Boundary Properties of Analytic Functions

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1. Introduction. This paper considers asymptotic values of single-valued meromorphic functions in general domains, metric properties of sets of angular limits of functions meromorphic in the unit disc, and schlicht functions.

Let D be an arbitrary domain, Γ its boundary, and let $w = f(z)$ be a single-valued meromorphic function in D. Suppose that there exists a sequence $\{z_n\} \subset D$ such that $z_n \to z_0$ ($z_0 \varepsilon \Gamma$) and $f(z_n) \to a$, and suppose that the value a is not assumed by f arbitrarily near z_0. Then at least one of the following three statements holds (Theorem 1): (i) z_0 is in a (nondegenerate) continuum that is contained in Γ and is the "limit" of a sequence $\{J_n\}$ of Jordan arcs in D on which $f \to a$ as $n \to \infty$. (ii) Every neighborhood of z_0 contains an asymptotic path (not necessarily tending to a point) of f for the value a. (iii) For any neighborhoods $N(z_0)$ and $N(a)$ of z_0 and a, respectively, there exists a closed subset of $N(z_0) \cap \Gamma$ of positive (logarithmic) capacity at each point of which f has an asymptotic value that is in $N(a)$, and the set of these asymptotic values contains a closed set of positive capacity. This result contains a theorem of Noshiro [11, 14] (see Corollary 1 of the present paper). A global analog of Theorem 1 is also given (Theorem 2).

Now let $w = f(z)$ be meromorphic in $\{|z| < 1\}$, and let $\mathcal{S}$ be the Riemann surface of f over the extended w-plane. Suppose that at each point ζ of a subset E_z of $\{|z| = 1\}$ of positive measure, f has a finite angular limit a_ζ, and let $E_w = \{a_\zeta : \zeta \varepsilon E_z\}$. Then E_w may have linear measure zero even when f is a schilcht function mapping $\{|z| < 1\}$ onto the interior of a Jordan curve (see Lavrentieff [5, 830]). For each $\zeta \varepsilon E_z$ and positive number h, let $\mathcal{S}(\zeta, h)$ be the component of $\mathcal{S}$ over $\{|w - a_\zeta| < h\}$ such that if r is sufficiently near 1 ($r < 1$), then $r\zeta$ corresponds under f to a point of $\mathcal{S}(\zeta, h)$, and let $P\mathcal{S}(\zeta, h)$ be the projection of $\mathcal{S}(\zeta, h)$ onto the w-plane. Suppose that to each $\zeta \varepsilon E_z$ there correspond a positive number h_ζ and a continuum K_ζ in the w-plane such that $a_\zeta \varepsilon K_\zeta$ and $K_\zeta \cap P\mathcal{S}(\zeta, h_\zeta) = \emptyset$. Then E_w contains a closed set that does not have $\frac{1}{2}$-dimensional measure zero (Theorem 3). This result extends theorems of Matsumoto [6, 133] and the author [9]. We apply Theorem 3 to refine Theorem 1 (see Theorem 6). It follows from Theorem 3 that if f is any schilcht function, then E_w contains a closed set that does not have $\frac{1}{2}$-dimensional measure zero.

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As a slight refinement of a theorem of J. Dufresnoy and M. Tsuji (see [13, 347]), we prove that if f is any schlicht function and E_z is any closed set of positive capacity (possibly of measure zero), then E_w contains a closed set of positive capacity.

Let $w = f(z)$ be a schlicht holomorphic function in $\{|z| < 1\}$. Suppose that at each point ζ of a subset E_z of $\{|z| = 1\}$, f has a finite radial limit $f(\zeta)$, and let $E_w = \{f(\zeta) : \zeta \in E_z\}$. Suppose that for each $\zeta \in E_z$ there exists a sequence $\{c_n\}$ of crosscuts at ζ such that $c_n \to \zeta$ (defined precisely below) and

$$|f(c_n)|/d(f(c_n), E_w) \to 0 \qquad (n \to \infty),$$

where $|f(c_n)|$ and $d(f(c_n), E_w)$ denote the Euclidean diameter of $f(c_n)$ and distance between $f(c_n)$ and E_w, respectively. Then E_z is a set of measure zero (Theorem 7). (For a related open question, see Remark 4 at the end of this paper.) Theorem 7 depends on a lemma, concerning a family of harmonic measures, which has a certain interest in its own right and which is proved by applying a method due to Matsumoto [6].

2. Meromorphic functions in general domains. Let D be an arbitrary domain, Γ its boundary, and let $w = f(z)$ be a single-valued meromorphic function in D. We say that a (nondegenerate) continuum K is a *Koebe continuum* of f for the value a if $K \subset \Gamma$ and there exists a sequence $\{J_n\}$ of Jordan arcs in D such that, as $n \to \infty$,

$$\max\{d(z, K): z \in J_n\} \to 0, \qquad \max\{d(z, J_n): z \in K\} \to 0,$$
$$\max\{\chi(f(z), a): z \in J_n\} \to 0,$$

where $d(z, S)$ denotes the Euclidean distance from the point z to the set S, and χ is the chordal metric. A simple curve α in D, described by a continuous function $z(t)$ $(0 \leqq t < 1)$, is an *asymptotic path* of f for the value a, and a is an *asymptotic value* of f, if for each compact subset A of D and neighborhood $N(a)$ of a, there exists t_0 $(0 < t_0 < 1)$ such that $z(t) \notin A$ and $f(z(t)) \in N(a)$ if $t_0 < t < 1$. If, in addition, $z(t) \to z_0$ as $t \to 1$ $(z_0 \in \Gamma)$, then we say that f has the *asymptotic value a at z_0*.

Theorem 1. *Let $z_0 \in \Gamma$ $(z_0 \neq \infty)$. Suppose that there exists a sequence $\{z_n\}$ of points in D such that $z_n \to z_0$ and $f(z_n) \to a$, and suppose that the value a is not assumed by f arbitrarily near z_0. Then at least one of the following three statements is true:*

(i) *A Koebe continuum of f for the value a contains z_0.*

(ii) *Every neighborhood of z_0 contains an asymptotic path of f for the value a.*

(iii) *For any neighborhoods $N(z_0)$ and $N(a)$ of z_0 and a, respectively, there exists a closed subset of $N(z_0) \cap \Gamma$ of positive (logarithmic) capacity at each point of which f has an asymptotic value that is in $N(a) - \{a\}$, and the set of these asymptotic values contains a closed set of positive capacity; moreover, $N(a)$ con-*

tains a closed set of positive linear measure (defined in [3, 105], *where* $\mathfrak{T}$ *is the family of open discs and* $\tau(T)$, $T \varepsilon \mathfrak{T}$, *is the diameter of* T*) each point of which is an asymptotic value of* f *along some path contained in* $N(z_0)$.

Proof. We suppose, without loss of generality, that $a = \infty$. Let h_0 be a positive number such that $f(z) \neq \infty$ if $|z - z_0| < h_0$ $(z \varepsilon D)$, and let D_n be the component of

$$\{z \varepsilon D: \chi(f(z), \infty) < \chi(f(z_n), \infty) + (1/n)\}$$

that contains z_n. We first consider the case where there exists a positive number $h < h_0$ such that

$$D_n \not\subset \{|z - z_0| < h\}$$

for all $n \geq 1$, and we prove that in this case (i) holds. We suppose, without loss of generality, that $|z_n - z_0| < h$ $(n \geq 1)$, and we associate with each n a Jordan arc J_n in D_n, with one endpoint z_n, that lies in $\{|z - z_0| < h\}$ except for one endpoint on $\{|z - z_0| = h\}$. By choosing a subsequence of $\{J_n\}$, we can suppose, without loss of generality, that the sequence $\{J_n\}$ is "convergent" in the following sense [14, 8]: If we let

$$K = \bigcap_{n=1}^{\infty} \left(\text{closure of } \bigcup_{m=n}^{\infty} J_m\right) \qquad (= \limsup \{J_n\}),$$

then K is the set of points z such that any neighborhood of z intersects all but finitely many J_n; that is, $K = \liminf \{J_n\}$ also.
Since

$$K = \bigcap_{n=1}^{\infty} \left(\text{closure of } \bigcup_{m=n}^{\infty} [J_m \cup \{|z - z_0| \leqq |z_m - z_0|\}]\right),$$

it is a continuum. Clearly,

$$\max \{d(z, K): z \varepsilon J_n\} \to 0 \qquad (n \to \infty),$$

and since $K = \liminf \{J_n\}$, it follows from a standard compactness argument that

$$\max \{d(z, J_n): z \varepsilon K\} \to 0 \qquad (n \to \infty).$$

Hence K is a Koebe continuum of f for the value ∞, and (i) holds.

The remaining case is that each neighborhood of z_0 contains some D_n, and we prove that in this case either (ii) or (iii) holds. Suppose that (ii) is false, and let $N(z_0)$ and $N(\infty)$ be any neighborhoods of z_0 and ∞, respectively. There exists D_n (n is fixed throughout the rest of the argument) such that $\overline{D}_n \subset N(z_0)$, $\overline{f(D_n)} \subset N(\infty)$ (the bar denotes closure), f is holomorphic in D_n, and no asymptotic path of f for the value ∞ is contained in D_n. Let M_0 be the positive number such that D_n is a component of

$$\{z \varepsilon D: f(z) = \infty \quad \text{or} \quad |f(z)| > M_0\}.$$

If, for each $M \geqq M_0$, f were unbounded in each component of $\{z \in D_n : |f(z)| > M\}$, then we could let Δ_1 be a component of $\{z \in D_n : |f(z)| > M_0 + 1\}$, let Δ_2 be a component of $\{z \in \Delta_1 : |f(z)| > M_0 + 2\}$, and in this way define a nested sequence $\{\Delta_m\}$ and obtain an asymptotic path in D_n of f for the value ∞. Therefore, there exists $M \geqq M_0$ and a component D_0 of $\{z \in D_n : |f(z)| > M\}$ such that f is bounded in D_0.

Let $z = g(\zeta)$ be a conformal mapping of $\{|\zeta| < 1\}$ onto D_0 whose Riemann surface is the universal covering surface of D_0, and let $F(\zeta) = f(g(\zeta))$ $(|\zeta| < 1)$. By Fatou's theorem, at almost all points $e^{i\theta}$ of $\{|\zeta| = 1\}$, the bounded functions g and F have angular limits $g(e^{i\theta})$ and $F(e^{i\theta})$, respectively. Since F has a Poisson integral representation in terms of the $F(e^{i\theta})$, and since $|F(\zeta)| > M$ $(|\zeta| < 1)$, there exists a subset E_0 of $\{|\zeta| = 1\}$ of positive measure such that $|F(e^{i\theta})| > M$ if $e^{i\theta} \in E_0$. There exists a subset E_ζ of E_0 of positive measure such that the angular limit $g(e^{i\theta})$ exists for each $e^{i\theta} \in E_\zeta$. Let

$$E_z = \{g(e^{i\theta}) : e^{i\theta} \in E_\zeta\}, \qquad E_w = \{F(e^{i\theta}) : e^{i\theta} \in E_\zeta\}.$$

Each of the sets E_z and E_w contains a closed set of positive capacity (see [12, 210] and [10, 126], or see [13, 339]). Also, $E_z \subset N(z_0)$ and $E_w \subset N(\infty) - \{\infty\}$. For any $e^{i\theta} \in E_\zeta$, the curve $\{g(re^{i\theta}) : 0 \leqq r < 1\}$, which is contained in D_0, tends to $g(e^{i\theta})$, which is on the boundary of D_0, and on this curve f tends to $F(e^{i\theta})$ at $g(e^{i\theta})$. Since $|F(e^{i\theta})| > M$ $(e^{i\theta} \in E_\zeta)$, E_z is contained in the boundary of D. Thus it is easy to see that f has the asymptotic value $F(e^{i\theta})$ at $g(e^{i\theta})$ (along a simple curve).

All that remains to be shown is that $N(\infty)$ contains a closed set of positive linear measure each point of which is an asymptotic value of f along an asymptotic path contained in $N(z_0)$. We suppose, without loss of generality, that F assumes a real value u_0. Let δ be a positive number such that the set $L = \{u_0 + iv : -\delta < v < \delta\}$ is contained in the range of F. If $-\delta < v < \delta$ and P is a point over $u_0 + iv$ of the Riemann surface $\mathcal{S}$ of F, then there exists a largest real number $u' > u_0$ such that $\{u + iv : u_0 < u < u'\}$ has a lifting onto $\mathcal{S}$ beginning at P, and $u' + iv$ is an asymptotic value of F. By projecting these asymptotic values onto L, we see that the set

$$E = \{F(e^{i\theta}) : |F(e^{i\theta})| > M\}$$

has positive exterior linear measure. The set A of all asymptotic values of F is analytic [7], and it is therefore linearly measurable (see [3, p. 106, 8.5.3; p. 84, 7.2.1; p. 83, 7.1.21]). Thus, since $E = A \cap \{|w| > M\}$, it has positive linear measure and therefore contains a closed set of positive linear measure (see [3, p. 106, 8.5.3; p. 85, 7.2.22]). By an argument similar to the one used above, we see that each point of E is an asymptotic value of f along a path contained in D_0. The proof of Theorem 1 is complete.

Remark 1. It is clear from the proof that in Theorem 1 the assumption that f does not assume the value a arbitrarily near z_0 can be replaced by the following weaker statement: There exists no sequence $\{J_n\}$ of Jordan arcs in D such that J_n has endpoints z_n and z_n', $f(z_n') = a$, the diameter of J_n tends to 0, and the diameter (in the chordal metric) of $f(J_n)$ tends to 0.

Remark 2. If $D = \{|z| < 1\}$, then it follows from an extension of Löwner's lemma [13, 322] applied to F (defined in the proof of Theorem 1) that E_z contains a closed set of positive linear measure, so that (iii) can be replaced by a corresponding stronger statement. In this case the theorem is known [8], but the proof in [8] is not this simple.

In order to state the following corollary of Theorem 1, which contains a theorem of Noshiro [11, 14], we use the notation in Noshiro's monograph [11]. Note that the boundary cluster set $C_{\Gamma-E}(f, z_0)$ ($z_0 \in \Gamma$) can be computed for an arbitrary set $E \subset \Gamma$ (and might possibly be empty).

Corollary 1. *Let E be a subset of Γ that contains no continuum, and let $z_0 \in \Gamma$. Suppose that*

$$a \in C_D(f, z_0) - C_{\Gamma-E}(f, z_0),$$

and $a \notin R_D(f, z_0)$. Then either every neighborhood of z_0 contains a point of E at which f has the asymptotic value a, or the following statement holds: For any neighborhoods $N(z_0)$ and $N(a)$ of z_0 and a, respectively, there exists a closed subset of $E \cap N(z_0)$ of positive capacity at each point of which f has an asymptotic value in $N(a) - \{a\}$, and there exists a closed subset of $N(a)$ of positive linear measure each point of which is an asymptotic value of f at a point of $E \cap N(z_0)$.

An argument similar to the one in the proof of Theorem 1 can be used to prove the following global theorem:

Theorem 2. *Suppose that there exists a sequence $\{z_n\}$ of points in D, having no cluster value in D, such that $f(z_n) \to a$, and suppose that f assumes the value a at most finitely many times. Then either f has the asymptotic value a or the following statement holds: For any neighborhood $N(a)$ of a, there exists a closed subset of Γ of positive capacity at each point of which f has an asymptotic value that is in $N(a) - \{a\}$, and the set of these asymptotic values contains a closed set of positive capacity; moreover, $N(a)$ contains a closed set of positive linear measure each point of which is an asymptotic value of f.*

3. Metric properties of sets of angular limits. Now let $w = f(z)$ be a meromorphic function in $\{|z| < 1\}$, and let $\mathcal{S}$ be the Riemann surface of f over the extended w-plane. Suppose that at each point ζ of a subset E_z of $\{|z| = 1\}$ of positive measure, f has a finite angular limit a_ζ, and let $E_w = \{a_\zeta : \zeta \in E_z\}$. For each $\zeta \in E_z$ and positive number h, let $\mathcal{S}(\zeta, h)$ be the component of $\mathcal{S}$ over $\{|w - a_\zeta| < h\}$ such that if r is sufficiently near 1 ($r < 1$), then $r\zeta$ corresponds under f to a point of $\mathcal{S}(\zeta, h)$, and let $P\mathcal{S}(\zeta, h)$ be the projection of $\mathcal{S}(\zeta, h)$ onto the w-plane. The purpose of this section is to prove

Theorem 3. *Suppose that to each* $\zeta \in E_z$ *there correspond a positive number* h_ζ *and a (nondegenerate) continuum* K_ζ *in the w-plane such that* $a_\zeta \in K_\zeta$ *and* $K_\zeta \cap P\mathcal{S}(\zeta, h_\zeta) = \emptyset$. *Then* E_w *contains a closed set that does not have* $\frac{1}{2}$*-dimensional measure zero (defined in* [10, 149]).

We first prove the following special case of Theorem 3.

Theorem 4. *If* f *is a schlicht function, then* E_w *contains a closed set that does not have* $\frac{1}{2}$*-dimensional measure zero.*

Proof. Suppose that f is schlicht. By Lusin's theorem there exists a closed subset $E_z^{(1)}$ of E_z of positive measure such that the restriction of the function a_ζ ($\zeta \in E_z$) to $E_z^{(1)}$ is continuous. If $\zeta = e^{i\theta}$, let

$$S_\zeta = \left\{\zeta + \rho e^{i\varphi} : \rho > 0,\ \theta + \frac{3\pi}{4} < \varphi < \theta + \frac{5\pi}{4}\right\} \cap \{2^{-1/2} < |z| < 1\},$$

so that the boundary of S_ζ consists of a circular arc and two rectilinear segments. It is shown in [1] that there exists a closed subset $E_z^{(2)}$ of $E_z^{(1)}$ of positive measure such that f tends to a_ζ in S_ζ uniformly for $\zeta \in E_z^{(2)}$; that is, for each positive number ϵ there exists a positive number δ such that for all $\zeta \in E_z^{(2)}$

$$|f(z) - a_\zeta| < \epsilon \quad \text{if} \quad z \in S_\zeta \cap \{|z - \zeta| < \delta\}.$$

Let Δ be a component of $\cup S_\zeta$, where the union is taken over all $\zeta \in E_z^{(2)}$, such that the closed set

$$E_z^{(3)} = \{\zeta \in E_z^{(2)} : S_\zeta \subset \Delta\}$$

has positive measure. If $f(\zeta) = a_\zeta$ for $\zeta \in E_z^{(3)}$, then f is continuous on $\bar{\Delta}$. Since the boundary of Δ is a rectifiable Jordan curve, under the conformal mapping $z = g(z')$ of $\{|z'| < 1\}$ onto Δ, $E_z^{(3)}$ corresponds to a closed set E' on $\{|z'| = 1\}$ of positive measure [12, 127]. Let $F(z') = f(g(z'))$ ($|z'| < 1$) and let $B = \{F(z') : |z'| = 1\}$. Since any point of B is one endpoint of a Jordan arc in B, we can apply the theorem in [9], or apply Matsumoto's argument [6, proof of Theorem 1] directly, to see that the closed set $\{F(z') : z' \in E'\}$ cannot have $\frac{1}{2}$-dimensional measure zero. The proof of Theorem 4 is complete.

Proof of Theorem 3. By Lusin's theorem there exists a closed subset $E_z^{(1)}$ of E_z of positive measure such that the restriction of the function a_ζ ($\zeta \in E_z$) to $E_z^{(1)}$ is continuous. Associate with each $\zeta \in E_z^{(1)}$ an open disc Δ_ζ, with rational radius and center with rational real and imaginary parts, such that

$$a_\zeta \in \Delta_\zeta \subset \{|w - a_\zeta| < h_\zeta\},$$

and let $\mathcal{S}_\zeta$ be the component of $\mathcal{S}$ over Δ_ζ such that if r is sufficiently near 1 ($r < 1$), then $r\zeta$ corresponds under f to a point of $\mathcal{S}_\zeta$. Then $K_\zeta \cap P\mathcal{S}_\zeta = \emptyset$ ($\zeta \in E_z^{(1)}$). Since there are only countably many distinct $\mathcal{S}_\zeta$, there exists $\zeta_0 \in E_z^{(1)}$ such that the set

$$E_z^{(2)} = \{\zeta \in E_z^{(1)} : \mathcal{S}_\zeta = \mathcal{S}_{\zeta_0}\}$$

has positive exterior measure. Let $\mathcal{S}_0 = \mathcal{S}_{\zeta_0}$ and $\Delta_0 = \Delta_{\zeta_0}$. There exists a positive number δ and a subset $E_z^{(3)}$ of $E_z^{(2)}$ of positive exterior measure such that for every $\zeta \in E_z^{(3)}$, the diameter of K_ζ is greater than δ. There exist an open disc Δ, with diameter less than δ, and a subset $E_z^{(4)}$ of $E_z^{(3)}$ of positive exterior measure such that $\bar{\Delta} \subset \Delta_0$ and $a_\zeta \in \Delta$ if $\zeta \in E_z^{(4)}$. If $\zeta = e^{i\theta}$ and $1/\sqrt{2} < r < 1$, let

$$S(\zeta, r) = \left\{\zeta + \rho e^{i\varphi} : \rho > 0,\ \theta + \frac{3\pi}{4} < \varphi < \theta + \frac{5\pi}{4}\right\} \cap \{r < |z| < 1\},$$

so that $S(\zeta, r)$ is bounded by an arc of $\{|z| = r\}$ and two rectilinear segments. There exist r $(1/\sqrt{2} < r < 1)$ and a subset $E_z^{(5)}$ of $E_z^{(4)}$ of positive exterior measure such that

$$f(S(\zeta, r)) \subset \Delta \quad \text{if} \quad \zeta \in E_z^{(5)}.$$

Let I be a component of $\cup S(\zeta, r)$, where the union is taken over all $\zeta \in E_z^{(5)}$, such that the set

$$E_z^{(6)} = \{\zeta \in E_z^{(5)} : S(\zeta, r) \subset I\}$$

has positive exterior measure. Let $E_z^* = \bar{E}_z^{(6)}$. Then $E_z^* \subset E_z^{(1)}$, $I = \cup S(\zeta, r)$, where the union is taken over all $\zeta \in E_z^*$, and I is the interior of a rectifiable Jordan curve whose intersection with $\{|z| = 1\}$ is E_z^*. Clearly, $f(I) \subset \Delta$, and it follows that I corresponds under f to a subset of $\mathcal{S}_0$. Thus $f(I) \subset P\mathcal{S}_0$ and $f(I) \cap K_\zeta = \emptyset$ if $\zeta \in E_z^{(6)}$. The closure K of the set

$$(\text{circumference of } \Delta) \cup \Big(\bigcup_{\zeta \in E_z^{(6)}} K_\zeta\Big)$$

is connected, since every K_ζ $(\zeta \in E_z^{(6)})$ intersects the circumference of Δ. Therefore, the component U of the complement of K that contains $f(I)$ is simply connected. Every point of the compact set

$$E_w^* = \{a_\zeta : \zeta \in E_z^*\}$$

is accessible through $f(I)$. Thus a_ζ is on the boundary of U if $\zeta \in E_z^{(6)}$, and since the restriction of the function a_ζ $(\zeta \in E_z)$ to $E_z^{(1)}$ is continuous, it follows that E_w^* is contained in the boundary of U.

We suppose that E_w^* has $\frac{1}{2}$-dimensional measure zero, and we prove that this assumption leads to a contradiction. Let $w = g(w')$ be a conformal mapping of $\{|w'| < 1\}$ onto U, and let g^{-1} be the inverse mapping. Let E' be the set of points $e^{i\theta} \in \{|w'| = 1\}$ such that the radial limit $g(e^{i\theta})$ of g at $e^{i\theta}$ exists and $g(e^{i\theta}) \in E_w^*$. Since the radial-limit function $g(e^{i\theta})$ of g is a function of the first Baire class defined on an $F_{\sigma\delta}$-set [4, 308], E' is measurable [4, 303]. It follows from Theorem 4 that the measure of E' is zero. Let $z = G(z')$ be a conformal mapping of $\{|z'| < 1\}$ onto I. Since the boundary of I is a rectifiable Jordan curve, E_z^* corresponds under G to a closed set $E_{z'}^*$ on $\{|z'| = 1\}$ of positive measure. Let

$$F(z') = g^{-1}(f(G(z'))) \qquad (|z'| < 1).$$

It follows from familiar arguments that for each $\zeta \in E_z^*$, the image under g^{-1} of the curve $\{f(\rho\zeta): r < \rho < 1\}$ tends to a point of E' (as $\rho \to 1$); that is, the limit

$$\lim_{\rho \uparrow 1} g^{-1}(f(\rho\zeta))$$

exists and is a point of E'. Thus, the radial limit $F(e^{i\theta})$ of F at $e^{i\theta}$ exists for every $e^{i\theta} \in E_{z'}^*$, and if we let

$$E_{w'}^* = \{F(e^{i\theta}): e^{i\theta} \in E_{z'}^*\},$$

then $E_{w'}^* \subset E'$. This contradicts an extension of Löwner's lemma [11, 34], and the proof of Theorem 3 is complete.

4. Remark on schlicht functions. In this section we point out that a theorem of J. Dufresnoy and M. Tsuji (see [13, 347]) can be sharpened slightly.

Theorem 5. *Let $w = f(z)$ be a schlicht holomorphic function in $\{|z| < 1\}$. Let E_z be a closed subset of $\{|z| = 1\}$ of positive capacity such that for each $\zeta \in E_z$ the radial limit $f(\zeta)$ of f at ζ exists, and let $E_w = \{f(\zeta): \zeta \in E_z\}$. Then E_w contains a closed set of positive capacity.*

This theorem is readily seen to be a consequence of the theorem of Dufresnoy and Tsuji, and the following lemma:

Lemma 1. *Let $y = f(x)$ be a (finite) real-valued function of a real variable defined on a compact set E of positive capacity (as a subset of the plane), and suppose that f is a function of the first Baire class on E. Then there exists a closed subset E^* of E of positive capacity such that the restriction of f to E^* is continuous.*

Proof. There exists a sequence $\{f_n\}$ of (finite) real-valued functions defined on E such that each f_n is a function of the first class on E with an isolated range, and $f_n \to f$ uniformly on E (see [4, 280]). Let $\{a_{n,j}\}_j$ be an enumeration of the (possibly finite) range of f_n, and let

$$H_{n,j} = \{x \in E: f_n(x) = a_{n,j}\}.$$

Since $a_{n,j}$ is an isolated point of the range of f_n, $H_{n,j}$ is an F_σ-set relative to the closed set E (see [4, 280, I]), and is therefore an F_σ-set. Let $\{F_{n,j,k}\}_k$ be a sequence of closed sets such that

$$F_{n,j,k} \subset F_{n,j,k+1} \qquad (k \geqq 1), \qquad H_{n,j} = \bigcup_k F_{n,j,k}.$$

Let μ be the equilibrium distribution [13, 55] of E ($\mu(E) = 1$). Associate with each n a natural number j_n such that

$$\sum_{j=1}^{j_n} \mu(H_{n,j}) > 1 - (1/2^{n+2}),$$

and associate with each (n, j) such that $1 \leqq j \leqq j_n$ a natural number $k_{n,j}$ such that

$$\mu(F_{n,j,k_{n,j}}) > \mu(H_{n,j}) - (1/2^{n+2+j}).$$

Let

$$E^* = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{j_n} F_{n,j,k_{n,j}} .$$

Then $\mu(E^*) > \frac{1}{2}$, and it follows from a definition of capacity [13, 54] that E^* is a set of positive capacity. Since each f_n is continuous on E^*, f is continuous on E^*, and the proof of Lemma 1 is complete.

5. Refinement of Theorem 1.

Theorem 6. *Let the hypotheses of Theorem* 1 *be satisfied, and define the subset* S *of* Γ *as follows:* $z \in S$ *if and only if* $\{z\}$ *is a component of* Γ. *Then at least one of the following four statements is true:*

(i) *A Koebe continuum of* f *for the value* a *contains* z_0 .

(ii) *Every neighborhood of* z_0 *contains an asymptotic path of* f *for the value* a.

(iii) *For any neighborhoods* $N(z_0)$ *and* $N(a)$ *of* z_0 *and* a, *respectively, there exists a closed subset of* $N(z_0) \cap S$ *of positive capacity at each point of which* f *has an asymptotic value that is in* $N(a) - \{a\}$.

(iv) *For any neighborhoods* $N(z_0)$ *and* $N(a)$ *of* z_0 *and* a, *respectively, there exists a closed subset of* $N(z_0) \cap \Gamma$, *that does not have* $\frac{1}{2}$*-dimensional measure zero, at each point of which* f *has an asymptotic value that is in* $N(a) - \{a\}$.

Proof. We suppose, without loss of generality, that $a = \infty$. Suppose that (iii) is false. Then there exist neighborhoods $N^*(z_0)$ and $N^*(\infty)$ of z_0 and ∞, respectively, such that $N^*(z_0) \cap S$ contains no closed subset of positive capacity at each point of which f has an asymptotic value that is in $N^*(\infty)$. Suppose now that neither (i) nor (ii) holds. To show that (iv) holds, we let $N(z_0)$ and $N(\infty)$ be any neighborhoods of z_0 and ∞, respectively, such that $N(z_0) \subset N^*(z_0)$ and $N(\infty) \subset N^*(\infty)$, and, as in the proof of Theorem 1, we define the conformal mapping $z = g(\zeta)$ of $\{|\zeta| < 1\}$ onto D_0 whose Riemann surface is the universal covering surface of D_0 ($\overline{D}_0 \subset N(z_0)$, $\overline{f(D_0)} \subset N(\infty)$).

Note that S is a G_δ-set, since for each natural number m, the set $F_m = \{\zeta \in \Gamma$: the component of Γ that contains ζ intersects $\{|z - \zeta| = 1/m\}\}$ is closed. If $z_n \in F_m$, $n = 1, 2, \cdots$, and $z_n \to z^*$, and if $0 < h < 1/m$, then [14, p. 8, (7.1); p. 12, (9.12)] a continuum in Γ contains z^* and intersects $\{|z - z^*| = h\}$, and it follows that $z^* \in F_m$.

Since the radial-limit function $g(e^{i\theta})$ of g is a function of the first class on the $F_{\sigma\delta}$-set of radial convergence of g, the set $\{e^{i\theta}: g(e^{i\theta}) \in S\}$ is measurable [4, 303]. It must therefore have measure zero, for otherwise the argument in the proof of Theorem 1 would contradict the choice of $N^*(z_0)$ and $N^*(\infty)$.

Thus, the set $\{e^{i\theta}: g(e^{i\theta})$ exists and $g(e^{i\theta}) \notin S\}$ has positive measure, and we can apply Theorem 3 and the argument in the proof of Theorem 1 to complete the proof of Theorem 6.

6. Schlicht functions. Applying a method due to Matsumoto [6], we prove

Lemma 2. Let $\mathfrak{F}$ be the family of all Jordan arcs γ such that γ joins 0 to $\{|z| = 1\}$ and lies in $\{|z| < 1\}$ except for its endpoint on $\{|z| = 1\}$. Let $\omega_\gamma(z)$ $(\gamma \in \mathfrak{F})$ be the harmonic measure of $\{|z| = 1\} - \gamma$ with respect to

$$D_\gamma = \{|z| < 1\} - \gamma.$$

Then the limit

$$\lim_{\substack{z \to 0 \\ z \in D_\gamma}} \omega_\gamma(z) = 0$$

is uniform for all $\gamma \in \mathfrak{F}$; that is, for any positive number ϵ there exists a positive number δ such that

$$\omega_\gamma(z) < \epsilon \quad \text{if} \quad \gamma \in \mathfrak{F} \quad \text{and} \quad z \in D_\gamma \cap \{|z| < \delta\}.$$

Proof. Let ϵ be a positive number, and suppose that $|z_0| < \frac{1}{2}$. Let $\omega(z; z_0, h)$ $(0 \leqq h < \frac{1}{2})$ be the harmonic measure of

$$\{|z - z_0| = \tfrac{1}{2}\} - \{z_0 + \tfrac{1}{2}\}$$

with respect to $\{|z - z_0| < \frac{1}{2}\} - \{z_0 + x: h \leqq x < \frac{1}{2}\}$, and let $\omega(z; z_0, h)$ be extended continuously to all of $\{|z - z_0| < \frac{1}{2}\}$. Note that

$$\omega(z; z_0, h) \downarrow \omega(z; z_0, 0) \quad \text{as} \quad h \downarrow 0 \qquad (0 < |z - z_0| < \tfrac{1}{2}),$$

and the convergence is uniform on each $\{|z - z_0| = r\}$ $(0 < r < \frac{1}{2})$. Let r be such that $(0 < r < \frac{1}{2})$ $\omega(z; z_0, 0) < \epsilon/2$ if $|z - z_0| = r$, and let δ be independent of z_0 $(|z_0| < \frac{1}{2})$ and be such that $(0 < \delta < \frac{1}{2})$ $\omega(z; z_0, \delta) < \epsilon$ if $|z - z_0| = r$. By the maximum principle, $\omega(z_0; z_0, \delta) < \epsilon$. Suppose now that $\gamma \in \mathfrak{F}$ and $z_0 \in D_\gamma \cap \{|z| < \delta\}$. Let γ_0 be a subarc of γ that joins $\{|z - z_0| = \delta\}$ to $\{|z - z_0| = \frac{1}{2}\}$ and lies, except for its endpoints, in $\{\delta < |z - z_0| < \frac{1}{2}\}$. Let $\omega(z)$ be the harmonic measure of

$$\{|z - z_0| = \tfrac{1}{2}\} - \gamma_0$$

with respect to $\{|z - z_0| < \frac{1}{2}\} - \gamma_0$. In the following paragraph we prove, using Matsumoto's argument [6], that

$$\omega(z_0) \leqq \omega(z_0; z_0, \delta).$$

Then we shall have

$$\omega_\gamma(z_0) < \omega(z_0) \leqq \omega(z_0; z_0, \delta) < \epsilon,$$

and the proof of Lemma 2 will be complete.

Let $D_r = \{|z - z_0| < r\}$, $C_r = \{|z - z_0| = r\}$ $(0 < r \leqq \frac{1}{2})$. For each r satisfying $\delta < r < \frac{1}{2}$ (we later let $r \uparrow \frac{1}{2}$), let γ_r be the subarc of γ_0 that joins C_δ to C_r and lies, except for its endpoints, in $\{\delta < |z - z_0| < r\}$; and let $\{J_n\}$ be a sequence of Jordan curves in $D_{1/2}$ such that J_{n+1} is contained in the interior domain I_n of J_n, $z_0 \notin I_n \cup J_n$, and $\gamma_r = \bigcap I_n$. Then the harmonic measure $\omega_n(z)$ of $C_{1/2}$ with respect to $D_{1/2} - (I_n \cup J_n)$ and the harmonic measure $\omega^*(z)$ of $C_{1/2}$ with respect to $D_{1/2} - \gamma_r$ satisfy $\omega_n(z_0) \uparrow \omega^*(z_0)$. For a fixed n, we choose rectilinear segments

$$L_j = \{z : r_j \leqq |z - z_0| \leqq r_{j+1} , \arg (z - z_0) = \theta_j\}$$

$(j = 1, \cdots , k; \delta = r_1 < r_2 < \cdots < r_{k+1} = r)$ that are contained in I_n. Then the harmonic measure $\omega_n^*(z)$ of $C_{1/2}$ with respect to $D_{1/2} - \bigcup_{j=1}^k L_j$ satisfies $\omega_n(z_0) < \omega_n^*(z_0)$. By a lemma of Matsumoto [6, 131, Lemma 1], the harmonic measure $\omega'(z)$ of $C_{1/2}$ with respect to

$$D_{1/2} - \{z_0 + x : \delta \leqq x \leqq r\}$$

satisfies $\omega_n^*(z_0) \leqq \omega'(z_0)$. Thus $\omega_n(z_0) < \omega'(z_0)$ and $\omega^*(z_0) \leqq \omega'(z_0)$. Letting $r \uparrow \frac{1}{2}$, we see that $\omega(z_0) \leqq \omega(z_0 ; z_0 , \delta)$. The proof of Lemma 2 is complete.

We say that a subset c of $\{|z| < 1\}$ is a *crosscut at the point* $e^{i\varphi}$ of $\{|z| = 1\}$ if $c \cup \{\zeta_1 , \zeta_2\}$ is a Jordan arc and

$$\zeta_1 \in \{e^{i\theta} : \varphi - (\pi/2) < \theta < \varphi\}, \qquad \zeta_2 \in \{e^{i\theta} : \varphi < \theta < \varphi + (\pi/2)\}.$$

If $\{c_n\}$ is a sequence of crosscuts at $e^{i\varphi}$, then the symbol $c_n \to e^{i\varphi}$ means that the diameter of the set $c_n \cup \{e^{i\varphi}\}$ tends to zero as $n \to \infty$.

Theorem 7. *Let $w = f(z)$ be a schlicht holomorphic function in $\{|z| < 1\}$. Suppose that at each point $e^{i\theta}$ of a subset E_z of $\{|z| = 1\}$, f has a finite radial limit $f(e^{i\theta})$, and let*

$$E_w = \{f(e^{i\theta}) : e^{i\theta} \in E_z\}.$$

Suppose that for each $e^{i\theta} \in E_z$ there exists a sequence $\{c_n\}$ of crosscuts at $e^{i\theta}$ such that $c_n \to e^{i\theta}$ and

$$|f(c_n)|/d(f(c_n), E_w) \to 0 \qquad (n \to \infty),$$

where $|f(c_n)|$ and $d(f(c_n), E_w)$ denote the Euclidean diameter of $f(c_n)$ and distance between $f(c_n)$ and E_w, respectively. Then E_z is a set of measure zero.

Proof. For each natural number n, let E_n be the set of points $e^{i\theta}$ such that there exists a crosscut c at $e^{i\theta}$ satisfying $|c| < 1/n$ and

$$|f(c)|/d(f(c), E_w) < 1/n,$$

and let $E = \{e^{i\theta} : f(e^{i\theta}) \in \bar{E}_w\}$. Since E_n is open and E is measurable, we see by considering the set $E \cap (\bigcap_{n=1}^\infty E_n)$ that we can suppose, without loss of generality, that E_z is measurable. Suppose, contrary to the assertion of the theorem, that E_z has positive measure. Using the argument in the proof of

Theorem 4, we see that we can suppose, without loss of generality, that f is continuous on $\{|z| \leqq 1\}$ and that E_z is a closed set of positive measure.

Let $u(z)$ be the harmonic measure of E_z with respect to $\{|z| < 1\}$. Let ϵ be a positive number, and let δ be as given by Lemma 2. Let $\zeta \in E_z$, and let c be a crosscut at ζ, that separates 0 and ζ, such that $|f(c)| < \delta h$, where $h = d(f(c), E_w)$. Let ζ_1 be one endpoint of c, and let $w_1 = f(\zeta_1)$. Since $f(\{|z| = 1\})$ is arcwise connected, we can let J be a Jordan arc in $f(\{|z| = 1\})$ that joins w_1 to $\{|w - w_1| = h\}$ and lies in $\{|w - w_1| < h\}$ except for one endpoint on $\{|w - w_1| = h\}$. Let $U(w)$ be the harmonic measure of $\{|w - w_1| = h\} - J$ with respect to the set

$$\Delta = \{|w - w_1| < h\} - J.$$

Because of the choice of δ,

$$U(w) < \epsilon \quad \text{if} \quad w \in \Delta \cap \{|w - w_1| < \delta h\}.$$

Thus $u(f^{-1}(w)) \leqq U(w) < \epsilon$ if $w \in f(c)$, where f^{-1} denotes the inverse function of f; that is $u(z) < \epsilon$ if $z \in c$. Since this can be done for every $\zeta \in E_z$, $u(0) < \epsilon$. Thus $u(0) = 0$. This is a contradiction, and the proof of Theorem 7 is complete.

Remark 3. For an example illustrating Theorem 7, see [2, Theorem 2].

Remark 4. Let D be the interior domain of a Jordan curve J in the w-plane. Let E_w be a subset of J such that for each $\zeta \in E_w$ there exists a sequence $\{c_n\}$ of crosscuts (of D) at ζ such that $c_n \to \zeta$ (let the meaning here be analogous to the corresponding statement in the disc) and

$$(*) \qquad |c_n|/d(c_n, E_w) \to 0 \qquad (n \to \infty).$$

Then by Theorem 7, under conformal mapping of D onto $\{|z| < 1\}$, E_w corresponds to a set E_z on $\{|z| = 1\}$ of measure zero. I do not know whether this statement remains valid if (*) is replaced by

$$|c_n|/d(c_n, \zeta) \to 0 \qquad (n \to \infty).$$

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