{\varepsilon}\}$$

around the separatrices with the Dirichlet boundary conditions (1.2) on the outer boundary $\partial\Omega$ and Neumann boundary conditions on the level set $\mathcal{L}(N\sqrt{\varepsilon}) = \{\mathbf{x} \in \Omega : \ |H(\mathbf{x})| = N\sqrt{\varepsilon}\}$ This problem has a computational cost independent of ε. We show that its solution ϕ_N^ε is close to ϕ^ε. Denote by $\chi(s)$ a smooth even function, monotonic on $s \geq 0$, so that

$$\chi(s) = \begin{cases} 1, & |s| \leq 1/2, \\ 0, & |s| \geq 1. \end{cases}$$

The following result describes the L^∞-approximation of the solution of the full problem by the solution of the water-pipe problem.

Theorem 1.3. *Let ϕ^ε solve (1.1) and let ϕ_N^ε be the solution of the water-pipe problem. Then there exist constants $\tilde{K}_{m,N}^\varepsilon$ so that ϕ_N^ε satisfies*

$$|\phi_N^\varepsilon(\mathbf{x}) - \tilde{K}_{m,N}^\varepsilon| \leq \frac{C}{N^{3/2}}, \ \mathbf{x} \in \mathcal{L}_j(N\sqrt{\varepsilon}) = \mathcal{L}(N\sqrt{\varepsilon}) \cap \mathcal{C}_m. \tag{1.8}$$

Let $\tilde{\phi}_N^\varepsilon$ be an extension ϕ_N^ε to the whole domain Ω as

$$\tilde{\phi}_N^\varepsilon(\mathbf{x}) = \chi\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\phi_N^\varepsilon(\mathbf{x}) + \tilde{K}_{m,N}^\varepsilon\left(1 - \chi\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\right), \ \mathbf{x} \in \mathcal{C}_m.$$

with the constants $\tilde{K}_{m,N}^\varepsilon$ defined above. Then we have

$$\|\phi^\varepsilon - \tilde{\phi}_N^\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{N^{3/2}}, \ |K_m^\varepsilon - \tilde{K}_{m,N}^\varepsilon| \leq \frac{C}{N^{3/2}}, \tag{1.9}$$

and

$$\sqrt{\varepsilon}\|\nabla\phi^\varepsilon - \nabla\tilde{\phi}_N^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{C}{N^4}. \tag{1.10}$$

Moreover, the constants converge to finite limits:

$$\lim_{\varepsilon \to 0} \tilde{K}_{m,N}^\varepsilon = K_{m,N}, \ \lim_{N \to \infty} K_{m,N} = K_m \tag{1.11}$$

and

$$\lim_{\varepsilon \to 0} \tilde{K}_m^\varepsilon = K_m, \tag{1.12}$$

with K_m as in (1.11).

The proofs of the convergence of the water-pipe solution to a constant as in (1.8) and of the error bound (1.9) are contained in Section 4: see Theorem 4.2 and Proposition 4.1. Convergence of the constants $\tilde{K}^{\varepsilon}_{m,N}$, K^{ε}_{m} to the corresponding limits in (1.11) and (1.12) is shown in Theorem 6.1.

The next result describes the approximation of the dissipation rate by the solution of the water-pipe problem.

Theorem 1.4. *The dissipation rate of the solution of the water-pipe system has a limit*

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{\Omega^{\varepsilon}_{N}}|\nabla\phi^{\varepsilon}_{N}(\mathbf{x})|^2 d\mathbf{x}=D_N,\ \lim_{N\to\infty}D_N=D^* \tag{1.13}$$

with D^* as in (1.6). Moreover, the error

$$Error^{\varepsilon}_{N}=\sqrt{\varepsilon}\left|\int_{\Omega^{\varepsilon}_{N}}|\nabla\phi^{\varepsilon}_{N}(\mathbf{x})|^2 d\mathbf{x}-\int_{\Omega}|\nabla\phi^{\varepsilon}(\mathbf{x})|^2 d\mathbf{x}\right|\le K(N)$$

is bounded by a constant $K(N)$ that depends on the flow u in Ω^{ε}_{N} only and such that $K(N)\to 0$ as $N\to\infty$.

The main observation of the second statement in Theorem 1.4 is that the error made in the approximation of the total dissipation rate by the water-pipe system is independent of the flow outside the boundary layer. This theorem is proved in Section 7 in Theorems 7.1 and 7.2 using variational methods.

The asymptotic Childress problem

Our final set of results concern the approximation of the solution of (1.1)-(1.2) by the solution of the asymptotic Childress problem on the graph of separatrices. As the formulation of the latter is rather lengthy we postpone its detailed discussion and the precise statement of the corresponding result until later, in Sections 5 and 6. Here we just mention that the Childress probelm for one cell, as has been introduced in [4] and also in [5, 6], involves finding a periodic solution for heat flow along the separatrices with the coordinate along the streamlines playing the role of time. The multi-cell version we consider here is a coupled system of one-cell problems, which seeks an equilibrium of such coupled heat flows with a prescribed boundary data.

We recall that

$$D^{\varepsilon}(T_0)=\int_{\Omega}|\nabla\phi^{\varepsilon}|^2 d\mathbf{x}=\int_{\partial\Omega}\phi^{\varepsilon}\frac{\partial\phi^{\varepsilon}}{\partial n}dn=\int_{\partial\Omega}T_0\frac{\partial\phi^{\varepsilon}}{\partial n}dn$$

is nothing but the quadratic form of the Dirichlet-to-Neumann map

$$T_0\to\left.\frac{\partial\phi^{\varepsilon}}{\partial n}\right|_{\partial\Omega}$$

of the advection-diffusion problem (1.1). We show that the limiting constant D^* in Theorem 1.1 is, in fact, the quadratic form of the Dirichlet-to-Neumann map of the asymptotic Childress problem, evaluated on the boundary data T_0. Moreover, Theorem 6.1 shows that the asymptotic Childress solution approximates the full solution in the H^1-norm as well.

The paper is organized as follows. The upper bound on the dissipation rate is presented in Theorem 2.1 in Section 2. Section 3 contains the proof of the corresponding lower bound in Proposition 3.3. Convergence of the solution to a constant is proved first, in Theorem 3.1 in the same section. The "water-pipe" boundary layer problem is introduced in Section 4, where we also prove in Theorem 4.2 that the solution of this problem approximates the solution of the full problem. The asymptotic Childress problem is introduced and studied in Theorem 5.2 in Section 5. We show that the solution of the Childress problem approximates the solution of the water-pipe model in Theorem 6.1 in Section 6. We also show in this section that the values of the constants inside each flow cell for the full problem converge to those given by the asymptotic Childress problem. Finally, the variational principles for the total dissipation rate and estimates on the error in the effective diffusivity of the water-pipe model are obtained in Section 7.

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2 A uniform upper bound on the dissipation rate

We prove in this section the uniform upper bound on the total dissipation rate in the inequality (1.5) in Theorem 1.1. The proof is based on the energy methods and uses the techniques of [10]. As a corollary we show in proposition 2.2 that the boundary layer has to extend at least for a distance $O(\sqrt{\varepsilon})$ inside the cells.

Theorem 2.1. *Let us assume that* $\partial\Omega$ *is a piecewise smooth curve and* T_0 *is sufficiently smooth. Let*

$$M = \sup_{x\in\Omega}\sup_{\mathbf{v}\in S^1}\left(\frac{\partial u_n}{\partial x_m}v_n v_m\right),$$

then there exists a constant $C = C(M, T_0, \Omega)$ *so that*

$$\int_\Omega |\nabla\phi(\mathbf{x})|^2 d\mathbf{x} \le \frac{C}{\sqrt{\varepsilon}}. \tag{2.1}$$

Proof. We use a modification of the proof of an upper bound for the effective diffusivity in [10]. Let ψ^ε be a test function to be specified later. We multiply (1.1) by the function $q^\varepsilon = \phi^\varepsilon - \psi^\varepsilon$ and obtain after integration by parts:

$$\varepsilon \int_{\partial\Omega} q^\varepsilon \frac{\partial \phi^\varepsilon}{\partial n} dS - \varepsilon \int_\Omega (\nabla\phi^\varepsilon - \nabla\psi^\varepsilon)\cdot\nabla\phi^\varepsilon d\mathbf{x} - \int_\Omega (\phi^\varepsilon - \psi^\varepsilon) u\cdot\nabla\phi^\varepsilon d\mathbf{x} = 0.$$

Using incompressibility of the flow u we get

$$\begin{aligned}(2.2)\quad \varepsilon \int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x} &\leq \varepsilon \int_{\partial\Omega} q^\varepsilon \frac{\partial\phi^\varepsilon}{\partial n} dS + \varepsilon\int_\Omega |\nabla\psi^\varepsilon|^2 d\mathbf{x} + \frac{\varepsilon}{4}\int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x} \\ &+\alpha \int_\Omega |\psi^\varepsilon|^2 d\mathbf{x} + \frac{1}{\alpha}\int_\Omega |u\cdot\nabla\phi^\varepsilon|^2 d\mathbf{x}\end{aligned}$$

with the constant α to be chosen. We now multiply (1.1) by $u\cdot\nabla\phi^\varepsilon$ and integrate to get

$$\begin{aligned}\int_\Omega |u\cdot\nabla\phi^\varepsilon|^2 d\mathbf{x} &= \varepsilon\int_\Omega (u\cdot\nabla\phi^\varepsilon)\Delta\phi^\varepsilon d\mathbf{x} \\ &= \varepsilon\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n} dS - \varepsilon\int_\Omega \nabla(u\cdot\nabla\phi^\varepsilon)\cdot\nabla\phi^\varepsilon d\mathbf{x}.\end{aligned}$$

Once again using incompressibility of u and the definition of the constant M we obtain from the above

$$\begin{aligned}\int_\Omega |u\cdot\nabla\phi^\varepsilon|^2 d\mathbf{x} &= \varepsilon\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n} dS - \frac{1}{2}\varepsilon\int_\Omega (u\cdot\nabla\left(|\nabla\phi^\varepsilon|^2\right))d\mathbf{x} \\ (2.3)\quad -\varepsilon\int_\Omega \frac{\partial u_n}{\partial x_m}\frac{\partial\phi^\varepsilon}{\partial x_m}\frac{\partial\phi^\varepsilon}{\partial x_n} d\mathbf{x} &\leq \varepsilon\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n} dS + \varepsilon M\int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x}.\end{aligned}$$

We insert (2.3) into (2.2) to get

$$\begin{aligned}\varepsilon \int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x} &\leq \varepsilon \int_{\partial\Omega} q^\varepsilon \frac{\partial\phi^\varepsilon}{\partial n} dS + \varepsilon\int_\Omega |\nabla\psi^\varepsilon|^2 d\mathbf{x} + \frac{\varepsilon}{4}\int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x} \\ &+\alpha \int_\Omega |\psi^\varepsilon|^2 d\mathbf{x} + \frac{\varepsilon}{\alpha}\left(\int_{\partial\Omega}(u\cdot\nabla\phi^\varepsilon)\frac{\partial\phi^\varepsilon}{\partial n} dS + M\int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x}\right).\end{aligned}$$

With the choice $\alpha = 4M$ the above becomes

$$\begin{aligned}\frac{\varepsilon}{2}\int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x} &\leq \varepsilon\int_{\partial\Omega}\left[q^\varepsilon + \frac{1}{4M}(u\cdot\nabla\phi^\varepsilon)\right]\frac{\partial\phi^\varepsilon}{\partial n} dS \\ &+\varepsilon\int_\Omega |\nabla\psi^\varepsilon|^2 d\mathbf{x} + 4M\int_\Omega |\psi^\varepsilon|^2 d\mathbf{x}.\end{aligned}$$

It remains to require that $q^\varepsilon + \frac{1}{4M}(u\cdot\nabla\phi^\varepsilon) = 0$ on the boundary $\partial\Omega$. However, $\partial\Omega$ is a streamline of u so that $u\cdot\nabla\phi^\varepsilon = u\cdot\nabla T_0$ is a given function. That imposes a boundary condition on the function ψ^ε:

$$(2.4)\qquad \psi^\varepsilon\,|_{\partial\Omega}(\mathbf{x}) = T_0(\mathbf{x}) + \frac{1}{4M}(u\cdot\nabla T_0(\mathbf{x})).$$

Then, provided that (2.4) holds we obtain

$$\frac{\varepsilon}{2}\int_\Omega |\nabla\phi^\varepsilon|^2 d\mathbf{x} \le \varepsilon\int_\Omega |\nabla\psi^\varepsilon|^2 d\mathbf{x} + 4M\int_\Omega |\psi^\varepsilon|^2 d\mathbf{x}. \tag{2.5}$$

We may choose a function ψ^ε so that it satisfies the boundary conditions (2.4), vanishes identically at distances larger that $\sqrt{\varepsilon}$ away from $\partial\Omega$ and satisfies the uniform bounds $\|\psi^\varepsilon\|_{L^\infty(\Omega)} \le C$, $\|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)} \le C/\sqrt{\varepsilon}$. Using such a test function in (2.5) we obtain the upper bound (2.1). □

Theorem 2.1 implies that the boundary layer along the boundary $\partial\Omega$ has to extend to the distance at least of the order of $O(\sqrt{\varepsilon})$. This is made precise in Proposition 2.2: oscillations of ϕ^ε have to be present at such distances from the boundary – we will later see that this is actually the correct boundary layer scale.

In order to make this precise we let $\mathcal{C}_0$ be a flow cell adjacent to the boundary such that T_0 is not constant along $\mathbf{l}_0 = \partial\mathcal{C}_0 \cap \partial\Omega$. Such a cell exists as T_0 is continuous and non-constant on $\partial\Omega$. We let $\tilde{\mathbf{l}}_0$ be a part of $\mathbf{l}_0$ that is separated away from the end-points of $\mathbf{l}_0$ and such that $T_0(\mathbf{x})$ is not constant on $\tilde{\mathbf{l}}_0$. We may then introduce the following orthogonal system of coordinates in a neighborhood of $\tilde{\mathbf{l}}_0$. The coordinate $H(x,y)$ is "the label of the streamline". The coordinate θ orthogonal to H is normalized so that $|\nabla\theta| = |\nabla H|$ on $\tilde{\mathbf{l}}_0$ and $\tilde{\mathbf{l}}_0$ may be represented as

$$\tilde{\mathbf{l}}_0 = \{H = 0,\ \theta_1 \le \theta \le \theta_2\}. \tag{2.6}$$

We may consider a sufficiently small tubular neighborhood $U_0 = \{|H| \le H_0,\ \theta_1 \le \theta \le \theta_2\}$ of $\tilde{\mathbf{l}}_0$ so as to have $|\nabla H|, |\nabla\theta| \ge C > 0$.

Proposition 2.2. *Let $\mathcal{C}_0$ be a flow cell as above adjacent to the boundary and $\mathcal{L}_0(\gamma) = \{(x,y) \in \mathcal{C}_0 :\ H(x,y) = \gamma\sqrt{\varepsilon}\}$ be the level set of $H(x,y)$ inside the cell $\mathcal{C}_0$. Then there exists a constant $C > 0$ so that we have an inequality*

$$\int_{\theta_1}^{\theta_2} |\phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta) - \bar{\phi}^\varepsilon(\gamma\sqrt{\varepsilon})|^2 d\theta \ge \int_{\theta_1}^{\theta_2} (T_0(\theta) - \bar{T}_0)^2 d\theta - C\gamma \tag{2.7}$$

for all $\gamma < H_0/\sqrt{\varepsilon}$ and with $\theta_{1.2}$ as in(2.6). Here

$$\bar{\phi}^\varepsilon(\rho) = (\theta_2 - \theta_1)^{-1}\int_{\theta_1}^{\theta_2} \phi^\varepsilon(\rho,\theta)d\theta$$

is the average of ϕ^ε over the corresponding part of the streamline and

$$\bar{T}_0 = (\theta_2 - \theta_1)^{-1}\int_{\theta_1}^{\theta_2} T_0(\theta)d\theta$$

is the average of T_0 along $\tilde{\mathbf{l}}_0$.

Proof. We have a simple bound

$$|\phi^\varepsilon(0,\theta) - \phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 \le \gamma\sqrt{\varepsilon}\int_0^{\gamma\sqrt{\varepsilon}} \left|\frac{\partial\phi^\varepsilon}{\partial H}(H,\theta)\right|^2 dH.$$

Integrating the above in θ and using the boundary data for ϕ^ε we obtain

$$\int_{\theta_1}^{\theta_2} |T_0(\theta)-\phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 d\theta \le \gamma\sqrt{\varepsilon}\int_{\theta_1}^{\theta_2}\int_0^{\gamma\sqrt{\varepsilon}} \left|\frac{\partial\phi^\varepsilon}{\partial H}(H,\theta)\right|^2 dHd\theta. \tag{2.8}$$

The Jacobian

$$J = D(H,\theta)/D(x,y) \tag{2.9}$$

is uniformly bounded from above and below away from zero in U_0. Hence we may re-write the right side as an integral over $U_\gamma = \{|H| \le \gamma\sqrt{\varepsilon},\ \theta_1 \le \theta \le \theta_2\}$:

$$\int_{\theta_1}^{\theta_2}\int_0^{\gamma\sqrt{\varepsilon}} \left|\frac{\partial\phi^\varepsilon}{\partial H}(H,\theta)\right|^2 dHd\theta \le C\int_{U_\gamma} |\nabla\phi(\mathbf{x})|^2 d\mathbf{x}.$$

Using Theorem 2.1 and (2.8) we obtain

$$\int_{\theta_1}^{\theta_2} |T_0(\theta)-\phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 d\theta \le C\gamma\sqrt{\varepsilon}\int_{U_\gamma} |\nabla\phi(x,y)|^2 dxdy \le C\gamma. \tag{2.10}$$

Therefore we have for any constant $a\in\mathbb{R}$:

$$\int\limits_{\theta_1}^{\theta_2} |\phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)-a|^2 d\theta \ge \int\limits_{\theta_1}^{\theta_2} |T_0(\theta)-a|^2 d\theta - \int\limits_{\theta_1}^{\theta_2} |T_0(\theta)-\phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 d\theta$$

$$\ge \int_{\theta_1}^{\theta_2} |T_0(\theta)-\bar{T}_0|^2 d\theta - C\gamma$$

so that (2.7) follows. □

3 Convergence to constants inside the cells

3.1 The main results

In this section we obtain the lower bound of the inequality (1.5) in Theorem 1.1 and prove the first statement in Theorem 1.2: we show that solution of (1.1)-(1.2) is close to a constant (that may depend on the Péclet number) inside each cell of the flow when ε is small.

As before, we denote by $\mathcal{L}_j(\gamma) = \{(x,y:\ H(x,y)=\gamma\}$ the level set of $H(x,y)$ inside a cell $\mathcal{C}_j$. We will usually omit the subscript j to simplify the notation as long as we consider one cell and this does not cause any confusion. We denote by $\mathcal{D}_j(\gamma)$ the region bounded by $\mathcal{L}_j(\gamma)$ inside each cell and by $\mathcal{D}_j(\alpha,\beta) = \mathcal{D}_j(\beta)\backslash\mathcal{D}_j(\alpha)$ the annulus between two level sets. The main result of this Section is the following theorem.

Theorem 3.1. *There exist constants K_j^ε so that we have inside each cell $\mathcal{C}_j$*

$$\sup_{\mathbf{x}\in\mathcal{D}(N\sqrt{\varepsilon})} \left|\phi^\varepsilon(\mathbf{x})-K_j^\varepsilon\right| \le \frac{C}{N^{3/2}}. \tag{3.1}$$

This shows that the function ϕ^ε is close to a constant inside each cell $\mathcal{C}_j$. An integral measure of the same phenomenon is provided by the following theorem.

Theorem 3.2. *We have an upper bound*

$$\int_{\mathcal{D}(H)} |\nabla\phi^\varepsilon|^2 d\mathbf{x} \le \frac{C}{H}\left(\frac{\varepsilon}{H^2}\right)^{3/8} \tag{3.2}$$

for $H \ge \sqrt{\varepsilon}$.

Theorem 3.1 implies a lower bound on the L^2-norm of the gradient of solution in Theorem 1.1.

Theorem 3.3. *There exists a constant* $C = C(T_0, \Omega, u)$ *so that*

$$\int_\Omega |\nabla\phi^\varepsilon(\mathbf{x})|^2 d\mathbf{x} \ge \frac{C}{\sqrt{\varepsilon}}. \tag{3.3}$$

The proofs of Theorems 3.1-3.3 are organized as follows. The heart of the matter is the proof of Theorem 3.1: to estimate the oscillation along the individual streamlines of the flow at distance $O(\sqrt{\varepsilon})$ away from the boundary. Theorems 3.2 and 3.3 are an easy consequence of this result. We first explain how they follow from Theorem 3.1 and then go to the proof of Theorem 3.1.

3.2 The lower bound on the dissipation rate: proof of Theorem 3.3

We choose the boundary cell $\mathcal{C}_0$ as in the proof of Proposition 2.2 and recall the first inequality in (2.10) (with the notation as in the same proof):

$$\int_{\theta_1}^{\theta_2} |T_0(\theta) - \phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 d\theta \le C\gamma\sqrt{\varepsilon}\int_\Omega |\nabla\phi(\mathbf{x})|^2 d\mathbf{x}.$$

The left side may be bounded from below by

$$\begin{aligned}
&\int_{\theta_1}^{\theta_2} |T_0(\theta) - \phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 d\theta \\
&\ge \int_{\theta_1}^{\theta_2} |T_0(\theta) - K_0^\varepsilon|^2 d\theta - \int_{\theta_1}^{\theta_2} |K_0^\varepsilon - \phi^\varepsilon(\gamma\sqrt{\varepsilon},\theta)|^2 d\theta \\
&\ge \int_{\theta_1}^{\theta_2} |T_0(\theta) - \bar{T}_0|^2 d\theta - C\gamma^{-3/4} \ge C(1-\gamma^{-3/4})
\end{aligned}$$

with the constant K_0^ε as in (3.1) in Theorem 3.1 for the cell $\mathcal{C}_0$. Combining the last two inequalities and using $\gamma > 1$ we obtain (3.3). □

3.3 The dissipation in the interior: proof of Theorem 3.2

Theorem 3.2 is proved as follows. Integrating (1.1) over $\mathcal{D}(H)$ we obtain

$$\int_{\mathcal{D}(H)} |\nabla\phi^\varepsilon|^2 d\mathbf{x} = \int_{\mathcal{L}(H)} \phi^\varepsilon \frac{\partial\phi^\varepsilon}{\partial n} ds.$$

Integrating this equation in $H \in (H_0, H_0 + l)$ we get

$$\int_{H_0}^{H_0+l} \int_{\mathcal{D}(H)} |\nabla\phi^\varepsilon|^2 d\mathbf{x}dH = \int_{H_0}^{H_0+l} \int_{\mathcal{L}(H)} \phi^\varepsilon \frac{\partial\phi^\varepsilon}{\partial n} dsdH. \tag{3.4}$$

The left side of (3.4) is bounded below by

$$\int_{H_0}^{H_0+l} \int_{\mathcal{D}(H)} |\nabla\phi^\varepsilon|^2 d\mathbf{x}dH \geq l \int_{\mathcal{D}(H_0+l)} |\nabla\phi^\varepsilon|^2 d\mathbf{x},$$

as $\mathcal{D}(H_0 + l) \subset \mathcal{D}(H)$ for $H_0 \leq H \leq H_0 + l$. The right side of (3.4) may be estimated as

$$\left| \int_{H_0}^{H_0+l} \int\limits_{\mathcal{L}(h)} \phi^\varepsilon \frac{\partial\phi^\varepsilon}{\partial n} dsdh \right| \leq C \left(M^\varepsilon(H_0) - m^\varepsilon(H_0) \right) l^{1/2} \left(\int\limits_{\mathcal{D}(H_0)} |\nabla\phi^\varepsilon|^2 d\mathbf{x} \right)^{1/2}.$$

We have introduced here $M_j^\varepsilon(\alpha) = \sup\limits_{\mathbf{x}\in\mathcal{L}_j(\alpha)} \phi^\varepsilon(\mathbf{x})$, and $m_j^\varepsilon(\alpha) = \inf\limits_{\mathbf{x}\in\mathcal{L}_j(\alpha)} \phi^\varepsilon(\mathbf{x})$.

We denote $F(H) = \int_{\mathcal{D}(H)} |\nabla\phi^\varepsilon|^2 d\mathbf{x}$. Then the above estimates with $H_0 = l = H$ imply that

$$HF(2H) \leq C \left(\frac{\varepsilon}{H^2} \right)^{3/4} (HF(H))^{1/2}.$$

That is, $\tilde{F}(H) = HF(H)$ satisfies $\tilde{F}(H) \leq C$ for $\sqrt{\varepsilon} \leq H \leq 2\sqrt{\varepsilon}$ and

$$\tilde{F}(2H) \leq \left(\frac{\varepsilon}{H^2} \right)^{3/4} \tilde{F}^{1/2}(H).$$

This implies that $\tilde{F}(H) \leq C \left(\frac{\varepsilon}{H^2} \right)^{3/8}$ for $H \geq \sqrt{\varepsilon}$ so that

$$\int_{\mathcal{D}(H)} |\nabla\phi^\varepsilon|^2 d\mathbf{x} \leq \frac{C}{H} \left(\frac{\varepsilon}{H^2} \right)^{3/8}$$

which is (3.2). □

3.4 Oscillation along the streamlines: proof of Theorem 3.1

Theorem 3.1 is an immediate corollary of the maximum principle and the following proposition.

Proposition 3.4. *Let $\phi^\varepsilon(\mathbf{x})$ be solution of (1.1)-(1.2) and let $M_j^\varepsilon(\alpha) = \sup_{\mathbf{x}\in\mathcal{L}_j(\alpha)} \phi^\varepsilon(\mathbf{x})$, and $m_j^\varepsilon(\alpha) = \inf_{\mathbf{x}\in\mathcal{L}_j(\alpha)} \phi^\varepsilon(\mathbf{x})$. Then there exists a constant $C > 0$ so that*

$$M_j^\varepsilon(\alpha) - m_j^\varepsilon(\alpha) \le C\left(\frac{\varepsilon}{\alpha^2}\right)^{3/4}. \tag{3.5}$$

This proposition states the converse of Proposition 2.2: while the meaning of the latter is that the width of the boundary layer is at least $O(\sqrt{\varepsilon})$, the former shows that it is not larger than $O(\sqrt{\varepsilon})$, as the oscillation on the level set $H = N\sqrt{\varepsilon}$ is bounded by $C/N^{3/2}$.

The proof of Proposition 3.4 is based on the following key lemma.

Lemma 3.5. *(The level-set oscillation inequality) Let $\mathcal{L}_j(\alpha)$ and $\mathcal{L}_j(\beta)$ be two level sets of the stream function $H(x)$ in a cell $\mathcal{C}_j$ with $\mathcal{D}_j(\alpha) \subset \mathcal{D}_j(\beta)$. Then we have*

$$(M_\varepsilon(\alpha) - m_\varepsilon(\alpha))|F(\alpha,\beta)| \le \varepsilon \int_{\mathcal{L}_j(\alpha)} \left|\frac{\partial\phi^\varepsilon}{\partial n}\right| ds + \varepsilon \int_{\mathcal{L}_j(\beta)} \left|\frac{\partial\phi^\varepsilon}{\partial n}\right| ds, \tag{3.6}$$

where $F(\alpha,\beta)$ is the flux between two level sets

(3.7)

$$F(\alpha,\beta) = \int_\gamma (u\cdot n)ds,\;\; \gamma=\gamma(t),\;\; t\in[0,1],\;\; \gamma(0)\in\mathcal{L}_j(\alpha),\;\; \gamma(1)\in\mathcal{L}_j(\beta).$$

Here γ is any smooth curve that connects the level sets $\mathcal{L}_j(\alpha)$ and $\mathcal{L}_j(\beta)$ and does not intersect itself.

We will assume without loss of generality that $F(\alpha,\beta) \ge 0$. Note that the flux between two level sets is independent of the choice of the curve γ because of the incompressibility of the flow u.

The proof of Lemma 3.5

We now prove the level-set oscillation inequality (Lemma 3.5). As we restrict our analysis to one cell we drop the subscript j in all the involved quantities. The idea of the proof is to construct a set $\mathcal{R}$ bounded by a pair of gradient curves of ϕ^ε and parts of the streamlines $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$ if possible. The gradient curves would be chosen so that the difference in the values of the function ϕ^ε between these curves is at least as large as the oscillation of ϕ^ε along $\mathcal{L}(\alpha)$. Integrating equation (1.1) over $\mathcal{R}$ we get then (3.6). The main technicality is the construction of the set $\mathcal{R}$: see Figures 3.1 and 3.2 below for a geometric depiction of $\mathcal{R}$.

We turn now to the construction of $\mathcal{R}$. Let us define the oscillation function $d(\gamma) = M^\varepsilon(\gamma) - m^\varepsilon(\gamma)$. The maximum principle implies that if the level set $\mathcal{L}(\gamma)$ is contained inside the region $\mathcal{D}(\gamma')$ bounded by the level set $\mathcal{C}(\gamma')$, then $d(\gamma) < d(\gamma')$. We denote by $\mathbf{x}_m(\alpha)$ and $\mathbf{x}_M(\alpha)$ the points where ϕ^ε attains its minimum and maximum on the level set $\mathcal{L}(\alpha)$: $M^\varepsilon(\alpha) = \phi^\varepsilon(\mathbf{x}_M(\alpha))$ and $m^\varepsilon(\alpha) = \phi^\varepsilon(\mathbf{x}_m(\alpha))$.

Consider the gradient curves

$$\frac{d\gamma_m}{dt} = -\nabla\phi^\varepsilon(\gamma_m(t)),\ \gamma_m(0) = \mathbf{x}_m(\alpha), \tag{3.8}$$

and

$$\frac{d\gamma_M}{dt} = \nabla\phi^\varepsilon(\gamma_M(t)),\ \gamma_M(0) = \mathbf{x}_M(\alpha). \tag{3.9}$$

The function ϕ^ε may have critical points in $\mathcal{D}(\alpha,\beta)$ and the gradient curves γ_M and γ_m potentially may tend to those points as $t \to +\infty$. However, all critical points of ϕ^ε are isolated saddle points as it may have neither internal maxima nor minima according to the maximum principle. Moreover, as ϕ^ε satisfies an elliptic problem in Ω it may have only finitely many critical points in the interior away from the boundary. Thus there are only finitely many critical points of ϕ^ε inside $\mathcal{D}(\alpha,\beta)$ that we denote by $\xi_1,\ldots,\xi_{N_\varepsilon}$. Note that both $\mathbf{x}_M(\alpha), \mathbf{x}_m(\alpha) \neq \xi_k$ for all k because of the strong maximum principle [9]. Let us consider the disks $U_j^r = \{|\mathbf{x} - \xi_j| \le r\}$, $j = 1,\ldots,N_\varepsilon$ centered at the singular points, and let $U^r = \cup_{j=1}^{N_\varepsilon} U_j^r$. Note also that $|\nabla\phi^\varepsilon(\mathbf{x})| > C(\varepsilon, r)$ for $\mathbf{x} \in \mathcal{D}^r(\alpha,\beta) = \mathcal{D}(\alpha,\beta)\backslash U^r$. Therefore $\phi^\varepsilon(\gamma_M(t)) > M^\varepsilon(\alpha) + C(\varepsilon)t$ if $\gamma_M(s) \in \mathcal{D}^r(\alpha,\beta)$ for $0 \le s \le t$ and hence the curve $\gamma_M(t)$ must leave the set $\mathcal{D}^r(\alpha,\beta)$ at a finite time since the function ϕ^ε is uniformly bounded. However, the curve $\gamma_M(t)$, $t > 0$ may not intersect the level set $\mathcal{L}(\alpha)$ because $\phi^\varepsilon(\gamma_M(t))$ is strictly increasing for $t < t_0$ provided that it stays inside $\mathcal{D}^r(\alpha,\beta)$ for all $t < t_0$. Hence there are two possibilities: either both γ_M and γ_m exit the set $\mathcal{D}^r(\alpha,\beta)$ at $\mathcal{L}(\beta)$ or one of them crosses $\partial\mathcal{D}^r(\alpha,\beta)$ at one of ∂U_j^r. We consider these two cases separately. First, we assume that we may choose $r > 0$ so small that the curves γ_M and γ_m do not intersect the circles $U_j^r = \{|x - \xi_j| = r\}$, $j = 1,\ldots,N_\varepsilon$, and then we treat the other case.

Case 1: There exists $r > 0$ so small that both $\gamma_m(t)$ and $\gamma_M(t)$ exit $\mathcal{D}^r(\alpha,\beta)$ at $\mathcal{L}(\beta)$.

We denote the corresponding exit times by t_m and t_M, that is $\gamma_m(t_m) \in \mathcal{L}(\beta)$ and $\gamma_M(t_M) \in \mathcal{L}(\beta)$, while $\gamma_m(s) \in \mathcal{D}^r(\alpha,\beta)$ for $0 \le s \le t_m$ and $\gamma_M(s) \in \mathcal{D}^r(\alpha,\beta)$ for $0 \le s \le t_M$. With a slight abuse of notation we denote $\gamma_m = \{\gamma_m(s),\ 0 \le s \le t_m\}$ and $\gamma_M = \{\gamma_M(s),\ 0 \le s \le t_M\}$. The curves γ_m and γ_M both have a finite length since $|\nabla\phi^\varepsilon|$ is uniformly bounded above and below in $D^r(\alpha,\beta)$ (by constants that may depend on ε and r). These curves may not intersect since $\phi^\varepsilon(\mathbf{x}) > M^\varepsilon > m^\varepsilon > \phi^\varepsilon(\mathbf{y})$ for all $\mathbf{x} \in \gamma_M$ and $\mathbf{y} \in \gamma_m$. Let $\mathcal{R}$ be a domain bounded by γ_m, γ_M and parts of the streamlines $\gamma_\alpha \in \mathcal{L}(\alpha)$ and $\gamma_\beta \in \mathcal{L}(\beta)$ (see Figure 3.1). There are two such domains, $\mathcal{R}$ and $\mathcal{D}(\alpha,\beta)\backslash\mathcal{R}$. We fix $\mathcal{R}$ so that $u \cdot n > 0$ on $\gamma_M(t)$ for t sufficiently small – this guarantees that “each streamline of u goes out of $\mathcal{R}$

when it intersects γ_M for the last time." Furthermore, we have $u \cdot n < 0$ on γ_m for t sufficiently small so that "each streamline of u goes into $\mathcal{R}$ when it intersects γ_m for the first time." Here n is the outward normal to $\partial\mathcal{R}$. Integrating (1.1) over $\mathcal{R}$ we obtain

$$\begin{aligned} 0 = \int_{\mathcal{R}} (\varepsilon\Delta\phi^\varepsilon - u \cdot \nabla\phi^\varepsilon) d\mathbf{x} &= \varepsilon \int_{\gamma_\alpha} \frac{\partial\phi^\varepsilon}{\partial n} ds + \varepsilon \int_{\gamma_\beta} \frac{\partial\phi^\varepsilon}{\partial n} ds \\ &- \int_{\gamma_m} (u \cdot n)\phi^\varepsilon ds - \int_{\gamma_M} (u \cdot n)\phi^\varepsilon ds, \end{aligned} \tag{3.10}$$

because $u \cdot n \equiv 0$ on γ_α, γ_β and $\dfrac{\partial\phi^\varepsilon}{\partial n} = 0$ on γ_m, γ_M since the latter are gradient curves of ϕ^ε.

We will use the following fact.

Lemma 3.6. *Let* $\gamma : [0,1] \to \mathcal{D}(\alpha,\beta)$ *be any non-self intersecting smooth curve that connects* $\mathcal{L}(\alpha)$ *and* $\mathcal{L}(\beta)$*:* $\gamma(0) \in \mathcal{L}(\alpha)$, $\gamma(1) \in \mathcal{L}(\beta)$, *has a finite length and is not tangent to* $\mathcal{L}(\alpha)$ *at* $t = 0$*. Fix the unit normal* n *to* γ *so that* $n(t)$ *is continuous and* $u \cdot n$ *is non-negative when a streamline of* u *intersects* γ *for the last time, that is,* $u \cdot n(\tau(\xi)) \geq 0$ *for all* ξ *between* α *and* β*, with* $\tau(\xi) = \sup\{t : \gamma(t) \in \mathcal{L}(\xi)\}$*. Let* $f(\mathbf{x}) \geq 0$ *be a continuous function monotonically increasing along* γ*. Then we have*

$$F(\alpha,\beta) \inf_{\mathbf{x}\in\gamma} f \leq \int_\gamma (u \cdot n) f ds \leq F(\alpha,\beta) \sup_{\mathbf{x}\in\gamma} f, \tag{3.11}$$

where $F(\alpha,\beta)$ *is the flux (3.7).*

Proof. First, we observe that $u \cdot n(\tau(\xi)) \geq 0$ for all $\xi \in [\alpha,\beta]$ provided that $u \cdot n(t) > 0$ for t sufficiently small. The inequality (3.11) is shown as follows. For any $N \in \mathbb{N}$ we may approximate f along γ by two piecewise constant (along γ) monotonically increasing functions $\bar{f}_N$ and $\tilde{f}_N$ so that

$$\begin{aligned} \int_\gamma (u \cdot n) f ds - \frac{1}{N} &\leq \int_\gamma (u \cdot n)\tilde{f}_N ds \leq \int_\gamma (u \cdot n) f ds \\ &\leq \int_\gamma (u \cdot n)\bar{f}_N ds \leq \int_\gamma (u \cdot n) f ds + \frac{1}{N} \end{aligned} \tag{3.12}$$

and $|f - \tilde{f}_N| \leq 1/N$, $|f - \bar{f}| \leq 1/N$. Therefore it suffices to prove (3.11) for a step function f that has finitely many discontinuities, the general case follows after passing to the limit $N \to \infty$ in (3.12). We assume below that f is a step function. Let $\alpha_1, \dots, \alpha_p$ be values of the stream function H such that f has jumps only on the level sets $\mathcal{L}(\alpha_k)$, $k = 1, \dots, p$. We order them so that $\mathcal{L}(\alpha_k) \subset \mathcal{D}(\alpha_{k+1})$. Then we may represent γ as the union

$$\gamma = \cup_{k=1}^p \gamma_k, \ \gamma_k \subset \mathcal{D}(\alpha_k, \alpha_{k+1}).$$

Here γ_k is the part of γ contained in the annulus $\mathcal{D}(\alpha_k, \alpha_{k+1})$. We may further split the subset γ_k as a union $\gamma_k = \gamma_k' \cup \gamma_k''$. Here the set $\gamma_k' = \cup_{l=1}^{s_k} \gamma_{kl}'$

is a union of finitely many curves γ'_{kl} that connect the level sets $\mathcal{L}(\alpha_k)$ and $\mathcal{L}(\alpha_{k+1})$. There can be only finitely many of such curves since γ has a finite length and the distance between $\mathcal{L}(\alpha_k)$ and $\mathcal{L}(\alpha_{k+1})$ is positive. The set $\gamma''_k = \cup_l \gamma''_{kl}$ consists of curves that start and end on the same level set $\mathcal{L}(\alpha_k)$ or $\mathcal{L}(\alpha_{k+1})$. We note that the function f is constant on each curve γ'_{kl} and γ''_{kl}. Therefore we have using incompressibility of u

$$\int_{\gamma''_k} f(u\cdot n)ds = \sum_l \int_{\gamma''_{kl}} f(u\cdot n)ds = 0.$$

We also have

$$\int_{\gamma'_{kl}} f(u\cdot n)ds = (-1)^{l+1} f_{kl} F(\alpha_k,\alpha_{k+1}),$$

where f_{kl} is the constant value of f on the curve γ'_{kl}. This implies that

$$\int_{\gamma_k} f(u\cdot n)ds = \int_{\gamma'_k} f(u\cdot n)ds = F(\alpha_k,\alpha_{k+1}) \sum_{l=1}^{s_k} (-1)^{l+1} f_{kl}.$$

However, f_{kl} is an increasing function of l and the total number of times s_k that γ crosses from $\mathcal{L}(\alpha_k)$ to $\mathcal{L}(\alpha_{k+1})$ must be odd. Thus the above may be bounded below by

$$\int_{\gamma_k} f(u\cdot n)ds \geq f_{k1} F(\alpha_k,\alpha_{k+1}) \geq F(\alpha_k,\alpha_{k+1}) \inf_{\gamma} f.$$

Summing the above over k we obtain the first inequality in (3.11). The second inequality is proved in the same way. □

We now apply (3.11) to the curves γ_m and γ_M with $f = \phi^\varepsilon$. Since $\max_{\gamma_m} \phi^\varepsilon = m^\varepsilon(\alpha)$ and $\min_{\gamma_M} \phi^\varepsilon = M^\varepsilon(\alpha)$, we have

$$\int_{\gamma_m} (u\cdot n)\phi^\varepsilon ds \geq -m^\varepsilon(\alpha) F(\alpha,\beta), \quad \int_{\gamma_M} (u\cdot n)\phi^\varepsilon ds \geq M^\varepsilon(\alpha) F(\alpha,\beta), \tag{3.13}$$

so that

$$\int_{\gamma_m} (u\cdot n)\phi^\varepsilon ds + \int_{\gamma_M} (u\cdot n)\phi^\varepsilon ds \geq (M^\varepsilon(\alpha) - m^\varepsilon(\alpha)) F(\alpha,\beta).$$

Clearly we also have

$$\left| \int_{\partial\gamma_\beta} \frac{\partial\phi^\varepsilon}{\partial n} ds \right| \leq \int_{\mathcal{L}(\beta)} \left| \frac{\partial\phi^\varepsilon}{\partial n} \right| ds,$$

and

$$\left| \int_{\partial\gamma_\alpha} \frac{\partial\phi^\varepsilon}{\partial n} ds \right| \leq \int_{\mathcal{L}(\alpha)} \left| \frac{\partial\phi^\varepsilon}{\partial n} \right| ds.$$

The claim of Lemma 3.5 now follows from the last three inequalities and (3.10) in the case when ϕ^ε has no critical points in $\mathcal{D}(\alpha,\beta)$ or when γ_m and γ_M exit $D^r(\alpha,\beta)$ along $\mathcal{L}(\beta)$.

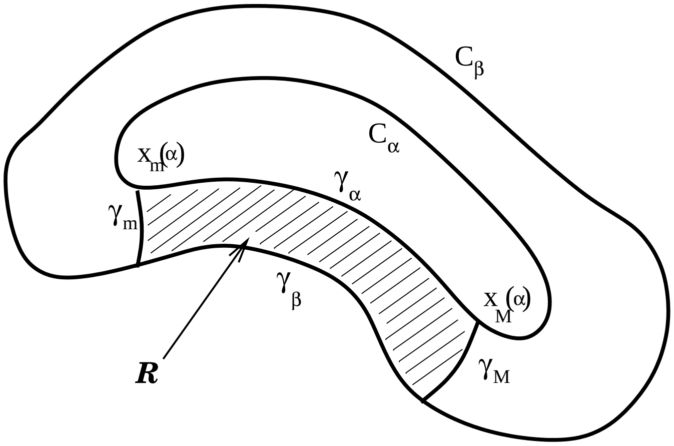

FIGURE 3.1. The non-critical case

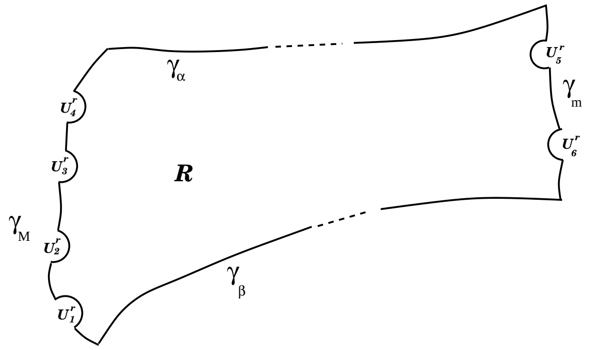

FIGURE 3.2. The critical case

Case 2: It remains to consider the second case when γ_m or γ_M exit the set $\mathcal{D}^r(\alpha,\beta)$ at the boundary ∂U^r for all $r > 0$.

We pick $r > 0$ sufficiently small to be specified below. In particular we require that the starting points $\mathbf{x}_M(\alpha)$ and $\mathbf{x}_m(\alpha)$ are not contained in any of U_j^r, $j = 1, \ldots, N_\varepsilon$ – this is possible since $\mathbf{x}_m(\alpha)$ and $\mathbf{x}_M(\alpha)$ are not critical points of ϕ^ε as implied by the strong maximum principle. Then one (or both) of the curves γ_m and γ_M defined by (3.8) and (3.9) should exit $\mathcal{D}^r(\alpha,\beta)$ at the boundary $\partial U^r = \cup_{j=1}^{N_\varepsilon} \partial U_j^r$. Let us assume that this happens to γ_M and that it exits $\mathcal{D}^r_{\alpha\beta}$ at a point on $\partial U^r_{j_1}$ at a time $\tilde{t}^1_M$. We continue γ_M past the time $\tilde{t}^1_M$ as follows (see Figure 3.2). Let $\tilde{\eta}^{j_1}_M = \gamma_M(\tilde{t}^1_M)$ be the point where γ_M intersected $\partial U^r_{j_1}$ and let also $\eta^{j_1}_M$ be the point where ϕ^ε

reaches its maximum over $\partial U^r_{j_1}$

$$\phi^\varepsilon(\eta^{j_1}_M) = \sup_{\mathbf{x}\in\partial U^r_{j_1}} \phi^\varepsilon(\mathbf{x}).$$

The vector $\nabla\phi^\varepsilon(\eta^{j_1}_M)$ points in the direction of the outer normal to $\partial U^r_{j_1}$ by the maximum principle. We stop γ_M at $\tilde\eta^{j_1}_M$ and continue it along the circle $\partial U^r_{j_1}$ to $\eta^{j_1}_M$ in either direction with the speed equal to one, so that $\gamma_M(t^1_M) = \eta^{j_1}_M$. Then γ_M follows the gradient curve going out of $\eta^{j_1}_M$ for $t \geq t^1_M$ until it hits either $\mathcal{L}(\beta)$ or another circle $\partial U^r_{j_2}$ at a point $\tilde\eta^{j_2}_M$ at a time $\tilde t^2_M$. In the former case we stop the curve γ_M, while in the latter we continue it in the same fashion as at $\partial U^r_{j_1}$, connecting γ_M to η^2_M, the maximum of ϕ^ε along $\partial U^r_{j_2}$, etc. Eventually γ_M has to cross the level set $\mathcal{L}(\beta)$ at some finite time t^β_M. Indeed, we have $\phi^\varepsilon(\tilde\eta^{j_k}_M) < \phi^\varepsilon(\eta^{j_k}_M) < \phi^\varepsilon(\tilde\eta^{j_{k+1}}_M) < \phi^\varepsilon(\eta^{j_{k+1}}_M)$ which implies that the curve γ_M may not hit the same circle ∂U^r_j twice. Given that the total number of critical points N_ε is finite and that γ_M may not stay inside $\mathcal{D}^r(\alpha,\beta)$ for an infinite time we conclude that the exit time t^β_M is finite. A similar construction may be applied to the curve γ_m with $\eta^{j_k}_m$ being the point where ϕ^ε attains its minimum on $U^r_{j_k}$.

In order to guarantee that the curves γ_m and γ_M constructed in such way do not intersect, we require that r is so small that

$$0 < \sup_{\partial U^r_j}\phi^\varepsilon - \inf_{\partial U^r_j}\phi^\varepsilon < \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))),\ j=1,\ldots,N_\varepsilon \tag{3.14}$$

where δ is a small parameter. Observe that the sequence $\phi^\varepsilon(\eta^{j_k}_M)$ is increasing in k, $\phi^\varepsilon(\eta^{j_1}_M) > \phi^\varepsilon(x_M(\alpha))$ and $\phi^\varepsilon(\gamma(s)) > \phi^\varepsilon(\eta^{j_k}_M)$ for $t^k_M < s < \tilde t^{k+1}_M$. We also have

$$\phi^\varepsilon(\gamma_M(s)) > \phi^\varepsilon(\eta^{j_k}_M) - \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha)))$$

for $\tilde t^k_M < s < t^k_M$. That implies a lower bound

$$\phi^\varepsilon(\gamma_M(s)) > \phi^\varepsilon(x_M(\alpha)) - \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))) \tag{3.15}$$

for all $0 < s < t^\beta_M$. Similarly we have

$$\phi^\varepsilon(\gamma_m(s)) < \phi^\varepsilon(x_m(\alpha)) + \frac{\delta}{1+N_\varepsilon}(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha)))$$

for all $0 < s < t^\beta_m$. That implies an estimate

$$\phi^\varepsilon(\gamma_M(s)) - \phi^\varepsilon(\gamma_m(s')) > \left(1 - \frac{2\delta}{1+N_\varepsilon}\right)(\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))) \tag{3.16}$$

for all s and s' so that γ_M and γ_m may not intersect provided that $\delta < 1/2$.

We may now proceed as in the first part of the proof. Let $\mathcal{R}$ be the domain bounded by γ_m, γ_M and parts of the level sets $\mathcal{L}(\alpha)$ and $\mathcal{L}(\beta)$, as depicted on Figure 3.2, chosen so that $u \cdot n > 0$ for t sufficiently small, that is, so that each streamline of u goes out of $\mathcal{R}$ when it crosses γ_M for the last time. Integrating (1.1) over $\mathcal{R}$ we now obtain instead of (3.10):

$$\begin{aligned} 0 &= \int_{\mathcal{R}} (\varepsilon \Delta \phi^\varepsilon + u \cdot \nabla \phi^\varepsilon) d\mathbf{x} \\ &= \varepsilon \int_{\gamma_\alpha} \frac{\partial \phi^\varepsilon}{\partial n} ds + \varepsilon \int_{\gamma_\beta} \frac{\partial \phi^\varepsilon}{\partial n} ds + \varepsilon \sum_{k=1}^{N_\varepsilon} \int_{\gamma_M \cap \, \partial U_k^r} \frac{\partial \phi^\varepsilon}{\partial n} ds \\ &+ \varepsilon \sum_{k=1}^{N_\varepsilon} \int_{\gamma_m \cap \, \partial U_k^r} \frac{\partial \phi^\varepsilon}{\partial n} ds \int_{\gamma_m} (u \cdot n) \phi^\varepsilon ds - \int_{\gamma_M} (u \cdot n) \phi^\varepsilon ds, \end{aligned} \tag{3.17}$$

where $\gamma_\alpha = \partial \mathcal{R} \cap \mathcal{L}(\alpha)$ and similarly for γ_β. The function $\phi^\varepsilon(\gamma_M(s))$ is no longer necessarily monotonically increasing in s, as monotonicity might be broken for $\tilde{t}_M^j < s < t_M^j$. However, we may adjust its values on these intervals, interpolating linearly between $\phi^\varepsilon(\tilde{\eta}_M^{j_k})$ and $\phi^\varepsilon(\eta_m^{j_k})$, to make the new function $\tilde{\phi}^\varepsilon(s)$ monotonic in s. The oscillation bound (3.14) implies that

$$\begin{aligned} \left| \int_{\gamma_M} (u \cdot n) \tilde{\phi}^\varepsilon ds - \int_{\gamma_M} (u \cdot n) \phi^\varepsilon ds \right| &\le \sum_{k=1}^{N_\varepsilon} \left| \int_{\gamma_M \cap \, \partial U_k^r} (u \cdot n)(\tilde{\phi}^\varepsilon - \phi^\varepsilon) ds \right| \\ &\le \|u\|_\infty N_\varepsilon \frac{\delta}{1 + N_\varepsilon} (\phi^\varepsilon(x_M(\alpha)) - \phi^\varepsilon(x_m(\alpha))) \le C\delta. \end{aligned} \tag{3.18}$$

The estimate (3.11) may be applied to $\tilde{\phi}^\varepsilon$ which together with (3.15) and (3.18) implies:

$$\begin{aligned} \int_{\gamma_M} (u \cdot n) \phi^\varepsilon ds \ge \int_{\gamma_M} (u \cdot n) \tilde{\phi}^\varepsilon ds - C\delta &\ge [M^\varepsilon(\alpha) - C\delta] F(\alpha, \beta) - C\delta \\ &= M^\varepsilon(\alpha) F(\alpha, \beta) - C\delta. \end{aligned} \tag{3.19}$$

Similarly we obtain

$$\begin{aligned} \int_{\gamma_m} (u \cdot n) \phi^\varepsilon ds \ge \int_{\gamma_m} (u \cdot n) \tilde{\phi}^\varepsilon ds - C\delta &\ge -[m^\varepsilon(\alpha) + C\delta] F(\alpha, \beta) - C\delta \\ &= -m^\varepsilon(\alpha) F(\alpha, \beta) - C\delta. \end{aligned} \tag{3.20}$$

Furthermore, we may choose $r < 1$ so small that $|\nabla \phi^\varepsilon| < \delta/(1 + N_\varepsilon)$ on all ∂U_j^r, $j = 1, \ldots, N_\varepsilon$ – this is possible since the centers of U_j^r are singular points of $\nabla\phi^\varepsilon$. Then we obtain

$$\left|\sum_{k=1}^{N_\varepsilon}\int\limits_{\gamma_M\cap\ \partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds\right|\le N_\varepsilon 2\pi r\frac{\delta}{1+N_\varepsilon}\le C\delta.$$

Using the above estimates in (3.17) we get

$$\varepsilon\int_{\mathcal{L}(\alpha)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds+\varepsilon\int_{\mathcal{L}(\beta)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds\ge\varepsilon\left|\int_{\mathcal{L}(\alpha)}\frac{\partial\phi^\varepsilon}{\partial n}ds+\int_{\mathcal{L}(\beta)}\frac{\partial\phi^\varepsilon}{\partial n}ds\right|$$

$$=\left|\varepsilon\sum_{k=1}^{N_\varepsilon}\int\limits_{\gamma_M\cap\ \partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds+\varepsilon\sum_{k=1}^{N_\varepsilon}\int\limits_{\gamma_m\cap\ \partial U_k^r}\frac{\partial\phi^\varepsilon}{\partial n}ds-\int\limits_{\gamma_m}(u\cdot n)\phi^\varepsilon ds-\int\limits_{\gamma_M}(u\cdot n)\phi^\varepsilon ds\right|$$

$$\ge(M^\varepsilon(\alpha)-m^\varepsilon(\alpha))F(\alpha,\beta)-C\delta.$$

This proves Lemma 3.5 in case 2, as δ is arbitrary, and thus the proof of this lemma is complete. □

The proof of Proposition 3.4

We now prove Proposition 3.4. We use inequality (3.6) for a pair of level sets $\mathcal{L}((\alpha+\beta)/2+H)$ and $\mathcal{L}(\beta+H)$ with $0\le H\le\dfrac{\alpha-\beta}{2}$ to obtain

$$\left(M^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)-m^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)\right)F\left(\frac{\alpha+\beta}{2}+H,\beta+H\right)$$
$$\le\varepsilon\int\limits_{\mathcal{L}(\frac{\alpha+\beta}{2}+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds+\varepsilon\int\limits_{\mathcal{L}(\beta+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds.$$

However, we have

$$M^\varepsilon(\alpha)-m^\varepsilon(\alpha)\le M^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)-m^\varepsilon\left(\frac{\alpha+\beta}{2}+H\right)$$

according to the maximum principle. Therefore we get

$$\begin{aligned}(M^\varepsilon(\alpha)-m^\varepsilon(\alpha))\,F\left(\frac{\alpha+\beta}{2}+H,\beta+H\right)\\ \le\varepsilon\int\limits_{\mathcal{L}(\frac{\alpha+\beta}{2}+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds+\varepsilon\int\limits_{\mathcal{L}(\beta+H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds.\end{aligned}\tag{3.21}$$

We integrate (3.21) with respect to H to obtain

$$\begin{aligned}(M^\varepsilon(\alpha)-m^\varepsilon(\alpha))\int_0^{(\alpha-\beta)/2}F\left(\frac{\alpha+\beta}{2}+H,\beta+H\right)dH\\ \le\varepsilon\int_\beta^\alpha\int\limits_{\mathcal{L}(H)}\left|\frac{\partial\phi^\varepsilon}{\partial n}\right|ds\ dH.\end{aligned}\tag{3.22}$$

The integral on the right side of inequality (3.22) may be re-written in the curvilinear coordinates as

$$\begin{aligned}
&\int_\beta^\alpha \int\limits_{\mathcal{L}(h)} \left|\frac{\partial \phi^\varepsilon}{\partial n}\right| ds dH = \int_\beta^\alpha \int \left|\frac{\partial \phi^\varepsilon}{\partial n}\right| \frac{d\theta dH}{|\nabla \theta|} \le \int_{\mathcal{D}(\alpha,\beta)} |\nabla \phi^\varepsilon| \frac{|J| dxdy}{|\nabla \theta|} \\
&= \int_{\mathcal{D}(\alpha,\beta)} |\nabla \phi^\varepsilon| |\nabla H| dxdy \\
&\le \left(\int_{\mathcal{D}(\alpha,\beta)} |\nabla H|^2 dxdy\right)^{1/2} \left(\int_{\mathcal{C}} |\nabla \phi^\varepsilon|^2 dxdy\right)^{1/2} \\
&\le \frac{C}{\varepsilon^{1/4}} \left(\int_{\mathcal{D}(\alpha,\beta)} |\nabla H| dxdy\right)^{1/2} \le \frac{C(\alpha-\beta)^{1/2}}{\varepsilon^{1/4}}
\end{aligned}$$

where $J = |\nabla H||\nabla \theta|$ is the Jacobian (2.9).

The left side of (3.22) satisfies

$$(M^\varepsilon(\alpha) - m^\varepsilon(\alpha)) \int\limits_0^{(\alpha-\beta)/2} F\left(\frac{\alpha+\beta}{2} + H, \beta + H\right) dH \ge C(M^\varepsilon(\alpha) - m^\varepsilon(\alpha))(\alpha-\beta)^2.$$

The above estimates imply that

$$M^\varepsilon(\alpha) - m^\varepsilon(\alpha) \le C \frac{\varepsilon(\alpha-\beta)^{1/2}}{(\alpha-\beta)^2 \varepsilon^{1/4}} \le C\left(\frac{\varepsilon}{\alpha^2}\right)^{3/4}$$

with the choice $\beta = \alpha/2$. This finishes the proof of Proposition 3.4. □

4 The water-pipe network

The previous arguments show that there exist constants K_j^ε so that solution of (1.1) is well approximated by solution of the following water-pipe problem (see Figures 4.1 and 4.2). As before, we denote by $\Omega_N^\varepsilon = \{|H(\mathbf{x})| \le N\sqrt{\varepsilon}\}$ the domain consisting of narrow pipes (boundary layers) near the separatrices. Its boundary consists of $\partial\Omega$ and finitely many level set curves $\boldsymbol{l}_k^N = \mathcal{L}_k(N\sqrt{\varepsilon})$, $k = 1, \dots, p$ so that $|H(x)| = N\sqrt{\varepsilon}$ on $\boldsymbol{l}_k^N$. The results of Section 3 show that ϕ^ε, solution of (1.1) is uniformly close to solution of

$$\varepsilon \Delta \psi^\varepsilon - u \cdot \nabla \psi^\varepsilon = 0, \ \mathbf{x} \in \Omega_N^\varepsilon \tag{4.1}$$

with the boundary conditions

$$\psi^\varepsilon \,|_{\partial\Omega} = T_0 \ , \ \psi^\varepsilon \Big|_{\boldsymbol{l}_m^N} = K_m^\varepsilon \ , \ m = 1, \dots, p \tag{4.2}$$

with the constants K_m^ε as in Theorem 3.1. More precisely we have a uniform bound

$$|\phi^\varepsilon(\mathbf{x}) - \psi^\varepsilon(\mathbf{x})| \le \frac{C}{N^{3/2}}. \tag{4.3}$$

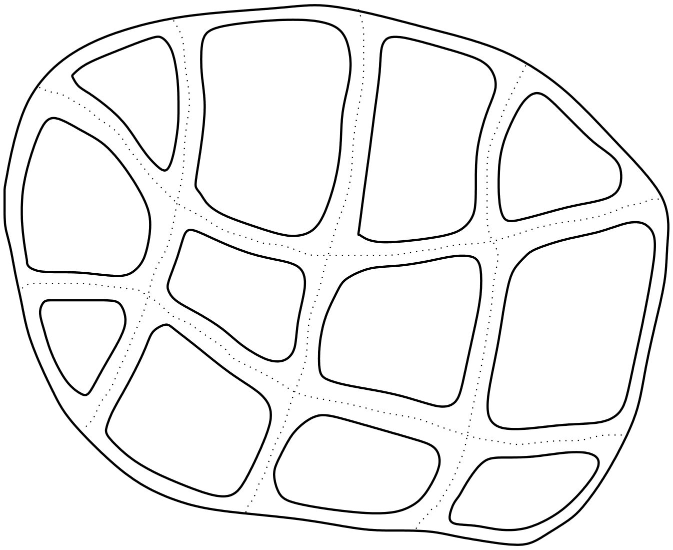

FIGURE 4.1. The water-pipe model

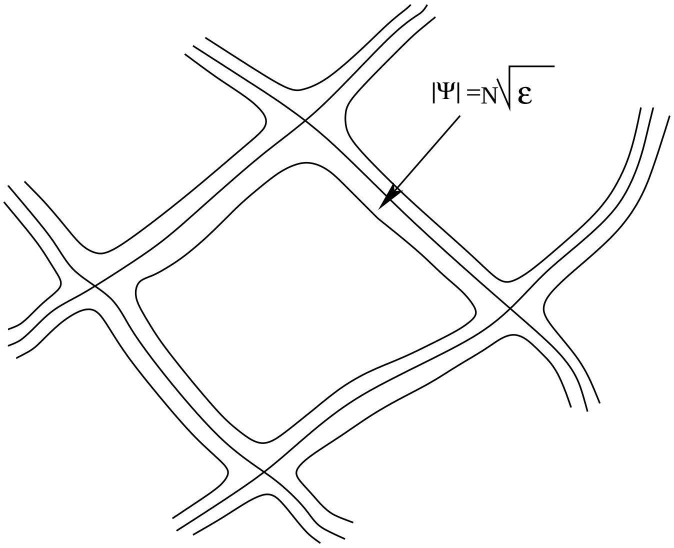

FIGURE 4.2. One cell

This shows that in a numerical computation of ϕ^ε it suffices to consider the pipe-problem (4.1)-(4.2) with the correct constants K_m^ε in order to obtain a good approximation of the solution. However, the constants K_m^ε are not known a priori and their computation is part of the problem. As we have seen the function ϕ^ε is very close to a constant near the level sets $\boldsymbol{l}_m^N$. Therefore we should expect that we may replace the Dirichlet boundary data on $\boldsymbol{l}_m^N$ by the homogeneous Neumann boundary conditions in the water-pipe problem (4.1) and obtain an approximation that has the same order of error. In particular this would provide an efficient numerical way to find the constants K_m^ε as the boundary value of the solution of (4.1) with the Neumann boundary conditions. This is confirmed by the following results.

Proposition 4.1. *Let ϕ_N^ε be solution of the water-pipe model:*

$$\varepsilon\Delta\phi_N^\varepsilon - u\cdot\nabla\phi_N^\varepsilon = 0,\ \mathbf{x}\in\Omega_N^\varepsilon \tag{4.4}$$

on the domain $\Omega_N^\varepsilon = \Omega\cap\{|H(\mathbf{x})|\le N\sqrt{\varepsilon}\}$ with the boundary conditions

$$\phi_N^\varepsilon\,|_{\partial\Omega} = T_0\ ,\ \left.\frac{\partial\phi_N^\varepsilon}{\partial n}\right|_{\boldsymbol{l}_m^N} = 0\ ,\ m = 1,\ldots,p. \tag{4.5}$$

Then there exist constants $\tilde{K}_{m,N}^\varepsilon$ so that

$$|\phi_N^\varepsilon(\mathbf{x}) - \tilde{K}_{m,N}^\varepsilon| \le \frac{C}{N^{3/2}}, \tag{4.6}$$

for all $x\in\boldsymbol{l}_m^N$.

Proof. The proof of this proposition is essentially the same as of Theorem 3.1. One only has to observe that the strong maximum principle implies that the maximum and minimum of the function ϕ_N^ε over any sub-domain $\{\alpha\le|H(\mathbf{x})|\le N\sqrt{\varepsilon}\}\cap\mathcal{C}_m$ is achieved on the boundary $\{|H(\mathbf{x})| = \alpha\}\cap\mathcal{C}_m$ and not on the interior level set $\boldsymbol{l}_N^m$. Therefore all arguments in the proof of the level-set oscillation inequality (Lemma 3.5) are applicable verbatim, and we do not repeat them. □

Theorem 4.2. *Let ϕ^ε solve (1.1) and let $\chi(s)$ be a smooth even function, monotonic on $s\ge 0$, so that*

$$\chi(s) = \begin{cases} 1, & |s|\le 1/2,\\ 0, & |s|\ge 1\end{cases}$$

Let us extend ϕ_N^ε to the whole domain Ω as

$$\tilde{\phi}_N^\varepsilon(\mathbf{x}) = \chi\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\phi_N^\varepsilon(\mathbf{x}) + \tilde{K}_{m,N}^\varepsilon\left(1-\chi\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\right),\ \mathbf{x}\in\mathcal{C}_m$$

with the constants $\tilde{K}_{m,N}^\varepsilon$ given by Proposition 4.1. Then we have

$$\|\phi^\varepsilon - \tilde{\phi}_N^\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{N^{3/2}}, \tag{4.7}$$

and

$$\sqrt{\varepsilon}\|\nabla\phi^\varepsilon - \nabla\tilde{\phi}_N^\varepsilon\|_{L^2(\Omega)}^2 \le \frac{C}{N^4}, \tag{4.8}$$

where ϕ^ε solves (1.1).

Proof. Let $\zeta^\varepsilon = \phi^\varepsilon - \phi_N^\varepsilon$ be the error that we need to estimate. It satisfies the equation

$$\varepsilon\Delta\zeta^\varepsilon - u\cdot\nabla\zeta^\varepsilon = g^\varepsilon,\ \mathbf{x}\in\Omega, \tag{4.9}$$

with

$$g^\varepsilon(\mathbf{x}) = (\tilde{K}^\varepsilon_{m,N} - \phi_N^\varepsilon)\left[\frac{\sqrt{\varepsilon}}{N}\Delta H(\mathbf{x})\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right) + \frac{1}{N^2}|\nabla H(\mathbf{x})|^2\chi''\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\right]$$
$$-\frac{2\sqrt{\varepsilon}}{N}\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\nabla\phi_N^\varepsilon(\mathbf{x})\cdot\nabla H(\mathbf{x})$$

and the boundary condition $\zeta^\varepsilon = 0$ on $\partial\Omega$. We multiply (4.9) by ζ^ε and integrate over Ω:

$$\varepsilon\int_\Omega|\nabla\zeta^\varepsilon(\mathbf{x})|^2 d\mathbf{x} = -\int_\Omega\zeta^\varepsilon(\mathbf{x})g^\varepsilon(\mathbf{x})d\mathbf{x} = I + II + III.$$

The first term on the right may be estimated using Proposition 4.1 as

$$I = -\int_\Omega\zeta^\varepsilon(\mathbf{x})(\tilde{K}_m^N - \phi_N^\varepsilon(\mathbf{x}))\frac{\sqrt{\varepsilon}}{N}\Delta H(\mathbf{x})\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)d\mathbf{x}$$
$$\le \frac{C\sqrt{\varepsilon}\|\zeta^\varepsilon\|_{L^\infty(\Omega_N^\varepsilon)}}{N^{5/2}}\int_\Omega\left|\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\right|d\mathbf{x} \le \frac{C\sqrt{\varepsilon}}{N^{5/2}}\|\zeta^\varepsilon\|_{L^\infty(\Omega_N^\varepsilon)}.$$

The second term is bounded in a similar way as

$$II = -\int_\Omega\zeta^\varepsilon(\mathbf{x})(\tilde{K}_m^N - \phi_N^\varepsilon(\mathbf{x}))\frac{1}{N^2}|\nabla H(\mathbf{x})|^2\chi''\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)d\mathbf{x}$$
$$\le \frac{C\sqrt{\varepsilon}}{N^{5/2}}\|\zeta^\varepsilon\|_{L^\infty(\Omega_N^\varepsilon)}.$$

The last term we bound integrating by parts as

$$III = \int_\Omega\zeta^\varepsilon(\mathbf{x})\frac{2\sqrt{\varepsilon}}{N}\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\nabla\phi_N^\varepsilon(\mathbf{x})\cdot\nabla H(\mathbf{x})d\mathbf{x}$$
$$= -\frac{2\sqrt{\varepsilon}}{N^2\sqrt{\varepsilon}}\int_\Omega\zeta^\varepsilon(\mathbf{x})(\phi_N^\varepsilon(\mathbf{x}) - \tilde{K}_m^N)\chi''\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)|\nabla H(\mathbf{x})|^2 d\mathbf{x}$$
$$-\frac{2\sqrt{\varepsilon}}{N}\int_\Omega\zeta^\varepsilon(\mathbf{x})(\phi_N^\varepsilon(\mathbf{x}) - \tilde{K}_m^N)\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\Delta H(\mathbf{x})d\mathbf{x}$$
$$-\frac{2\sqrt{\varepsilon}}{N}\int_\Omega(\phi_N^\varepsilon(\mathbf{x}) - \tilde{K}_m^N)\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\nabla\zeta^\varepsilon(\mathbf{x})\cdot\nabla H(\mathbf{x})d\mathbf{x}$$
$$\le \frac{C\sqrt{\varepsilon}}{N^{5/2}}\|\zeta^\varepsilon\|_{L^\infty(\Omega_N^\varepsilon)} + \frac{C\sqrt{\varepsilon}}{N^{5/2}}\|\zeta^\varepsilon\|_{L^\infty(\Omega_N^\varepsilon)}$$
$$+\frac{C\sqrt{\varepsilon}}{N^{5/2}}\left[A\int_\Omega|\nabla\zeta^\varepsilon(\mathbf{x})|^2 d\mathbf{x} + \frac{1}{A}\int_\Omega\left|\chi'\left(\frac{H(\mathbf{x})}{N\sqrt{\varepsilon}}\right)\right|^2|\nabla H(\mathbf{x})|^2 d\mathbf{x}\right].$$

We choose $A = \sqrt{\varepsilon} N^{5/2}/(2C)$ to obtain the bound

$$\varepsilon \int_{\Omega} |\nabla \zeta^{\varepsilon}(\mathbf{x})|^2 d\mathbf{x} \leq \frac{C\sqrt{\varepsilon}}{N^{5/2}} \|\zeta^{\varepsilon}\|_{L^{\infty}(\Omega_N^{\varepsilon})} + \frac{C\sqrt{\varepsilon}}{N^4}. \tag{4.10}$$

Recall that $\zeta^{\varepsilon} = 0$ on $\partial\Omega$ and

$$|\zeta^{\varepsilon}(\mathbf{x}) - (K_m^{\varepsilon} - \tilde{K}_m^{\varepsilon})| \leq \frac{C}{N^{3/2}}$$

on the level set $\boldsymbol{l}_m^N$. Then if $|K_m^{\varepsilon} - \tilde{K}_m^{\varepsilon}| = \delta > \dfrac{2C}{N^{3/2}}$ we have, on one hand,

$$\varepsilon \int_{\Omega_N^{\varepsilon}} |\nabla \zeta^{\varepsilon}(\mathbf{x})|^2 d\mathbf{x} \geq \frac{C\sqrt{\varepsilon}}{N} \left(\delta - \frac{C}{N^{3/2}}\right)^2,$$

while on the other $\|\zeta\|_{L^{\infty}(\Omega_N^{\varepsilon})} \leq \delta + \dfrac{C}{N^{3/2}}$. Putting these bounds into (4.10) we obtain

$$\frac{C\sqrt{\varepsilon}}{N} \left(\delta - \frac{C}{N^{3/2}}\right)^2 \leq \frac{C\sqrt{\varepsilon}}{N^{5/2}} \left(\delta + \frac{C}{N^{3/2}}\right) + \frac{C\sqrt{\varepsilon}}{N^4}.$$

We denote $\gamma = \delta - \dfrac{C}{N^{3/2}}$ and rewrite the above as

$$\frac{C\sqrt{\varepsilon}}{N} \gamma^2 \leq \frac{C\sqrt{\varepsilon}}{N^{5/2}} \gamma + \frac{C\sqrt{\varepsilon}}{N^4} + \frac{C\sqrt{\varepsilon}}{N^4}$$

so that

$$\gamma \leq \frac{C}{N^{3/2}}.$$

Therefore

$$|K_m^{\varepsilon} - \tilde{K}_m^{\varepsilon}| = \delta \leq \frac{2C}{N^{3/2}}.$$

An application of the maximum principle on Ω_N^{ε} finishes the proof of Theorem 4.2. □

Note that Proposition 4.1 and Theorem 4.2 *do not* imply existence of the limits

$$\lim_{\varepsilon \to 0} K_m^{\varepsilon} = K_m. \tag{4.11}$$

The proof of (4.11) requires a separate argument based on the analysis of the asymptotic limit $\varepsilon \to 0$ in the next two Sections. We will present first the asymptotic analysis, and then return to the proof of (4.11) at the end of Section 6.

5 The asymptotic problem

It turns out that in the limit $\varepsilon \to 0$ the asymptotic behavior of the solution to the advection-diffusion problem may be described in terms of a model that is essentially a system of one-dimensional heat equations on a graph. This section is concerned with the construction of this model.

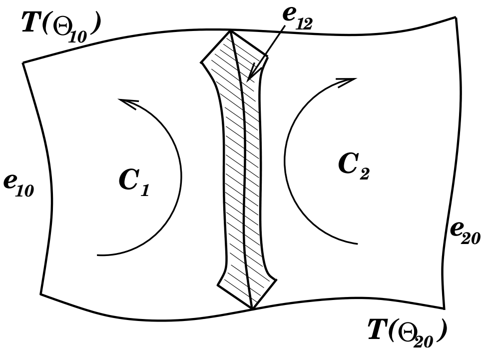

Figure 5.1. The two-cell problem

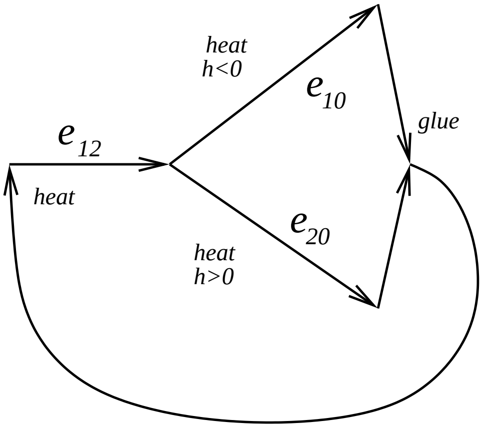

Figure 5.2. The gluing procedure

5.1 The two-cell case

We describe the asymptotic problem first on the simplest example of a domain Ω that consists of two cells $\mathcal{C}_1$ and $\mathcal{C}_2$ depicted in Figure 5.1. We denote by $e_{j0} = \partial\Omega \cap \partial\mathcal{C}_j$, $j = 1, 2$, the part of the boundary of Ω along the cell $\mathcal{C}_j$ and by e_{12} the common edge of the two cells. We also introduce the boundary layer coordinates h and θ_{12}, θ_{j0}, $j = 1, 2$. The coordinate θ_{12} represents parameterization along the edge $e_{12} = \{h = 0\} \cap \{0 \le \theta_{12} \le l_{12}\}$, while the coordinates θ_{j0} parameterize along the boundaries $e_{j0} = \{h = 0\} \cap \{l_{12} \le \theta_{j0} \le l_{j0}\}$. We first solve the heat equation "along e_{12}":

$$\frac{\partial f_{12}}{\partial \theta_{12}} = \frac{\partial^2 f_{12}}{\partial h^2},\ h \in [-N, N],\ 0 \le \theta_{12} \le l_{12} \tag{5.1}$$

with a prescribed initial data f_{12}^0 and the Neumann boundary conditions at $h = \pm N$:

$$\frac{\partial f_{12}(\theta_{12}, \pm N)}{\partial h} = 0. \tag{5.2}$$

Then we solve two half-space problems "along the outer boundaries e_{j0}" with the prescribed Dirichlet data that comes from (1.2):

$$\frac{\partial f_{10}}{\partial \theta_{10}} = \frac{\partial^2 f_{10}}{\partial h^2}, \ -N \le h \le 0, \ 0 \le \theta_{10} \le l_{10} \tag{5.3}$$

and

$$\frac{\partial f_{20}}{\partial \theta_{20}} = \frac{\partial^2 f_{20}}{\partial h^2}, \ 0 \le h \le N, \ 0 \le \theta_{20} \le l_{20} \tag{5.4}$$

with the Neumann boundary condition (5.2) at $h = -N$, and $h = N$, respectively, and with the Dirichlet data $f_{j0}(\theta_{j0}, 0) = T_0(\theta_{j0})$ at $h = 0$. The initial data for (5.3) and (5.4) comes from (5.1):

$$\begin{aligned} f_{10}(l_{12}, h) &= f_{12}(l_{12}, h), \ -N \le h \le 0, \\ f_{20}(l_{12}, h) &= f_{12}(l_{12}, h), \ 0 \le h \le N. \end{aligned} \tag{5.5}$$

Finally we glue together the functions $f_{10}(l_{10}, h)$, $h \le 0$ and $f_{20}(l_{20}, h)$, $h \ge 0$:

$$f_{12}^g(h) = \begin{cases} f_{10}(l_{10}, h), \ -N \le h \le 0 \\ f_{20}(l_{20}, h), \ 0 \le h \le N \end{cases} \tag{5.6}$$

The asymptotic problem is to construct a periodic solution of the above, that is, find a function $f_{12}^0(h)$ so that $f_{12}^0(h) = f_{12}^g(h)$, $h \in [-N, N]$. This problem is described schematically in Figure 5.2.

Proposition 5.1. *There exists a unique function* $f_{12}^0 \in L^2(-N, N)$ *such that* $f_{12}^0 = f_{12}^g$.

Proof. Let us define the operator $L_{12} : \ L^2(-N, N) \to L^2(-N, N)$ by $L_{12} : \ f_{12}^0 \to f_{12}(l_{12})$, that is, the solution operator of (5.1). The operator L_{12} is bounded and compact, since $\|f_{12}(l_{12})\|_{H^1(-N,N)} \le C\|f_{12}^0\|_{L^2(-N,N)}$. We also let L_{10} and L_{20} be solution operators for (5.3) and (5.4), respectively with homogeneous boundary data $T_0 = 0$. The operators $\mathcal{R}_\pm$ restrict a function defined on $[-N, N]$ to the positive and negative semi-axes, respectively, while the gluing operator $\mathcal{G}$ glues together two functions defined on those axes:

$$\mathcal{G}[f_-, f_+](h) = \begin{cases} f_-(h), \ h \le 0, \\ f_+(h), \ h > 0, \end{cases}$$

as in (5.6). We denote by $g(h)$ the function obtained by solving (5.1)–(5.6) with $f_{12}^0 = 0$ and inhomogeneous boundary conditions. Then equation $f_{12}^0 = f_{12}^g$ is equivalent to:

$$\mathcal{G}(L_{10}\mathcal{R}_- L_{12} f_{12}^0, L_{20}\mathcal{R}_+ L_{12} f_{12}^0) + g = f_{12}^0, \tag{5.7}$$

or

$$\mathcal{K}f_{12}^0 - f_{12}^0 = -g,\ \mathcal{K}f_{12}^0 = \mathcal{G}(L_{10}\mathcal{R}_-L_{12}f_{12}^0, L_{20}\mathcal{R}_+L_{12}f_{12}^0). \tag{5.8}$$

The operator $\mathcal{K}$ is a compact operator on $L^2(-N,N)$. Furthermore, we have $\|L_{10}\|_{L^2\to L^2} < 1$ and $\|L_{20}\|_{L^2\to L^2} < 1$, while $\|L_{12}\|_{L^2\to L^2} = 1$. This implies easily that $\|\mathcal{K}\|_{L^2\to L^2} < 1$ so that solution of (5.8) exists and is unique by the Fredholm alternative since $\mathcal{K}$ is compact.

An alternative approach to the proof of existence of a periodic solution of (5.1)-(5.6), that is somewhat less transparent in the two-cell case but is easier to generalize to the case of N cells is as follows. We introduce an operator $\mathcal{L} = L_{12}\otimes L_{10}\otimes L_{20}$ defined on $L^2(\mathbb{R})\times L^2(\mathbb{R}_-)\times L^2(\mathbb{R}_+)$ as

$$\mathcal{L}\begin{pmatrix} f_{12}\\ f_{10}\\ f_{20}\end{pmatrix} = \begin{pmatrix} L_{12}f_{12}\\ L_{10}f_{10}\\ L_{20}f_{20}\end{pmatrix}.$$

We also define a re-distribution operator $\mathcal{R}$ on the same space $L^2(\mathbb{R})\times L^2(\mathbb{R}_-)\times L^2(\mathbb{R}_+)$ as

$$\mathcal{R}\begin{pmatrix} f_{12}\\ f_{10}\\ f_{20}\end{pmatrix} = \begin{pmatrix} \mathcal{G}[f_{10}, f_{20}]\\ \mathcal{R}_-f_{12}\\ \mathcal{R}_+f_{12}\end{pmatrix}.$$

Then we may re-write (5.7) as

$$\mathcal{R}\mathcal{L}\begin{pmatrix} f_{12}^0(h)\\ f_{10}(l_{12},h)\\ f_{20}(l_{12},h)\end{pmatrix} + \begin{pmatrix} g(h)\\ 0\\ 0\end{pmatrix} = \begin{pmatrix} f_{12}^0(h)\\ f_{10}(l_{12},h)\\ f_{20}(l_{12},h)\end{pmatrix}. \tag{5.9}$$

In a sense, (5.9) views (5.1)-(5.6) as a boundary value problem while (5.7) treats it is a periodic "in time" solution. The operator $\mathcal{Q} = \mathcal{R}\mathcal{L}$ is compact since $\mathcal{L}$ is compact. Observe that $\mathcal{Q}^2$ may be written as

$$\begin{aligned}\mathcal{Q}^2\begin{pmatrix} f_{12}\\ f_{10}\\ f_{20}\end{pmatrix} &= \mathcal{R}\mathcal{L}\begin{pmatrix} \mathcal{G}[L_{10}f_{10}, L_{20}f_{20}]\\ \mathcal{R}_-(L_{12}f_{12})\\ \mathcal{R}_+(L_{12}f_{12})\end{pmatrix}\\ &= \begin{pmatrix} \mathcal{G}[L_{10}(\mathcal{R}_-(L_{12}f_{12})), L_{20}(\mathcal{R}_+(L_{12}f_{12}))]\\ \mathcal{R}_-(L_{12}(\mathcal{G}[L_{10}f_{10}, L_{20}f_{20}]))\\ \mathcal{R}_+(L_{12}(\mathcal{G}[L_{10}f_{10}, L_{20}f_{20}]))\end{pmatrix}.\end{aligned} \tag{5.10}$$

The norms $\|L_{10}\|_{L^2\to L^2}$ and $\|L_{20}\|_{L^2\to L^2}$ are both less than one, as we have noted before. This implies immediately that $\|\mathcal{Q}^2\| < 1$ and thus (5.9) has a unique solution by the Fredholm alternative. This approach has a straightforward generalization to the case of more than two cells.

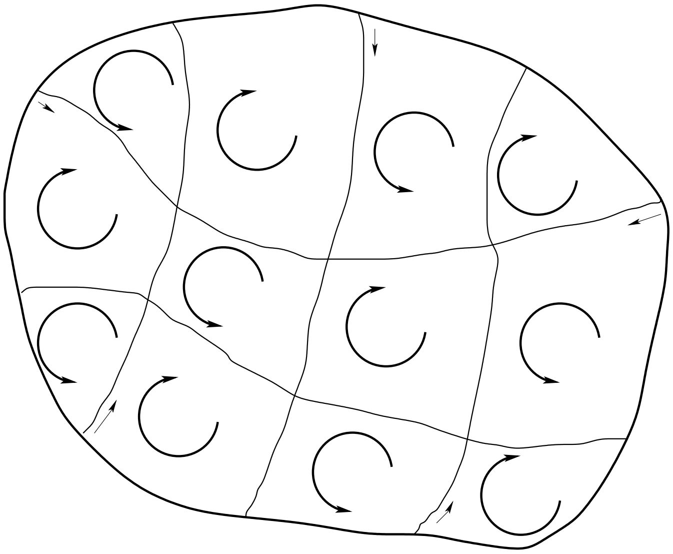

FIGURE 5.3. The velocity profile

The Dirichlet-to-Neumann map for the Childress problem

The Dirichlet-to-Neumann map for the Childress problem is defined as the mapping of $L^2(e_{10} \cup e_{20})$ onto itself by

$$\mathcal{D}^*(T_0)\big|_{e_{j0}} = \frac{\partial f_{j0}}{\partial h}\bigg|_{e_{j0}}.$$

It is straightforward to compute using (5.1), (5.3) and (5.4) that its quadratic form is given by

$$(5.11) D^*(T_0) = \langle \mathcal{D}^*(T_0), T_0 \rangle = \int_{e_{10}} T_0 \frac{\partial f_{10}}{\partial h} d\theta + \int_{e_{10}} T_0 \frac{\partial f_{10}}{\partial h} d\theta$$
$$= \int_0^N \int \left| \frac{\partial f_{10}}{\partial h} \right|^2 d\theta dh + \int_0^N \int \left| \frac{\partial f_{10}}{\partial h} \right|^2 d\theta dh + \int_{-N}^N \int \left| \frac{\partial f_{12}}{\partial h} \right|^2 d\theta dh.$$

Note that only the gradient in the direction perpendicular to the streamlines enters the Dirichlet-to-Neumann map – this is a natural consequence of the narrow boundary layer phenomenon.

5.2 The general N-cell case

We now consider the general case when the domain Ω consists of a finite number of cells. The asymptotic model is described in terms of an oriented graph constructed using the stream function H as shown on Figures 5.3 and 5.4. The vertices of this graph are associated with the saddle points of H. The edges e_{ij} of the graph are associated with the separatrices of the the stream function. The direction of an edge is determined by the direction of

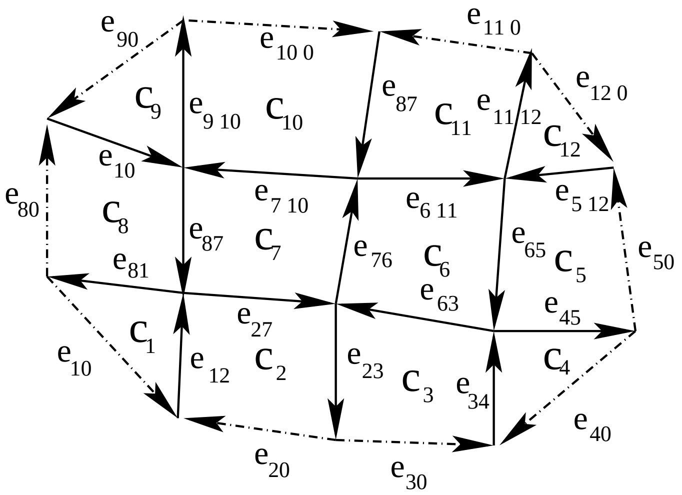

FIGURE 5.4. The graph

the velocity field on the corresponding separatrix. The length of an edge is determined by the length of the separatrix in the boundary layer coordinate θ associated with H. The boundary edges are those that are associated with the separatrices at the boundary of the domain. The cells $\mathcal{C}_i$ are quadrangles bounded by minimal cycles of the graph. The interior edges (drawn as solid arrows on Figure 5.4) are indexed so that a common edge of two cells $\mathcal{C}_i$ and $\mathcal{C}_j$ is denoted by e_{ij}. The boundary edges (drawn as dotted arrows on Figure 5.4) are indexed so that the outer part of a boundary cell $\mathcal{C}_i$ is denoted by e_{i0}. The boundary value problem is:

- [i] Given the values of the temperature T_0 on the boundary edges e_{i0}, determine the values of the temperature f_{ij} on all the edges. Note that the value of f_{ij} may vary along each edge.
- [ii] Given the values of f_e on all the edges, find the solutions f_i of the Childress' problem for each cell $\mathcal{C}_i$:

$$
\begin{cases}
\frac{\partial^2 f_i}{\partial h^2} - \frac{\partial f_i}{\partial \theta} = 0, \\
h \in [0, N], \quad \theta \in]-\infty, +\infty[, \\
f_i(h=0,\theta) = f_{ik}(\theta), \\
\frac{\partial f_i}{\partial h}(h=N,\theta) = 0, \\
f_i(h,\theta) = f(h, \theta + l_i),
\end{cases}
\tag{5.12}
$$

where the index k takes four values of the adjacent cells, $l_i = l_{ik_1} + \cdots + l_{ik_4}$ is the length in θ of the four edges $e_{ik_1}, \ldots\ e_{ik_4}$, bounding $\mathcal{C}_i$ and $f_{ik}(\theta) = f_{ik_1}(\theta)$, $\ldots$, $f_{ik}(\theta) = f_{ik_4}(\theta)$ are the values of the temperature on respective edges.

- [iii] When any two cells $\mathcal{C}_i$ and $\mathcal{C}_j$ share a common edge, the normal derivatives from the left and from the right match point-wise on this edge:
$$\frac{\partial f_i}{\partial h}\Big|_{h=0} + \frac{\partial f_j}{\partial h}\Big|_{h=0} = 0 \text{ on } e_{ij}.$$

We note that it is straightforward to extend the Dirichlet-to-Neumann map (5.11) for the two-cell Childress problem to the N-cell case formulated above.

Theorem 5.2. *There exists a unique solution of the boundary value problem [i], [ii], [iii].*

Proof. The proof generalizes the construction in two-cell case considered in Proposition 5.1 to the general situation in a fairly straightforward albeit somewhat tedious manner. Assume that a solution to the boundary value problem [i],[ii],[iii] is found. Then the solutions f_i and f_j on two adjacent cells $\mathcal{C}_i$ and $\mathcal{C}_j$ are such that they can be glued together into one function $f_{ij}(\theta, h)$, $h \in [-N, N]$, $\theta \in [0, l_{ij}]$ so that (possibly after an appropriate shift of θ by a constant)

$$f_{ij}(\theta, h) = f_i(\theta, h) \text{ for } h > 0, \text{ and} f_{ij}(\theta, h) = f_j(\theta, -h) \text{ for } h \leq 0.$$

The function f_{ij} satisfies the heat equation

$$\begin{aligned} \frac{\partial^2 f_{ij}}{\partial h^2} - \frac{\partial f_{ij}}{\partial \theta} &= 0, \\ \frac{\partial f_{ij}}{\partial h}(h = \pm N, \theta) &= 0 \end{aligned} \tag{5.13}$$

on $(h, \theta) \in [-N, N] \times [0, l_{ij}]$. Equation (5.13) can be solved uniquely as a Cauchy problem, provided that the initial data

$$f_{ij}^0(h) = f_{ij}(h, \theta = 0) \tag{5.14}$$

is given. Therefore, we may define a linear operator

$$L_{ij} : f_{ij}^0(h) \to f_{ij}^1(h),$$

which maps the function $f_{ij}^0(h)$, assigned to the beginning of an interior edge e_{ij}, to its value $f_{ij}^1(h) = f_{ij}(l_{ij}, h)$ at the end of this edge by solving the heat equation (5.13),(5.14). For boundary edges the operator L_{i0} and, hence, $f_{i0}^1(h)$ are defined by solving the homogeneous heat equation in half-space:

$$\begin{aligned} \frac{\partial^2 \bar{f}_{i0}}{\partial h^2} - \frac{\partial \bar{f}_{i0}}{\partial \theta} &= 0, \\ \frac{\partial \bar{f}_{i0}}{\partial h}(h = N, \theta) &= 0, \\ \bar{f}_{i0}(h = 0, \theta) &= 0, \\ \bar{f}_{i0}(h = 0, \theta) &= f_{i0}^0(h), \end{aligned} \tag{5.15}$$

on $(h,\theta)\in[0,N]\times[0,l_{i0}]$. We denote by $g_{i0}(h)$ $h\in[0,+\infty)$ solutions of the inhomogeneous heat equation "along the boundary edge e_{i0}"

$$\begin{aligned}\frac{\partial^2 g_{i0}}{\partial h^2}-\frac{\partial g_{i0}}{\partial\theta}&=0,\\ \frac{\partial g_{i0}}{\partial h}(h=N,\theta)&=0,\\ g_{i0}(h=0,\theta)&=f_{i0}(\theta),\\ g_{i0}(h=0,\theta)&=0,\end{aligned}\tag{5.16}$$

on $(h,\theta)\in[0,N]\times[0,l_{i0}]$. Hence, if f solves the boundary value problem [i], [ii], [iii], then the corresponding vector-valued function

$$f^0=(f^0_{10},\dots,f^0_{ij},\dots,f^0_{km})$$

solves

$$\mathcal{RL}f^0+g=f^0,\tag{5.17}$$

similar to (5.9) where $g=(g_{10},g_{20},g_{30},\dots,g_{m0},0,\dots,0)$ and $\mathcal{L}=\otimes L_{ij}$. The first (non-zero) components of the vector g (and those of f) correspond to the vertices at the boundary where the flow u is incoming: there is only one such vertex in the two-cell case and hence g has only one non-zero component in (5.9). The operator $\mathcal{R}$

$$\mathcal{R}:f^1\to f^0$$

is a linear redistribution operator. Given the values f^1_{ij} at the ends of the edges the operator $\mathcal{R}$ constructs the values $f^0_{i'j'}$ at the beginnings of the edges at each vertex in a natural way: f must be a continuous function in each cell. Given the problem (5.17) is solved uniquely, the boundary value problem [i],[ii],[iii] is equivalent to (5.17) as both amount to solving the heat equations (5.13), (5.15), (5.16). Therefore it remains to show that

$$(\mathcal{RL}-I)f^0=-g,\tag{5.18}$$

has a unique solution. However, the unique solvability of (5.18) follows from the Fredholm alternative. Indeed, the operator $\mathcal{R}$ is clearly bounded on $[L^2([-N,N])]^k$ (here k is the number of edges) by construction. The operator $\mathcal{L}$ is compact on $[L^2([-N,N])]^k$ for the same reason as in the case of two cells; it is associated with the solution of the heat equation. Moreover, $\lambda=1$ is not an eigenvalue of the compact operator $\mathcal{RL}$. Indeed, each boundary operator L_{i0} has norm less than one: $\|L_{i0}\|<1$. Therefore, if we let M be the total number of edges, we have $\|(\mathcal{RL})^M\|<1$ and thus $\mathcal{RL}$ may not have eigenvalue equal to one. □

6 Approximation by the asymptotic problem

We now compare the function ϕ_N^ε, the solution of the approximate water-pipe problem (4.4), to the stretched asymptotic boundary layer solution

$$f^\varepsilon(x,y) = f(H(\mathbf{x})/\sqrt{\varepsilon}, \theta(\mathbf{x})).$$

Here $f(h,\theta)$ is the unique solution of the Childress' problems described in Section 5.2 and Theorem 5.2. The function $f(h,\theta)$ is smooth except at the points $(h=0,\theta_{jk})$ that correspond to saddle points of the stream function H, where f is discontinuous. This necessitates a careful local analysis near the corners. Hence, we postpone the formulation of the main result of this Section, Theorem 6.1, until we define all the local coordinates. Here we just mention that we build our approximation as close to the Chilcress solution f away from the corners – at distances larger than $M\varepsilon^{1/4}$ with $M \gg N$. We will use an orthogonal system $(h = H/\sqrt{\varepsilon}, \theta_{jk})$ along each edge e_{jk} that separates cells $\mathcal{C}_j$ and $\mathcal{C}_k$, and at indicated distances away from the corners. However, a different coordinate system and a different approximation are needed near the corners. We begin with the introduction of suitable local coordinate systems.

6.1 The local coordinates

Observe that the advection-diffusion equation (1.1) has the following form in an orthogonal system of coordinates of the form $(h = H/\sqrt{\varepsilon}, \theta)$:

$$|\nabla H|^2 \frac{\partial^2 f}{\partial h^2} + \sqrt{\varepsilon}\Delta H \frac{\partial f}{\partial h} + \varepsilon \Delta\theta \frac{\partial f}{\partial \theta} + \varepsilon |\nabla\theta|^2 \frac{\partial^2 f}{\partial \theta^2} - J\frac{\partial f}{\partial \theta} = 0 \tag{6.1}$$

with $J = \nabla^\perp H \cdot \nabla\theta = |\nabla H||\nabla\theta|$. Therefore, in order to have at least a formal approximation of (6.1) by (5.12) as $\varepsilon \to 0$ we should have $J \approx |\nabla H|^2$, or, equivalently, $|\nabla H| \approx |\nabla\theta|$ in the boundary layer $|H| \le N\sqrt{\varepsilon}$. We impose the condition $|\nabla H| = |\nabla\theta_{jk}|$ along the edge e_{jk}. However, the coordinate θ_{jk} introduced in such way may have a singularity at the end-points of e_{jk}. Therefore we will use these coordinates only away from the corners.

In order to perform a local analysis near the corners we may introduce the local orthogonal coordinates (X,Y) in a δ-neighborhood of a corner that we fix at $\mathbf{x}=0$, so that near the saddle point we have

$$H = X^2 - kY^2. \tag{6.2}$$

Moreover, we may assume that the change of variables satisfies

$$D_{\mathbf{x}}\mathbf{X} = U + O(\mathbf{x}), \ \ \Delta_{\mathbf{x}}\mathbf{X} = O(\mathbf{x}) \tag{6.3}$$

with U a unitary matrix. Such change of coordinates always exists according to the Morse lemma in a ball $|\mathbf{x}| \le \delta$ near the saddle point with $\delta > 0$ sufficiently small. We may assume without loss of generality that the constant $k \ge 1$. Then the separatrices are given by $X = \pm\sqrt{k}Y$ in the variables

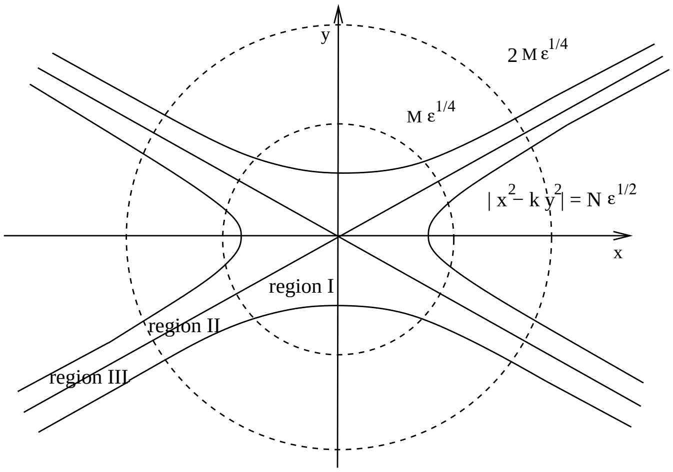

FIGURE 6.1. The regions near the corner

(X, Y). In order to simplify the notation we will assume that actually at the corner the function H has the form (6.2) in the old coordinate system (x, y) and no change of variables is required. Extension to the general case using the coordinates (X, Y) is straightforward, with the help of the estimates (6.3), at the expense of slightly lengthier calculations. We omit them for the sake of readability. Under our assumptions, the coordinate θ, orthogonal to H, is defined along the whole edge e_{jk}, and is given explicitly near the corner by

$$\theta = B_k (x^k y)^{\frac{2}{k+1}}. \tag{6.4}$$

The normalizing constant is chosen to be $B_k = (k+1)k^{-(k-1)/(2(k+1))}$. It is fixed by the requirement that we have $|\nabla\theta| = |\nabla H|$ along the separatrices $|x| = \sqrt{k}|y|$. With such a choice of B_k we obtain

$$\nabla\theta = 2\left(\frac{x}{\sqrt{k}y}\right)^{\frac{k-1}{k+1}}(ky, x). \tag{6.5}$$

We will use the following three regions inside the boundary layer (see Figure 6.1):

$$I = \left\{(x, y) \in \Omega_N^\varepsilon : \ \theta(x, y) \le M^2\sqrt{k\varepsilon}\right\} \tag{6.6}$$

is the region around the corner. The region

$$II = \left\{(x, y) \in \Omega_N^\varepsilon : \ M^2\sqrt{k\varepsilon} \le \theta(x, y) \le 4M^2\sqrt{k\varepsilon}\right\} \tag{6.7}$$

is the next closest, and

$$III = \left\{(x, y) \in \Omega_N^\varepsilon : \ 4M^2\sqrt{k\varepsilon} \le \theta(x, y)\right\} \tag{6.8}$$

is the farthest from the corner. Region III extends all the way to the adjacent corner along the edge. The constant $\sqrt{k}$ is included for convenience in the definition of these regions, because

$$|\theta| = \frac{(k+1)}{\sqrt{k}} \left(x^k \sqrt{k} y\right)^{\frac{2}{k+1}} \approx \sqrt{k}(x^2+y^2) \tag{6.9}$$

inside the boundary layer $\{|H| \leq N\sqrt{\varepsilon}\}$, as $x \approx \sqrt{k}y$. Hence the boundaries of the three regions are approximately parts of the circles: $\sqrt{x^2+y^2} \approx M\varepsilon^{1/4}$ and $\sqrt{x^2+y^2} \approx 2M\varepsilon^{1/4}$.

We now show that for distances larger than $M\varepsilon^{1/4}$ away from the corner inside the boundary layer (regions II and III) the desired approximation $J = \nabla\theta \cdot \nabla^\perp H \approx |\nabla H|^2$ is valid. An elementary geometric calculation shows that in region $II \cup III$ we have

$$\left|\frac{x^2}{ky^2} - 1\right| \leq \frac{Ch}{M^2 - h} \leq C\frac{h}{M^2}, \ h \leq N, \tag{6.10}$$

as $M \gg N$. Combining the last inequality with (6.5), and using the form (6.2) of the stream function H near the corner we have

$$\left||\nabla H|^2 - J\right| \leq C\left(x^2 + k^2y^2\right)\frac{h}{M^2}. \tag{6.11}$$

Similarly, we have that $\Delta\theta$ is uniformly bounded in the same region (regions II and III):

$$|\Delta\theta| \leq C\frac{N}{M^2}. \tag{6.12}$$

Observe also the following uniform bounds:

$$\frac{|J - |\nabla H|^2|}{|\theta|} \leq C\frac{h}{M^2}, \ \frac{|\nabla H|^2}{|\theta|} \leq C, \ \frac{|J|}{|\theta|} \leq C \tag{6.13}$$

that we will need later. Here $\theta = 0$ is the coordinate of the saddle point. Indeed, the inequalities (6.13) are trivially true, when $|\theta| > \delta$. In the δ-neighborhood of the saddle point we have (6.13) by using (6.11) and $\theta > C(x^2+y^2)$ in the boundary layer. Note that these estimates may not be pushed all the way to the corner $\mathbf{x} = 0$, that is, inside region I, as (6.10) breaks down, and $|\nabla\theta|$ blows up at the saddle point except in the special case $k = 1$. This is another reason why the Childress solution may not be used at the corner.

6.2 The approximate solution

We may now present the main result of this section, Theorem 6.1. The approximation to the solution of the full problem is constructed as follows.

Let χ be a smooth cut-off function such that $\chi(r) = 0$ for $0 \le r \le 1$ and $\chi(r) = 1$ for $r \ge 4$. We denote by $\mathbf{x}_{jk}$ the saddle points of H and let

$$\phi_N^{\varepsilon,app}(\mathbf{x}) = \sum_{j,k} f^\varepsilon(\mathbf{x})\chi\left(\frac{|\theta(\mathbf{x})|}{M^2\varepsilon^{1/2}}\right) + \sum_{j,k}\left[1 - \chi\left(\frac{|\theta(\mathbf{x})|}{M^2\varepsilon^{1/2}}\right)\right]\bar{f}_{ij}^\varepsilon(\mathbf{x}),$$

(6.14) $\Phi^\varepsilon(\mathbf{x}) = \phi_N^{\varepsilon,app}(\mathbf{x}) - \phi_N^\varepsilon(\mathbf{x})$.

Here ϕ_N^ε is the solution of the water-pipe problem (4.4), the function $f^\varepsilon(\mathbf{x}) = f(H(\mathbf{x})/\sqrt{\varepsilon}, \theta(\mathbf{x}))$ is the stretched solution of the Childress problem, and the function $\bar{f}_{ij}^\varepsilon$ satisfies the exact problem

$$\varepsilon\Delta\bar{f}_{jk}^\varepsilon - u \cdot \nabla\bar{f}_{jk}^\varepsilon = 0 \tag{6.15}$$

on the domain $G = I \cup II$ near the corner (see (6.6), (6.7)), that we again fix to be at $\mathbf{x}_{jk} = 0$ in the local analysis that follows, so that

$$G = \left\{\mathbf{x} :\ \chi\left(\frac{|\theta|}{M^2\varepsilon^{1/2}}\right) \neq 1\right\}.$$

The boundary ∂G consists of two parts: ∂G_n that is part of the level set $|H(\mathbf{x})| = N\sqrt{\varepsilon}$, and ∂G_d that consists of pieces of the curve $|\theta| = 4M^2\sqrt{k\varepsilon}$, which is close to the circle $|\mathbf{x}| = 2M\varepsilon^{1/4}$. We prescribe the homogeneous Neumann boundary conditions for $\bar{f}^\varepsilon$ on ∂G_n and the Dirichlet boundary condition $\bar{f}_{jk}^\varepsilon(\mathbf{x}) = f^\varepsilon(\mathbf{x})$ on ∂G_d. That is, $\bar{f}^\varepsilon$ coincides with $f^\varepsilon(\mathbf{x})$ on ∂G_d.

Qualitatively, since the Childress solution is not smooth near the corners, we cut the approximation f at a distance $M\varepsilon^{1/4}$, $M \gg N$ away from the corners and glue into the corners solution of the true original equation that coincides with the approximation on the gluing set. For the distances between $M\varepsilon^{1/4}$ and $2M\varepsilon^{1/4}$ we interpolate the two functions.

Theorem 6.1. *The boundary layer approximation $\phi_N^{\varepsilon,app}$ given by (6.14) approximates the water-pipe solution ϕ_N^ε of (4.4) in the sense that there exists a constant $C > 0$ so that*

$$\begin{aligned}\int_{\mathcal{D}(N\sqrt{\varepsilon})} &|\nabla\phi_N^\varepsilon(\mathbf{x}) - \nabla\phi_N^{\varepsilon,app}|^2\, d\mathbf{x} \\ &\le \frac{C}{\sqrt{\varepsilon}}\left(\varepsilon^{2\alpha} + \varepsilon^{1/4+\alpha} + \sqrt{N\varepsilon^{2\alpha} + N^3(\varepsilon^{1/4-\alpha} + \varepsilon^{1/2-2\alpha})}\right)\end{aligned} \tag{6.16}$$

Moreover, the interior constants $\tilde{K}_{m,N}^\varepsilon$ for the water-pipe solution ϕ^ε and the constants $K_{m,N}$ obtained from the asymptotic problem satisfy

(6.17)

$$|\tilde{K}_{m,N}^\varepsilon - K_{m,N}| \le C\sqrt{\varepsilon}\left(\varepsilon^{2\alpha} + \varepsilon^{1/4+\alpha} + \sqrt{N\varepsilon^{2\alpha} + N^3(\varepsilon^{1/4-\alpha} + \varepsilon^{1/2-2\alpha})}\right).$$

Finally, the asymptotic constants $K_{m,N}$ converge to certain constants K_m as $N \to \infty$, and, moreover, the interior constants K_m^ε of the true solution converge as $\varepsilon \to 0$ to K_m.

We note, first, that the right side in the gradient bound (6.16) is of the order smaller than $O(\varepsilon^{-1/2})$, the size of the gradient of the solution itself. Second, (6.17) shows that for each N fixed the interior value of the solution of the water-pipe converges as $\varepsilon \to 0$ to that given by the asymptotic solution of the Childress problem at this N.

6.3 The proof of the approximation Theorem 6.1

The proof of Theorem 6.1 is fairly straightforward if long. The main difficulty is the analysis near the corners – this is overcome with the help of an explicit solution, the function f_2 constructed below in (6.34)-(6.35). It mimics the behavior of the exact solution quite precisely and allows us to obtain the necessary bounds. The rest of the proof is fairly routine. We point out that many technical difficulties disappear when the corner is at a right angle. This is best seen from the explicit expression (6.4) for the local coordinate θ: when $k = 1$ (the right angle case) it is smooth, otherwise it is not. That is one of the reasons why the variational methods and test functions used in [5, 6] for the estimates on the effective diffusivity worked so well in the case of a square cell.

Bounds for the Childress solution

We begin with some bounds for the Childress solution. We may decompose the function f at the corner $\mathbf{x}_{jk}$ into a smooth and a discontinuous component as

$$f(\theta_{jk}, h) = f_{sm}(\theta_{jk}, h) + B_{jk} s(h), \;\; s(h) = \begin{cases} 0, & \text{for } h \le 0, \\ 1, & \text{for } h > 1. \end{cases} \tag{6.18}$$

With the convention of Section 6.1 we have $\theta_{jk} = 0$. Here $s(h)$ is the Heaviside function, B_{jk} is the magnitude of the jump of f that appears because of gluing together of two solutions that come from different cells, and f_{sm} is a smooth function, except for the corners, where f_{sm} is continuous. Hence

$$\int_{-N}^{N} \left(\frac{\partial f_{sm}}{\partial h}\right)^2 dh \le C. \tag{6.19}$$

The function f solves the boundary value Childress' problem inside each cell, hence f_{ij} converges exponentially to the corresponding constants K_i and K_j away from the separatrix

$$|f_{ij}(h) - K_i| \le \exp(-c|h|), \; h \ge 0, \;\; |f_{ij}(h) - K_j| \le \exp(-c|h|), \; h \le 0.$$

Decomposition (6.18) implies that f satisfies the following bounds:

$$\left|\frac{\partial f}{\partial \theta}\right| \le \frac{C}{|\theta|}, \;\; \left|\frac{\partial^2 f}{\partial \theta^2}\right| \le \frac{C}{|\theta|^2}. \tag{6.20}$$

These estimates follow from the explicit expression for the solution of the heat equation on the interval $-N \leq h \leq N$ with the Neumann boundary conditions at $h = \pm N$, and with the initial data $f(h, 0)$ as in (6.18). We can also estimate in a similar fashion, for θ close to zero,

$$\|f(\theta) - f(0)\|^2_{L^2(-N,N)} \leq C\sqrt{\theta}, \tag{6.21}$$

where the main contribution comes from the discontinuous part of f in (6.18). Similar considerations lead to a better bound for f_{sm}:

$$|f_{sm}(\theta, h) - f_o(h)| \leq C\sqrt{\theta}, \text{ where } f_o(h) = f_{sm}(h, 0). \tag{6.22}$$

for all $h \in (-N, N)$.

Estimates on the corner solution

We will use the following bounds on the solution at the corner. The first on provides a poor but sufficient for our purposes bound on its H^1-norm.

Lemma 6.2. *Solution of (6.15) with the boundary conditions as above satisfies the following bound:*

$$\varepsilon \int_G |\nabla \bar{f}|^2 d\mathbf{x} \leq CN\sqrt{\varepsilon}. \tag{6.23}$$

The second lemma shows that the corner solution is close to the Childress solution.

Lemma 6.3. *Solution of (6.15) satisfies the following bound:*

$$\|\bar{f} - f\|^2_{L^2(II)} \leq C\frac{N^2\sqrt{\varepsilon}}{M^2} + CMN\varepsilon^{3/4} + C\varepsilon M^2. \tag{6.24}$$

where f is the Childress solution.

The proof of Lemma 6.3 is the main difficulty in the proof of Theorem 6.1. We postpone the proof of both of the above lemmas until after the proof of Theorem 6.1.

Bounds for the remainder

The error function Φ^ε defined by (6.14) satisfies an equation inside the boundary layer $\Omega^\varepsilon_N = \{|H(\mathbf{x})| \leq N\sqrt{\varepsilon}\}$ of the form

$$\varepsilon\Delta\Phi^\varepsilon - u \cdot \nabla\Phi^\varepsilon = g^\varepsilon. \tag{6.25}$$

The function $g^\varepsilon = 0$ for distances less than $M\varepsilon^{1/4}$ away from the corner, that is, in region I:

$$g^\varepsilon = 0 \text{ in region } I \tag{6.26}$$

as both ϕ_N^ε and $\bar{f}_{ij}$ are exact solutions of (6.25). Furthermore, for distances larger than $2M\varepsilon^{1/4}$ away from the corner, that is, in region III, the function g^ε may be written in the $h = H/\sqrt{\varepsilon}$, θ coordinates as

$$\begin{aligned} g^\varepsilon &= \left[|\nabla H|^2\frac{\partial^2 f}{\partial h^2} + \sqrt{\varepsilon}\Delta H\frac{\partial f}{\partial h} + \varepsilon\Delta\theta\frac{\partial f}{\partial\theta} + \varepsilon|\nabla\theta|^2\frac{\partial^2 f}{\partial\theta^2} - J\frac{\partial f}{\partial\theta}\right] \\ &= \left[(|\nabla H|^2 - J)\frac{\partial f}{\partial\theta} + \sqrt{\varepsilon}\Delta H\frac{\partial f}{\partial h} + \varepsilon\Delta\theta\frac{\partial f}{\partial\theta} + \varepsilon|\nabla\theta|^2\frac{\partial^2 f}{\partial\theta^2}\right]. \end{aligned}$$

It may be now estimated as follows. Using the first inequalities in (6.13) and (6.20) we bound the first term in the first bracket as

$$\left|(J - |\nabla H|^2)\frac{\partial f}{\partial\theta}\right| \le C\frac{h}{M^2}.$$

Similarly, using the bound $\sqrt{|\theta|} > CM\varepsilon^{1/4}$ we estimate the other terms in the first bracket:

$$\left|\sqrt{\varepsilon}\Delta H\frac{\partial f}{\partial h}\right| \le C\frac{\varepsilon^{1/4}}{M},\ \left|\varepsilon\Delta\theta\frac{\partial f}{\partial\theta}\right| \le C\frac{\sqrt{\varepsilon}}{M^2}$$

and

$$\varepsilon|\nabla\theta|^2\left|\frac{\partial^2 f}{\partial\theta^2}\right| \le C\frac{\sqrt{\varepsilon}}{M^2}.$$

Therefore we have in region III

$$|g^\varepsilon| \le C\left[\frac{h}{M^2} + \frac{\varepsilon^{1/4}}{M}\right] \le C\left[\frac{N}{M^2} + \frac{\varepsilon^{1/4}}{M}\right]. \tag{6.27}$$

It remains to estimate the error term in region II. There we have

$$\begin{aligned} g^\varepsilon &= \chi\left[\varepsilon\Delta f^\varepsilon - u\cdot\nabla f^\varepsilon\right] + 2\varepsilon\left[\nabla f^\varepsilon\cdot\nabla\chi - \nabla\bar{f}^\varepsilon\cdot\nabla\chi\right] \\ &+(f^\varepsilon - \bar{f}^\varepsilon)\left[\varepsilon\Delta\chi - u\cdot\nabla\chi\right] = g_1 + g_2 + g_3 \end{aligned} \tag{6.28}$$

The first term can be estimated as above:

$$|g_1| \le C\left[\frac{h}{M^2} + \frac{\varepsilon^{1/4}}{M}\right] \le C\left[\frac{N}{M^2} + \frac{\varepsilon^{1/4}}{M}\right] \text{ in region } II \tag{6.29}$$

since estimates (6.13) hold in region II as well. In order to estimate the second term we use Lemma 6.2. Lemma 6.2 and the estimate (6.33) below in region II allow us to bound the second term in g^ε in this region. Indeed, we again have that the area where $\nabla\chi \neq 0$ is bounded by $CN\sqrt{\varepsilon}$. Hence g_2 is bounded as

$$\|g_2\|_{L^2(II)} \le \frac{C\varepsilon}{M\varepsilon^{1/4}}\frac{1}{\sqrt{\varepsilon}}\left(N\sqrt{\varepsilon}\right)^{1/2} + \frac{C\varepsilon^{1/2}}{M\varepsilon^{1/4}}\left(N\sqrt{\varepsilon}\right)^{1/2} \le C\frac{\sqrt{N\varepsilon}}{M}. \tag{6.30}$$

It remains to estimate g_3, the third term in region II. This is done using Lemma 6.3. Lemma 6.3 implies that the third term in (6.28) may be estimated as

$$\|g_3\|_{L^2(II)} \le \|f-\bar{f}\|_{L^2(II)}\|[\varepsilon\Delta\chi + u\cdot\nabla\chi]\|_{L^\infty} \le C\sqrt{\frac{N^2\sqrt{\varepsilon}}{M^2} + CMN\varepsilon^{3/4} + C\varepsilon M^2}. \tag{6.31}$$

By construction Φ^ε is approximately constant (within $C/N^{3/2}$) on each level set $|H(\mathbf{x})| = N\sqrt{\varepsilon}$ and it satisfies homogeneous boundary conditions. Our goal is to show that these constants are small. Using (6.27), (6.29) , (6.30), (6.31) we obtain for Φ^ε:

$$\varepsilon\int_{\mathcal{D}(N\sqrt{\varepsilon})}|\nabla\Phi^\varepsilon|^2 d\mathbf{x} \le CN\sqrt{\varepsilon}\left(\frac{N}{M^2}+\frac{\varepsilon^{1/4}}{M}\right) + C\frac{N}{M}\varepsilon^{3/4} + C\sqrt{N\sqrt{\varepsilon}}\sqrt{\frac{N^2\sqrt{\varepsilon}}{M^2} + CMN\varepsilon^{3/4} + C\varepsilon M^2}.$$

Choosing

$$M = \varepsilon^{-\alpha}N, 0 < \alpha < 1/4,$$

we have

$$\sqrt{\varepsilon}\int\limits_{\mathcal{D}(N\sqrt{\varepsilon})}|\nabla\Phi^\varepsilon|^2 d\mathbf{x} \le C\left(\varepsilon^{2\alpha} + \varepsilon^{1/4+\alpha} + \sqrt{N\varepsilon^{2\alpha} + N^3(\varepsilon^{1/4-\alpha} + \varepsilon^{1/2-2\alpha})}\right).$$

This implies the first two statements in Theorem 6.1.

It remains only to verify the last statement in Theorem 6.1. However, it follows immediately from (6.17) and the uniform in ε error bounds (4.7). Indeed, we have from these estimates

$$|K_m^\varepsilon - K_{m,N}| \le \frac{C}{N^{3/2}} + o(\varepsilon). \tag{6.32}$$

This implies that the sequence $K_{m,N}$ converges as $N \to \infty$ in an elementary way. Indeed, if it has two limit points A_m and B_m then given any $\delta > 0$ we may choose ε so small that $|K_m^\varepsilon - K_{m,N}| \le \delta$ for all $N \ge N_0$. This in particular implies that $|A_m - B_m| \le 2\delta$ and hence $K_{m,N}$ converges to a limit K_m as $N \to \infty$. Then (6.32) implies that K_m^ε converges to the same limit as $\varepsilon \to 0$. $\Box$

The proof of Lemma 6.2

We write $\bar{f} = q + f^\varepsilon\eta(\theta/(M^2\sqrt{\varepsilon}/2))$, that is we cut-off f at distance $M\varepsilon^{1/4}/2$ from the corner. Here η is a cut-off function of the same kind as

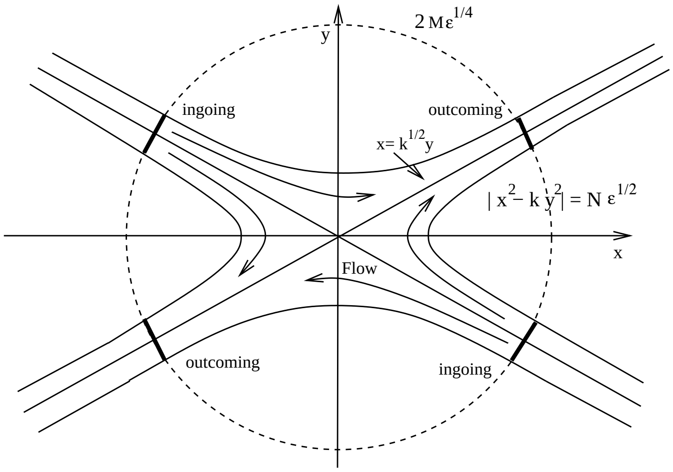

FIGURE 6.2. The incoming and outgoing parts at the corner

χ. Then the function q satisfies

$$\varepsilon\Delta q - u\cdot\nabla q = -p^\varepsilon,$$
$$p^\varepsilon = -\eta[\varepsilon\Delta f^\varepsilon - u\cdot\nabla f^\varepsilon] - f^\varepsilon[\varepsilon\Delta\eta - u\cdot\nabla\eta] - 2\varepsilon\nabla f^\varepsilon\cdot\nabla\eta = p_1+p_2+p_3$$

with the homogeneous Dirichlet boundary conditions on G_d and the homogeneous Neumann boundary conditions on G_n. Note that $|p_1|\le CN/M^2$ - this term is estimated as the first term in g^ε.The second term is bounded as $|p_2|\le C$, while the last one is estimated by $|p_3|\le C\varepsilon^{3/4}|\nabla f^\varepsilon|/M$, because $|\nabla\eta|\le C/(M\varepsilon^{1/4})$. However, we have in the region where $\nabla\eta\neq 0$:

$$|\nabla f^\varepsilon| \le \frac{|\nabla H|}{\sqrt{\varepsilon}}\left|\frac{\partial f}{\partial h}\right| + |\nabla\theta|\left|\frac{\partial f}{\partial\theta}\right| \le \frac{CM\varepsilon^{1/4}}{\sqrt{\varepsilon}}\frac{1}{M\varepsilon^{1/4}} + \frac{C}{M\varepsilon^{1/4}} \le \frac{C}{\sqrt{\varepsilon}} \tag{6.33}$$

so that $|p_3|\le C\varepsilon^{1/4}/M$. Observe that the area of the region where $\eta\neq 0$ is bounded by $CN\sqrt{\varepsilon}$, where the constant C is independent of M. Therefore we obtain, since q is uniformly bounded as a difference of two bounded functions:

$$\varepsilon\int_G |\nabla q|^2 d\mathbf{x} = \int_{G,\eta\neq 0} qp^\varepsilon d\mathbf{x} \le C\left(\frac{N}{M^2}+1+\frac{\varepsilon^{1/4}}{M}\right)N\sqrt{\varepsilon}\le CN\sqrt{\varepsilon}.$$

This estimate, combined with the bound (6.33) in the region where $\eta\neq 0$ proves (6.23). □

The proof of Lemma 6.3

The boundary ∂G_d consists of four parts: ∂G_d^j, $j=1,\dots,4$, one in each of the coordinate quadrants. The flow $u=(2ky,2x)$ is incoming on $\partial G_d^{in}=\partial G_d^{2,4}$ and outgoing on $\partial G_d^{out}=\partial G_d^{1,3}$ (see Figure 6.2). We will show that the first term in (6.24) comes from the Heaviside function $s(h)$ in decomposition (6.18) while the second and the third terms in (6.24) come from the continuous piecewise smooth part in (6.18). Hence we first prove inequality (6.24) in the special case when $f=1$ on ∂G_d^2 and $f=-1$ on ∂G_d^4. The values of f on ∂G_d^{out} are determined by solving the heat equation with the Neumann boundary conditions at $h=\pm N$, for a time $\theta=4M^2\sqrt{k\varepsilon}$ and the initial data $f_{in}(\theta=0,h)=\mathrm{sgn}(h)$.

We claim that both the function f^ε and the function $\bar{f}$ for such data are very well approximated by an exact solution of (6.15) with $u=(2ky,2x)$ in the form $f_2=f_2(t)$, $t=x-\sqrt{k}y$. It mimics very precisely the behavior of f^ε with the discontinuous data as we are considering. The function f_2 satisfies

$$(1+k)\varepsilon f_2''+2\sqrt{k}tf_2'=0 \tag{6.34}$$

so that

$$f_2(t)=-1+\alpha(k)\int_{-\infty}^{t}\exp\left(-\frac{\sqrt{k}s^2}{(1+k)\varepsilon}\right)ds. \tag{6.35}$$

The constant $\alpha(k)$ is chosen so that $f_2(+\infty)=1$. Observe that f_2 approximately satisfies the Neumann boundary conditions on the ∂G_n part of the boundary:

$$\left|\frac{\partial f_2}{\partial n}\right|\le C\exp\left(-C\frac{N}{M^2\sqrt{\varepsilon}}\right)\ \text{ on }\partial G_n. \tag{6.36}$$

Note also that

$$|f_2-f^\varepsilon|\le C\exp\{-CM^2\varepsilon^{-1/2}\} \tag{6.37}$$

on the inflow boundary, as follows immediately from (6.35), as $|t|\sim CM\varepsilon^{1/4}$ on G_d^{in}. In order to show that f_2 is close to f^ε on the outflow boundary $G_d^{1,3}$ we first observe that the value of f^ε on $G_{1,3}$ are very well approximated by the anti-derivative of the heat-kernel on the whole real line. Let

$$\tilde{f}(\theta,h)=\frac{1}{\sqrt{2\pi\theta}}\int_{\mathbb{R}}e^{-(\xi-h)^2/(4\theta)}\mathrm{sgn}(\xi)d\xi$$

be the solution of

$$\frac{\partial f}{\partial\theta}=\frac{\partial^2\tilde{f}}{\partial h^2},\ h\in\mathbb{R},\ \tilde{f}(0,h)=\mathrm{sgn}(h).$$

Then we have

$$|f(\theta=4M^2\sqrt{k\varepsilon},h)-\tilde{f}(\theta=4M^2\sqrt{k\varepsilon},h)|\le C\exp\left(-\frac{CN^2}{M^2\sqrt{\varepsilon}}\right),\ |h|\le N$$

on the outgoing boundary. The function $\tilde{f}$ satisfies an equation along a curve $\theta = \text{const}$ of the form

$$\frac{\partial^2}{\partial h^2}\tilde{f} = -\frac{h}{2\theta}\frac{\partial}{\partial h}\tilde{f},\ \tilde{f}(-\infty) = -1,\ \tilde{f}(+\infty) = 1. \tag{6.38}$$

Now, in order to show that f_2 is uniformly close to $\tilde{f}$ (and hence to f^ε and $\bar{f}$) on the curve $\{|\theta| = 4M^2\sqrt{\varepsilon}\}$ we observe that f_2 also satisfies an equation along this curve of the form

$$\frac{\partial^2}{\partial h^2}f_2 = -\frac{c_1(h)\sqrt{\varepsilon} + h(1 + c_2(h))}{8M^2\sqrt{k\varepsilon}}\frac{\partial}{\partial h}f_2,\ h = H/\sqrt{\varepsilon} \tag{6.39}$$

with

$$|c_1(h)| \le C,\ |c_2(h)| \le C\frac{N}{M^2}. \tag{6.40}$$

This is shown as follows. Introducing the variable $s = x + \sqrt{k}y$ we note that along the outflow boundary $\theta = \text{const}$. Parametrizing ∂G_d^{out} as $s = s(t)$ we have

$$\frac{ds}{dt} = \frac{(1+k)t - (k-1)s}{(1+k)s - (k-1)t}.$$

A straightforward estimate shows that

$$|t| \le CN\varepsilon^{1/4}/M,\ |s| \sim CM\varepsilon^{1/4} \text{ along } \partial G_d^{out}. \tag{6.41}$$

Hence we obtain

$$C_1 \le \left|\frac{ds}{dt}\right| \le C_2,\ \text{along } \partial G_d^{out}. \tag{6.42}$$

We also verify by a direct calculation

$$C_1 \le s\frac{d^2 s}{dt^2} \le C_2 \text{ along } \partial G_d^{out}. \tag{6.43}$$

Parametrizing now ∂G_d^{out} in terms of $H = H(t)$ we may re-write (6.34) along ∂G_d^{out} as

$$\varepsilon(1+k)\left(\frac{dH}{dt}\right)^2\frac{d^2 f_2}{dH^2} + \left(2\sqrt{k}t\frac{dH}{dt} + \varepsilon(1+k)\frac{d^2H}{dt^2}\right)\frac{df_2}{dH} = 0.$$

Using the relation $H = ts(t)$ we obtain

$$\begin{aligned}&\varepsilon(1+k)\left(s + t\frac{ds}{dt}\right)^2\frac{d^2 f_2}{dH^2}\\ &+\left[2\sqrt{k}t\left(s + \frac{ds}{dt}\right) + \varepsilon(1+k)\left(2\frac{ds}{dt} + t\frac{d^2 s}{dt^2}\right)\right]\frac{df_2}{dH} = 0.\end{aligned}$$

This may re-stated as

$$\varepsilon\frac{d^2 f_2}{dH^2} + \frac{2\sqrt{k}}{(1+k)s^2}\left(H\frac{s}{s + ts_t} + \zeta^\varepsilon(t)\right)\frac{df_2}{dH} = 0 \tag{6.44}$$

with

$$\zeta^\varepsilon(t) = \frac{\varepsilon}{2\sqrt{k}}\left(\frac{s}{s+ts_t}\right)^2(2s_t + ts_{tt}).$$

Using the estimates (6.41), (6.42) and (6.43) we obtain

$$\left|\frac{s}{s+ts_t} - 1\right| \le C\frac{N}{M^2},\ \ |\zeta^\varepsilon(t)| \le \varepsilon C$$

However, we have along the outflow boundary, using (6.9) and (6.10)
(6.45)

$$\frac{\sqrt{k}}{(1+k)s^2} = \frac{\sqrt{k}}{(1+k)(x+\sqrt{k}y)^2} = \frac{1}{4\theta}(1+c_o(H)) = \frac{1}{16M^2\sqrt{k\varepsilon}}(1+c_o(H))$$

with $|c_o(H)| \le CN/M^2$. Then (6.39) and (6.40) follow from (6.44), (6.45) and the bounds on c_o and ζ^ε above.

Equations (6.38), (6.39) and the bounds (6.40), together with the boundary conditions for $\tilde{f}$ and f_2 at infinity imply that

$$|f_2 - f^\varepsilon| \le C\left[\frac{N}{M^2} + \frac{\varepsilon^{1/4}}{M}\right] \tag{6.46}$$

on the outflow boundary. We now let $\eta(\mathbf{x})$ be a function such that

$$\left.\frac{\partial\eta}{\partial n}\right|_{\partial G_n} = \left.\frac{\partial f_2}{\partial n}\right|_{\partial G_n},\ \ \|\eta\|_{C^2(\bar{G})} \le \frac{C\varepsilon^{100}}{N}.$$

This is possible because the bound in (6.36) is exponentially small in ε. Then the function $s = f_2 - f - \eta$ satisfies

$$|\varepsilon\Delta s - u\cdot\nabla s| = |-\varepsilon\Delta\eta + u\cdot\nabla\eta| \le \frac{C\varepsilon^{100}}{N},\ \ |s|_{\partial G_d}$$
$$\le C\left[\frac{N}{M^2} + \frac{\varepsilon^{1/4}}{M}\right],\ \ \left.\frac{\partial s}{\partial n}\right|_{G_n} = 0.$$

The maximum principle implies that then $|s(\mathbf{x})| \le C\left[N/M^2 + \varepsilon^{1/4}/M\right]$ for all $\mathbf{x} \in G$. This is the first contribution in (6.24).

Let us now discuss the contribution of f_{sm}. We assume that

$$\bar{f}^\varepsilon_{jk}|_{\partial G_d} = f_{sm}.$$

Inequality (6.22) implies that the boundary conditions for (6.15) on the Dirichlet's part differ from $f_o(h)$ no more than $CM\varepsilon^{1/4}$ point-wise. Hence by the maximum principle

$$\|\bar{f}^\varepsilon_{jk} - f_3\|_{L^2(G)} \le CMN\varepsilon^{3/4}.$$

where f_3 solves (6.15) with the boundary conditions

$$f_3(\mathbf{x}) = f_o(\mathbf{x}),\ \ \mathbf{x} \in \partial G_d.$$

The function $f_o(h)$ is well-defined on the whole G so that $u \cdot \nabla f_o = 0$ and it satisfies the homogeneous Neumann boundary conditions everywhere on ∂G_n. This allows to estimate the $\dot{H}^1$ norm of f_3. Multiplying the equation

$$\varepsilon \Delta f_3 - u \cdot \nabla f_3 = 0 \tag{6.47}$$

by $f_3 - f_o$ and integrating by parts we have, using (6.11),

$$\begin{aligned} \|\nabla f_3\|^2_{L^2(G)} &= \int_G \nabla f_3 \cdot \nabla f_o dxdy \le C||\nabla f_o||^2_{L^2(G)} \\ &= C \int_0^{4M^2\varepsilon^{1/2}} \int_{-N}^{N} \frac{|\nabla H|^2}{\varepsilon} \left(\frac{\partial f_o}{\partial h}\right)^2 \frac{\sqrt{\varepsilon}}{J} dh d\theta \\ &\le \frac{C}{\sqrt{\varepsilon}} \int_0^{4M^2\sqrt{\varepsilon}} \int_{-N}^{N} \left(\frac{\partial f_o}{\partial h}\right)^2 dh d\theta \le \frac{C}{\sqrt{\varepsilon}} \int_0^{4M^2\sqrt{\varepsilon}} d\theta \le CM^2. \end{aligned} \tag{6.48}$$

We now once again multiply (6.47) by $f_3 - f_o$ and integrate over each of the four disconnected parts G^i_δ , $i = 1, 2, 3, 4$, of the domain

$$G_\delta = \{(x, y) \in \Omega^\varepsilon_N : \ 4M^2\sqrt{\varepsilon} - \delta \le \theta \le 4M^2\sqrt{\varepsilon}\} = \cup^4_{i=1} G^i_\delta \subseteq II$$

On each G^i_δ we have

$$\begin{aligned} &\varepsilon \int_{l^i_\delta} (f_3 - f_o) \frac{\partial f_3}{\partial n} dS - \varepsilon \int_{G^i_\delta} |\nabla f_3|^2 d\mathbf{x} + \varepsilon \int_{G^i_\delta} \nabla f_3 \cdot \nabla f_o d\mathbf{x} \\ &+ \int_{l^i_\delta} \frac{(u \cdot n)(f_3 - f_o)^2}{2} dS = 0, \end{aligned} \tag{6.49}$$

where $l^i_\delta = \{(x, y) \in \Omega^\varepsilon_N \cap \partial G^i_\delta : \ \theta(x, y) = 4M^2\sqrt{\varepsilon} - \delta\}$. Since $(u \cdot n) = \pm|u|$ with the same sign in each of the four connected components, (6.48) implies

$$\int_{l_\delta} \frac{|u|(f_3 - f_o)^2}{2} dS \le \varepsilon \int_{l^i_\delta} |f_3 - f_o| \left|\frac{\partial f}{\partial n}\right| dS + C\varepsilon M^2. \tag{6.50}$$

Changing variables, (6.50) may be re-written as

$$\begin{aligned} &\int_{-N\sqrt{\varepsilon}}^{N\sqrt{\varepsilon}} \frac{|u|(f_3 - f_o)^2}{2} (\rho, 4M^2\sqrt{\varepsilon} - \delta) \frac{d\rho}{|\nabla H|} \\ &\le \varepsilon \int_{-N\sqrt{\varepsilon}}^{N\sqrt{\varepsilon}} |f_3 - f_o| \left|\frac{\partial f_3}{\partial n}\right| (\rho, 4M^2\sqrt{\varepsilon} - \delta) \frac{d\rho}{|\nabla H|} + C\varepsilon M^2. \end{aligned} \tag{6.51}$$

Integrating in $\delta \in (0, 3M^2\sqrt{\varepsilon})$ and adding up the resulting four inequalities we obtain

(6.52)

$$\int_{II} \frac{|u|(f_3 - f_o)^2}{2} (\rho, \theta) \frac{d\rho d\theta}{|\nabla H|} \le \int_{II} |f_3 - f_o| \left|\frac{\partial f_3}{\partial n}\right| (\rho, \theta) \frac{d\rho d\theta}{|\nabla H|} + C\varepsilon^{3/2} M^4.$$

Once again using (6.11) we may re-write this as

(6.53)

$$\int_{II} \frac{|u|(f_3 - f_o)^2}{2}(\mathbf{x})|\nabla H|d\mathbf{x} \leq \varepsilon \int_{II} |f_3 - f_o| \left|\frac{\partial f_3}{\partial n}\right| (\mathbf{x})|\nabla H|d\mathbf{x} + C\varepsilon^{3/2}M^4.$$

However, we have $C_1 M\varepsilon^{1/4} \leq |u| = |\nabla H| \leq C_2 M\varepsilon^{1/4}$ in region II so that the above together with (6.48) imply

$$M^2\sqrt{\varepsilon}\int_{II} |f_3 - f_o|^2 d\mathbf{x} \leq C\varepsilon M\varepsilon^{1/4}\left[\alpha \int_{II} |f - f_o|^2 d\mathbf{x} + \frac{1}{\alpha}\int_{II} |\nabla f|^2 d\mathbf{x}\right]$$
$$+C\varepsilon^{3/2}M^4.$$

We choose $\alpha = M/(2C\varepsilon^{3/4})$ and obtain

$$\int_{II} |f_3 - f_o|^2 d\mathbf{x} \leq C\varepsilon M^2 + C\varepsilon^{3/2} \leq C\varepsilon M^2 \tag{6.54}$$

which is the third contribution in (6.24). This finishes the proof of Lemma 6.3 since $\|f_o - f^\varepsilon\|^2_{L^2(II)} \leq CM\varepsilon^{1/4}N\sqrt{\varepsilon}$ - that contribution is included in the second term in (6.24). □

7 Approximation of the dissipation rate by the water-pipe network

We show in this section that the total dissipation rate of the full advection-diffusion problem

$$\left\{ \begin{array}{l} \varepsilon\Delta T^\varepsilon - u\cdot\nabla T^\varepsilon = 0, \text{ in } \Omega\subset\mathbb{R}^2, \\ T^\varepsilon(\mathbf{x}) = T_0(\mathbf{x}), \ \mathbf{x}\in\partial\Omega, \end{array}\right. \tag{7.1}$$

may be approximated by the dissipation rate for the water-pipe model

$$\left\{ \begin{array}{l} \varepsilon\Delta T^\varepsilon_N - u\cdot\nabla T^\varepsilon_N = 0, \text{ in } \Omega^\varepsilon_N\subset\Omega, \\ T^\varepsilon_N(\mathbf{x}) = T_0(\mathbf{x}), \ \mathbf{x}\in\partial\Omega, \\ \partial T^\varepsilon_N(\mathbf{x})/\partial n = 0, \ \mathbf{x}\in\mathcal{L}(N\sqrt{\varepsilon}), \\ \mathcal{L}(N\sqrt{\varepsilon}) = \{\mathbf{x}\in\Omega: \ |H(\mathbf{x})| = N\sqrt{\varepsilon}\}, \end{array}\right. \tag{7.2}$$

posed in the smaller domain:

$$\Omega^\varepsilon_N = \{\mathbf{x}\in\Omega: \ |H(\mathbf{x})| \leq N\sqrt{\varepsilon}\}.$$

While this result is not surprising in itself, given that the water-pipe network provides an L^∞-approximation of the full problem, remarkably, the error in the approximation of the dissipation rate is independent of the flow inside the cell, that is, outside the water-pipe model itself. This is the main result of this section.

Recall that for the solution of the advection-diffusion problem (7.1) the dissipation rate is defined as

$$D^\varepsilon(u, T_0) = \varepsilon\langle|\nabla T^\varepsilon|^2\rangle_\Omega$$

where

$$\langle f\rangle_\Omega = \int_\Omega f(\mathbf{x})d\mathbf{x}.$$

Similarly, we define the dissipation rate for the solution of the water-pipe network problem (7.2) as the total dissipation rate:

$$D_N^\varepsilon(u,T_0) = \varepsilon\langle|\nabla T^\varepsilon|^2\rangle_{\Omega_N^\varepsilon}.$$

The dissipation rates for the full advection-diffusion problem and for the water-pipe model have the same limit:

Theorem 7.1. *For any u, T_0 sufficiently regular, there exists a finite limit*

$$\lim_{\varepsilon\to 0} D^\varepsilon(u,T_0)/\sqrt{\varepsilon} = D^*(T_0). \tag{7.3}$$

Moreover, if $N = N(\varepsilon)\to\infty$, as $\varepsilon\to 0$ then

$$\lim_{\varepsilon\to 0} D_N^\varepsilon(u,T_0)/\sqrt{\varepsilon} = D^*(T_0). \tag{7.4}$$

Here $D^*(T_0)$ is the quadratic form of the Dirichlet-to-Neumann map for the asymptotic Childress problem. It is defined explicitly in (5.11) in the two-cell case – this definition may be extended to the general case in a straightforward manner. In the case of a smooth boundary $\partial\Omega$ Theorem 7.1 implies the convergence of the Dirichlet-to-Neumann map $\mathcal{D}^*: H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)$ in the limit $\varepsilon\to 0$. However, unfortunately, we do not have the bounds of the form $D^\varepsilon(T_0)\sim\|T_0\|_{H^{1/2}}/\sqrt{\varepsilon}$ for a finite $\varepsilon>0$ and it is not clear if this is because of the proof or if there exists a real obstacle to such a bound.

The existence and equality of finite limits (7.3) (7.4) can be obtained from the construction of the approximate solution in Section 6. The main result of this section is the following statement about the error. Theorem 7.1 implies that

$$|D_N^\varepsilon - D^\varepsilon|/\sqrt{\varepsilon} \le C^\varepsilon(T_0,u,N), \tag{7.5}$$

with $C^\varepsilon\to 0$ as $\varepsilon\to 0$ and $N = N(\varepsilon)\to\infty$. However, a priori the error C^ε may depend on the flow inside the cell, away from the separatrices. The next theorem shows that this is not the case.

Theorem 7.2. *The water-pipe model approximates the dissipation rate with an error that is independent of the flow u outside of Ω_N^ε.*

The proof relies on variational techniques. We construct variational minimum and maximum principles for the dissipation rate. Using solutions of the water-pipe model, we construct trial fields which depend on the flow u only in Ω_N^ε. These trial fields give upper and lower bounds on the dissipation rate, and as $\varepsilon\to 0$, $N\to\infty$, these bounds have the limit D^*. We conclude that the error of the water-pipe model is determined by the flow u in Ω_N^ε only. For example, if we choose $N=\varepsilon^{-\alpha}$, $0<\alpha<1/2$, then the error is determined by u in the neighborhood of the separatrices $|H| \le \varepsilon^{\beta}$, $\beta = 1/2 - \alpha > 0$.

The above results may interpreted as the approximation of the quadratic form associated to the Dirichlet-to-Neumann map of the full advection-diffusion problem by that of the water-pipe problem. The latter, in turn, converges to that of the Childress problem (defined in (5.11) for the two-cell case).

We now turn to the two main technical details of the argument: the variational principles and the trial fields.

7.1 Variational principles

We derive here saddle-point variational principles for the dissipation rate D^{ε}. The method follows the general ideas of [3, 5, 6, 13]. The first step is to introduce the adjoint problems for (7.1) and (7.2), which are [5, 6, 14] the same advection-diffusion equations but with the reversed advection: u is replaced by $-u$. The adjoint problem for the advection-diffusion problem (7.1) is:

$$\left\{\begin{array}{l} \varepsilon\Delta\tilde{T}^{\varepsilon} + u\cdot\nabla\tilde{T}^{\varepsilon} = 0, \text{ in } \Omega\subset\mathbb{R}^2, \\ \tilde{T}^{\varepsilon}(\mathbf{x}) = T_0(\mathbf{x}),\ \mathbf{x}\in\partial\Omega. \end{array}\right. \tag{7.6}$$

Let us use the "symmetrization" [3, 5]:

$$T^{\pm} = \frac{T^{\varepsilon} \pm \tilde{T}^{\varepsilon}}{2} \tag{7.7}$$

and define $E^{\pm} = \nabla T^{\pm}$. We dropped the superscript ε in the notation of the symmetrized temperature to simplify the notation. The functions T^{+} and T^{-} satisfy the boundary conditions

$$T^{+}(\mathbf{x}) = T_0(\mathbf{x}),\ T^{-}(\mathbf{x}) = 0,\ \mathbf{x}\in\partial\Omega.$$

The gradients $E^{\pm}$ obey

$$\nabla\cdot\left(E^{\pm} + \mathbf{H}^{\varepsilon}E^{\mp}\right) = 0 \tag{7.8}$$

where

$$\mathbf{H}^{\varepsilon} = \frac{1}{\varepsilon}\begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}.$$

It is easy to check that (7.8) are the Euler-Lagrange equations of the functional

$$W^{\varepsilon}(E^{+},E^{-}) = \langle|E^{+}|^2\rangle_{\Omega} - 2\langle E^{-}\cdot\mathbf{H}^{\varepsilon}E^{+}\rangle_{\Omega} - \langle|E^{-}|^2\rangle_{\Omega}. \tag{7.9}$$

The dissipation rate can be determined as the the value of this functional at its saddle-point:

$$\begin{aligned} &D^\varepsilon = \varepsilon \min_{E^+\in\mathbf{V}^+} \max_{E^-\in\mathbf{V}^-} W^\varepsilon(E^+, E^-), \\ &\mathbf{V}^+ = \Big\{E^+ = \nabla T^+,\ T^+ \in H^1(\Omega),\ T^+(\mathbf{x}) = T_0(\mathbf{x}),\ \mathbf{x} \in \partial\Omega\Big\}, \\ &\mathbf{V}^- = \Big\{E^- = \nabla T^-,\ T^- \in H^1(\Omega),\ T^-(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\Omega\Big\}. \end{aligned} \tag{7.10}$$

Indeed, if $E^\pm$ solve (7.8), then

$$D^\varepsilon = \varepsilon\left(\langle |E^+|^2\rangle_\Omega + \langle |E^-|^2\rangle_\Omega\right), \tag{7.11}$$

and

$$\langle |E^-|^2\rangle_\Omega = -\langle E^- \cdot \mathbf{H}^\varepsilon E^+\rangle_\Omega,$$

hence for such $E^\pm$

$$D^\varepsilon = \varepsilon W^\varepsilon(E^+, E^-).$$

Following the technique of [3, 5] we use the partial Legendre transform to reformulate the min-max variational principle (7.10) as a min-min and a max-max principles. The max-max principle is

$$\begin{aligned} &D^\varepsilon = \varepsilon \max_{J^+\in\mathbf{W}^+} \max_{E^-\in\mathbf{V}^-} W^\varepsilon_{\max}(J^+, E^-), \\ &W^\varepsilon_{\max}(J^+, E^-) = 2\int_{\partial\Omega} T_0 J^+ \cdot n ds - \langle |J^+ - \mathbf{H}^\varepsilon E^-|^2\rangle_\Omega - \langle |E^-|^2\rangle_\Omega, \\ &\mathbf{W}^+ = \Big\{J^+,\ \nabla \cdot J^+ = 0,\ J^+ \in L^2(\Omega)\Big\}, \end{aligned} \tag{7.12}$$

while the min-min variational principle is

$$\begin{aligned} &D^\varepsilon = \varepsilon \min_{E^+\in\mathbf{V}^+} \min_{J^-\in\mathbf{W}^-} W_{\min}(E^+, J^-), \\ &W^\varepsilon_{\min}(E^+, J^-) = \langle |E^+|^2\rangle_\Omega + \langle |J^- - \mathbf{H}^\varepsilon E^+|^2\rangle_\Omega, \\ &\mathbf{W}^- = \Big\{J^-,\ \nabla \cdot J^- = 0,\ J^- \in L^2(\Omega)\Big\}. \end{aligned} \tag{7.13}$$

The former allows us to obtain the lower bounds for D^ε while the latter produces the upper bounds. As a consequence we have

$$\varepsilon W^\varepsilon_{\max}(J^+_{lower}, E^-_{\text{lower}}) \le D^\varepsilon \le \varepsilon W^\varepsilon_{\min}(E^+_{\text{upper}}, J^-_{\text{upper}}) \tag{7.14}$$

for any trial fields $E^+_{\text{upper}} \in \mathbf{V}^+$, $E^-_{lower} \in \mathbf{V}^-$, $J^+_{\text{lower}} \in \mathbf{W}^+$, and $J^-_{\text{upper}} \in \mathbf{W}^-$.

7.2 The trial fields

The classical approach to variational bounds is to find some “good” trial functions E^+_{upper}, E^-_{lower}, J^+_{lower}, J^-_{upper} and apply inequality (7.14). A successful choice of the trial functions leads to tight bounds, and it usually relies on specific features of the problem. We construct the trial fields based on the solution of the water-pipe problem.

Let T^ε_N solve (7.2) and $\tilde{T}^\varepsilon_N$ be the solution of the adjoint water-pipe network problem:

$$\left\{\begin{array}{l} \varepsilon\Delta\tilde{T}^\varepsilon_N + u\cdot\nabla\tilde{T}^\varepsilon_N = 0,\ \text{in } \Omega^\varepsilon_N \subset \Omega, \\ \tilde{T}^\varepsilon_N(\mathbf{x}) = T_0(\mathbf{x}),\ \mathbf{x}\in\partial\Omega, \\ \partial\tilde{T}^\varepsilon_N(\mathbf{x})/\partial n = 0,\ \mathbf{x}\in\mathcal{L}(N\sqrt{\varepsilon}). \end{array}\right. \tag{7.15}$$

We define the constants K^ε_j and $\tilde{K}^\varepsilon_j$ as the averages of T^ε_N and $\tilde{T}^\varepsilon_N$ over the streamline $\mathcal{L}_j(N\sqrt{\varepsilon}) = \mathcal{L}(N\sqrt{\varepsilon})\cap\mathcal{C}_j$:

$$K^\varepsilon_j = \frac{1}{|\mathcal{L}_j(N\sqrt{\varepsilon})|}\oint_{\mathcal{L}_j(N\sqrt{\varepsilon})} T_N(\mathbf{x})dl,\ \tilde{K}^\varepsilon_j = \frac{1}{|\mathcal{L}_j(N\sqrt{\varepsilon})|}\oint_{\mathcal{L}_j(N\sqrt{\varepsilon})} \tilde{T}_N(\mathbf{x})dl.$$

As we have shown previously, $T_N(\mathbf{x})$ and $\tilde{T}_N(\mathbf{x})$ are uniformly close to K^ε_j and $\tilde{K}^\varepsilon_j$, respectively, along $\mathcal{L}_j(N\sqrt{\varepsilon})$. Let T^ε_K and $\tilde{T}^\varepsilon_K$ be the solutions of the Poisson equation in Ω^ε_N with constant Dirichlet boundary conditions on the interior boundaries:

$$\begin{array}{l} \left\{\begin{array}{l} \varepsilon\Delta T^\varepsilon_K = \varepsilon\Delta T^\varepsilon_N \equiv u\cdot\nabla T^\varepsilon_K,\ \mathbf{x}\in\Omega^\varepsilon_N, \\ T^\varepsilon_K(\mathbf{x}) = T_0(\mathbf{x}),\ \mathbf{x}\in\partial\Omega, \\ T^\varepsilon_K(\mathbf{x}) = K_j,\ \mathbf{x}\in\mathcal{L}_j(N\sqrt{\varepsilon}), \end{array}\right. \\ \left\{\begin{array}{l} \varepsilon\Delta\tilde{T}^\varepsilon_K = \varepsilon\Delta\tilde{T}^\varepsilon_N \equiv -u\cdot\nabla\tilde{T}^\varepsilon_N,\ \mathbf{x}\in\Omega^\varepsilon_N, \\ \tilde{T}^\varepsilon_K(\mathbf{x}) = T_0(\mathbf{x}),\ \mathbf{x}\in\partial\Omega, \\ \tilde{T}^\varepsilon_K(\mathbf{x}) = \tilde{K}_j,\ \mathbf{x}\in\mathcal{L}_j(N\sqrt{\varepsilon}). \end{array}\right. \end{array} \tag{7.16}$$

Let us denote the symmetrized temperatures as

$$T^\pm_N = \frac{T^\varepsilon_N \pm \tilde{T}^\varepsilon_N}{2},\ \ T^\pm_K = \frac{T^\varepsilon_K \pm \tilde{T}^\varepsilon_K}{2}.$$

We can now define the trial fields for the upper and lower bounds. For the upper bound we take

$$\left\{\begin{array}{l} E^+_{\text{upper}} = \nabla T^+_K,\ \ J^-_{\text{upper}} = \nabla T^-_N + \mathbf{H}^\varepsilon\nabla T^+_K,\ \text{in } \Omega^\varepsilon_N, \\ E^+_{\text{upper}} = 0, J^-_{\text{upper}} = 0,\ \ \text{otherwise}. \end{array}\right. \tag{7.17}$$

and for the lower one

$$\left\{\begin{array}{l} E^-_{\text{lower}} = \nabla T^-_K,\ \ J^+_{\text{lower}} = \nabla T^+_N + \mathbf{H}^\varepsilon\nabla T^-_K,\ \text{in } \Omega^\varepsilon_N, \\ E^+_{\text{lower}} = 0, J^-_{\text{lower}} = 0,\ \ \text{otherwise}. \end{array}\right. \tag{7.18}$$

By construction, the trial fields E and J given by (7.17) and (7.18) satisfy $E^\pm\in\mathbf{V}^\pm$, $J^\pm\in\mathbf{W}^\pm$ (here we dropped the subscripts upper/lower). Indeed, the only nontrivial property we have to check is that $\nabla\cdot J^\pm = 0$ weakly. Equations (7.16) imply that $J^\pm$ are indeed divergence-free away from the level set $|H(\mathbf{x})| = N\sqrt{\varepsilon}$. We have to verify that the normal components of

$J^{\pm}$ agree on the two sides of this level set. The inner normal component $n \cdot J^{\pm} \equiv 0$. The outer normal component is

$$n{\cdot}J^{\pm} = n{\cdot}\Big(\nabla T_N^{\pm}(\mathbf{x}) + \mathbf{H}^{\varepsilon}\nabla T_K^{\mp}(\mathbf{x})\Big) = n{\cdot}\mathbf{H}^{\varepsilon}\nabla T_K^{\mp}(\mathbf{x}) = \frac{H(\mathbf{x})}{|\nabla H(\mathbf{x})|}u{\cdot}\nabla T_K^{\mp} = 0,$$

as $T_K^{\pm}$ is constant on the level set. Hence $J^+_{\text{lower}} \in W^+$, and $J^-_{\text{upper}} \in W^-$.

Lemma 7.3. *There exist the finite limits*

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}W_{\max}(J^+_{lower}, E^-_{lower}) = D^*,\ \lim_{\varepsilon\to 0}\sqrt{\varepsilon}W_{\min}(E^+_{upper}, J^-_{upper}) = D^*.$$

where $N = N(\varepsilon) \to \infty$.

The proof of Lemma 7.3 again follows from our previous asymptotic analysis and we do not repeat the details. □

It remains to show that the error between D_N^{ε} and D^{ε} depends on the flow near the separatrices only. However, since

$$|W_{\min}(E^+_{\text{upper}}, J^-_{\text{upper}}) - D^{\varepsilon}| \le W_{\min}(E^+_{\text{upper}}, J^-_{\text{upper}}) - W_{\max}(J^+_{\text{lower}}, E^-_{\text{lower}}),$$

and all J^+_{lower}, E^-_{lower}, E^+_{upper}, J^-_{upper} depend only on the flow inside Ω_N^{ε}, the error

$$|W_{\min}(E^+_{\text{upper}}, J^-_{\text{upper}}) - D^{\varepsilon}|$$

also has this property. Finally, multiplying the equation

$$\varepsilon\Delta T_N^+ - u\cdot\nabla T_N^- = 0$$

by T_N^- and integrating by parts we obtain

$$\langle \nabla T_N^+ \cdot \nabla T_N^- \rangle_{\Omega_N^{\varepsilon}} = 0,$$

and therefore

$$D_N^{\varepsilon} = \varepsilon\left(\langle\left|\nabla T_N^+\right|^2\rangle_{\Omega_N^{\varepsilon}} + \langle\left|\nabla T_N^-\right|^2\rangle_{\Omega_N^{\varepsilon}}\right) = \varepsilon W^{\varepsilon}_{\min}(E^+_{\text{upper}}, J^-_{\text{upper}})$$

$$+2\varepsilon\langle\nabla T_K^+\cdot(\nabla T_N^+ - \nabla T_K^+)\rangle_{\Omega_N^{\varepsilon}} + \varepsilon\langle\left|\nabla T_N^+ - \nabla T_K^+\right|^2\rangle_{\Omega_N^{\varepsilon}}.$$

Hence we have

$$|D^{\varepsilon} - D_N^{\varepsilon}|/\sqrt{\varepsilon} \le |\sqrt{\varepsilon}W_{\min}(E^+_{\text{upper}}, J^-_{\text{upper}}) - \frac{D^{\varepsilon}}{\sqrt{\varepsilon}}|$$

$$+\sqrt{\frac{D_N^{\varepsilon}}{\sqrt{\varepsilon}}}\sqrt{\sqrt{\varepsilon}\langle\left|\nabla T_N^+ - \nabla T_K^+\right|^2\rangle_{\Omega_N^{\varepsilon}}} + \sqrt{\varepsilon}\langle\left|\nabla T_N^+ - \nabla T_K^+\right|^2\rangle_{\Omega_N^{\varepsilon}}.$$

Since all the terms on the right-hand side of the last inequality tend to zero as $\varepsilon \to 0$ and depend only on the flow u inside Ω_N^{ε}, Theorem 7.2 holds. □

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