BOUNDARY PROBLEMS FOR PSEUDO-DIFFERENTIAL OPERATORS

BY

LOUIS BOUTET DE MONVEL

Faculté des sciences, Nice, France

Introduction

Let Ω be a C^∞ manifold, with boundary $\partial\Omega$. Let E, E' (resp. F, F') be vector bundles on Ω (resp. $\partial\Omega$). Our aim is to construct an "algebra" of operators:

$$(0.1) \qquad \begin{pmatrix} A & K \\ T & Q \end{pmatrix}: \begin{matrix} C^\infty(\overline{\Omega}, E) \\ \oplus \\ C^\infty(\partial\Omega, F) \end{matrix} \rightarrow \begin{matrix} C^\infty(\overline{\Omega}, E') \\ \oplus \\ C^\infty(\partial\Omega, F) \end{matrix}$$

which will contain at least the operator describing a classical boundary problem, and also its parametrix in the elliptic case. In fact what we construct there is one of the smallest possible "algebras" that will work. In that respect, our result is less general than that of Višik and Eskin [10]. The difference lies in the fact that in our problem, the pseudo-differential appearing in (0.1) (coefficient A) has to satisfy a supplementary condition along the boundary: the transmission property.(1) The operators that arise in (0.1) have already been described in [6] (where we also require analyticity).

In this work, we only require that the operators preserve locally C^∞ functions. The symbolic calculus is developed further than in [6], and we derive an index formula for elliptic problems, extending that of [3].

Roughly speaking, the coefficient A in (0.1) is a sum $A = P + G$, where P is a pseudo-differential operator satisfying the transmission condition (§ 2), and G (which we call a singular Green operator —§ 3) is an operator which takes any distribution into a function which is C^∞ inside Ω (but may be irregular at the boundary): such operators arise for

(1) With Višik's notations, the partial indices $\chi_i(X', \xi')$ have to be integers. This is an important restriction, since the operators that arise in mixed boundary problems do not usually satisfy it. However, many problems can already be reduced to this case.

example to describe the change in the solution of an elliptic boundary problem when the boundary conditions are modified.

The coefficient K in the matrix (0.1) is called a Poisson operator (§ 3). It serves in particular to describe the solutions of the homogeneous equation $Pf=0$, where P is an elliptic (pseudo) differential operator, in terms of boundary data.

The coefficient T is called a trace operator (§ 3).

It is the sum of the adjoint of a Poisson operator and of classical trace operators $f \to Q(\partial_n^k f/\partial\Omega)$, where Q is a pseudo-differential operator on the boundary, and ∂_n a normal derivative. (The adjoints of Poisson operators do not seem to have been systematically used anywhere yet, although of course they arise implicitly in many places. They may be of interest for boundary problems since they extend continuously to much larger spaces than the classical boundary conditions—e.g. they are continuous for the L^2 topology).

The last term Q is a pseudo-differential operator on the boundary.

These operators form an "algebra"—i.e. the sum and the composition of two matrices such as (0.1) is another one if it is defined.

They also give rise to a symbolic calculus. In fact this is done in two steps. First an interior symbol is defined: this is just the symbol of the pseudo-differential operator that appears in (0.1). It is a continuous matrix on the set of non vanishing covectors on Ω.

Secondly a boundary symbol is defined. This is a Wiener–Hopf operator depending continuously on a non vanishing covector on the boundary $\partial\Omega$.

The Wiener–Hopf operators that we will use are described in § 1, and the other paragraphs depend rather heavily on that. The symbolic calculus is otherwise described in § 4. The general boundary problem is discussed in § 5, where we prove an index formula extending that of [3], [4]. The idea of the proof of the index formula the following: first we check that the index of some "key" elliptic systems are zero (§ 5, no. 2). In the general case, it turns out that the composition of an elliptic system and one of these can be deformed into a new system which splits as the direct sum of an elliptic operator on the boundary, and of an elliptic operator on the interior which coincides with the identity operator near the boundary ((5.15), (5.18), (5.19)). Then the index formula of [4] can be applied (§ 5, no. 8).

This work is the final version of lecture notes to the Nordic Summer School of Mathematics 1969. I wish to express here my warmest thanks to the organizers, T. Ganelius, L. Gårding and L. Hörmander, for the very profitable time I spent there.

Notations

Since the notations concerning function spaces on a manifold with boundary are not very uniform in the existing litterature, we begin by describing those that are used here.

$\bar{R}_+$ is the closed half line $x \geq 0$

$\bar{R}^n_+$ is the closed half-space $x_n \geq 0$ in R^n.

Let $\bar{\Omega}$ be a C^∞ manifold with boundary. We denote the boundary by $\partial\Omega$, and the interior by Ω (thus $\bar{\Omega}$ is the disjoint union $\bar{\Omega} = \Omega \cup \partial\Omega$).

We will usually suppose that $\bar{\Omega}$ is embedded in some C^∞ neighboring manifold of the same dimension (e.g. the double of $\bar{\Omega}$). All the constructions hereafter do not depend on the choice of this manifold, and we will refer to it as "a neighborhood of $\bar{\Omega}$".

We denote by $C^\infty(\bar{\Omega})$ (resp. $C_0^\infty(\bar{\Omega})$) the space of functions which are C^∞ up to the boundary on Ω (resp. and of compact support), i.e. every derivative has a limit on the boundary: such a function can be extended into a C^∞ function near $\bar{\Omega}$.

Similarly, if E is a C^∞ vector bundle on $\bar{\Omega}$, $C^\infty(\bar{\Omega}, E)$ (resp. $C_0^\infty(\bar{\Omega}, E)$) is the space of sections of E which are C^∞ up to the boundary (resp. and of compact support).

We denote by $\mathcal{D}'(\bar{\Omega})$ (resp. $\mathcal{E}'(\bar{\Omega})$) the space of distributions defined in a neighborhood of $\bar{\Omega}$, and supported by $\bar{\Omega}$ (resp. and of compact support). Thus $\mathcal{D}'(\bar{\Omega})$ is the dual space of $C_0^\infty(\bar{\Omega})$, and $\mathcal{E}'(\bar{\Omega})$ is the dual space of $C^\infty(\bar{\Omega})$.(1) (As usual $\mathcal{D}'(\Omega)$ denotes the space of distributions on the interior Ω. Let us recall that the restriction map $\mathcal{D}'(\bar{\Omega}) \to \mathcal{D}'(\Omega)$ is neither onto nor one to one.)

Throughout this work, $T\Omega$ will denote the cotangent bundle (this is of course isomorphic to the tangent bundle—e.g. through the choice of a C^∞ metric on Ω). $T\bar{\Omega}$ is the restriction to $\bar{\Omega}$ of the cotangent bundle of a neighborhood of $\bar{\Omega}$.

In § 5 we use K-theory. For the definitions and main theorems, we refer to [1], [2]. We will be concerned with K-theory with compact supports only. Thus if $B_{\bar{\Omega}}$ and $S_{\bar{\Omega}}$ are the unit ball and unit sphere of $T\bar{\Omega}$ for some metric, there is a natural identification

$$K(T\bar{\Omega}) = K(B_{\bar{\Omega}})$$

$$K(T\Omega) = K(B_{\bar{\Omega}}, \partial B_{\bar{\Omega}}) = K(B_{\bar{\Omega}}, S_{\bar{\Omega}} \cup (B_{\bar{\Omega}}/\partial\Omega)).$$

---

(1) In the present article, we are really concerned with the space of currents of order 0 (generalized functions), which is the natural extension of the space of functions. Naturally this is identified with the space of distributions once a measure with C^∞ density has been chosen. And we will not make the distinction further on.

1. The Wiener–Hopf algebra

The results in this paragraph are not particularly new or difficult. But it was convenient to group them here.

1. *Notations.*

(1.1) Let H be the vector space of all complex valued functions $f(t)$ on the real line, which are C^∞ and have a regular pole at infinity, i.e. $(z+1)^p f\left(\frac{1}{i}\,\frac{1-z}{1+z}\right)$ is a C^∞ function on the unit circle $|z|=1$ (including at the point $z=-1$) for large integral p. Or equivalently f is C^∞ and has an asymptotic expansion

$$f \sim \sum_{k \geq -N} a_k t^{-k} \quad (t \to \infty)$$

and this expansion still holds after any number of differentiations.

(1.2) Let H^+ be the subspace consisting of those functions $f \in H$ which can be extended analytically in the lower complex half plane $\operatorname{Im} t \geq 0$, and vanish at infinity (for such functions, the asymptotic expansion (1.1) holds when $t \to \infty$, $\operatorname{Im} t \leq 0$, and $N=-1$).

(1.3) Let H^- be the supplementary of H^+ in H consisting of those functions which can be extended analytically in the upper half plane $\operatorname{Im} t \geq 0$: for such functions, the asymptotic expansion (1.1) holds when $t \to \infty$, $\operatorname{Im} t \geq 0$.

(These spaces have a topology in a natural way: H^+ is Fréchet, H and H^- are LF.)

(1.4) We will denote by h^+ (resp. h^-) the projection on H^+ parallel to H^- (resp. $h^- = 1 - h^+$).

Thus if f is analytic on the real line, meromorphic at infinity, we have

$$(1.5) \qquad \begin{aligned} h^+ f(t) &= -\frac{1}{2i\pi}\int_\gamma \frac{f(\tau)}{\tau - t}\, d\tau \quad (\text{if } \operatorname{Im} t < 0) \\ h^- f(t) &= \frac{1}{2i\pi}\int_\gamma \frac{f(\tau)}{\tau - t}\, d\tau \quad (\text{if } \operatorname{Im} t > 0), \end{aligned}$$

where γ is a large circle in the upper half plane $\operatorname{Im} \tau > 0$, oriented in the usual way (γ is required to contain t in its interior for the second formula).

If f vanishes at infinity, we also have

$$(1.5)' \qquad \begin{aligned} h^+ f(t) &= \lim_{\varepsilon \to +0} -\frac{1}{2i\pi}\int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t + i\varepsilon}\, d\tau \\ h^- f(t) &= \lim_{\varepsilon \to +0} \frac{1}{2i\pi}\int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t - i\varepsilon}\, d\tau. \end{aligned}$$

If f is analytic on the real line, and meromorphic at infinity, we set

$$\int^+ f = \int^+ f(t)\,dt = \int_\gamma f(\tau)\,d\tau, \tag{1.6}$$

where γ is, as before, a large circle in the upper half plane $\operatorname{Im}\tau > 0$.

This linear operation extends continuously to H. Of course $\int^+ f = \int f$ is just the ordinary integral if f is integrable (i.e. vanishes to the second order at infinity).

Let us notice that $\int^+ f$ vanishes if f belongs to H^+ and is integrable or if f belongs to H^-. So $\int^+ fg$ only depends on $h^- g$ (resp. $h^+ g$) if $f \in H^+$ (resp. $f \in H^-$).

Let $p \in H$. Then p has a unique expansion

$$p(t) = \sum_{s=1}^{N} \alpha_s t^s + \sum_{-\infty}^{+\infty} a_k \left(\frac{1-it}{1+it}\right)^k, \tag{1.7}$$

where the coefficients a_k form a rapidly decreasing sequence If $f \in H$ vanishes at infinity, it also has a unique expansion

$$f(t) = \sum_{-\infty}^{+\infty} a_k \frac{(1-it)^k}{(1+it)^{k+1}}, \tag{1.8}$$

where the coefficients a_k form a rapidly decreasing sequence. In this case, f belongs to H^+ (resp. H^-) if and only if $a_k = 0$ when $k < 0$ (resp. $k \geqslant 0$).

In formulas (1.7), (1.8), one can replace $\left(\frac{1-it}{1+it}\right)^k$ $\left(\text{resp. } \frac{(1-it)^k}{(1+it)^{k+1}}\right)$ by $\left(\frac{\lambda-it}{\lambda+it}\right)^k$ $\left(\text{resp. } \frac{(\lambda-it)^k}{(\lambda+it)^{k+1}}\right)$, where λ is any positive number.

We also have the following result:

(1.9) *H^+ is the space of Fourier transforms of functions $\varphi(x)$ which vanish for $x<0$ and are $C^\infty(\bar{R}_+)$, rapidly decreasing at infinity for $x>0$ (i.e. every derivative tends to zero at infinity, faster than any power of x, and has a limit when $x \to +0$).*

Proof. First let $f \in H^+$, and let φ be its inverse Fourier transform. Then φ is square integrable and vanishes on the negative half line. Moreover, the distribution $(d/dx)^p x^q \varphi$ coincides on the open half line $x>0$ with the inverse Fourier transform of $h^+(i^{p-q}\xi^p(d/d\xi)^q f)$ which lies in H^+, so it is also square integrable. It follows that every derivative of φ has a limit when $x \to +0$, and is of rapid decrease when $x \to +\infty$.

Now let φ be as in proposition (1.9), and let f be its Fourier transform. Then f is holomorphic for $\operatorname{Im}\xi \leqslant 0$, C^∞ up to the boundary, and admits the asymptotic expansion

$$f \sim \sum_{n\geqslant 0} \varphi^{(n)}(0)\,(i\,\xi)^{-n-1} \quad (\xi \to \infty,\ \operatorname{Im}\xi \leqslant 0).$$

Since $x^p\varphi$ has the same properties as φ, this expansion still holds after any number of differentiations. This ends the proof.

Example; $\dfrac{(\lambda-it)^k}{(\lambda+it)^{k+1}}$ $(k\geqslant 0)$ is the Fourier transform of the product of a Laguerre polynomial by an exponential:

$$\varphi(x)=\begin{cases}\left(\lambda-\dfrac{d}{dx}\right)^k\left(\dfrac{x^k}{k!}e^{-\lambda x}\right) & \text{if } x>0\\ 0 & \text{if } x<0.\end{cases}$$

In a similar way, a function $f\in H^-$ is the Fourier transform of the sum of the symmetric of a function φ as above, and of a linear combination of derivatives of the Dirac measure at the origin.

2. *The Wiener–Hopf algebra.* We proceed now to describe a family of operators on H^+—more generally of matrices:

$$\begin{pmatrix}p+g & k\\ t & q\end{pmatrix}:\ \begin{matrix}H^+\otimes E\\ \oplus\\ F\end{matrix}\ \rightarrow\ \begin{matrix}H^+\otimes E'\\ \oplus\\ F',\end{matrix}\tag{1.10}$$

where E, E', F, F' are finite dimensional vector spaces.

a. Let $p\in H\otimes L(E,E')$. We will denote by $h^+\circ p$ (or p as in (1.10) if this does not lead to confusion) the operator

$$f\in H^+\otimes E\rightarrow h^+(p\cdot f)\in H^+\otimes E'$$

b. $q\in L(F,F')$ is any linear operator.

Let $k\in H^+\otimes L(F,E')$. We denote by the same letter k the operator

$$u\in F\rightarrow k\cdot u\in H^+\otimes E'.$$

Let $t\in H^-\otimes L(E,F')$. We denote by the same letter t $\left(\text{or } \dfrac{1}{2\pi}\displaystyle\int^+\circ\, t \text{ if there may be any confusion}\right)$ the operator

$$f\in H^+\otimes E\rightarrow\frac{1}{2\pi}\int^+ t\cdot f\in F'.$$

c. Finally let $g\in H_\xi^+\hat{\otimes}H_\eta^-\otimes L(E,E')$ i.e. g has a series expansion

$$g(\xi,\eta)=\sum_{s=0}^{N}k_s(\xi)\,\eta^s+\sum_{\substack{p\geqslant 0\\ q\geqslant 0}}a_{pq}\,\frac{(1-i\xi)^p}{(1+i\xi)^{p+1}}\,\frac{(1+i\eta)^q}{(1-i\eta)^{q+1}},\tag{1.11}$$

where $$k_s(\xi)\in H^+\otimes L(E,E')\quad (s=0,\ldots N)$$

and the $a_{pq}\in L(E,E')$ form a rapidly decreasing double sequence of matrices.

We denote by the same letter g $\left(\text{or } \frac{1}{2\pi}\int^{+} \circ g \text{ if there may be any confusion}\right)$ the operator

$$f \in H^{+} \otimes E \to \frac{1}{2\pi}\int^{+} g(\xi, \eta)\, f(\eta)\, d\eta \in H^{+} \otimes E'$$

p will be called a pseudo differential symbol

g will be called a singular Green symbol

$p+g$ will be called a Green symbol

k will be called a Poisson symbol

t will be called a trace symbol.

(1.12) THEOREM. *The operators* $\begin{pmatrix} p+g & k \\ t & q \end{pmatrix}$ *above form an algebra—i.e. the sum and composition of two such operators is another one if it is defined.*

We have the following formulas

(1.13) 1. $h^{+}p \circ k = h^{+}(p \cdot k)$

2. $g \circ k = \frac{1}{2\pi}\int^{+} g(\xi, \eta) \cdot k(\eta)\, d\eta$

3. $k \circ q = k \cdot q$

are Poisson symbols

4. $t \circ h^{+}p = h^{-}(t \cdot p)$

5. $t \circ g = \frac{1}{2\pi}\int^{+} t(\eta) \cdot g(\eta, \xi)\, d\eta$

6. $q \circ t = q \cdot t$

are trace symbols

7. $t \circ k = \frac{1}{2\pi}\int^{+} t(\xi) \cdot k(\xi)\, d\xi$

is an operator of finite rank

8. $p \circ g = h_{\xi}^{+}[p(\xi) \cdot g(\xi, \eta)]$

9. $g \circ p = h_{\eta}^{-}[g(\xi, \eta) \cdot p(\eta)]$

10. $g_1 \circ g_2 = \frac{1}{2\pi}\int^{+} g_1(\xi, s)\, g_2(s, \eta)\, ds$

11. $k \circ t = k(\xi) \cdot t(\eta)$

12. $h_0^+ p_1 p_2 - h^+ p_1 \circ h^+ p_2 = L(p_1, p_2)$

are singular Green symbols.

We have set

13. $L(p_1, p_2) = h_\xi^+ h_\eta^- [(p_1^+(\xi) - p_1^+(\eta))(p_2^-(\xi) - p_2^-(\eta))(i\xi - i\eta)^{-1}]$

with $p_1^+ = h^+(p_1)$, $p_2^- = h^-(p_2)$.

Theorem 1.12 is a consequence of formulas (1.13) 1.... 13. These formulas are all obvious, except maybe the last one, which we prove now. Let L be the operator of (1.13) 12:

$$Lf = h^+(p_1 p_2 f) - h^+(p_1 h^+(p_2 f)) = h^+(p_1 h^-(p_2 f)).$$

Since f belongs to H^+, it follows that Lf only depends on $p_1^+ = h^+(p_1)$ and $p_2^- = h^-(p_2)$.

In a first step, we will suppose that p_1 and p_2 both vanish at infinity. Then we have

$$Lf = \lim_{\varepsilon \to +0} -\frac{1}{2i\pi} \int \frac{p_1^+(t)\,dt}{(t-\xi+i\varepsilon)} \lim_{\delta \to +0} \int \frac{p_2^-(\eta)\, f(\eta)\, d\eta}{(\eta - t - i\delta)} = \frac{1}{2\pi} \int g(\xi, \eta)\, p_2^-(\eta)\, f(\eta)\, d\eta$$

with

$$g(\xi, \eta) = \lim_{\varepsilon \to +0} \lim_{\delta \to +0} -\frac{1}{i} \frac{1}{2i\pi} \int \frac{p_1^+(t)\,dt}{(t-\xi+i\varepsilon)(\eta - t - i\delta)}$$

$$= \lim_{\varepsilon \to +0} \lim_{\delta \to +0} -\frac{1}{i} \left(\frac{p_1^+(\xi - i\varepsilon) - p_1^+(\eta - i\delta)}{\xi - i\varepsilon - \eta + i\delta} \right) = -(p_1^+(\xi) - p_1^+(\eta))(i\xi - i\eta)^{-1}.$$

We have used the identity

$$\frac{1}{(t-\xi+i\varepsilon)(\eta - t - i\delta)} = -\frac{1}{(\xi - i\varepsilon - \eta + i\delta)} \left(\frac{1}{t - \xi + i\varepsilon} - \frac{1}{\eta - t - i\delta} \right).$$

Since we have obviously

$$h_\xi^+ h_\eta^- [(p_1^+(\xi) - p_1^+(\eta))\, p_2^-(\xi)\, (i\xi - i\eta)^{-1}] = 0$$

we finally get as announced

$$Lf = \frac{1}{2\pi} \int^+ l(\xi, \eta)\, f(\eta)\, d\eta$$

with $$l(\xi, \eta) = h_\xi^+ h_\eta^- [(p_1^+(\xi) - p_1^+(\eta))(p_2^-(\xi) - p_2^-(\eta))(i\xi - i\eta)^{-1}].$$

Next we prove formula (1.13) 12, when p_2 is a polynomial:

$$p_2(\xi) = \xi^q.$$

Then if $f \sim \sum a_k \xi^{-k-1}$ when $\xi \to \infty$ we have

$$h^-(\xi^q f) = \sum_{k \leq q-1} a_k \xi^{q-k-1} = \sum_{k \leq q-1} \xi^{q-k-1} \frac{1}{2i\pi} \int^+ \eta^k f(\eta)\, d\eta$$

(because of the formula $a_k = (1/2i\pi)\int^+ \eta^k f(\eta)\,d\eta$). Using the identity $\sum \xi^{q-k-1}\eta^k (\xi^q - \eta^q)/(\xi - \eta)$ we finally get

$$Lf = h_\xi^+ \left[p_1^+(\xi) \cdot \frac{1}{2\pi} \int^+ \frac{(p_2^-(\xi) - p_2^-(\eta))}{i\xi - i\eta} f(\eta)\,d\eta \right] = \frac{1}{2\pi} \int^+ l(\xi, \eta) f(\eta)\,d\eta$$

with
$$l(\xi, \eta) = h_\xi^+ [p_1^+(\xi)\,(p_2^-(\xi) - p_2^-(\eta))\,(i\xi - i\eta)^{-1}]$$
$$= h_\xi^+ h_\eta^- [(p_1^+(\xi) - p_1^+(\eta))\,(p_2^-(\xi) - p_2^-(\eta))\,(i\xi - i\eta)^{-1}].$$

(The last equality comes from the fact that $(\xi^q - \eta^q)/(\xi - \eta)$ is a polynomial with respect to ξ and η, so $h_\xi^+ [p_1^+(\eta)\,(\xi^q - \eta^q)\,(i\xi - i\eta)^{-1}] = 0$ and $h_\eta^- [p_1^+(\xi)(\xi^q - \eta^q)\,(i\xi - i\eta)^{-1}] = p_1^+(\xi)\,(\xi^q - \eta^q)\,(i\xi - i\eta)^{-1}$).

In the general case, p_1^+ vanishes at infinity anyway, and p_2^- is the sum of a polynomial and of a function which vanishes at infinity. This proves formula (1.13) 12, 13.

It remains to check that the symbol $L(p_1, p_2)$ that we have just obtained satisfies conditon (1.11). This is obvious when p_2^- is a polynomial: then $L(p_1, p_2)$ is a finite sum

$$L(p_1, p_2) = \sum_{p,q} h_\xi^+ (\xi^p p_1^+(\xi))\,\eta^q.$$

On the other hand, if p_1 and p_2 are both bounded:

$$p_1 = \sum_{-\infty}^{+\infty} a_p \left(\frac{1 - i\xi}{1 + i\xi}\right)^p$$

$$p_2 = \sum_{-\infty}^{+\infty} b_q \left(\frac{1 - i\xi}{1 + i\xi}\right)^q$$

we get

$$L(p_1, p_2) = \sum_{\substack{p \geq 0 \\ q \geq 0}} c_{pq} \frac{(1 - i\xi)^p}{(1 + i\xi)^{p+1}} \frac{(1 + i\eta)^q}{(1 - i\eta)^{q+1}} \tag{1.14}$$

$$c_{pq} = 2 \sum_{k=1}^{\infty} a_{p+k}\, b_{-q-k}$$

so that c_{pq} is a rapidly decreasing double sequence. We end this paragraph with

(1.15) PROPOSITION. *Let* $a = \begin{pmatrix} p + g & k \\ t & q \end{pmatrix}$ *be an invertible Wiener–Hopf operator. Then the inverse* a^{-1} *is also a Wiener–Hopf operator.*

Proof. Notice first that a "singular" operator $\begin{pmatrix} g & k \\ t & q \end{pmatrix}$ behaves very much like a compact operator (in fact it extends continuously into a compact operator on the completion of

H^+ for the Fourier transform of the H_s norm if s is large enough). Then if $a = \begin{pmatrix} p+g & k \\ t & q \end{pmatrix}$ is invertible, p is already an invertible C^∞ matrix i.e. $p^{-1} \in H$, and the index of $\begin{pmatrix} p^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is zero. Now we can add to $\begin{pmatrix} p^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ an operator of an finite rank of the form $\begin{pmatrix} g' & k' \\ t' & q' \end{pmatrix}$ so that $a' = \begin{pmatrix} p^{-1}+g' & k' \\ t' & t' \end{pmatrix}$ is invertible. Multiplying a by a', we are thus reduced to the case where $p = 1$, $a = 1 + \begin{pmatrix} g & k \\ t & q \end{pmatrix} = 1 + G$.

The reader will check readily that proposition (1.15) is true if $G = \begin{pmatrix} g & k \\ t & q \end{pmatrix}$ is very small: then the inverse is $a' = 1 + G'$ where G' is the sum of the geometric series $G' = \sum_0^\infty (-1)^s G^s$, and its coefficients satisfy conditions b, c of the beginning of this section (the product rules giving G^s are those of (1.13)).

In the general case, $G = \begin{pmatrix} g & k \\ t & q \end{pmatrix}$ can be approximated: $G = K'T + G'$ where K' is a column matrix $K' = \begin{pmatrix} k' \\ q' \end{pmatrix}$, T a row matrix $T = (t\ p)$, and G' is arbitrarily small (this follows immediately from the series expansion (1.11)). Then we get

$$a = 1 + G = (1 + G')(1 + KT)$$

with $K = (1 + G')^{-1} K'$.

So we are reduced to the case where $G = KT = \begin{pmatrix} g & k \\ t & q \end{pmatrix}$ is an operator of finite rank. In this last case, if $a = 1 + G$ is invertible, its inverse a^{-1} is a polynomial of a so the result follows from theorem (1.12).

In view of the symbolic calculus developed in § 4, we introduce the following notation:

(1.16) If N is an oriented 1-dimensional real line, H_N^+ is the space of measures on N whose density lies in H^+.

Of course, N can depend continuously on a parameter, i.e. be a one dimensional oriented real vector bundle. In § 4, N will be the normal cotangent bundle (oriented by the inward normal).

All the constructions of this paragraph can be repeated with H_+^N instead of H^+.

2. Pseudo-differential operators. The transmission property

In what follows, we restrict our attention to pseudo-differential operators of type 1, 0 (i.e. the symbol $p(x, \xi)$ lies in $S_{1,0}^d$ for some d, with the notations of [7]). In § 4 and § 5, we restrict ourselves further to those pseudo-differential operators whose symbol admits an asymptotic expansion in homogeneous functions of integral degree of ξ (this is a special case of [9]).

Let $p(x, D)$ be a pseudo-differential operator defined in a neighborhood of the closed half space $\bar{R}^n_+$. We are interested in those pseudo-differential operators for which all the derivatives of the symbol admit an expansion such as (1.7) when x lies in the boundary:

$$(2.1)\qquad (\partial/\partial x)^\alpha p(x,\xi)=\sum_0^d \alpha_s(x,\xi')\,\xi_n^s+\sum_0^\infty a_k(x,\xi')\,\frac{(\langle\xi'\rangle-i\xi_n)^k}{(\langle\xi'\rangle+i\xi_n)^{k+1}}\ \text{ if }\ x_n=0,$$

where $\alpha_s \in S^{d-s}_{1,0}$, and a_k is a rapidly decreasing sequence in $S^{d+1}_{1,0}$.

We have set $\langle\xi'\rangle=(1+|\xi'|^2)^{\frac{1}{2}}$.

(2.2) *Definition.* We will say that $p(x, D)$ has the *transmission property with respect to the boundary* R^{n-1} if every derivative of its symbol admits the series expansion (2.1) along the boundary.

More generally, we will say that a pseudo-differential operator P has the transmission property, if it is the sum of an operator with C^∞ distribution kernel (negligible operator), and of a pseudo-differential operator $p(x, D)$ as in definition (2.2). Such an operator admits a Fourier integral representation (as in [8]):

$$Pf(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}p(x,y,\xi)\,f(y)\,dy\,d\xi,$$

where the function $p(x, y, \xi)$ and all its derivatives admit a series expansion such as (2.1) on the set $x=y$, $x_n=0$ (because this is true if P is negligible—e.g. if P is defined by the C^∞ kernel $\varphi(x, y)$, we have

$$Pf(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}p(x,y,\xi)\,f(y)\,dy\,d\xi$$

if we set $p(x,y,\xi)=e^{i(x-y)\cdot\xi}p_0(\xi)\varphi(x,y)$ where $p_0\in C_0^\infty(R^n)$, and $\int p_0(\xi)d\xi=(2\pi)^n$).

Condition (2.1) is equivalent to the following:

(2.1)′ *For all derivation indices* α, β, *the function*

$$(z+1)^d\,p^\alpha_\beta\left(x,\xi',\,-i\langle\xi'\rangle\frac{z+1}{z-1}\right)$$

is C^∞ *on the set* $x_n=0$, $|z|=1$, *and we have*

$$\left|(\partial/\partial z)^m\left[(z+1)^d\,p^\alpha_\beta\left(x,\xi',\,-i\langle\xi'\rangle\frac{z+1}{z-1}\right)\right]\right|\leqslant c_{m,\alpha,\beta}(x')\,\langle\xi'\rangle^{d-|\alpha|},$$

where $c_{m,\alpha,\beta}$ *is a continuous positive function of* x' *alone (we have set* $p^\alpha_\beta=(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta p$, *and* d *is the degree of* p*) or equivalently*

(2.1)″ *For all derivation indices α, β, p^α_β admits an asymptotic expansion*

$$p^\alpha_\beta \sim \sum_{k \geqslant -d} b_k(x, \xi')\, \xi_n^{-k}$$

when $\xi_n \to \infty$, $x_n = 0$, and the other variables are fixed, and we have

$$\Big| p^\alpha_\beta - \sum_{-d \leqslant k \leqslant N} b_k(x, \xi')\, \xi_n^{-k} \Big| \leqslant c_{N,\alpha,\beta}(x')\, \langle \xi' \rangle^{N+d-|\alpha|}\, \xi_n^{-N}$$

(*here again, d is the degree of p*)

(We leave the proof of these equivalences to the reader.)

Let us also notice that the coefficients α_s, a_k of (2.1) cannot be arbitrary if $p \in S^d_{1,0}$. First α_s has to be a polynomial of degree $d-s$ with respect to ξ': this is seen by descending recursion on s—knowing it when $t < s \leqslant d$, we get

$$(\partial/\partial\xi)^\alpha (\partial/\partial\xi_n)^t p = t!\, (\partial/\partial\xi')^\alpha \alpha_t + O(|\xi'|^{d-|\alpha|} \xi_n^{-t}) = O(|\xi|^{d-|\alpha|-t})$$

so that $(\partial/\partial\xi')^\alpha \alpha_t = 0$ if $|\alpha| > d-t$

Also if p is the sum of the series:

$$p = \sum a_k(x', \xi')\, (\langle \xi' \rangle - i\xi_n)^k\, (\langle \xi' \rangle + i\xi_n)^{-k-1}$$

where a_k is a rapidly decreasing sequence in $S^{d+1}_{1,0}$, we only have

$$\left| \left(\frac{\partial}{\partial \xi'} \right)^\alpha p \right| \leqslant c_\alpha(x')\, \langle \xi' \rangle^{d-|\alpha|+1}\, |\xi|^{-1}.$$

So in general we do not have $p \in S^d_{1,0}$, which shows that the sequence a_k cannot be arbitrary either.

Suppose now that $p(x, \xi)$ has an asymptotic expansion (as in (7))

$$p(x, \xi) \sim \sum p_k(x, \xi)$$

i.e. $p - \sum_{k<N} p_k \in S^{d_N}_{1,0}$ where $d_N \to -\infty$.

Then if every p_k has the transmission property with respect to the boundary, so has p. (In particular, if $p(x, y, \xi)$ is as above, and if P is the Fourier integral operator it defines, we have $P \sim P'(x, D)$ where

$$p'(x, \xi) \sim \sum \frac{i^{-|\alpha|}}{\alpha!} \left(\frac{\partial}{\partial y} \right)^\alpha \left(\frac{\partial}{\partial \xi} \right)^\alpha p(x, y, \xi)_{y=x}$$

—cf. [8]—so P has the transmission property.)

Suppose now that the p_k hereabove are homogeneous functions of ξ, as in [9], and that the degree of p_k is d_k. Then by considering Taylor expansions of $p_k(x, \xi)$ near the normal covectors $\xi'=0$, and using homogeneity to reflect this on the behavior of $p(x, \xi)$ and the $p_k(x, \xi)$ when $\xi_n \to \infty$, we see that condition (2.1) is equivalent to

(2.3) *for every* k, $p_k(x, -\xi) - e^{i\pi d_k} p_k(x, \xi)$ *vanishes to the infinite order on the set of non zero normal covectors* $(x_n=0, \xi'=0, \xi_n \neq 0)$.

(This condition is in fact stronger than necessary for theorem (2.9) to hold (cf. [5]), except when the degree d_k is an integer for every k. If P is elliptic, $p_0(x, \xi)$ cannot vanish, so the degree d_0 has to be integral. In § 4 and 5, we assume that the d_k are all integers, but this is not an essential restriction.)

We deduce immediately from the formulas of symbolic calculus developed in [7] or [9].

(2.4) PROPOSITION. *If P and Q have the transmission property and are properly supported, then the composition $P \circ Q$ also has the transmission property.*

If P is elliptic and has the transmission property, then so has any parametrix of P.

(2.5) PROPOSITION. *A partial differential operator with C^∞ coefficients has the transmission property. In particular the multiplication by a C^∞ function has the transmission property.*

(2.6) PROPOSITION. *Definition* (2.2) *is invariant under a change of coordinates that preserves the boundary.*

As a consequence of these propositions, we see that the transmission property with respect to the boundary makes sense for pseudo-differential operators acting on the sections of a C^∞ vector bundle on a C^∞ manifold with boundary.

As a first useful example, we describe the pseudo-differential operators that have the transmission property in dimension one:

(2.7) THEOREM. *Let $P(x, D)$ be a pseudo-differential operator defined in a neighborhood of the half line $x \geqslant 0$. In order that the transmission property with respect to the origin hold for P, it is necessary and sufficient that P admits a decomposition $P = P_0 + P_1 + P_2$, where the symbol of P_0 vanishes to the infinite order at the origin $x=0$, P_1 is a differential operator with C^∞ coefficients, and the distribution kernel of P_2 is a function $f(x, y)$ which is C^∞ up to the diagonal for $x > y$, and also for $x < y$.*

Proof. Let us first choose P' so that the symbol of P' satisfies condition (2.1) at every point, and the symbol of $P_0 = P - P'$ vanishes to the infinite order at the origin. This can be done in any dimension: take for instance

$$p'(x,\xi)=\sum \frac{x_n^k}{k!}\left(\frac{\partial}{\partial x_n}\right) p(x',0,\xi)\,\varphi(\lambda_k x_n),$$

where $\varphi \in C_0^\infty(R)$ is equal to 1 near the origin, and the sequence λ_k increases sufficiently rapidly.

Next define $p_1(x, \xi)$ to be the polynomial part of $p'(x, \xi)$. Finally let $p_2(x, \xi)$ be the remainder $p_2 = p' - p_1$ so that $p_2(x, \xi)$ admits a series expansion as in (1.8)

$$p_2(x,\xi)=\sum_{-\infty}^{+\infty} a_k(x)\,\frac{(1-i\xi)^k}{(1+i\xi)^{k+1}},$$

where the $a_k(x)$ form a rapidly decreasing sequence in $C^\infty(R)$. Then $P_1(x, D)$ is a differential operator with C^∞ coefficients. The distribution kernel of P_2 is the function $f(x, y) = g(x, x-y)$ where $g(x, z)$ is the inverse Fourier transform of $p_2(x, \xi)$ with respect to ξ. It follows from proposition (1.9) that $g(x, z)$ is C^∞ up to the boundary $z = 0$ for $z > 0$, and also for $z < 0$. Conversely an operator $P_0(x, D)$ whose symbol vanishes to the infinite order at the origin obviously has the transmission property. So has a differential operator with C^∞ coefficients. That a pseudo-differential operator $P_2(x, D)$ as in theorem (2.7) has the transmission property follows from proposition (1.9) exactly as above. This ends the proof.

Now we introduce other notations.

Let $\overline{\Omega}$ be a C^∞ manifold with boundary $\partial\Omega$, and let V be a neighboring manifold.

If $f \in C^\infty(\Omega)$ (and more generally if $f \in C^\infty(\overline{\Omega}, E)$ where E is a C^∞ vector bundle on V) we denote by $\tilde{f}$ the extension of f by 0 outside Ω.

Now let P be a pseudo-differential operator on V. We define a new operator P_Ω: $C_0^\infty(\overline{\Omega}) \to C^\infty(\Omega)$ by

(2.8) $$P_\Omega f = P\tilde{f}/\Omega$$

P_Ω obviously depends only on the restriction of P to Ω (this is a pseudo-differential operator on the interior Ω, that can be extended as a pseudo-differential operator in a neighborhood of $\overline{\Omega}$).

(2.9) THEOREM. *Let P have the transmission property with respect to $\partial\Omega$. Then P_Ω is continuous $C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ (i.e. if f is C^∞ up to the boundary, then so is $P_\Omega f$).*

Proof. The theorem is local, so we will suppose that $\overline{\Omega}$ is the half space $\overline{R}_+^n$ $(x_n \geq 0)$. We can also suppose that P is properly supported, since the theorem is obviously true if P has a C^∞ distribution kernel. So P admits a Fourier integral representation. Finally since $P \cdot e^{x_n} \cdot f$ also has the transmission property, we can suppose $f = e^{-x_n}$. Then we have

$$P\tilde{f} = (2\pi)^{-1}\int e^{ix_n\xi_n}\,p(x,0,\xi_n)\,(1+i\xi_n)^{-1}\,d\xi_n,$$

where the integral represents the one dimensional inverse Fourier transform.

Now $p(x, 0, \xi_n) = p(x', x_n, 0, \xi_n)$, considered as a function of x_n, ξ_n alone, is the symbol of a one dimensional pseudo-differential operator which has the transmission property with respect to the origin, and as such depends C^∞ on x'.

To prove theorem (2.9), it only remains to prove it in dimension one. Then we use the decomposition of theorem (2.7): the result is obviously true for all three terms.

To end this paragraph, let us mention without proof the following result: if P has the transmission property, and its degree is d, then P_Ω extends continuously $H_s^{\text{comp}}(\overline{\Omega}) \to H_{s-d}^{\text{loc}}(\overline{\Omega})$ if $s > -\frac{1}{2}$. If the degree is negative, $d \leq 0$, then P_Ω also extends continuously $\mathcal{E}'(\overline{\Omega}) \to \mathcal{D}'(\overline{\Omega})$ (cf. [5]).

3. Poisson operators, trace operators, singular Green operators

0. *Negligible operators.* Let $\overline{\Omega}$ be a C^∞ manifold with boundary $\partial\Omega$. Let dy (resp. dy') be a measure on $\overline{\Omega}$ (resp. $\partial\Omega$) whose density is positive and C^∞ up to the boundary.

A negligible Poisson operator is an operator K: $C_0^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ which extends continuously: $\mathcal{E}'(\partial\Omega) \to C^\infty(\overline{\Omega})$. Equivalently K is defined by:

$$Kf(x) = \int_{\partial\Omega} k(x, y')\, f(y')\, dy',$$

where $k(x, y')$ is C^∞ up to the boundary on $\overline{\Omega} \times \partial\Omega$.

A negligible trace operator of class r is an operator T: $C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ defined by:

$$Tf(x') = \int_{\Omega} t(x', y)\, f(y)\, dy + \sum_{0}^{r-1} \int_{\partial\Omega} q_k(x', y')\, f^{(k)}(y')\, dy',$$

where t (resp. q_k, $k = 0, \ldots, r-1$) is C^∞ up to the boundary on $\partial\Omega \times \overline{\Omega}$ (resp. $\partial\Omega \times \partial\Omega$), and $f^{(k)}$ is the kth derivative of f with respect to some C^∞ normal vector $\partial/\partial n$.

If $r = 0$, T has a continuous extension $\mathcal{E}'(\overline{\Omega}) \to C^\infty(\partial\Omega)$. Otherwise T only extends continuously $C_0^r(\overline{\Omega}) \to C^\infty(\partial\Omega)$ (or $H_s^{\text{comp}}(\overline{\Omega}) \to C^\infty(\partial\Omega)$ when $s > r - \frac{1}{2}$).

A negligible pseudo-differential operator (or Green operator) of class r is an operator G: $C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ defined by

$$Gf(x) = \int_{\Omega} g(x, y)\, f(y)\, dy + \sum_{0}^{r-1} \int_{\partial\Omega} k_p(x, y')\, f^{(p)}(y')\, dy',$$

where g (resp. the k_p, $p = 0, \ldots, r-1$) is C^∞ up to the boundary (including corners) on $\overline{\Omega} \times \overline{\Omega}$ (resp. $\overline{\Omega} \times \partial\Omega$).

If $r = 0$, G extends continuously: $\mathcal{E}'(\overline{\Omega}) \to C^\infty(\overline{\Omega})$. Otherwise it only extends continuously $C_0^{r-1}(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ (or $H_s^{\text{comp}}(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ when $s > r - \frac{1}{2}$).

Finally a negligible pseudo-differential operator on the boundary is an operator Q: $C_0^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ with C^∞ kernel.

If in the matrix $A = \begin{pmatrix} G & K \\ T & Q \end{pmatrix}$ all the coefficients negligible of class r, and if $\bar{\Omega}$ is compact, then there exists a negligible matrix, of class r, A', such that $(1-A)(1-A')$ is a projector on the range of $(1-A)$ (resp. or such that $(1-A')(1-A)$ is a projector on a supplementary of the kernel of $(1-A)$, containing the range of $(1-A)^N$ if N is large enough); the range (resp. kernel) of $(1-A)$ is closed and finite codimensional (resp. finite dimensional). The proof is easy and left to the reader.

1. *Poisson operators.* A typical example is the operator that solves the Dirichlet problem in the half space:

$$\begin{cases} (\sum_0^{n-1} (\partial/\partial x_k)^2 + (t\partial/\partial x_n)^2) F = 0 \\ F(x', 0) = f(x'), \end{cases}$$

where t is a complex number of positive real part $\operatorname{Re} t > 0$.

If $f \in C_0^\infty(R^{n-1})$, the unique bounded solution is:

$$\begin{aligned} F(x) &= \pi^{-n/2}\Gamma(n/2)\int_{R^{n-1}} tx_n(t^2x_n^2 + |x'-y'|^2)^{-n/2} f(y')\,dy' \\ &= (2\pi)^{1-n}\int_{R^{n-1}} e^{ix'\cdot\xi'} e^{-tx_n|\xi'|} \hat{f}(\xi')\,d\xi' \\ &= (2\pi)^{-n}\int_{R^n} e^{ix\cdot\xi}(t|\xi'| + i\xi_n)^{-1}\hat{f}(\xi')\,d\xi \end{aligned}$$

(the last integral represents the inverse Fourier transform, i.e. it is $\int^+$ with respect to ξ_n).

Here we will be interested in the last formula.

Now let $k(x', \xi)$ be a C^∞ function on $R^{n-1}\times R^n$, admitting a series expansion:

$$k(x', \xi) = \sum_0^\infty a_p(x', \xi')(\langle\xi'\rangle - i\xi_n)^p(\langle\xi'\rangle + i\xi_n)^{-p-1}, \tag{3.1}$$

where $a_p(x', \xi')$ is a rapidly decreasing sequence in $S^d_{1,0}$, and where we have set as in § 2 $\langle\xi'\rangle = (1+|\xi'|^2)^{\frac{1}{2}}$.

(3.2) *Definition. The Poisson operator K of degree d and symbol $k(x', \xi)$* is the operator $K: C_0^\infty(R^{n-1}) \to C^\infty(\bar{R}^n_+)$ defined by:

$$Kf(x) = (2\pi)^{-n}\int d\xi_n \int e^{ix\cdot\xi} k(x', \xi)\,\hat{f}(\xi')\,d\xi \qquad (^1)$$

(1) We have written $x = (x', x_n)$ where $x' \in R^{n-1}$ is the tangential component of x, and $x_n \in R_+$ its normal component.

(the formula hereabove only defines K as a continuous operator: $C_0^\infty(R^{n-1}) \to C^\infty(R_+^n)$, where R_+^n is the open half space; but we show in (3.8) that K is in fact continuous $C_0^\infty(R^{n-1}) \to C^\infty(\bar{R}_+^n)$).

More generally, we will call Poisson operator the sum of a negligible Poisson operator as in definition (3.2).

We will also denote by $\mathcal{K}^d$ the set of functions satisfying (3.1); by $\mathcal{K}^{-\infty}$ the intersection $\mathcal{K}^{-\infty} = \bigcap \mathcal{K}^d$. We write $k \sim \sum k_j$ if $k - \sum_0^{m-1} k_j \in \mathcal{K}^{d_m}$ where $d_m \to -\infty$. It will follow from (3.8) that the Poisson operator of definition (3.2) is negligible if and only if $k \sim 0$, i.e. $k \in \mathcal{K}^{-\infty}$.

(3.3) *Example;* Let P be a pseudo-differential operator defined near $\bar{R}_+^n$, and satisfying the transmission property. The operator K_P: $C_0^\infty(R^{n-1}) \to C^\infty(R_+^n)$ defined by

$$K_P(f) = P(f \cdot \delta(x_n))/R_+^n$$

is a Poisson operator.

(Here $\delta(x_n)$ represents the Lebesgue measure on the boundary R^{n-1}.)

Proof. We can always write P as a sum $P = P_0 + P_1 x_n + P_2$, where P_0 is negligible, and $P_2 = p_2(x\ D)$, where the symbol $p_2(x, \xi)$ does not depend on x_n (so both P_1 and P_2 have the transmission property).

Then K_{P_0} is a negligible Poisson operator, $K_{P_1} = 0$ (because $x_n \delta(x_n) = 0$), and we have

$$\text{(3.4)} \qquad K_{P_2} f(x) = (2\pi)^{-n} \int_{R^n} e^{ix \cdot \xi} p_2(x, \xi) \hat{f}(\xi')\, d\xi$$

$$= (2\pi)^{-n} \int^+ d\xi_n \int_{R^{n-1}} p_2^+(x', \xi) \hat{f}(\xi')\, d\xi'$$

where we have set $p_2^+(x', \xi) = h_\xi^+\ p_2(x, \xi)$: since p_2 satisfies (2.1), p_2^+ obviously satisfies (3.1).

Conversely we have the following result:

(3.5) PROPOSITION. *Every Poisson operator can be defined as in the example above.*

Proof. Let R be the Seeley extension operator (R. T. Seeley—Extension of C^∞ functions defined in a half space. Proc. Amer. Math. Soc., 15 (1964), 625–626):

$$Ef(x) = \sum_1^\infty a_n f(-nx) \quad \text{when } x < 0,$$

where a_n is a rapidly decreasing sequence, and $\Sigma\, n^k a_n = (-1)^k$ for every k.

Let $\hat{E}$ be the Fourier transform of E: $\hat{E}\hat{f} = \widehat{Ef}$. Then $\hat{E}$ is continuous: $H^+ \to \mathcal{S}$; we have $h^+(\hat{E}\varphi) = \varphi$; and $\hat{E}$ commutes with homotheties.

Let K be the Poisson operator of definition (3.2), and set

$$p(x, \xi) = E_{\xi_n} k(x', \xi).$$

Then p belongs to $S^{d-1}_{1,0}$ and has the transmission property (it is a rapidly decreasing function of ξ_n when the other variables are fixed). Since we have $k(x', \xi) = h^+_{\xi_n} p(x, \xi)$, it follows from (3.4) that $K_{p(x,D)} = K$.

Now proposition (3.5) is obvious for negligible Poisson operators and this ends the proof.

As a first consequence we see that any Poisson operator K admits a Fourier integral representation:

$$Kf(x) = (2\pi)^{-n} \int^{+} d\xi_n \iint e^{i(x-y')\cdot\xi} k(x, y', \xi) f(y') \, dy' \, d\xi,$$

where $k(x, y', \xi)$ admits a series expansion such as (3.1).

Conversely if in the formula hereabove, $k(x, y', \xi)$ admits a series expansion such as (3.1) for all x, y', then the Fourier integral operator K it defines is a Poisson operator; to see this, we just repeat the proof of proposition (3.5). In particular, modulo a negligible Poisson operator, K always admits a Fourier integral representation as above, where the function $k(x, y', \xi)$ only depends on y'.

We also have the following result:

(3.6) $x_n^p k(x', \xi)$ *and* $(i\partial/\partial\xi_n)^p k(x', \xi)$ *define the same Poisson operator.*

(3.7) COROLLARY. *Definition (3.2) is invariant under a change of coordinates that preserves the boundary.*

(3.8) COROLLARY. *A Poisson operator K is continuous:* $C_0^\infty(R^{n-1}) \to C^\infty(\bar{R}^n_+)$ *and extends continuously* $\mathcal{E}'(R^{n-1}) \to \mathcal{D}'(\bar{R}^n_+) \cap C^\infty(R^n_+)$ (1).

If $f \in \mathcal{E}'(R^{n-1})$ *is* C^∞ *near a point* $x \in R^{n-1}$, *then* Kf *is* C^∞ *up to the boundary near* x.

Proof of (3.8). Let P be a pseudo-differential operator satisfying the transmission property, such that $Kf = P(f\delta(x_n))/\bar{R}^n_+$. First we see that Kf is well defined and is C^∞ in the open half space $x_n > 0$ if $f \in \mathcal{E}'(R^{n-1})$. Since corollary (3.8) is obvious for negligible Poisson operators, we can always assume that P is properly supported. Then we have

$$Kf = (P \cdot \partial/\partial x_n)_\Omega \cdot \varphi,$$

(1) $\mathcal{D}'(\bar{R}^n_+) \cap C^\infty(R^n_+)$ denotes the space of distributions supported by the closed half space $x_n \geqslant 0$, which are C^∞ in the open half space $x_n > 0$.

where $\varphi(x)=\varphi(x', x_n)=f(x')$ (φ belongs to $C^\infty(\bar{R}^n_+)$, and its restriction to R^{n-1} is f). So it follows from theorem (2.9) that Kf is C^∞ up to the boundary if f is C^∞ (or at any point near which f is C^∞).

Now let us suppose that P is precisely the pseudo-differential operator constructed in the proof of (3.5). Then the symbol of P does not depend on x_n, and is a rapidly decreasing function of ξ_n when $\xi_n \to \infty$ (the other variables remaining fixed). It follows that if $f \in \mathcal{E}'(R^{n-1})$, $P(f\delta(x_n)) = g(x', x_n)$ is a distribution of x' which depends C^∞ on x_n. So if we set $Y(x_n)=1$ if $x_n \geqslant 0$, 0 if $x_n < 0$, the product $Y(x_n)g(x', x_n)$ makes sense, and $f \to Y(x_n)\, g(x', x_n)$ is the continuous extension of K announced in (3.8).

It follows from corollary (3.7) that we can define a Poisson operator acting on the sections of C^∞ vector bundles on a C^∞ manifold $\bar{\Omega}$ with boundary $\partial\Omega$. It is a continuous operator

$$K\colon C^\infty(\partial\Omega, E) \to C^\infty(\bar{\Omega}, F)$$

(E is a C^∞ bundle on $\partial\Omega$, F a C^∞ bundle on $\bar{\Omega}$). It extends continuously as in (3.8).

To end this section, let us mention without proof the following result (cf. (5)):

If K is a Poisson operator of degree d, then K extends continuously: $H^{\text{comp}}_{s-1/2}(\partial\Omega) \to H^{\text{loc}}_{s-d}(\bar{\Omega})$.

2. *Trace operators.* The classical trace operators are those which take $f \in C_0^\infty(\bar{\Omega})$ into $Tf = Q(f^{(\alpha)}/\partial\Omega)$, where Q is a pseudo-differential operator on the boundary, and $f^{(\alpha)}$ some derivative of f. To these we add the adjoints of Poisson operators, and we finally get the following definition.

Let $t(x', \xi)$ be a C^∞ function on $R^{n-1} \times R^n$ admitting the following series expansion:

$$t(x', \xi) = \sum_0^{r-1} \alpha_s(x', \xi')\, \xi_n^s + \sum_0^{\infty} a_p(x', \xi')\, (\langle \xi' \rangle + i\xi_n)^p (\langle \xi' \rangle - i\xi_n)^{p+1} \tag{3.9}$$

where α_s belongs to $S^{d-s}_{1,0}$, and the a_p form a rapidly decreasing sequence in $S^{d+1}_{1,0}$.

(3.10) *Definition. The trace operator T of degree d and symbol $t(x', \xi)$ is the continuous* operator: $C_0^\infty(\bar{R}^n_+) \to C^\infty(R^{n-1})$ defined by

$$Tf(x') = (2\pi)^{-n} \int e^{ix'\cdot\xi'} d\xi' \int^+ t(x', \xi)\, \hat{f}(\xi)\, d\xi_n$$

(here $\hat{f}$ is the Fourier transform of the extension of f by 0 outside of the half space R^n_+).

We will say that T is of class r if r is the integer limiting the first sum in (3.9).

To these operators we add the negligible trace operators.

The following assertions are immediate consequences of those of section 1:

(3.11) PROPOSITION. *T is a trace operator if and only if it can be written as a sum*

$$Tf = \Sigma\, Q_i(P_i \cdot f / \partial\Omega),$$

where the Q_i are pseudo-differential operators on the boundary R^{n-1}, P_i are pseudo-differential operators satisfying the transmission condition on $\bar{\Omega} = R^n$.

(If T is of class 0, then there exists a pseudo-differential operator satisfying the transmission condition P such that $Tf = P_\Omega f/\partial\Omega$. But this is not the case for a "classical" trace operator: $Tf = Q(f^{(\alpha)}/\partial\Omega)$ can be represented in this way only if Q is a partial differential operator.)

(3.12) PROPOSITION. *Definition (3.9) is invariant by a change of coordinates which preserves the boundary.*

As a consequence, we can define trace operators acting on the sections of C^∞ bundles on a C^∞ manifold with boundary.

(3.13) PROPOSITION. *A trace operator T is continuous $C_0^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$. It extends continuously $H_s^{\mathrm{comp}}(\bar{\Omega}) \to H_{s-d-\frac{1}{2}}(\partial\Omega)$ if $s > r - \frac{1}{2}$ (where d is the degree of T, and r its class). If $r = 0$, it also extends continuously $\mathcal{E}'(\bar{\Omega}) \to \mathcal{D}'(\partial\Omega)$, and if f is C^∞ up to the boundary near a point $x \in \partial\Omega$, then Tf is C^∞ near x.*

The limitation on s comes from the "classical" trace operator contained in T (which involves the restriction to the boundary of normal derivatives of order smaller than $r-1$).

We also have

(3.14) *$t(x', \xi) \cdot x_n^p$ and $(-i\partial/\partial\xi_n)^p t(x', \xi)$ define the same trace operator.*

3. *Singular Green operators.* Let $g(x', \xi', \xi_n, \eta_n)$ be a C^∞ function on $R^{n-1} \times R^{n-1} \times R \times R$ admitting a series expansion:

$$(3.15) \qquad g(x', \xi', \xi_n, \eta_n) = \sum_0^{r-1} k_s(x', \xi', \xi_n)\, \eta_n^s \sum_{\substack{p \geq 0 \\ q \geq 0}} a_{pq}(x', \xi') + \frac{(\langle\xi'\rangle - i\xi_n)^p}{(\langle\xi'\rangle + i\xi_n)^{p+1}} \frac{(\langle\xi'\rangle + i\eta_n)^q}{(\langle\xi'\rangle - i\eta_n)^{q+1}},$$

where $k_s \in \mathcal{K}^{d-s}$ is the symbol of a Poisson operator of degree $d-s$, and a_{pq} a rapidly decreasing double sequence in $S_{1,0}^{d+1}$.

(3.16) *Definition. The singular Green operator G of degree d and class r defined by the symbol g* is the operator $G: C_0^\infty(\bar{R}_+^n) \to C^\infty(\bar{R}_+^n)$ defined by:

$$Gf(x) = (2\pi)^{-n-1} \int d\xi' \int^+ e^{ix\cdot\xi} d\xi_n \int^+ g(x', \xi', \xi_n, \eta_n)\, \hat{f}(\xi', \eta_n)\, d\eta_n$$

(here $\hat{f}$ is the Fourier transform of the extension of f by 0 for $x_n < 0$,

To these operators we add the negligible Green operators.

An equivalent definition is the following: there exists a rapidly decreasing sequence of Poisson operators K_j of degree d, and a rapidly decreasing sequence of trace operators T_j of degree 0 and class r, one of the two sequences beeing uniformly properly supported, such that

$$G-\Sigma\, K_j T_j \text{ is a negligible Green operator,}$$

Using formula (3.15), we see that we can in fact choose T_j so that its symbol does not depend on x', and so that only a finite number of them are not of class 0. In this case the composition $K_j T_j$ is clearly a singular Green operator That KT is a singular Green operator for general Poisson operators and trace operators (one of which is properly supported) follows from the fact that the composition of two pseudo-differential operators on the boundary is another one, and of formulas (3.1), (3.6).

The following assertions are immediate consequences of sections 1 and 2 of this paragraph:

(3.17) PROPOSITION. *Definition (3.16) is invariant under a change of coordinates which preserves the boundary.*

Since the composition of a Poisson operator (or trace operator) with the multiplication by a C^∞ function is another one, it follows that we can define a singular Green operator acting on the sections of a C^∞ vector bundle on a C^∞ manifold with boundary.

(3.18) PROPOSITION. *A singular Green operator of degree d and class r is continuous* $C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ *and extends continuously;* $H_s^{\text{comp}}(\overline{\Omega}) \to H_{s-d}^{\text{loc}}(\overline{\Omega})$ *if* $s > r + \frac{1}{2}$.

If $r=0$, *it also extends continuously* $\mathcal{E}'(\overline{\Omega}) \to \mathcal{D}'(\overline{\Omega})$. *If f is* C^∞ *up to the boundary near a point x, then so is Gf.*

(3.19) $x_n^p g(x', \xi', \xi_n, \eta_n) x_n^q$ *and* $i^{p-q}(\partial/\partial\xi_n)^p(\partial/\partial\eta_n)^q g(\ldots)$ *define the same singular Green operator.*

4. Symbolic calculus

Let Ω be a C^∞ manifold with boundary as before. A general Green operator on Ω is a matrix (as (0.1) in the introduction):

$$A = \begin{pmatrix} P_\Omega + G & K \\ T & Q \end{pmatrix}, \tag{4.1}$$

where P is a pseudo-differential operator (defined in a neighborhood of $\overline{\Omega}$), satisfying the transmission property with respect to $\partial\Omega$; G is a singular Green operator, K a Poisson operator, T a trace operator, and Q a pseudo-differential operator on the boundary.

These operators form an "algebra" as was announced in the introduction (i.e. if A and B are two such operators, the sum $A+B$ is another if it is defined, and the composition $A\circ B$ is another if A or B is properly supported and the composition is defined). To prove this we examine first the case where $\bar{\Omega}$ is the half space $\bar{R}^n_+$, and the symbols of the coefficients of A and B do not depend on x (or x'): in this case we get:

(4.2)

1. $\widehat{P_\Omega f} = h^+_{\xi_n}(p(\xi)\hat{f}(\xi))$

2. $\widehat{Gf} = (2\pi)^{-1}\int^+ g(\xi',\xi_n,\eta_n)\hat{f}(\xi',\eta_n)\,d\eta_n$

3. $\widehat{Ku} = k(\xi',\xi_n)\,\hat{u}(\xi')$

4. $\widehat{Tf} = (2\pi)^{-1}\int^+ t(\xi',\xi_n)\hat{f}(\xi',\xi_n)\,d\xi_n$

5. $\widehat{Qu} = q(\xi')\,\hat{u}(\xi')$.

(In these formulas, if f is a function on the half space $\bar{R}^n_+$, $\hat{f}$ denotes the Fourier transform of the extension of f by 0 on the complementary half space $x_n<0$.)

So in this case the assertion follows from § 1.

With this as starting point, the fact that the composition of two Green operators such as (4.1) is another one is proved exactly in the same way as the fact that the composition of two pseudo-differential operators is another one, where the starting point is the case of two translation invariant operators (cf. [7], [8], [9]), and we will omit the proof.

From now on, we will suppose that the symbols of the coefficients of A in (4.1) admit asymptotic expansions in homogeneous functions of integral degree of ξ.

Then we define a principal symbol corresponding to the leading term in the expansion. In fact we define two symbols: first we define the interior symbol $\sigma_\Omega(A)$. This is just the principal symbol of the pseudo-differential coefficient in A:

$$\sigma_\Omega(A) = p_0(x,\xi).$$

It is a bundle homomorphism on the unit cotangent sphere S_Ω of $\bar{\Omega}$, the coefficients of whose matrix are C^∞ up to the boundary and satisfy the symmetry condition (2.3).

Conversely, by examining first the case of an operator on the half space $\bar{R}^n_+$, and then patching together by means of a partition of the unity, we see that any bundle homomorphism $p_0(x,\xi)$ on S whose coefficients are as above is the interior symbol of a pseudo-differential operator satisfying the transmission property on Ω.

The interior symbol map is a homomorphism, i.e. we have

$$\sigma_\Omega(A+B) = \sigma_\Omega(A) + \sigma_\Omega(B)$$

$$\sigma_\Omega(A \circ B) = \sigma_\Omega(A) \circ \sigma_\Omega(B)$$

whenever the sum or product is defined.

Next we define the boundary symbol $\sigma_{\partial\Omega}(A)$. This is a Wiener–Hopf operator (as in § 1) depending C^∞ on a unit cotangent vector on the boundary, whose matrix is (with the notations of § 1)

$$\sigma_{\partial\Omega}(A) = \begin{pmatrix} p(\xi_n) + g(\xi_n, \eta_n) & k(\xi_n) \\ t(\eta_n) & q \end{pmatrix} \tag{4.3}$$

(we have written $p(\xi_n)$ instead of $p_0(x', \xi', \xi_n)$ etc. ...)

It operates from $E \otimes H_N^+ \oplus F$ to $E' \otimes H_N^+ \oplus F'$, where H_+^N is defined by (1.16) (N is the normal cotangent bundle, oriented by the inward normal). (The resaon for interpreting H_N^+ as a space of measures rather than a space of functions is the following: if $\overline{\Omega}$ is the half space $\overline{R}^n_+$, the Fourier transforms $\hat{f}$ of $f \in C_0^\infty(\overline{R}^n_+)$ and $\hat{u}$ of $u \in C^\infty(R^{n-1})$ are really measures on the dual space of R^n (resp. R^{n-1}); the reader can check by making a linear change of coordinates preserving the boundary that the interpretation of H_N^+ as a space of measures on N leads to the right formula for the behaviour of the symbol under a change of coordinates).

The interior symbol map is also a homomorphism, i.e.

$$\sigma_{\partial\Omega}(A+B) = \sigma_{\partial\Omega}(A) + \sigma_{\partial\Omega}(B)$$

$$\sigma_{\partial\Omega}(A \circ B) = \sigma_{\partial\Omega}(A) \circ \sigma_{\partial\Omega}(B)$$

whenever the sum or product is defined.

This is proved exactly as for the principal symbol of pseudo-differential operators, the starting point being the case of operators on the half space $\overline{R}^n_+$ whose symbols do not depend on x or x': in this case the assertion follows from (4.2) and § 1.

Of course, the boundary and interior symbols of a Green operator A are related: the coefficient $p(\xi_n)$ in the matrix $\sigma_{\partial\Omega}(A) = \begin{pmatrix} p+g & k \\ t & q \end{pmatrix}$ is the restriction to the boundary of the interior symbol $\sigma_\Omega(A)$.

(4.4) Conversely, if $p(\xi)$ and $a(\xi') = \begin{pmatrix} p'+g & k \\ t & q \end{pmatrix}$ are respectively an interior and a boundary symbol, we will say that they are compatible if p' is the restriction of p to $\partial\Omega$: if p and a are compatible, there exists a Green operator A such that $\sigma_\Omega(A) = p$, $\sigma_{\partial\Omega}(A) = a$.

(4.5) PROPOSITION. *Let P (resp. G, K, T, Q) be a pseudo-differential operator of degree d on $\overline{\Omega}$, satisfying the transmission property (resp. ...). Then P (resp. ...) is compact:*

$H_s^{\text{comp}}(\bar{\Omega}) \to H_{s-d}^{\text{loc}}(\bar{\Omega})$ *for large* s *(resp. same, resp.* $H_{s-1/2}^{\text{comp}}(\partial\Omega) \to H_{s-d}^{\text{loc}}(\bar{\Omega})$, $H_s^{\text{comp}}(\bar{\Omega}) \to H_{s-d-1/2}^{\text{loc}}(\partial\Omega)$ *for large* s, $H_s^{\text{comp}}(\partial\Omega) \to H_{s-d}^{\text{loc}}(\partial\Omega)$ *if and only if its principal symbol is* 0.

Proof. The condition is sufficient because the operator is really of degree $d-1$ if the principal symbols vanish identically. That the condition is necessary is known for P and Q (then the principal symbol must vanish inside Ω resp. $\partial\Omega$). The assertion then follows for the other operators: if the principal symbol of K (resp. T, G) does not vanish identically, it follows from the composition formulas (1.13) that there exists a compactly supported trace operator T of degree 0 and class 0 (resp. K of degree 0, resp. K and T of degree 0) such that the principal symbol of $Q = T \circ K$ (resp. same, resp. $T \circ G \circ K$) does not vanish identically.

5. Boundary problems. The index formula

In this paragraph, we suppose $\bar{\Omega}$ compact. Otherwise, results concerning symbolic calculus only hold locally.

1. Let $A = \begin{pmatrix} P+G & K \\ T & Q \end{pmatrix}$ be a Green operator as in (4.1).

As in § 4, we will suppose that (in some coordinate patch) the complete symbols have asymptotic expansions in homogeneous functions of integral degree of ξ, so that the degree d and the (principal) interior and boundary symbols are well defined.

We will say that A is elliptic if it admits a both sided parametrix A' of degree $-d$ (i.e. $1-AA'$ and $1-A'A$ are negligible operators in the sense of § 3.0). (We do not investigate here the case where there exists a both sided parametrix of the wrong degree. One such case is easily reduced to the case studied here: this is when the bundles on the sections of which A operates are split in direct sums, and A has different degrees in different directions but is elliptic in the sense of Agmon, Douglis and Nirenberg (Comm. Pure Appl. Math., 17 (1964), 32–92); in fact the results of this paragraph remain valid in this case, provided the principal symbols are redefined conveniently).

(5.1) THEOREM. *A is elliptic if and only if both its interior and boundary symbols are invertible.*

Proof. The condition is obviously necessary. Conversely if $\sigma_\Omega(A)$ and $\sigma_{\partial\Omega}(A)$ are both invertible, their inverses are compatible, so there exists a Green operator A'' such that $\sigma_\Omega(A'') = \sigma_\Omega(A)^{-1}$, $\sigma_{\partial\Omega}(A'') = \sigma_{\partial\Omega}(A)^{-1}$ (cf. (4.4)).

Then $B = 1 - A \circ A''$ is of degree -1.

Now let A' be a Green operator such that

$$A' \sim A'' \circ \sum_0^\infty B_k$$

(where as in (7), $\sim$ means that the degree of $A' - A'' \circ \sum_0^N B^k$ tends to $-\infty$ when $N \to \infty$. That such an operator exists is proved just as in (7)). Then $1 - A \circ A'$ is a negligible operator, so that A' is a right parametrix to A. In the same way one proves that there exists a left parametrix, and it follows that A' is already a both sided parametrix.

2. *An example.*

Let $A = \begin{pmatrix} P+G & K \\ T & Q \end{pmatrix}$ and suppose that P is elliptic near $\bar{\Omega}$, i.e. the interior symbol is invertible. Then $\sigma_{\partial\Omega}(A)$ is a Fredholm operator depending continuously on $\xi' \in S_{\partial\Omega}$ ($S_{\partial\Omega}$ denotes the unit contangent sphere on $\partial\Omega$). So it has an index bundle: (1)

(5.2) *Definition:* $j(A) \in K(S_{\partial\Omega})$ *is the index bundle of the Fredholm operator* $\sigma_{\partial\Omega}(A)$ *(when* $\sigma_\Omega(A)$ *is invertible).*

(This depends only on the boundary symbol a of A, and we shall also write it $j(a)$ when convenient.)

Quite obviously we have

(5.3) $$j(A \oplus B) = j(A) + j(B)$$
$$j(A \circ B) = j(A) + j(B) \text{ when } A \circ B \text{ is defined.}$$

In particular, if A operates on the bundles E, E', F, F' as in (0.1), and if π is the projection of the cotangent bundle onto $\partial\Omega$, we have:

(5.4) $$j(A) = j(P_\Omega) + \pi^* F - \pi^* F'$$

Example. Let us first choose a metric on $\bar{\Omega}$, so that $\bar{\Omega}$ is isometric to $\partial\Omega \times \bar{R}_+$ near the boundary. Let α be a smooth function near $\partial\Omega$, equal to 1 near $\partial\Omega$ and to 0 out of a small neighborhood of $\partial\Omega$. Finally let $\varphi(t)$ be a smooth function of one variable, such that $0 \leqslant \varphi \leqslant 1$, $\varphi(t) = 0$ near $t = 0$, and $\varphi(t) = 1$ if $t > \varepsilon$ (ε will be chosen small later on).

Now let us set:

$$\zeta = (\xi_n + i|\xi'|\varphi(|\xi'|/|\xi|))^{-1}(\xi_n - i|\xi'|\varphi(|\xi'|/|\xi|)).$$

(1) This bundle is also used in [11] in a more general situation. It is the bundle M^+ of [3] when A is a partial differential operator. For the definition of the index bundle, we refer to [2] and its bibliography.

(The normal and tangential components ξ_n, ξ' of ξ correspond to the isometry between $\overline{\Omega}$ and $\partial\Omega \times \overline{R}_+$ near $\partial\Omega$, so that ζ is well defined and smooth when $\xi \neq 0$ near $\partial\Omega$ — say near the support of α.)

Let E be a vector bundle on $\partial\Omega$, and let $E^\perp$ be a complementary bundle, so that $E \oplus E^\perp \approx C^N$ is a trivial bundle. Then we define an interior symbol γ_E by

$$\gamma_E = \zeta^\alpha p_E + 1 - p_E \tag{5.5}$$

(p_E is the orthogonal projection on E, $1 - p_E$ is the orthogonal projection on $E^\perp$. γ_E is at first only defined near $\partial\Omega$, and we extend it by the identity of $C^N = E \oplus E^\perp$ where $\alpha = 0$).

If ε is small, ζ is very close to

$$\zeta_0 = (\xi_n + i|\xi'|)^{-1}(\xi_n - i|\xi'|).$$

Now the kernel of the Wiener Hopf operator on H^+ defined by ζ_0 is the line of functions proportional to $(\xi_n - i|\xi'|)^{-1}$, and the cokernel is 0. So it follows that we have

$$j(\gamma_E) = E. \tag{5.6}$$

(Notice that we can always choose a pseudo-differential operator Γ_E with interior symbol $\sigma_\Omega(\Gamma_E) = \gamma_E$ so that it has the transmission property (2.3), and so that it coincides with the identity operator out of a small neighborhood of $\partial\Omega$.)

Since γ_E is an elliptic symbol, it follows that given any virtual bundle F on $\partial\Omega$, there exists an elliptic pseudo-differential operator P satisfying the transmission property, such that $j(P_\Omega) = \pi^* F$.

Next let E be a bundle on $\overline{\Omega}$. Then there exist two elliptic pseudo-differential operators of degree 1: Λ_E^+ and Λ_E^- with interior symbol:

$$\begin{aligned} \sigma_\Omega(\Lambda_E^+) = \lambda_E^+ &= (\xi_n - i|\xi'|\varphi(|\xi'|/|\xi|))^\alpha |\xi|^{1-\alpha} \quad 1_E \\ \sigma_\Omega(\Lambda_E^-) = \lambda_E^- &= (\xi_n + i|\xi'|\varphi(|\xi'|/|\xi|))^\alpha |\xi|^{1-\alpha} \quad 1_E \end{aligned} \tag{5.7}$$

(φ, α, and the metric on $\overline{\Omega}$ are as above).

For the same reason as in (5.6) we get:

$$j(\lambda_E^+) = E \tag{5.8}$$

$$j(\lambda_E^-) = 0. \tag{5.9}$$

Now let T_0 denote the Dirichlet data: $T_0 f = f/\partial\Omega$. We introduce the four following systems (Green operators):

$$\begin{pmatrix} (\Lambda_E^+)_\Omega \\ T_0 \cdot 1_E \end{pmatrix} \quad (\Lambda_E^-)_\Omega \quad \begin{pmatrix} (\Delta_E)_\Omega \\ T_0 \cdot 1_E \end{pmatrix} \quad \begin{pmatrix} (\Gamma_E)_\Omega \\ T_0 \cdot p_E \end{pmatrix} \tag{5.10}$$

(Δ_E is the Laplace operator on the sections of E. Its interior symbol is $-|\xi|^2 \cdot 1_E$. In the last case, E is a bundle on $\partial\Omega$ as in (5.4), not necessarily the restriction of a bundle on $\bar{\Omega}$, p_E is the projection on E, parallel to $E^\perp$ and Γ_E is as above).

If ε has been chosen very small, λ^+ (resp. λ^-, γ) is very close to $\xi_n - i|\xi'|$ (resp. $\xi_n + i|\xi'|$, $(\xi_n + i|\xi'|)^{-1}(\xi_n + i|\xi'|)$), and it follows from theorem (5.1) that these four systems are elliptic. We also have:

(5.11)
$$\begin{pmatrix}(\Delta_E)_\Omega \\ T_0 \cdot 1_E\end{pmatrix} \text{ is homotopic to } \begin{pmatrix}(\Lambda_E^-)_\Omega & 0 \\ 0 & 1\end{pmatrix} \circ \begin{pmatrix}(\Lambda_E^+)_\Omega \\ T_0 \cdot 1_E\end{pmatrix}$$
$$\begin{pmatrix}(\Gamma_E)_\Omega \\ T_0 \cdot p_E\end{pmatrix} \text{ is homotopic to } 1_{E^\perp} \oplus \begin{pmatrix}(\Lambda_E^-)_\Omega & 0 \\ 0 & 1\end{pmatrix}^{-1} \circ \begin{pmatrix}(\Lambda_E^+)_\Omega \\ T_0 \cdot 1_E\end{pmatrix}$$

(in these formulas, A^{-1} represents a parametrix of A).

Since we suppose that Ω is compact, the four elliptic operators of (5.10) are Fredholm, and have an index.

(5.12) THEOREM. *The four elliptic systems of (5.10) have index 0.*

Proof. The result is known for the Dirichlet problem $\begin{pmatrix}(\Delta_E)_\Omega \\ T_0 \cdot 1_E\end{pmatrix}$. So in view of (5.11) it is sufficient to prove that the index of $\begin{pmatrix}(\Gamma_E)_\Omega \\ T_0 \cdot p_E\end{pmatrix}$ is 0. We can choose Γ_E so that it coincides with the identity operator out of a small neighborhood of $\partial\Omega$, so we will suppose $\bar{\Omega} = \partial\Omega \times \bar{R}_+$. We will also forget the bundle E (the index of the identity operator on the sections of E is 0).

Let Δ'_E be a second order self adjoint operator on the sections of E on $\partial\Omega$, with symbol $-|\xi'|^2 \cdot 1_E$, all of whose eigenvalues are negative (we suppose that C^∞ metrics on $\partial\Omega$ and E have been chosen). Define Γ'_E (on $\Omega \times R$) by

$$\Gamma'_E = (\partial/\partial x_n + \sqrt{-\Delta'_E})^{-1}\ (\partial/\partial x_n - \sqrt{-\Delta'_E}).$$

Taking expansions with respect to the eigenfunctions of Δ'_E and a Fourier transform with respect to the normal variable x_n, one checks immediately that $\begin{pmatrix}(\Gamma'_E)_\Omega \\ T_0 \cdot 1_E\end{pmatrix}$ is an isomorphism: $H_1(\bar{\Omega}, E) \to H_1(\bar{\Omega}, E) \oplus H_{\frac{1}{2}}(\partial\Omega, E)$.

To prove theorem (5.12), we will construct a homotopy of Fredholm operators from $H_1(\bar{\Omega}, E)$ to $H_1(\bar{\Omega}, E) \oplus H_{\frac{1}{2}}(\partial\Omega, E)$ connecting $\begin{pmatrix}(\Gamma_E)_\Omega \\ T_0 \cdot 1_E\end{pmatrix}$ to $\begin{pmatrix}(\Gamma'_E)_\Omega \\ T_0 \cdot 1_E\end{pmatrix}$.

First let us set

$$P = (\partial/\partial x_n + \sqrt{-\Delta'_E} \cdot \varphi(\sqrt{\Delta'_E/\Delta_E}))\ (\partial/\partial x_n - \sqrt{-\Delta'_E} \cdot \varphi(\sqrt{\Delta'_E/\Delta_E}))$$

(we have set $\Delta_E = (\partial/\partial x_n)^2 + \Delta'_E$ and φ is as in (5.5)) i.e. if $-\lambda_k^2 (\lambda_k > 0)$ and $f_k(x')$ are the eigenvalues and eigenfunctions of Δ'_E, and if $\hat{u}(\xi_n)$ is the Fourier transform of $u(x_n)$ with respect to x_n, P is defined by its partial Fourier transform:

$$\hat{P}[f_k(x')\, u(x_n)] = (\xi_n + i\lambda_k \varphi(\lambda_k (\lambda_k^2 + \xi_n^2)^{-\frac{1}{2}} (\xi_n - i\lambda_k \varphi(\lambda_k (\lambda_k^2 + \xi_n^2)^{-\frac{1}{2}}))\, f_k(x')\, \hat{u}(\xi_n)$$

P is an x_n-translation invariant pseudo differential operator on the sections of E on $\partial\Omega \times R$. Its symbol is $\sigma(P) = \zeta$, and P induces an isomorphism of $H_1(\partial\Omega \times R,\, E)$ onto itself.

We choose the function α of (5.5) so that it only depends on x_n. Let $\beta, \gamma \in C^\infty(R)$ be such that $0 \leqslant \beta \leqslant 1$, $0 \leqslant \gamma \leqslant 1$ $\beta = 1$ near supp α, $\beta = 0$ for large x_n, and $\beta^2 + \gamma^2 = 1$. We will suppose that Γ_E is defined by

$$\Gamma_E = \beta\, P(x, D)^{\alpha(x_n)} \beta + \gamma^2$$

(where the power P^α is defined by taking the determination of ζ^α which is real positive when $\xi' = 0$, $\xi_n > 0$).

Our first homotopy is given by:

$$P_t = \beta P^{\alpha(x_n) + t(1-\alpha(x_n))} \beta + \gamma P^t \gamma \quad (0 \leqslant t \leqslant 1).$$

Our second homotopy consists in translating P_1 to the left:

$$P_t = P_1(x_n - t + 1) \quad (1 \leqslant t \leqslant t_2)$$

so that P_{t_2} and P coincide near the half line $x_n \geqslant 0$.

Our third homotopy consists in replacing φ by 1 in the formula defining P:

$$\varphi_t(u) = (t - t_2) + (1 - t + t_2)\varphi(u) \quad (t_2 \leqslant t \leqslant t_2 + 1).$$

The Green operator $A_t = \begin{pmatrix} (P_t)_\Omega \\ T_0 \cdot 1_E \end{pmatrix}$ depends continuously on t in the norm topology. That it remains a Fredholm operator in the third homotopy is seen by taking expansions with respect to the eigenfunctions of Δ'_E and a Fourier transform with respect to x_n, as above. It remains a Fredholm operator during the second homotopy: then P_t remains unchanged for large x_n, and its symbol is constant, so that if B_{t_2} is the inverse of A_{t_2}, it is also a quasi inverse of A_t $(1 \leqslant t \leqslant t_2)$ (i.e. $1 - A_t B_{t_2}$ and $1 - B_{t_2} A_t$ are compact operators). That it remains a Fredholm operator during the first homotopy is a consequence of the fact that we have chosen P so that it induces an isomorphism of $H_1(\partial\Omega \times R)$ and of the following lemma (the proof of which we leave to the reader):

LEMMA. *Let P be an x_n-translation invariant pseudo-differential operator of degree 0 on $\partial\Omega \times R$, and let φ be constant for large x_n. Then $P\varphi - \varphi P$ induces a compact operator on $H_1(\partial\Omega \times R)$.*

Knowing this, let B'_t be any parametrix of A_t, and let β', γ' be two C^∞ functions such that $\beta'=1$ near $\operatorname{supp}\beta$, $\beta'=0$ for large x_n, and $\beta'^2+\gamma'^2=1$.

Then $B_t=\beta' B'_t \beta' + \gamma'(P^{-t}, 0)\gamma'$ is such that $1-A_tB_t$ and $1-B_tA_t$ are compact operators on $H_1(\overline{\Omega}, E)\oplus H_{\frac{1}{2}}(\partial\Omega, E)$ and $H_1(\overline{\Omega}, E)$ respectively (because $\beta'^2 - A_t\bar{\beta}' B'_t \beta$ and $\gamma'^2 - A'_t\gamma'(P^{-t}, 0)\gamma'$, and also $\beta'^2-\beta' B'_t \beta' A_t$ and $\gamma'^2-\gamma'(P^{-t}, 0)\gamma' A_t$ are compact operators. We have written as a row matrix an operator from $H_1(\overline{\Omega})\oplus H_{\frac{1}{2}}(\partial\Omega)$ to $H_1(\overline{\Omega})$).

3. *The index bundle* $j(A)$. We investigate the bundle defined in (5.2) a little further.

Let P be an elliptic pseudo-differential operator satisfying the transmission property. If $A=\begin{pmatrix} P_\Omega+G & K \\ T & Q\end{pmatrix}$ is any elliptic system associated to P, then the boundary symbol of A is invertible, so we must have $j(A)=0$. Then, by (5.4) $j(P_\Omega)=\pi^*F'-\pi^*F$ has to be the pull back of a virtual bundle on $\partial\Omega$.

From now on we will suppose that A is of degree 0. If this is not the case, we replace A by $A'=A\circ\begin{pmatrix}(\Lambda_E^-)_\Omega & 0 \\ 0 & (\Delta'_F)^{\frac{1}{2}}\end{pmatrix}^{-d}$ where d is the degree of A: by (5.9) and (5.12) this does not change the index bundle $j(A)$, nor the index.

The interior symbol $p=\sigma_\Omega(A)$ is an isomorphism of bundles: $E \simeq E'$ over the cotangent sphere S_Ω, and it satisfies the symmetry condition $p(\nu)=p(-\nu)$ on the normal bundle $\nu\in N$, so that it can be canonically extended on the normal bundle N: Then it defines a virtual bundle

$$d(p)\in K(T\overline{\Omega}, N)=K(B_{\overline{\Omega}}, S_{\overline{\Omega}}\cup N)$$

($B_{\overline{\Omega}}$ is the unit ball of $T\overline{\Omega}$, $S_{\overline{\Omega}}$ the unit sphere, for some metric).

Now let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & K(\Omega\times R^2) & \overset{\beta^{-1}}{\simeq} & K(\partial\Omega) \\
 & & & & & & \downarrow & & \downarrow \\
K(T\Omega) & \to & K(T\overline{\Omega}, N) & \to & K(T\overline{\Omega}/\partial\Omega, N) & \simeq & K(S_{\partial\Omega}\times R^2) & \overset{\beta^{-1}}{\simeq} & K(S_{\partial\Omega}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(T\Omega) & \to & K(T\overline{\Omega}) & \to & K(T\overline{\Omega}/\partial\Omega) & \simeq & K^{-1}(T\Omega) & \simeq & K^1(T\partial\Omega).
\end{array} \tag{5.13}$$

In this diagram, we identify $T\overline{\Omega}/\partial\Omega$ and $T\partial\Omega\times R$ by taking the inward orientation of the normal bundle. The isomorphism $K(T\overline{\Omega}/\partial\Omega, N)\simeq K(S_{\partial\Omega}\times R^2)$ is defined by the map which takes (ξ', ξ_n) into $(\xi'/|\xi'|, \xi_n, \operatorname{Log}|\xi'|)$. β is the Bott isomorphism, i.e. the multiplication by the difference bundle $d(C, C, \xi_n+i\operatorname{Log}|\xi'|)\in K(R^2)=K^{-2}$ (point).

In the diagram, the columns are exact, and so are the rows at the second place $(K(T\overline{\Omega}, N), K(T\overline{\Omega}))$ (this is just the exact sequence of K-theory).

Now the image of $d(p) \in K(T\bar{\Omega}, N)$ in $K(S_{\partial\Omega})$ is precisely $j(P_\Omega)$ (cf. (2), where it is shown that the map which takes $f \in K^{-1}(X \times R) = K^{-2}(X)$ (i.e. f is a continuous matrix on $X \times R$, such that $f(x, \pm\infty) = 1$) into the index bundle $j(f)$ of the Wiener–Hopf operator defined by f is inverse to the Bott isomorphism: here the matrix f corresponding to $d(p)$ is precisely the interior symbol $p(\xi', \xi_n)$).

This proves the necessary part of

(5.14) THEOREM. *In order that there exist an elliptic Green operator with interior symbol $\sigma_\Omega(P_\Omega)$, it is necessary and sufficient that $j(P_\Omega)$ be the pull back of a virtual bundle on $\partial\Omega$, or equivalently that for large N, $1_{C^N} \oplus \sigma_\Omega(P)$ may be extended as an isomorphism on the full boundary of $B\bar{\Omega}$ (i.e. $\partial B_{\bar{\Omega}} = S_{\bar{\Omega}} \cup (B_{\bar{\Omega}}/\partial\Omega)$, where $S_{\bar{\Omega}}$ is the unit sphere, $B_{\bar{\Omega}}$ the unit ball of $T\bar{\Omega}$).*

The sufficiency will follow from section 4.

Theorem 5.14 remains true if the degree of P is not 0, since the symbol of Λ^- can obviously be extended.

The second condition is equivalent to the following: there exists a homotopy of elliptic pseudo-differential operators connecting $1_{C^n} \oplus P$ to an operator P' which coincides with the identity near $\partial\Omega$ (we identify the two bundles E and E' on the sections of which P operates by $\sigma_\Omega(P)(\nu)$ on $\partial\Omega$). However, such a homotopy will usually not preserve the symmetry condition $\sigma(\nu) = \sigma(-\nu)$. In view of the exactness of the middle row in (5.13), we have:

(5.15) PROPOSITION. *There exists a homotopy of elliptic pseudo-differential operators satisfying the transmission condition, connecting $1_{C^n} \oplus P$ to an operator P' which coincides with 1 near $\partial\Omega$ if and only if $j(P_\Omega) = 0$.*

(The exactness of the middle row of (5.13) only insures that there exists a continuous family of elliptic symbols p_t satisfying the symmetry condition $p_t(\nu) = p_t(-\nu)$, and such that $p_0 = 1_{C^n} \oplus \sigma_\Omega(P)$, $p_1 = 1$ near $\partial\Omega$. But then it is immediate to regularize p_t so that it is C^∞ and satisfies the transmission condition.)

4. *Reduction to the boundary.* In this section, we show how one can reduce the study of the ellipticity conditions to a problem concerning pseudo-differential operators on the boundary. (This technique has been used number of times—cf. A. P. Calderon: Boundary value problems for elliptic equations, *Outlines of the joint Soviet-American symposium on partial differential equations*, Novosibirsk 1963, 303–304; L. Hörmander—Non elliptic boundary problems, *Ann. of Math.*, 83 (1966), 129–209; R. T. Seeley—Singular integrals and boundary value problems, *Amer. J. Math.*, 88 (1966), 781–809.)

We begin first by studying the symbols: let P be an elliptic pseudo-differential operator, satisfying the transmission property, and let G be a singular Green operator (of the same degree), operating both from the sections of E to the sections of E'. Then the boundary symbol $p+g=\sigma_\Omega(P_\Omega+G)$ is a Fredholm operator depending C^∞ on ξ'. It follows that given any compact set $\Xi\subset S_{\partial\Omega}$ (if $\partial\Omega$ is compact we will take $\Xi=S_{\partial\Omega}$, otherwise we only have a local result), there exists a Poisson symbol (depending C^∞ on ξ') k: $F\to E'\otimes H_N^+$ such that $(p+g)\oplus k$ is onto if $\xi'\in\Xi$. The null space of $(p+g)\oplus k$ is a C^∞ bundle $\Phi\subset E\otimes H_N^+\oplus F$. If we replace k by $k\oplus k_1$ where k_1 is a Poisson symbol: $F_1\to E'\otimes H_N^+$, then Φ is replaced by $\Phi_1\approx\Phi\oplus F_1$.

Now let $a=\begin{pmatrix}p+g & k\\ t & q\end{pmatrix}$ be the boundary symbol of a Green operator associated to P.

Notice that a and $a'=\begin{pmatrix}p+g & k & k_1\\ t & q & 0\\ 0 & 0 & 1\end{pmatrix}$ are simultaneously one to one or onto. So we can always suppose that the Poisson symbol k is precisely the one we have just constructed (i.e. the first line of a is onto), Then the Green symbol a is an isomorphism if and only if its second row is an isomorphism of Φ onto F' (F' is the bundle in which t and q take their values).

It is usually practical to realise Φ as a subbundle of a trivial bundle C^N (or of the pull back to $S_{\partial\Omega}$ of a bundle on $\partial\Omega$). This one can do as follows: in any case, there always exists a surjection k_0: $C^N\to\Phi$, where k is a Green symbol of the form $\begin{pmatrix}k\\ q\end{pmatrix}$, and a Green operator t_0 (of the form $(t q)$): $E\otimes H_N^+\oplus F\to C^N$ such that k_0t_0 is a projector on Φ, and $h_0=t_0k_0$ is the projector on $\Phi_0=t_0\Phi$, parallel to ker k_0.

Now let A be a Green operator with interior symbol $\sigma_\Omega(P)$, and boundary symbol a.

(5.16) Proposition. *With the notations above, A is elliptic if and only if the boundary symbol $\sigma_{\partial\Omega}((T\ Q))\circ k_0$ is an isomorphism of $\Phi_0=$ range* of h_0 *onto F'.*

(Of course, if $\mathcal{K}_0$ is a Green operator of the form $\begin{pmatrix}K\\ Q\end{pmatrix}$ whose boundary symbol is k_0, $(T\ Q)\mathcal{K}_0$ is a pseudo-differential operator on $\partial\Omega$.)

When $\overline{\Omega}$ is compact, the same constructions can be carried out globally with the operators themselves (and not the symbols):

(5.17) Proposition. *Let P be a an elliptic-pseudo differential operator satisfying the transmission property, and G a singular Green operator. Then*

1) *There exists a Poisson operator K such that $(P_\Omega+G)\oplus K=A_1$ is onto.*

2) *We can choose the left inverse B_1 of A_1 in such a way that* (i) $1-B_1A_1=\mathcal{K}_0\mathcal{T}_0$ *is a projector on* Ker A_1, *which is the composition of two (vector valued) Green operators $\mathcal{K}_0$ of the*

form $\begin{pmatrix} K \\ Q \end{pmatrix}$ $\mathcal{J}_0$ *of the form* $(T\ \ Q)$. (ii) $A_1\mathcal{K}_0 = 0$, *so* $\mathcal{H}_0 = \mathcal{J}_0\mathcal{K}_0$ *is a pseudo-differential projector on* $\partial\Omega$.

3) *With these notations,* $A = \begin{pmatrix} P_\Omega + G & K \\ T & Q \end{pmatrix}$ *is elliptic if and only if the symbol of* $(T\ Q)\,\mathcal{K}_0$ *(which is a pseudo-differential operator on the boundary) is an isomorphism of the range of* $\sigma_{\partial\Omega}(\mathcal{H}_0)$ *onto* F' *(*F' *is the bundle in the space of the sections of which* T *and* Q *take their values).*

This could be deduced from (5.16), but since it is not longer, we give an independent proof.

We will use several times the two following facts (the proof of which we leave to the reader):

—if A is a very small Green operator of degree 0, then $1-A$ is invertible, and the inverse is also a Green operator.

—any singular Green matrix $\mathcal{G} = \begin{pmatrix} G & K \\ T & Q \end{pmatrix}$ can be arbitrarily approximated by a composition $\mathcal{K}\mathcal{J}$, where $\mathcal{K}$ is a column matrix of the form $\begin{pmatrix} K \\ Q \end{pmatrix}$, and $\mathcal{J}$ a row matrix of the form $(T\ \ Q)$.

Now let P' be a parametrix to P. Then we can approximate

$$G_1 = 1 - (P_\Omega + G)\,P'_\Omega$$

by a composite operator of the form $K_1\,T_1$ so that

$$1 - G_2 = 1 - G_1 + K_1\,T_1$$

is invertible. Then

$$B'_1 = \begin{pmatrix} P'_\Omega \\ T_1 \end{pmatrix} (1 - G_2)^{-1}$$

is a right inverse to

$$A_1 = (P_\Omega + G\,K_1).$$

Now $L_1 = 1 - B'_1 A_1$ is a projector on Ker A_1, of the form $\begin{pmatrix} G & K \\ T & Q \end{pmatrix}$. We can approximate it by a composition $\mathcal{K}_1\mathcal{J}_0$ as above, so that $1 - \mathcal{G}_1 = 1 - L_1 - \mathcal{K}_1\mathcal{J}_0$ is invertible.

Let us set

$$\mathcal{K}_2 = (1 - \mathcal{G}_1)^{-1}\mathcal{K}_1$$

so that we have

$$(1 - L_1) = 1(-\mathcal{G}_1)\,(1 - \mathcal{K}_2\mathcal{J}_0).$$

Finally let us set

$$\mathcal{K}_0 = L_1\,\mathcal{K}$$

$$B_1 = (1 - \mathcal{K}_0\mathcal{J}_0)\,B'_1.$$

Then we get as announced

$$A_1 B_1 = A_1(1 - \mathcal{K}_0\mathcal{J}_0)\,B'_1 = 1 - A_1 L_1 \dot{\mathcal{K}}_1 \mathcal{J}_0 B'_1 = 1$$

(because $A_1 L_1 = 0$). It follows that

$$\mathcal{K}_0 \mathcal{T}_0 = 1 - B_1 A_1$$

is a projector on ker A_1. Finally we have

$$A_1 \mathcal{K}_0 = 0, \quad \text{so} \quad \mathcal{K}_0 \mathcal{T}_0 \mathcal{K}_0 = (1 - B_1 A_1) \mathcal{K}_0 = \mathcal{K}_0$$

and

$$\mathcal{H}_0^2 = \mathcal{K}_0 \mathcal{T}_0 \mathcal{K}_0 \mathcal{T}_0 = \mathcal{K}_0 \mathcal{T}_0 = \mathcal{H}_0$$

which ends the proof.

We will use (5.16) to end the proof of theorem (5.14). Notice that if Φ is the kernel of $\sigma_{\partial\Omega}((P_\Omega + G\ K))$, we have $j(P_\Omega) = \Phi - \pi^* F$ (as above π is the projection $S_{\partial\Omega} \to \partial\Omega$). Suppose now that $j(P_\Omega)$ is a pull back: then there exists a bundle F' on $\partial\Omega$ and a C^∞ isomorphism $q(\xi'): \Phi \oplus C^M \to F'$ (if M is large enough). Then (with the notations of (5.16)), a Green operator with same interior symbol as P_Ω and with boundary symbol:

$$\sigma_{\partial\Omega}(A) = \sigma_{\partial\Omega}((P_\Omega + G\ \ K)) \oplus q(\xi')\,(t_0 \oplus 0_{C^M})$$

satisfies the condition of (5.16), so it is an elliptic Green operator associated to P.

All these constructions can be done exactly in the same way when the operators depend continuously on a parameter $x \in X$ (where X is a compact space). Then the index bundle $j(P_\Omega^x)$ is a virtual bundle on $S_{\partial\Omega} \times X$. Theorem (5.14) can be restated as: there exists an elliptic Green system A^x (depending continuously on $x \in X$) associated to P^x if and only if $j(P_\Omega^x) \in K(S_{\partial\Omega} \times X)$ is the pull back of a virtual bundle on $\partial\Omega \times X$.

We will be specially interested in the case where X is the interval $[0, 1]$ (or the square $[0, 1]^2$). Then we get:

(5.18) *Let P^t $(0 \leqslant t \leqslant 1)$ be a continuous family of elliptic pseudo-differential operators (satisfying the transmission property) on $\overline{\Omega}$, and let* $A = \begin{pmatrix} P_\Omega + G & K \\ T & Q \end{pmatrix}$ *be an elliptic system associated to P^0. Then there exists a continuous family of elliptic systems A^t, with $\sigma_\Omega(A^t) = \sigma_\Omega(P^t)$, and $A^0 = A \oplus 1_{C^M}$.*

In other words, an elliptic "boundary condition" associated to P^0 can be continued to P^t.

Proof. Let us first continue K^0, G^0 arbitrarily into K^t, G^t. Choose K as in (5.17) so that it works for every t (i.e. $\sigma_{\partial\Omega}((P_\Omega^t + G^t\ K^t\ K))$ is a surjection for every t). Let Φ^t be the kernel of $\sigma_{\partial\Omega}((P_\Omega^t + G^t\ K^t\ K))$: this is a C^∞ bundle on $S_{\partial\Omega} \times [0, 1]$. Now $\sigma_{\partial\Omega}((T^0\ Q^0\ 1_{C^M}))$ is an isomorphism of Φ^0 onto $\pi^* F' \oplus C^M$. But this can be extended as a C^∞ isomorphism of Φ^t

onto the pull back of $F' \oplus C^M$ to $S_{\partial\Omega} \times [0, 1]$, and because Φ^t is finite dimensional, such an isomorphism is the restriction to Φ^t of a boundary symbol $\sigma_{\partial\Omega}((T^t\, Q^t))$, where we can choose

$$T^0 = \begin{pmatrix} T \\ 0 \end{pmatrix}, \quad Q^0 = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}.$$

6. *The Agranovič–Dynin formula.* We will say that two elliptic systems A and B are equivalent and write $A \sim B$ if there exists a homotopy of elliptic systems connecting $A \oplus 1$ to $B \oplus 1$. The class of A for this equivalence relation depends only on its interior and boundary symbols, so we can speak of the class of a symbol as well as of the class of an operator. Quite obviously A and B are equivalent if their symbols are close enough to each other. Also we have

$$A \sim \begin{pmatrix} A & A' \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ A'' & 1 \end{pmatrix}.$$

Now let A_1 and A_2 be elliptic and have the same interior symbol. Then the interior symbol of $B = A_2 A_1^{-1}$ is 1 (we write A_1^{-1} for any parametrix of A_1). So the boundary symbol of B is of the form:

$$\sigma_{\partial\Omega}(B) = \begin{pmatrix} 1+g & k \\ t & q \end{pmatrix}.$$

We can always approximate g by a composition $k't'$. So we deduce all information on the difference between the classes of A_1 and A_2 from the following result, which generalizes the Agranovič–Dynin formula (Dokl. Akad. Nauk SSSR 145, no. 3, 511–514).

(5.19) PROPOSITION. *We have*

$$\begin{pmatrix} 1-K'T' & K \\ T & Q \end{pmatrix} \sim \begin{pmatrix} 1-T'K' & -T'K \\ -TK' & Q-TK \end{pmatrix}$$

where the second operator is a pseudo-differential operator on the boundary.

Proof. We have

$$\begin{pmatrix} 1-K'T' & K \\ T & Q \end{pmatrix} \sim \begin{pmatrix} 1-K'T' & K' & K \\ 0 & 1 & 0 \\ T & 0 & Q \end{pmatrix} \sim \begin{pmatrix} 1 & K' & K \\ T' & 1 & 0 \\ T & 0 & Q \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ T & 1-T'K' & -T'K \\ T' & -T\,K' & Q-T\,K \end{pmatrix}.$$

The first equivalence is obvious. The second follows by multiplying on the right by

$$\begin{pmatrix} 1 & 0 & 0 \\ T' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim 1,$$

and the third by multiplying on the right by

$$\begin{pmatrix} 1 & -K' & -K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim 1.$$

This construction can be done just as well when $A = \begin{pmatrix} 1+G & K \\ T & Q \end{pmatrix}$ depends continuously on a parameter $x \in X$; then we get $\begin{pmatrix} 1+G^x & K^x \\ T^x & Q^x \end{pmatrix} \sim Q'^x$, where Q'^x is a pseudo-differential operator on $\partial\Omega$ depending continuously on x. In particular we get

(5.20) *Let* $A^t = \begin{pmatrix} 1+G^t & K^t \\ T^t & Q^t \end{pmatrix}$ *be an elliptic Green operator depending continuously on* $t \in [0,1]$, *where* $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is the identity operator and* $A^1 = \begin{pmatrix} 1 & 0 \\ 0 & Q^1 \end{pmatrix}$ (*i.e.* $G^1 = 0, K^1 = 0, T^1 = 0$). *Then there exists a homotopy of elliptic pseudo-differential operators on* $\partial\Omega$ *connecting* $Q^1 \oplus 1$ *with* 1.

7. *The classical case.* The constructions of the preceding sections become fairly simple when $A = \begin{pmatrix} P_\Omega \\ T \end{pmatrix}$, where P is an elliptic partial differential operator. We repeat these simplified constructions here.

Let m be the degree of P, and let E and E' be the bundles on the sections of which P operates. Define $\mathcal{G}: C^\infty(\partial\Omega, E)^m \to \mathcal{D}'(\bar{\Omega}, E')$

$$\mathcal{G}\gamma_m f = P(\tilde{f}) - \widetilde{P_\Omega f},$$

where $\tilde{f}$ is the extension of f by 0 outside of $\bar{\Omega}$, and we have set $\gamma_m f = ((\partial/\partial x_n)^k f/\partial\Omega)_{k=0,\dots,m-1}$, $\partial/\partial x_n =$ normal derivative near $\partial\Omega$.

By Green's formula $P\tilde{f} - \widetilde{P_\Omega f}$ is a distribution supported by $\partial\Omega$, of the form

$$\sum_{0 \leq k+l \leq m-1} P_{k,l}(x', \partial/\partial x')\,((\partial/\partial x_n)^k f/\partial\Omega)\,\delta(x_n)^{(l)}$$

(the definition of the product is relative to an isomorphism $\bar{\Omega} \approx \partial\Omega \times \bar{R}^+$ near $\partial\Omega$, and $x_n \in \bar{R}^+$ is the normal variable in this isomorphism).

It follows that $\mathcal{G}$ is well defined and ranges in the space of distributions supported by $\partial\Omega$.

Now define K_0 by

$$K_0 u = P'(\mathcal{G}u)/\Omega.$$

where P' is a parametrix of P, so that K_0 is a Poisson operator.

Then we get $\quad \sigma_\Omega(P_\Omega\, P'_\Omega)=1 \quad \sigma_\Omega(P'_\Omega\, P_\Omega)=1$

$$\sigma_{\partial\Omega}(P_\Omega\, P'_\Omega)=1 \quad \sigma_{\partial\Omega}(P'_\Omega\, P_\Omega)=1-\sigma_{\partial\Omega}(K_0\;\gamma_m).$$

So that in this case, we can take the symbols k_0 and t_0 of (5.16) to be those of K_0 and γ_m.

$h_0=\sigma_{\partial\Omega}(K_0\gamma_m)$ is then the symbol of the Calderon–Seeley projector on the Cauchy data of the kernel of P_Ω.

Application of (5.16) to $A=\begin{pmatrix}P_\Omega\\ T\end{pmatrix}$, of course, gives back the Lopatinsky covering conditions.

If $A_1=\begin{pmatrix}P_\Omega\\ T_1\end{pmatrix}$ and $A_2=\begin{pmatrix}P_\Omega\\ T_2\end{pmatrix}$ are two elliptic systems associated to the same elliptic operator P, and if $B_1=(P'_\Omega+G_1\,K_1)$ is a parametrix to A_1, we get

$$\sigma_{\partial\Omega}(A_2B_1)=\begin{pmatrix}1 & 0\\ t & q\end{pmatrix}$$

with $q=\sigma_{\partial\Omega}(T_2\,K_1)$.

So we have $A_2B_1\sim Q=T_2K_1$, and in particular

$$\text{index}\,(A_2)-\text{index}\,(A_1)=\text{index}\,(Q)$$

which was the formula stated by Agranovitch and Dynin (loc. cit.).

8. *The index of an elliptic boundary problem.* Let us first recall the index formula when there is no boundary (cf. (4)): if $f\colon X\to Y$ is an embedding of C^∞ manifolds, we have the Thom map $f_!\colon K(TX)\to K(TY)$. In particular the Thom map $t\colon K(T\partial\Omega)\to K(T\Omega)$ is well defined if Ω is the interior of a C^∞ manifold with boundary $\overline{\Omega}$ (because Ω is isomorphic to a neighborhood of $\overline{\Omega}$, and we can choose the isomorphism to be homotopic to the identity of Ω).

The Bott periodicity theorem states that the Thom map is an isomorphism if X is a point, and $Y=R^n$.

If $\xi\in K(TX)$, the topological index of ξ is defined by

$$\chi(\xi)=i_!^{-1}f_!(\xi)\in Z=K\ (\text{point}),$$

where f is an embedding $X\to R^n$, and i the inclusion point $\subset R^n$.

If P is an elliptic operator on X, the index formula states that

$$\text{index}\,(P)=\chi([p]),$$

where $[p]=d(E,E',p)\in K(TX)$ is the difference bundle defined by the symbol p of P.

(5.21) THEOREM. *There exists one and only one map $A \to [A]$ which to an elliptic Green operator A on $\overline{\Omega}$ assigns a virtual bundle $[A] \in K(T\Omega)$ in such a way that*

(1) *$[A]$ depends continuously on A (i.e. $[A]=[B]$ if there exists a homotopy of elliptic operators connecting A and B).*

(2) *$A \to [A]$ is a homomorphism, i.e. $[A \oplus B]=[A]+[B]$, and $[A \circ B]=[A]+[B]$ if the composition is defined.*

(3) *$[A]=[P_\Omega]+t[Q]$ if $A=\begin{pmatrix} P_\Omega & 0 \\ 0 & Q \end{pmatrix}$ where P coincides with the identity near $\partial\Omega$ (so that $[P]$ is well defined as an element of $K(T\Omega)$), and t is the Thom map: $K(T\partial\Omega) \to K(T\Omega)$.*

(4) *$[A]=0$ if A is one of the elliptic operators of* (5.10).

Proof. We prove the uniqueness first: suppose that

$$A=\begin{pmatrix} P_\Omega + G & K \\ T & Q \end{pmatrix}: C^\infty(\overline{\Omega}, E) \oplus C^\infty(\partial\Omega, F) \to C^\infty(\overline{\Omega}, E') \oplus C^\infty(\partial\Omega, F')$$

and the degree of A is d. Then we must have

$$[A]=\left[A \circ \begin{pmatrix} \Lambda_E & 0 \\ 0 & (\Delta_F)^{\frac{1}{2}} \end{pmatrix}^{-d}\right]. \text{ So we can suppose } d=0.$$

Next let $j(P_\Omega)=\pi^*(\Phi-\Phi')$, where Φ and Φ' are two C^∞ bundles on $\partial\Omega$ (and π is the projection $S_{\partial\Omega} \to \partial\Omega$). Then we must have

$$[A]=\left[A \oplus \begin{pmatrix} (\Gamma_{\Phi'})_\Omega \\ T_0\, p_{\Phi'} \end{pmatrix} \oplus \begin{pmatrix} (\Gamma_\Phi)_\Omega \\ T_0\, p_\Phi \end{pmatrix}^{-1}\right]$$

(we have taken the same notations as in (5.10)). So we can suppose $j(P_\Omega)=0$.

Now in view of lemma (5.15), there exists a homotopy connecting P to P', where $P'=1$ near $\partial\Omega$. Then (5.18) and (5.19) show that there exists a homotopy connecting $A \oplus 1$ to a Green operator of the form $\begin{pmatrix} P'_\Omega & 0 \\ 0 & Q' \end{pmatrix}$, where $P'=1$ near $\partial\Omega$. So by (1) and (3) we must have

$$[A]=[P']+t[Q].$$

To prove the existence, we have to show that the bundle just constructed does not depend on the choice of homotopies hereabove, or equivalently: if A^t $(0 \leqslant t \leqslant 1)$ is a homotopy of elliptic systems connecting $A^0=\begin{pmatrix} P^0 & 0 \\ 0 & Q^0 \end{pmatrix}$ where $P^0=1$ near $\partial\Omega$ to $A^1=1$, then we have $[P^0]+t[Q^0]=0$. This will be proved in section 9 (it is useless for the proof of the index formula).

(5.22) THEOREM. *Let A be an elliptic Green operator on $\bar{\Omega}$. Then we have*

$$\operatorname{index}(A)=\chi([A])$$

(*where χ: $K(T\Omega)\to Z$ is the topological index*).

Proof. This is just the index formula of (4) when $A=\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$, where P is the identity near $\partial\Omega$.

If A is one of the operators of (5.10), the formula is true by section 2 (then index $(A)=0$). The formula in the general case then follows immediatetly: every construction in the proof of theorem (5.21) preserves the index. This ends the proof.

Let us also observe that this index formula still holds if $\bar{\Omega}$ is not compact, but A coincides with the identity operator out of a compact set.

9. *End of the proof of Theorem (5.21)*. Let $A^t=\begin{pmatrix} P^t_\Omega+G^t & K^t \\ T^t & Q^t \end{pmatrix}$ $(t>0)$ be a continuous family of elliptic systems, such that $A^t=1$ for $t\geqslant 1$, and

$$A^0=\begin{pmatrix} P^0 & 0 \\ 0 & Q^0 \end{pmatrix}$$

where $P^0=1$ *near* $\partial\Omega$.

We first choose an isomorphism $x\to(x',x_n)$ of a neighborhood of $\partial\Omega$ in $\bar{\Omega}$ onto $\partial\Omega\times\bar{R}^+$. Then we define a new symbol p'^t by

$$p'^t(x,\xi)=\begin{cases} p^{t\varphi}(x,\xi) & \text{if } t\leqslant 1 \\ p^{(t-1)+(2-t)\varphi}(x,\xi) & \text{if } 1\leqslant t\leqslant 2 \\ 1 & \text{if } t\geqslant 2. \end{cases}$$

where p^t is the interior symbol of P^t, and $\varphi\in C^\infty(\bar{\Omega})$ is such that $0\leqslant\varphi\leqslant 1$, $\varphi=0$ near $\partial\Omega$, and $\varphi=1$ when $x_n\geqslant 1$ and away from $\partial\Omega$.

Then we obviously have $[P^0]=[p'^0]=[p'^1]$.

Next define a new elliptic system A'^t by

$$A'^t=\begin{cases} \begin{pmatrix} P'^t+G^0 & K^0 \\ T^0 & Q^0 \end{pmatrix} & \text{if } 0\leqslant t\leqslant 1 \\ \begin{pmatrix} P'^t+G^{t-1} & K^{t-1} \\ T^{t-1} & Q^{t-1} \end{pmatrix} & \text{if } t\geqslant 1 \end{cases}$$

where p'^t is any pseudo-differential operator with interior symbol p'^t. Then $\sigma_\Omega(A'^t)$ is constantly equal to 1 when $t\geqslant 1$ and $x_n\geqslant 1$.

Replacing A^t by A''^{t-1}, we see that we can always suppose that the interior symbol of A^t is the identity when $x_n \geqslant 1$ or $t \geqslant 1$, and so we will also suppose $\bar{\Omega} = \partial\Omega \times \bar{R}^+$.

Now let us introduce the following commutative diagram (where $\bar{\Omega} = \partial\Omega \times \bar{R}^+$)

$$\begin{array}{lllll}
K(\bar{\Omega} \times \bar{R}_t^+, (T\bar{\Omega}/\partial\Omega \times \{0\}) \cup (N \times \bar{R}_t^+)) & & & & \\
\downarrow a & & & & \\
K(T\bar{\Omega}/\partial\Omega \times R_t^+, N \times R_t^+) & \overset{d}{\simeq} & K(S_{\partial\Omega} \times R^+_{|\xi'|} \times R_{\xi_n} \times R_t^+) & \overset{\beta_1^{-1}}{\simeq} & K(S_{\partial\Omega} \times R_t^+) \\
\downarrow b & & \downarrow \partial & & \downarrow \partial \\
K(T\bar{\Omega}/\partial\Omega \times R_t^+) & \overset{e}{\simeq} & K(T\partial\Omega \times R_{\xi_n} \times R_t^+) & \overset{\beta_2^{-1}}{\simeq} & K(T_{\partial\Omega}) \\
\downarrow c & & & & \\
K(T\Omega). & & & &
\end{array} \tag{5.23}$$

In this diagram, N is as usual the cotangent normal bundle.

a is the restriction to $\partial\Omega$ (it is an isomorphism because the inclusion

$$(T\bar{\Omega}/\partial\Omega \times \bar{R}^+,\ T\bar{\Omega}/\partial\Omega \times \{0\}) \subset (T\bar{\Omega} \times \bar{R}^+,\ (T\bar{\Omega}/\partial\Omega \times \{0\}) \cup (N \times \bar{R}^+))$$

is a homotopy equivalence of pairs: the second can be retracted on the first by rotations)

b is the natural restriction map

c is induced by the map $t \to x_n$.

d is induced by the map $(\xi, t) \to (\xi'/|\xi'|, |\xi'|, \xi_n, t)$

e is induced by the map $\xi \to (\xi', \xi_n)$

∂ is the boundary map of the exact sequence of K-theory:

$$K^{-1}(S_{\partial\Omega}) = K(S_{\partial\Omega} \times R^+) \to K(T\partial\Omega)$$

β_1 is the multiplication by the difference bundle $d(C, C, \xi_n + i \operatorname{Log} |\xi'|) \in K(R^+_{|\xi'|} \times R_{\xi_n})$

β_2 is the multiplication by the difference bundle $d(C, C, \operatorname{Log} t + i\xi_n) \in K(R_{\xi_n} \times R_t^+)$

$\bar{R}^+$ is the closed half line, R^+ is the open half line. The subscript ($R^+_{|\xi'|}$, R_{ξ_n}, etc.) indicates which variable varies in which factor.

The last square on the right would be commutative if we had replaced β_1 by β_2 in the second row of (5.23). It still commutes as it stands because the permutation $(|\xi'|, \xi_n, t) \to (e^{\xi_n}, \operatorname{Log} t, |\xi'|)$ is homotopic to the identity.

The composition $c \circ b \circ a$ equals the restriction to $\{0\} \subset \bar{R}^+$:

$$K(T\bar{\Omega} \times \bar{R}^+, \ldots) \to K(T\bar{\Omega} \times \{0\},\ T\bar{\Omega}/\partial\Omega \times \{0\}) = K(T\Omega)$$

because the maps $t \in \bar{R}^+ \to (0, t) \in \bar{R}^2$ or $(t, 0) \in \bar{R}^{+2}$ are homotopic.

The composition $c \circ e^{-1} \circ \beta^2$ is the Thom map: $K(T\partial\Omega) \to K(T\Omega)$.

Now the interior symbol $p^t = \sigma_\Omega(A^t)$ is a bundle homomorphism satisfying the symmetry condition $p^t(\nu) = p^t(-\nu)$ (where ν is any normal covector on $\partial\Omega$), so it defines a virtual bundle

$$d(p^t) \in K(T\bar{\Omega} \times \bar{R}_t^+,\ T\bar{\Omega} \times \{0\} \cup N \times \bar{R}_t^+).$$

The image q of $d(p^t)$ in $K(S_{\partial\Omega} \times R_t^+) = K^{-1}(S_{\partial\Omega})$ in diagram (5.23) is the index bundle $j(p^t)$ as in section 3. So we have

(5.24) $[P^0] = t[q]$ $(q = j(p^t) \in K(S\partial\Omega \times R_t^+)$ and $[q] = \partial q$ = difference bundle defined by q).

Let us first suppose $j(p^t) = 0 \in K(S_{\partial\Omega} \times R_t^+)$. Then by proposition (5.15) (with parameter) we can deform the continuous family $A^t \oplus 1$ into a new family A''^t (without changing $A^0 \oplus 1$ or $A^\infty \oplus 1$) so that $\sigma_\Omega(A''^t) = 1$ near $\partial\Omega$ for every t. In this case $[\sigma_\Omega(A''^t)]$ is defined, and equal to 0, for every t, and it follows from (5.20) that we also have $[Q^0] = 0$.

It remains to check $[P^0] + t[Q^0] = 0$ for one continuous family A^t as above, with given $q = j(p^t)$.

So let $q \in K(S_{\partial\Omega} \times R^+) = K^{-1}(S_{\partial\Omega})$. We represent q by a C^∞ matrix $q(\xi'): S_{\partial\Omega} \to GL(N)$.

Let M_u $(0 \leqslant u < \infty)$ be a homotopy connecting $\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ and 1 $\Big($i.e. M_u is a C^∞ $2N \times 2N$ matrix on $S_{\partial\Omega}$, depending continuously on u, and such that $M_0 = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, $M_u = 1$ for large $u\Big)$.

We define a continuous family of elliptic symbols p^t by

$$p^t(x, \xi) = M_{t+x_n}(\xi'/|\xi'|) \begin{pmatrix} \zeta^\alpha & 0 \\ 0 & 1 \end{pmatrix} M_{t+x_n}^{-1} \tag{5.26}$$

(where ζ, α are as in section 2) (p^t is at first only defined for $\xi \neq 0$, but we extend it by 1 when $\xi' = 0$. It is the identity when ξ' is small, or when $x_n + t$ is large).

Then $p^t(p^0)^{-1}$ is a continuous family as above, and we have

$$j(p^t(p^0)^{-1}) = j(p^t) - j(p^0) = (M_t \cdot C^N) - (C^N) \in K(S_{\partial\Omega} \times R^+),$$

(where we write C^N for the first factor in C^{2N}); this is precisely the bundle defined by q.

Now we follow the boundary condition: as in section 2, if T_0 is the Dirichlet data, and Π the projector on the first factor C^n in C^{2N}, $B^t = \begin{pmatrix} P^t \\ \Pi\, M_t^{-1} \cdot T_0 \end{pmatrix}$ is an elliptic column matrix (P^t is such that $\sigma_\Omega(P^t) = p^t$).

Then $A^t = B^t(B^\infty)^{-1}$ is as required: we have $A^\infty = 1$, and $A^0 = \begin{pmatrix} P_0 & 0 \\ 0 & Q^0 \end{pmatrix}$ with $\sigma_{\partial\Omega}(Q^0) = q^{-1}$.

Finally, by (5.24), we get in this case $[p^0] + [Q^0] = t[q] + t[q^{-1}] = 0$. This ends the proof.

References

[1]. ATIYAH, M. F., *K-theory*. Benjamin, New York 1967.
[2]. —— *Bott periodicity and the index of elliptic operators*. *Quart. J. Math.*, 19 (1968), 113–140.
[3]. ATIYAH, M. F. & BOTT, R., The index problem for manifolds with boundary. *Coll. Differential Analysis*, Tata Institute, Bombay, Oxford University Press 1964, 175–186.
[4]. ATIYAH, M. F. and SINGER, I. M., The index of elliptic operators. *Ann. of Math.*, 87 (1968) 434–530.
[5]. BOUTET DE MONVEL, L., Comportement d'un opérateur pseudo différential sur une variété à bord. *J. Anal. Math.*, 17 (1966), 241–304.
[6]. —— Opérateurs pseudo différentiels analytiques et problèmes aux limites elliptiques. *Ann. Inst. Fourier*, 19 (1969), 163–268.
[7]. HÖRMANDER, L., Pseudo-differential operators and hypoelliptic equations. *Amer. Math. Soc. Proc. Symp. Pure Appl. Math.*, 10 (1968), 138–183.
[8]. —— Fourier integral operators. *Acta Math.* (to appear).
[9]. KOHN, J. J. & NIRENBERG, L., An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.*, 18 (1965), 269–305.
[10]. SHAMIR, E., Elliptic systems of singular integral operators. *Trans. Amer. Math. Soc.*, 127 (1967), 107–124.
[11]. VIŠIK, M. I. & ESKIN, G. I., Normally solvable problems for elliptic systems of equations in convolutions. *Mat. Sbornik*, 74 (116) (1967), 326–356.

Received January 19, 1970