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Boundary observers for a reaction-diffusion system under time-delayed and sampled-data measurements

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***Abstract*—We construct finite-dimensional observers for a 1D reaction-diffusion system with boundary measurements subject to time-delays and data sampling. The system has a finite number of unstable modes approximated by a Luenberger-type observer. The remaining modes vanish exponentially. For a given reaction coefficient, we show how many modes one should use to achieve a desired rate of convergence. The finite-dimensional part is analyzed using appropriate Lyapunov–Krasovskii functionals that lead to LMI-based convergence conditions feasible for small enough time-delay and sampling period. The LMIs can be used to find appropriate injection gains.**

I. Introduction

Time-delays and data sampling are inevitable in practice due to finite speed of signal processing/transmission and digital nature of most controllers. Since the delay may lead to instability in the reaction-diffusion systems (see the examples in [1] and in Section IV below), these phenomena should be carefully studied.

Reaction-diffusion systems with various types of *in-domain* measurements/actuators subject to time-delays and sampling have been considered in [1]–[3]. These papers proposed observers/controllers that work if the delay, sampling period, and the distances between adjacent sensors/actuators are small enough. That is, the system should have enough high-frequency sensors/actuators.

The case of only one *boundary* sensor/actuator is more difficult to study. For diffusion-reaction systems, boundary controllers can be constructed using the backstepping approach [4], [5] or modal decomposition technique [6]–[9]. It has been shown in [10] that both approaches are robust to data sampling. In [11], modal decomposition technique was combined with a predictor to compensate a constant delay in the boundary controller. Robustness to small delays of general linear PDEs was studied in [12].

In this paper, we construct finite-dimensional observers for a 1D reaction-diffusion system with boundary measurements subject to time-delays and data sampling. Due to diffusion, there is a finite number of unstable modes, which we approximate by a Luenberger-type observer. The remaining modes vanish exponentially. For a given reaction coefficient, we show how many modes one should use to achieve a desired rate of convergence. Similar constructions have been proposed in [13], where a "lifting" technique and singular perturbation theory were used to obtain qualitative results. To obtain quantitative conditions, we use Lyapunov-Krasovskii functionals that lead to LMIs, which are feasible for small enough delay and sampling period and allow to find admissible upper bounds of these quantities.

Lemma 1 (Cauchy-Schwarz inequality): For $f \in L^2(0,1)$,

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \int_0^1 (f(x))^2\,dx. \tag{1}$$

Lemma 2 (Wirtinger inequality [14]): If $f \in \mathcal{H}^1(a,b)$ is such that $f(a)=0$ or $f(b)=0$ then

$$\|f\|_{L^2} \le \frac{2(b-a)}{\pi}\|f'\|_{L^2}. \tag{2}$$

II. Time-delayed boundary measurements

Consider the reaction-diffusion system

$$z_t(x,t) = z_{xx}(x,t) + az(x,t), \tag{3a}$$
$$z_x(0,t) = z(1,t) = 0, \tag{3b}$$
$$z(x,0) = z_0(x) \tag{3c}$$

with the state $z\colon [0,1]\times[0,\infty)\to\mathbb{R}$, reaction coefficient $a\in\mathbb{R}$, and initial function $z_0\colon[0,1]\to\mathbb{R}$.

In this section, we construct an observer for the system (3) under the time-delayed boundary measurements

$$y(t) = \begin{cases} z(0,t-\tau(t)), & t-\tau(t)\ge 0,\\ 0, & t-\tau(t)<0,\end{cases} \tag{4}$$

where $\tau(t)\in[\tau_m,\tau_M]\subset(0,\infty)$ is a known delay such that

$$\exists\, t_* \in [\tau_m,\tau_M]\colon \qquad \begin{cases} t-\tau(t)\ge 0, & t\ge t_*,\\ t-\tau(t)<0, & t<t_*.\end{cases} \tag{5}$$

The condition $0<\tau_m\le\tau(t)$ allows to use the step method for the well-posedness analysis (see Lemma 3). We perform robustness analysis with respect to the time delay, that is, the observer will converge to the system state for any $\tau(t)\le\tau_M$ with a small enough τ_M. Following [15], we require (5) to simplify the analysis on the interval where $t-\tau(t)<0$.

Remark 1: The results of this paper can be extended to a more general system

$$\begin{aligned} &\tfrac{\partial z}{\partial t}(x,t) = \tfrac{\partial}{\partial x}\left(p(x)\tfrac{\partial}{\partial x}z(x,t)\right) + q(x)z(x,t),\\ &a_1 z(0,t) + a_2 z_x(0,t) = 0,\\ &b_1 z(1,t) + b_2 z_x(1,t) = 0,\end{aligned} \tag{6}$$

where $p\in C^1([0,1];(0,\infty))$, $q\in C([0,1];\mathbb{R})$, $a_2\ne 0$, $|b_1|+|b_2|\ne 0$. We consider the simplified system (3) to avoid some technical details.

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A *strong solution* of (3) is a function

$$\begin{aligned} &z \in L^2((0,\infty);\mathcal{H}^2(0,1)) \cap C([0,\infty);\mathcal{H}^1(0,1)),\\ &z_t \in L^2((0,\infty);L^2(0,1)) \end{aligned} \tag{7}$$

that satisfies (3c) for $t=0$ and (3a), (3b) for almost all $t>0$. By [16, Theorem 7.7], (3) has a unique strong solution for

$$z_0 \in \mathcal{H}^1(0,1) \quad \text{s.t.} \quad z_0(1)=0. \tag{8}$$

To construct a finite-dimensional observer, note that (3) has a finite number of unstable modes, while the remaining modes converge to zero. Namely, the system (3) can be presented as

$$\frac{dz}{dt} + \mathcal{A}z = 0, \quad z(0)=z_0, \tag{9}$$

where $z\colon [0,\infty) \to L^2(0,1)$ and

$$\begin{aligned} &\mathcal{A}\colon D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1),\\ &\mathcal{A}w = -w'' - aw \end{aligned} \tag{10}$$

is a symmetric operator with the domain

$$D(\mathcal{A}) = \{w \in \mathcal{H}^2(0,1) \,|\, w'(0)=w(1)=0\} \tag{11}$$

dense in $L^2(0,1)$. The eigenfunctions of $\mathcal{A}$, given by

$$\begin{aligned} &\phi_n(x) = \sqrt{2}\cos\left(x\sqrt{\lambda_n + a}\right),\\ &\lambda_n = \tfrac{(2n-1)^2\pi^2}{4} - a, \end{aligned} \qquad n \in \mathbb{N}, \tag{12}$$

form an orthonormal basis in $L^2(0,1)$ [16, Corollary 3.26]. Thus, the solution of (3) can be presented as

$$z(\cdot,t) = \textstyle\sum_{n=1}^{\infty} z_n(t)\phi_n(\cdot) \tag{13}$$

with $z_n(t) = \langle z(\cdot,t), \phi_n\rangle$. Using the symmetry of $\mathcal{A}$,

$$\begin{aligned} \dot z_n(t) &= \langle z_t(\cdot,t),\phi_n\rangle \overset{(9)}{=} -\langle \mathcal{A}z(\cdot,t),\phi_n\rangle\\ &= -\langle z(\cdot,t), \mathcal{A}\phi_n\rangle = -\lambda_n\langle z(\cdot,t),\phi_n\rangle = -\lambda_n z_n(t). \end{aligned} \tag{14}$$

That is,

$$\dot z_n(t) = -\lambda_n z_n(t), \quad n \in \mathbb{N}. \tag{15}$$

Let $\delta > 0$ be a desired decay rate of the observer estimation error. Since $\lim_{n\to\infty}\lambda_n = +\infty$, there exists $N \in \mathbb{N}$ such that

$$-\lambda_n \le -\delta, \quad \forall n > N. \tag{16}$$

We will show that (16) implies the exponential convergence of $\sum_{n>N} z_n(t)\phi_n(\cdot)$ with the decay rate δ. Thus, it can be approximated by zero. The term $\sum_{n=1}^{N} z_n(t)\phi_n(\cdot)$ is approximated using the Luenberger-type observer

$$\hat z(x,t) = \textstyle\sum_{n=1}^{N} \hat z_n(t)\phi_n(x), \tag{17a}$$

$$\tfrac{d}{dt}\hat z_n(t) = -\lambda_n \hat z_n(t) - l_n[\hat z(0,t-\tau(t)) - y(t)], \tag{17b}$$

$$\hat z_n(t) = 0, \quad t \le 0, \quad n = 1,\dots,N \tag{17c}$$

with the injection gains $l_1,\dots,l_N \in \mathbb{R}$.

Remark 2: Our results can be easily extended to arbitrary initial conditions $\hat z_n(t) = z_n^0$, $n=1,\dots,N$. We consider (17c) to avoid some technical details.

Introduce the estimation error

$$e(x,t) = \hat z(x,t) - z(x,t). \tag{18}$$

If $e(\cdot,t) \in L^2(0,1)$, it can be presented as

$$e(\cdot,t) = \textstyle\sum_{n=1}^{\infty} e_n(t)\phi_n(\cdot), \tag{19}$$

where, in view of (13) and (17a),

$$e_n(t) = \hat z_n(t) - z_n(t), \quad n \le N, \tag{20a}$$

$$e_n(t) = -z_n(t), \qquad n > N. \tag{20b}$$

In view of (15) and (17b), relation (20a) implies

$$\dot e_n(t) = -\lambda_n e_n(t) - l_n e(0, t-\tau(t)), \quad n \le N, \tag{21}$$

which can be presented as

$$\dot{\bar e}(t) = A\bar e(t) - LC\bar e(t-\tau(t)) + L\zeta(t-\tau(t)) \tag{22}$$

with

$$\begin{aligned} &\bar e = (e_1,\dots,e_N)^T,\\ &A = \operatorname{diag}\{-\lambda_1,\dots,-\lambda_N\},\\ &L = (l_1,\dots,l_N)^T,\\ &C = (\phi_1(0),\dots,\phi_N(0)) = (\sqrt{2},\dots,\sqrt{2}),\\ &\zeta(t) = \textstyle\sum_{n=1}^{N} e_n(t)\phi_n(0) - e(0,t). \end{aligned} \tag{23}$$

Since $\lambda_1,\dots,\lambda_N$ are different, the pair (A,C) is observable. Therefore, we can choose $L = (l_1,\dots,l_N)^T \in \mathbb{R}^N$ such that

$$\exists P > 0\colon \quad P(A-LC) + (A-LC)^T P < -2\delta P. \tag{24}$$

If $\tau(t) \equiv 0$, then (24) guarantees ISS of (22) with respect to $\zeta(t)$, which decays exponentially (we show this below). Thus, (22) is exponentially stable for $\tau(t) \equiv 0$ and remains so for $\tau(t) \le \tau_M$ with a small enough τ_M. The next theorem allows to find admissible τ_M.

Theorem 1: Consider the system (3) with the measurements (4) subject to (5) and the boundary observer (17) with λ_n, ϕ_n from (12), N satisfying (16) with an arbitrary decay rate $\delta > 0$, and $L = (l_1,\dots,l_N)^T \in \mathbb{R}^N$. Let there exist matrices $P_2, P_3, G \in \mathbb{R}^{N\times N}$ and positive-definite matrices $P, S, R \in \mathbb{R}^{N\times N}$ such that[1]

$$\Phi < 0 \quad \text{and} \quad \begin{bmatrix} R & G \\ G^T & R \end{bmatrix} \ge 0, \tag{25}$$

where $\Phi = \{\Phi_{ij}\}$ is the symmetric matrix composed from

$$\begin{aligned} &\Phi_{11} = A^T P_2 + P_2^T A + 2\delta P + S - e^{-2\delta\tau_M}R,\\ &\Phi_{12} = P - P_2^T + A^T P_3,\ \Phi_{13} = e^{-2\delta\tau_M}(R-G) - P_2^T LC,\\ &\Phi_{14} = e^{-2\delta\tau_M}G,\ \Phi_{22} = -P_3 - P_3^T + \tau_M^2 R,\\ &\Phi_{23} = -P_3^T LC,\ \Phi_{24} = 0,\ \Phi_{33} = -e^{-2\delta\tau_M}(2R - G - G^T),\\ &\Phi_{34} = e^{-2\delta\tau_M}(R-G),\ \Phi_{44} = -e^{-2\delta\tau_M}(S+R) \end{aligned} \tag{26}$$

with A and C from (23). Then there exists $M > 0$ such that

$$\|\hat z(\cdot,t) - z(\cdot,t)\|_{L^2} \le Me^{-\delta t}\|z_0\|_{\mathcal{H}^1}, \quad t \ge 0 \tag{27}$$

for any initial function z_0 from (8).

Proof: Since ϕ_n and λ_n defined in (12) are eigenfunctions and eigenvalues of the operator $\mathcal{A}$ defined in (10),

$$\begin{aligned} \hat z_t(x,t) &\overset{(17a)}{=} \textstyle\sum_{n=1}^{N} \tfrac{d}{dt}\hat z_n(t)\phi_n(x)\\ &\overset{(17b)}{=} -\textstyle\sum_{n=1}^{N} \lambda_n \hat z_n(t)\phi_n(x)\\ &\quad -\textstyle\sum_{n=1}^{N} l_n[\hat z(0,t-\tau(t)) - z(0,t-\tau(t))]\phi_n(x)\\ &= -\textstyle\sum_{n=1}^{N} \hat z_n(t)\mathcal{A}\phi_n\\ &\quad -\textstyle\sum_{n=1}^{N} l_n[\hat z(0,t-\tau(t)) - z(0,t-\tau(t))]\phi_n(x)\\ &\overset{(10)}{=} \hat z_{xx}(x,t) + a\hat z(x,t)\\ &\quad -l(x)[\hat z(0,t-\tau(t)) - z(0,t-\tau(t))], \end{aligned} \tag{28}$$

[1] MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/TAC18a

where $l(x) = \sum_{n=1}^N l_n \phi_n(x)$. The latter, (3), and (18) imply

$$e_t(x,t) = e_{xx}(x,t) + ae(x,t) - l(x)e(0, t-\tau(t)), \quad (29a)$$
$$e_x(0,t) = e(1,t) = 0, \quad (29b)$$
$$e(\cdot, 0) = -z_0, \quad e(\cdot, t) = 0, \quad t < 0. \quad (29c)$$

Lemma 3: There exists a unique strong solution of (29) for any initial function z_0 satisfying (8).

Proof is given in Appendix A.

The strong solution $e(\cdot, t)$ of (29) can be presented as the series (19) and, by Parseval's identity,

$$\|e(\cdot,t)\|_{L^2}^2 = \sum_{n=1}^N e_n^2(t) + \sum_{n>N} e_n^2(t). \quad (30)$$

The second term can be bounded as

$$\begin{aligned}\sum_{n>N} e_n^2(t) &\overset{(20b)}{=} \sum_{n>N} z_n^2(t) \overset{(15)}{=} \sum_{n>N} e^{-2\lambda_n t} z_n^2(0) \\ &\overset{(16)}{\le} e^{-2\delta t} \sum_{n>N} z_n^2(0) \le e^{-2\delta t} \|z(\cdot,0)\|_{L^2}^2 \\ &\overset{(29c)}{=} e^{-2\delta t} \|e(\cdot,0)\|_{L^2}^2 \overset{\text{Lem.2}}{\le} e^{-2\delta t} \tfrac{4}{\pi^2} \|e_x(\cdot,0)\|_{L^2}^2.\end{aligned} \quad (31)$$

To bound the first summand of (30), i.e., the state of (22), we first show that $\zeta(t)$ exponentially converges to zero. Since $\phi_n(1) = e(1,t) = 0$ and $\|\phi_n'\|_{L^2}^2 = \lambda_n + a$, we have

$$\begin{aligned}\zeta^2(t) &= \left(\sum_{n=1}^N e_n(t)\phi_n(0) - e(0,t)\right)^2 \\ &= \left(\int_0^1 \left(\sum_{n=1}^N e_n(t)\phi_n'(x) - e_x(x,t)\right) dx\right)^2 \\ &\overset{\text{Lem.1}}{\le} \left\|\sum_{n=1}^N e_n(t)\phi_n'(\cdot) - e_x(\cdot,t)\right\|_{L^2}^2 \\ &= \left\|\sum_{n>N} e_n(t)\phi_n'\right\|_{L^2}^2 = \sum_{n>N}(\lambda_n + a)e_n^2(t) \\ &\le e^{-2\delta t} \sum_{n=1}^\infty (\lambda_n + a)e_n^2(0) = e^{-2\delta t}\|e_x(\cdot,0)\|_{L^2}^2.\end{aligned} \quad (32)$$

The last inequality is obtained in a manner similar to (31). Consequently,

$$\begin{aligned}\zeta^2(t-\tau(t)) &\le e^{-2\delta(t-\tau(t))}\|e_x(\cdot,0)\|_{L^2}^2 \\ &\le e^{2\delta\tau_M} e^{-2\delta t}\|e_x(\cdot,0)\|_{L^2}^2.\end{aligned} \quad (33)$$

Consider the functional $V_\tau = V_0 + V_S + V_R$ with

$$\begin{aligned} V_0 &= \bar{e}^T(t) P \bar{e}(t), \\ V_S &= \int_{t-\tau_M}^t e^{-2\delta(t-s)} \bar{e}^T(s) S \bar{e}(s)\, ds, \\ V_R &= \tau_M \int_{-\tau_M}^0 \int_{t+\theta}^t e^{-2\delta(t-s)} \dot{\bar{e}}^T(s) R \dot{\bar{e}}(s)\, ds\, d\theta. \end{aligned} \quad (34)$$

We consider $V_\tau(t)$ on $[t_*, \infty)$ with t_* from (5). On this interval, (22) does not depend on $\bar{e}(t)$ with $t < 0$. Thus, we formally set $\bar{e}(t) = \bar{e}(0)$ for $t < 0$ to define V_τ on $[t_*, \tau_M)$ (see [15]). We have

$$\begin{aligned} &\dot{V}_0 + 2\delta V_0 = 2\bar{e}^T P \dot{\bar{e}} + 2\delta \bar{e}^T P \bar{e}, \\ &\dot{V}_S + 2\delta V_S = \bar{e}^T S \bar{e} - e^{-2\delta\tau_M} \bar{e}^T(t-\tau_M) S \bar{e}(t-\tau_M), \\ &\dot{V}_R + 2\delta V_R = \tau_M^2 \dot{\bar{e}}^T R \dot{\bar{e}} - \tau_M \int_{t-\tau_M}^t e^{-2\delta(t-s)} \dot{\bar{e}}^T(s) R \dot{\bar{e}}(s)\, ds. \end{aligned} \quad (35)$$

Using Jensen's inequality [17, Proposition B.8] and reciprocally convex approach [18, Theorem 1], we have

$$\begin{aligned} &-\tau_M \int_{t-\tau_M}^t e^{-2\delta(t-s)} \dot{\bar{e}}^T(s) R \dot{\bar{e}}(s)\, ds \le -\tau_M e^{-2\delta\tau_M} \times \\ &\quad \left[\int_{t-\tau(t)}^t \dot{\bar{e}}^T(s) R \dot{\bar{e}}(s)\, ds + \int_{t-\tau_M}^{t-\tau(t)} \dot{\bar{e}}^T(s) R \dot{\bar{e}}(s)\, ds\right] \\ &\le -e^{-2\delta\tau_M} \tfrac{\tau_M}{\tau(t)} \left[\int_{t-\tau(t)}^t \dot{\bar{e}}(s)\, ds\right]^T R \left[\int_{t-\tau(t)}^t \dot{\bar{e}}(s)\, ds\right] \\ &\quad - e^{-2\delta\tau_M} \tfrac{\tau_M}{\tau_M - \tau(t)} \left[\int_{t-\tau_M}^{t-\tau(t)} \dot{\bar{e}}(s)\, ds\right]^T R \left[\int_{t-\tau_M}^{t-\tau(t)} \dot{\bar{e}}(s)\, ds\right] \\ &\le -e^{-2\delta\tau_M} \begin{bmatrix} \bar{e}(t) - \bar{e}(t-\tau(t)) \\ \bar{e}(t-\tau(t)) - \bar{e}(t-\tau_M) \end{bmatrix}^T \begin{bmatrix} R & G \\ G^T & R \end{bmatrix} \begin{bmatrix} \bar{e}(t) - \bar{e}(t-\tau(t)) \\ \bar{e}(t-\tau(t)) - \bar{e}(t-\tau_M) \end{bmatrix}. \end{aligned} \quad (36)$$

Similarly to [19], we use the descriptor representation of (22)

$$0 = 2[\bar{e}^T P_2^T + \dot{\bar{e}}^T P_3^T][-\dot{\bar{e}} + A\bar{e} - LC\bar{e}(t-\tau(t)) + L\zeta(t-\tau(t))]. \quad (37)$$

Summing up (35) and (37), for $\gamma > 0$ we obtain

$$\dot{V}_\tau(t) + 2\delta V_\tau(t) - \gamma\zeta^2(t-\tau(t)) \le \psi^T(t) \Psi \psi(t), \quad (38)$$

where $\psi = \mathrm{col}\{\bar{e}(t), \dot{\bar{e}}(t), \bar{e}(t-\tau(t)), \bar{e}(t-\tau_M), \zeta(t-\tau(t))\}$,

$$\Psi = \left[\begin{array}{ccc|c} & & & P_2^T L \\ & \Phi & & P_3^T L \\ & & & 0_{2N\times 1} \\ \hline L^T P_2 & L^T P_3 & 0_{1\times 2N} & -\gamma \end{array}\right] \quad (39)$$

Since $\Phi < 0$, the inequality $\Psi < 0$ holds for a large enough $\gamma \in \mathbb{R}$. Moreover, $\Phi < 0$ holds with δ replaced by $\delta + \epsilon$ if $\epsilon > 0$ is small enough. Thus,

$$\begin{aligned} \dot{V}_\tau(t) &\le -2(\delta+\epsilon)V_\tau(t) + \gamma\zeta^2(t-\tau(t)) \\ &\overset{(33)}{\le} -2(\delta+\epsilon)V_\tau(t) + \gamma e^{2\delta\tau_M} e^{-2\delta t}\|e_x(\cdot,0)\|_{L^2}^2. \end{aligned} \quad (40)$$

The comparison principle implies:

$$V_\tau(t) \le e^{-2\delta(t-t_*)} V_\tau(t_*) + \frac{\gamma e^{2\delta\tau_M}}{2\epsilon} e^{-2\delta t}\|e_x(\cdot,0)\|_{L^2}^2. \quad (41)$$

Due to (5), $\dot{\bar{e}}(t) = A\bar{e}(t)$ for $t \in [0, t_*)$, thus, $|\bar{e}(t)| \le e^{\kappa t}|\bar{e}(0)|$ for $t \in [0, t_*)$ with some $\kappa > 0$. Therefore, for some $C > 0$,

$$\begin{aligned} V_\tau(t_*) &\le C \max_{t\in[t_*-\tau_M, t_*]} |\bar{e}(t)|^2 \\ &\le Ce^{2\kappa t_*}|\bar{e}(0)|^2 \le Ce^{2\kappa t_*} \sum_{n=1}^\infty e_n^2(0) \\ &= Ce^{2\kappa t_*}\|e(\cdot,0)\|_{L^2}^2 \overset{\text{Lem.2}}{\le} Ce^{2\kappa t_*} \tfrac{4}{\pi^2}\|e_x(\cdot,0)\|_{L^2}^2. \end{aligned} \quad (42)$$

The latter and (41) imply

$$\sum_{n=1}^N e_n^2(t) \le \lambda_{\min}^{-1}(P) V_\tau(t) \le M_1 e^{-2\delta t}\|e_x(\cdot,0)\|_{L^2}^2 \quad (43)$$

with some $M_1 > 0$. Finally, we have

$$\begin{aligned} &\|\hat{z}(\cdot,t) - z(\cdot,t)\|_{L^2}^2 = \|e(\cdot,t)\|_{L^2}^2 \\ &= \sum_{n=1}^N e_n^2(t) + \sum_{n=N+1}^\infty e_n^2(t) \overset{(43),(31)}{\le} M^2 e^{-2\delta t}\|e_x(\cdot,0)\|_{L^2}^2 \end{aligned} \quad (44)$$

with some $M > 0$. Thus, (27) is true. ∎

Remark 3: We have to use the $\mathcal{H}^1$-norm in the right-hand side of (27), since the L^2-norm does not take into account the point values that we use as measurements (4). Namely, we cannot bound ζ without using the space derivative as in (33).

Corollary 1: The observer (17) with $L = (l_1, \ldots, l_N)^T$ satisfying (24) converges to (3) with the decay rate δ in the sense of (27) if the delay bound τ_M is small enough.

Proof: Take P from (24), $P_2 = P$, $P_3 = \varepsilon I > 0$, $R = \mu^{-1}I > 0$, $G = S = 0$, and $\tau_M = 0$. Then

$$\Phi \overset{(26)}{=} \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_2^T & M_3 \end{array}\right]$$

with

$$M_1 = \begin{bmatrix} A^TP + PA + 2\delta P - \mu^{-1}I & \varepsilon A^T \\ * & -2\varepsilon I \end{bmatrix},$$
$$M_2 = \begin{bmatrix} \mu^{-1}I - PLC & 0 \\ -\varepsilon LC & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -2\mu^{-1}I & \mu^{-1}I \\ * & -\mu^{-1}I \end{bmatrix}.$$

Clearly,

$$M_3 < 0 \quad \text{and} \quad M_3^{-1} = -\mu \begin{bmatrix} I & I \\ I & 2I \end{bmatrix}.$$

By Schur's complement lemma, $\Phi < 0$ is equivalent to

$$M_1 - M_2M_3^{-1}M_2^T = \begin{bmatrix} P(A-LC) + (A-LC)^TP + 2\delta P & \varepsilon(A-LC)^T \\ \varepsilon(A-LC) & -2\varepsilon I \end{bmatrix} + \mu \begin{bmatrix} PLC \\ \varepsilon LC \end{bmatrix} \begin{bmatrix} PLC \\ \varepsilon LC \end{bmatrix}^T < 0. \quad (45)$$

In view of (24), the later holds for small $\varepsilon > 0$ and $\mu > 0$. Thus, $\Phi < 0$ is feasible for $\tau_M = 0$. By continuity, it remains so for a small $\tau_M > 0$. Then Theorem 1 implies (27). ■

The well-posedness of (8), (29) with $\tau(t) \equiv 0$ can be proved using [20, Theorem 6.3.1]. Then Theorem 1 and Corollary 1 imply the following result.

Corollary 2: For $\tau(t) \equiv 0$, the observer (17) with $L = (l_1, \dots, l_N)^T$ satisfying (24) exponentially converges to (3) with the decay rate δ in the sense of (27).

Remark 4: The LMIs of Theorem 1 allow to find appropriate injection gain $L = (l_1, \dots, l_N)^T$. Following [21, Section 5.2], one can take $P_3 = \varepsilon P_2$, where ε is a tuning parameter, and use $Y = P_2^T L$ as a new decision variable. After solving the resulting LMIs, the injection gain can be found as $L = (P_2^T)^{-1}Y$.

III. Sampled-data boundary measurements

In this section, we construct an observer for the system (3) under the sampled in time boundary measurements

$$y(t) = z(0, t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}, \quad (46)$$

where $0 = t_1 < t_2 < t_3 < \cdots$ are sampling instants satisfying

$$0 < t_{k+1} - t_k \le h, \quad \lim_{k\to\infty} t_k = \infty. \quad (47)$$

Remark 5: The output (46) can be presented as (4) with

$$\tau(t) = t - t_k, \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \quad (48)$$

such that $0 \le \tau(t) \le \tau_M = h$ and (5) is satisfied with $t_* = 0$. The condition $0 < \tau_m \le \tau(t)$ was imposed only to establish the well-posedness of (29) (see Lemma 3) and we will show that it is not required for the measurements (46). Therefore, the results of Theorem 1 can be applied. However, we will perform a more subtle analysis using the ideas of [22], which take into account the saw-tooth shape of $\tau(t)$ and lead to simpler convergence conditions.

Similarly to (17), the boundary observer is constructed as

$$\begin{aligned} &\hat{z}(x,t) = \textstyle\sum_{n=1}^N \hat{z}_n(t)\phi_n(x), \\ &\tfrac{d}{dt}\hat{z}_n(t) = -\lambda_n \hat{z}_n(t) - l_n[\hat{z}(0,t_k) - y(t)], \\ &\qquad\qquad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}, \\ &\hat{z}_n(0) = 0, \quad n = 1, \dots, N. \end{aligned} \quad (49)$$

Theorem 2: Consider the system (3) with the measurements (46) subject to (47) and the boundary observer (49) with λ_n, ϕ_n from (12), N satisfying (16) with an arbitrary decay rate $\delta > 0$, and $L = (l_1, \dots, l_N)^T \in \mathbb{R}^N$. Let there exist matrices $P_2, P_3 \in \mathbb{R}^{N\times N}$ and positive-definite matrices $P, W \in \mathbb{R}^{N\times N}$ such that[2] $\Upsilon < 0$, where $\Upsilon = \{\Upsilon_{ij}\}$ is the symmetric matrix composed from

$$\begin{aligned} &\Upsilon_{11} = (A - LC)^TP_2 + P_2^T(A - LC) + 2\delta P, \\ &\Upsilon_{12} = P - P_2^T + (A - LC)^TP_3, \quad \Upsilon_{13} = -P_2^TLC, \\ &\Upsilon_{22} = -P_3 - P_3^T + h^2e^{2\delta h}W, \quad \Upsilon_{23} = -P_3^TLC, \\ &\Upsilon_{33} = -\tfrac{\pi^2}{4}W \end{aligned} \quad (50)$$

with A and C from (23). Then there exists $M > 0$ such that (27) holds for any initial function z_0 from (8).

Proof: Similarly to (29), the estimation error $e(x,t) = \hat{z}(x,t) - z(x,t)$ satisfies

$$\begin{aligned} &e_t(x,t) = e_{xx}(x,t) + ae(x,t) - l(x)e(0,t_k), \\ &\qquad\qquad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}, \\ &e_x(0,t) = e(1,t) = 0, \\ &e(\cdot, 0) = -z_0, \end{aligned} \quad (51)$$

where $l(x) = \sum_{n=1}^N l_n\phi_n(x)$. Similarly to Lemma 3, the well-posedness of (8), (51) is established considering $f(x,t) = -l(x)e(0,t_k)$ as constant inhomogeneities on every step $[t_k, t_{k+1})$, $k \in \mathbb{N}$. Presenting e as (19), we obtain (cf. (22))

$$\dot{\bar{e}}(t) = (A - LC)\bar{e}(t) - LCv(t) + L\zeta(t_k), \; t \in [t_k, t_{k+1}), \quad (52)$$

where $v(t) = \bar{e}(t_k) - \bar{e}(t)$ for $t \in [t_k, t_{k+1})$ and the other notations are from (23). Consider the functional $V_h = V_0 + V_W$ with $V_0 = \bar{e}^T(t)P\bar{e}(t)$ and

$$V_W = h^2e^{2\delta h}\int_{t_k}^t e^{-2\delta(t-s)}\dot{\bar{e}}^T(s)W\dot{\bar{e}}(s)\,ds - \frac{\pi^2}{4}\int_{t_k}^t e^{-2\delta(t-s)}v^T(s)Wv(s)\,ds, \quad t \in [t_k, t_{k+1}). \quad (53)$$

Note that $V_W \ge 0$ due to the exponential Wirtinger inequality [23, Lemma 1]. Moreover, V_h does not increase in the jumps at t_k and is continuous elsewhere. We have

$$\begin{aligned} &\dot{V}_0 + 2\delta V_0 = 2\bar{e}^TP\dot{\bar{e}} + 2\delta\bar{e}^TP\bar{e}, \\ &\dot{V}_W + 2\delta V_W = h^2e^{2\delta h}\dot{\bar{e}}^T(t)W\dot{\bar{e}}(t) - \tfrac{\pi^2}{4}v^T(t)Wv(t), \\ &0 = 2[\bar{e}^TP_2^T + \dot{\bar{e}}^TP_3^T]\times \\ &[-\dot{\bar{e}} + (A - LC)\bar{e}(t) - LCv(t) + L\zeta(t_k)], \quad t \in [t_k, t_{k+1}). \end{aligned} \quad (54)$$

Summing up, we obtain

$$\dot{V}_h + 2\delta V_h - \gamma\zeta^2(t_k) = \xi^T\Xi\xi, \quad (55)$$

[2] MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/TAC18a

where $\xi = \mathrm{col}\{\bar{e}, \dot{\bar{e}}, v, \zeta(t_k)\}$ and

$$\Xi = \left[\begin{array}{c|c} \Upsilon & \begin{array}{c} P_2^T L \\ P_3^T L \\ 0_{N\times 1} \end{array} \\ \hline L^T P_2 \; L^T P_3 \; 0_{1\times N} & -\gamma \end{array}\right]. \tag{56}$$

The rest of the proof is similar to that of Theorem 1. ■

Corollary 3: The observer (49) with $L = (l_1, \ldots, l_N)^T$ satisfying (24) converges to (3) with the decay rate δ in the sense of (27) if the sampling period h is small enough.

Proof: Take P from (24), $P_2 = P$, $P_3 = \varepsilon I > 0$, $W = \mu^{-1} I > 0$, and $h = 0$. Calculating the Schur complement, we find that $\Upsilon < 0$ is equivalent to (45), which, in view of (24), holds for small $\varepsilon > 0$ and $\mu > 0$. Thus, $\Upsilon < 0$ is feasible for $h = 0$ and, by continuity, remains so for a small $\tau_M > 0$. Then Theorem 2 implies (27). ■

Remark 6: The LMIs of Theorem 2 can be transformed to solve the design problem in a manner similar to Remark 4.

Remark 7: If the sampling is uniform, i.e., $t_k = kh$, the system (52) can be studied using the discretization [21, Section 7.1.1]. Combining it with the modal decomposition technique, one will obtain necessary and sufficient conditions for (3), (46), (49) to satisfy (27). The advantage of the Lyapunov-Krasovskii approach developed here is that it leads to simple conditions under variable sampling (47).

IV. Example

Consider the system (3) with $a = 25$ and sampled in time boundary measurements (46) subject to (47). We consider an unstable plant since otherwise $\hat{z}(x,t) = 0$ is an exponentially converging estimate. Let $\delta = 1$ be the desired rate of convergence of the observation error. Since (16) holds with $N = 2$, the observer (49) with appropriate injection gains l_1, l_2 provides exponentially converging state estimate for a small enough sampling period h. To find l_1, l_2, and h, we take small h and increase it while the design LMIs with $\varepsilon = 0.5$ (see Remarks 4 and 6) remain feasible. This gives

$$h = 0.048, \quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \approx \begin{bmatrix} 23.2 \\ -1.1 \end{bmatrix}. \tag{57}$$

The analytical bound for the uniform sampling is $h \approx 0.081$, which we found using the method described in Remark 7. Note that we used the Lyapunov functional with the Wirtinger-based term (53) that leads to simple LMIs on the account of some conservatism. Less conservative conditions may be derived using other types of Lyapunov functionals (see, e.g., [24]).

The results of numerical simulations for the initial function

$$z_0(x) = \sin(2\pi x), \quad x \in [0,1] \tag{58}$$

are given in Figs. 1 and 2. For comparison, Fig. 2 also shows the error under the continuous measurements $y(t) = z(0,t)$.

The observer (49) coincides with (17) for $\tau(t)$ defined in (48). Thus, it can be studied using Theorem 1 and Remark 4. In the considered example, these conditions lead to a smaller sampling period $h = 0.031$ with approximately the same injection gains l_1, l_2.

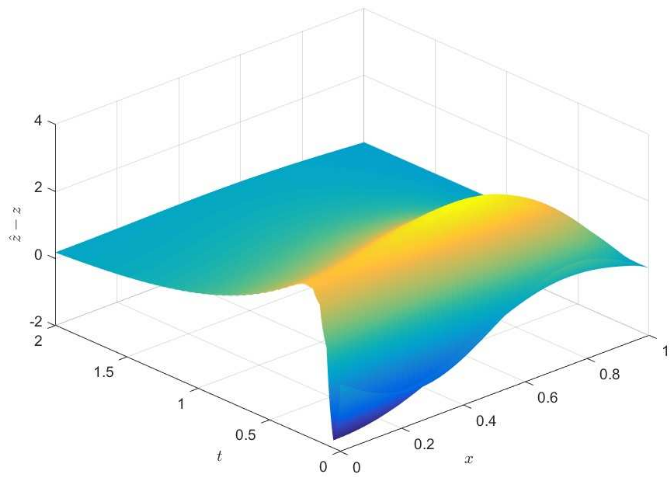

Fig. 1. Estimation error $\hat{z}(x,t) - z(x,t)$ of the observer (49) under the sampled-data measurements (46)

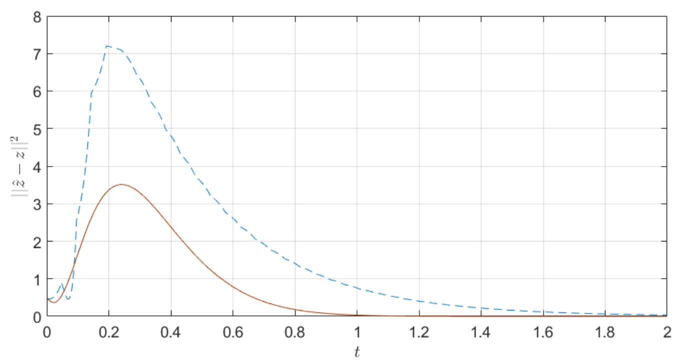

Fig. 2. Evolution of $\|\hat{z}(\cdot,t) - z(\cdot,t)\|^2_{L^2}$ for sampled-data (dashed blue line) and continuous-time (solid red line) measurements

V. Conclusion

We have designed finite-dimensional observers for a 1D reaction-diffusion system under delayed and sampled in time boundary measurements. We showed how to choose the observer injection gains and proved that it provides exponentially converging estimate if the time-delay or sampling period are small enough. The obtained LMIs allow to find admissible bounds on the delay and sampling period. The proposed observers can be used to design network-based controllers for parabolic systems. This may be a subject of the future research.

Appendix A
Proof of Lemma 3

The proof is based on [16, Theorem 7.7] and the step method. Since $t - \tau(t) \le 0$ for $t \in [0, \tau_m]$,

$$f(x,t) = -l(x) e(0, t - \tau(t)), \quad t \in [0, \tau_m] \tag{59}$$

can be viewed as inhomogeneity $f \colon [0, \tau_m] \to L^2(0,1)$ and

$$\int_0^{\tau_m} \|f(s)\|^2_{L^2}\, ds \overset{(29c)}{\le} \int_0^{\tau_m} \|l(\cdot) z_0(0)\|^2_{L^2}\, ds = \tau_m z_0^2(0) \|l\|^2_{L^2} < \infty. \tag{60}$$

Therefore, $f \in L^2((0,\tau_m); L^2(0,1))$ and [16, Theorem 7.7] guarantees the existence of a unique strong solution $e \in C([0,\tau_m]; \mathcal{H}^1)$.

Since $t - \tau(t) \leq \tau_m$ for $t \in [\tau_m, 2\tau_m]$,

$$f(x,t) = -l(x)e(0, t-\tau(t)), \quad t \in [\tau_m, 2\tau_m] \tag{61}$$

can be viewed as inhomogeneity $f\colon [\tau_m, 2\tau_m] \to L^2(0,1)$. Since $e(\cdot,t)$ is continuous on $[0,\tau_m]$ in $\mathcal{H}^1$, $e(0,t)$ is also continuous on $[0,\tau_m]$:

$$\begin{aligned} |e(0,t_1) - e(0,t_2)| &= \left| \int_0^1 \left(e_x(y,t_1) - e_x(y,t_2)\right) dy \right| \\ &\leq \|e_x(\cdot,t_1) - e_x(\cdot,t_2)\|_{L^2}. \end{aligned} \tag{62}$$

Thus, there exists $M_e \in \mathbb{R}$ such that $\sup_{t\leq\tau_m} |e(0,t)| \leq M_e$. Clearly,

$$\int_{\tau_m}^{2\tau_m} \|f(s)\|_{L^2}^2 \, ds \leq \tau_m M_e^2 \|l\|_{L^2}^2 < \infty. \tag{63}$$

Therefore, $f \in L^2((\tau_m, 2\tau_m); L^2(0,1))$ and [16, Theorem 7.7] guarantees the existence of a unique strong solution $e \in C([\tau_m, 2\tau_m]; \mathcal{H}^1)$. Repeating the same reasoning consequently on every interval $[j\tau_m, (j+1)\tau_m]$ with $j = 2, 3, \ldots$, we obtain the existence of a unique strong solution on $[0,\infty)$.

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