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Boundary regularity for the Monge-Ampère and affine maximal surface equations

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Abstract

In this paper, we prove global second derivative estimates for solutions of the Dirichlet problem for the Monge-Ampère equation when the inhomogeneous term is only assumed to be Hölder continuous. As a consequence of our approach, we also establish the existence and uniqueness of globally smooth solutions to the second boundary value problem for the affine maximal surface equation and affine mean curvature equation.

1. Introduction

In a landmark paper [4], Caffarelli established interior $W^{2,p}$ and $C^{2,\alpha}$ estimates for solutions of the Monge-Ampère equation

$$\det D^2 u = f \tag{1.1}$$

in a domain Ω in Euclidean n-space, $\mathbf{R}^n$, under minimal hypotheses on the function f. His approach in [3] and [4] pioneered the use of affine invariance in obtaining estimates, which hitherto depended on uniform ellipticity, [2] and [19], or stronger hypotheses on the function f, [9], [13], [18]. If the function f is only assumed positive and Hölder continuous in Ω, that is $f \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, then one has interior estimates for convex solutions of (1.1) in $C^{2,\alpha}(\Omega)$ in terms of their strict convexity. When f is sufficiently smooth, such estimates go back to Calabi and Pogorelov [9] and [18]. The estimates are not genuine interior estimates as assumptions on Dirichlet boundary data are needed to control the strict convexity of solutions [4] and [18].

Our first main theorem in this paper provides the corresponding global estimate for solutions of the Dirichlet problem,

$$u = \varphi \quad \text{on } \partial\Omega. \tag{1.2}$$

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Theorem 1.1. *Let Ω be a uniformly convex domain in $\mathbf{R}^n$, with boundary $\partial\Omega \in C^3$, $\varphi \in C^3(\overline{\Omega})$ and $f \in C^\alpha(\overline{\Omega})$, for some $\alpha \in (0,1)$, satisfying* $\inf f > 0$. *Then any convex solution u of the Dirichlet problem* (1.1), (1.2) *satisfies the a priori estimate*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C, \tag{1.3}$$

where C is a constant depending on n, α, $\inf f$, $\|f\|_{C^\alpha(\overline{\Omega})}$, $\partial\Omega$ *and* φ.

The notion of solution in Theorem 1.1, as in [4], may be interpreted in the generalized sense of Aleksandrov [18], with $u = \varphi$ on $\partial\Omega$ meaning that $u \in C^0(\overline{\Omega})$. However by uniqueness, it is enough to assume at the outset that u is smooth. In [22], it is shown that the solution to the Dirichlet problem, for constant $f > 0$, may not be C^2 smooth or even in $W^{2,p}(\Omega)$ for large enough p, if either the boundary $\partial\Omega$ or the boundary trace φ is only $C^{2,1}$. But the solution is C^2 smooth up to the boundary (for sufficiently smooth $f > 0$) if both $\partial\Omega$ and φ are C^3 [22]. Consequently the conditions on $\partial\Omega$, φ and f in Theorem 1.1 are optimal.

As an application of our method, we also derive global second derivative estimates for the second boundary value problem of the affine maximal surface equation and, more generally, its inhomogeneous form which is the equation of prescribed affine mean curvature. We may write this equation in the form

$$L[u] := U^{ij} D_{ij} w = f \quad \text{in } \Omega, \tag{1.4}$$

where $[U^{ij}]$ is the cofactor matrix of the Hessian matrix D^2u of the convex function u and

$$w = [\det D^2 u]^{-(n+1)/(n+2)}. \tag{1.5}$$

The second boundary value problem for (1.4) (as introduced in [21]), is the Dirichlet problem for the system (1.4), (1.5), that is to prescribe

$$u = \varphi, \qquad w = \psi \quad \text{on } \partial\Omega. \tag{1.6}$$

We will prove

Theorem 1.2. *Let Ω be a uniformly convex domain in $\mathbf{R}^n$, with $\partial\Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\overline{\Omega})$, $\psi \in C^{3,1}(\overline{\Omega})$,* $\inf_\Omega \psi > 0$ *and $f \le 0, \in L^\infty(\Omega)$. Then there is a unique uniformly convex solution $u \in W^{4,p}(\Omega)$ (for all $1 < p < \infty$) to the boundary value problem* (1.4)–(1.6). *If furthermore $f \in C^\alpha(\overline{\Omega})$, $\varphi \in C^{4,\alpha}(\overline{\Omega})$, $\psi \in C^{4,\alpha}(\overline{\Omega})$, and $\partial\Omega \in C^{4,\alpha}$ for some $\alpha \in (0,1)$, then the solution $u \in C^{4,\alpha}(\overline{\Omega})$.*

The condition $f \le 0$, corresponding to nonnegative prescribed affine mean curvature [1] and [17], is only used to bound the solution u. It can be relaxed to $f \le \delta$ for some $\delta > 0$, but it cannot be removed completely.

The affine mean curvature equation (1.4) is the Euler equation of the functional

$$J[u] = A(u) - \int_\Omega fu, \tag{1.7}$$

where

$$A(u) = \int_\Omega [\det D^2 u]^{1/(n+2)} \tag{1.8}$$

is the affine surface area functional. The natural or variational boundary value problem for (1.4), (1.7) is to prescribe u and ∇u on $\partial\Omega$ and is treated in [21]. Regularity at the boundary is a major open problem in this case.

Note that the operator L in (1.4) possesses much stronger invariance properties than its Monge-Ampère counterpart (1.1) in that L is invariant under unimodular affine transformations in $\mathbf{R}^{n+1}$ (of the dependent and independent variables).

Although the statement of Theorem 1.1 is reasonably succinct, its proof is technically very complicated. For interior estimates one may assume by affine transformation that a section of a convex solution is of good shape; that is, it lies between two concentric balls whose radii ratio is controlled. This is not possible for sections centered on the boundary and most of our proof is directed towards showing that such sections are of good shape. After that we may apply a similar perturbation argument to the interior case [4]. To show sections at the boundary are of good shape we employ a different type of perturbation which proceeds through approximation and extension of the trace of the inhomogeneous term f. The technical realization of this approach constitutes the core of our proof. Theorem 1.1 may also be seen as a companion result to the global regularity result of Caffarelli [6] for the natural boundary value problem for the Monge-Ampère equation, that is the prescription of the image of the gradient of the solution, but again the perturbation arguments are substantially different.

The organization of the paper is as follows. In the next section, we introduce our perturbation of the inhomogeneous term f and prove some preliminary second derivative estimates for the approximating problems. We also show that the shape of a section of a solution at the boundary can be controlled by its mixed tangential-normal second derivatives. In Section 3, we establish a partial control on the shape of sections, which yields $C^{1,\alpha}$ estimates at the boundary for any $\alpha \in (0, 1)$ (Theorem 3.1). In order to proceed further, we need a modulus of continuity estimate for second derivatives for smooth data and here it is convenient to employ a lemma from [8], which we formulate in Section 4. In Section 5, we conclude our proof that sections at the boundary are of good shape, thereby reducing the proof of Theorem 1.1 to analogous perturbation considerations to the interior case [4], which we supply in Section 6 (Theorem 6.1). Finally in Section 7, we consider the application of our preceding arguments to the affine maximal surface and affine mean curvature equations, (1.4). In these cases, the global second derivative estimates follow from a variant of the condition $f \in C^{\alpha}(\overline{\Omega})$ at the boundary, namely

$$|f(x) - f(y)| \le C|x - y|, \tag{1.9}$$

for all $x \in \Omega, y \in \partial\Omega$. This is satisfied by the function w in (1.5). The uniqueness part of Theorem 1.2 is proved directly (by an argument based on concavity), and the existence part follows from our estimates and a degree argument. The solvability of (1.4)–(1.6) without boundary regularity was already proved in [21] where it was used to prove interior regularity for the first boundary value problem for (1.4).

2. Preliminary estimates

Let Ω be a uniformly convex domain in $\mathbf{R}^n$ with C^3 boundary, and φ be a C^3 smooth function on $\overline{\Omega}$. For small positive constant $t > 0$, we denote $\Omega_t = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > t\}$ and $D_t = \Omega - \overline{\Omega}_t$. For any point $x \in \overline{\Omega}$, we will use ξ to denote a unit tangential vector of $\partial\Omega_\delta$ and γ to denote the unit outward normal of $\partial\Omega_\delta$ at x, where $\delta = \operatorname{dist}(x, \partial\Omega)$.

Let u be a solution of (1.1), (1.2). By constructing proper sub-barriers we have the gradient estimate

$$\sup_{x \in \Omega} |Du(x)| \le C. \tag{2.1}$$

We also have the second order tangential derivative estimates

$$C^{-1} \le u_{\xi\xi}(x) \le C \tag{2.2}$$

for any $x \in \partial\Omega$. The upper bound in (2.2) follows directly from (2.1) and the boundary condition (1.2). For the lower bound, one requires that φ be C^3 smooth, and $\partial\Omega$ be C^3 and uniformly convex [22]. For (2.1) and (2.2) we only need f to be a bounded positive function.

In the following we will assume that f is positive and $f \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Let f_τ be the mollification of f on $\partial\Omega$, namely $f_\tau = \eta_\tau * f$, where η is a mollifier on $\partial\Omega$. If $t > 0$ is small, then for any point $x \in D_t$, there is a unique point $\hat{x} \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - \hat{x}|$ and $\gamma = (\hat{x} - x)/|\hat{x} - x|$. Let

$$f_t(x) = \begin{cases} f(x) & \text{in } \Omega_{2t}, \\ f_\tau(\hat{x}) - C\tau^{\alpha} & \text{in } D_t, \end{cases} \tag{2.3}$$

where

$$\tau = t^{\varepsilon_0}, \quad \varepsilon_0 = 1/4n.$$

We define f_t properly in the remaining part $\Omega_t - \Omega_{2t}$ such that, with a proper choice of the constant $C = C_t > 0$, $f_t \le f$ in Ω and f_t is Hölder continuous in $\overline{\Omega}$ with Hölder exponent $\alpha' = \varepsilon_0 \alpha$,

$$|f_t - f| \le C\tau^\alpha = Ct^{\alpha'} \quad \text{in} \quad \Omega,$$
$$\|f_t\|_{C^{\alpha'}(\overline{\Omega})} \le C\|f\|_{C^\alpha(\overline{\Omega})}$$

for some $C > 0$ independent of t. From (2.3), f_t is smooth in D_t,

$$|Df_t| \le C\tau^{\alpha-1}, \quad |D^2 f_t| \le C\tau^{\alpha-2}, \quad \text{and} \quad |\partial_\gamma f_t| = 0 \quad \text{in} \quad D_t. \tag{2.4}$$

Let u_t be the solution of the Dirichlet problem,

$$\begin{aligned} \det D^2 u &= f_t \quad \text{in} \quad \Omega, \\ u &= \varphi \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.5}$$

First we establish some *a priori* estimates for u_t in D_t. Note that by the local strict convexity [3] and the *a priori* estimates for the Monge-Ampère equation [18], u_t is smooth in D_t.

For any given boundary point, we may suppose it is the origin such that $\Omega \subset \{x_n > 0\}$, and locally $\partial\Omega$ is given by

$$x_n = \rho(x') \tag{2.6}$$

for some C^3 smooth, uniformly convex function ρ satisfying $\rho(0) = 0$, $D\rho(0) = 0$, where $x' = (x_1, \cdots, x_{n-1})$. By subtracting a linear function we may also suppose that

$$u_t(0) = 0, \quad Du_t(0) = 0. \tag{2.7}$$

We make the linear transformation $T : x \to y$ such that

$$\begin{aligned} y_i &= x_i/\sqrt{t}, \quad i = 1, \cdots, n-1, \\ y_n &= x_n/t, \\ v &= u_t/t. \end{aligned} \tag{2.8}$$

Then v satisfies the equation

$$\det D^2 v = t f_t \quad \text{in} \quad T(\Omega). \tag{2.9}$$

Let $G = T(\Omega) \cap \{y_n < 1\}$. In G we have $0 \le v \le C$ since v is bounded on $\partial G \cap \{y_n < 1\}$. Observe that the boundary of G in $\{y_n < 1\}$ is smooth and uniformly convex. Hence

$$|v_\gamma| \le C \quad \text{in } \partial G \cap \left\{ y_n < \frac{7}{8} \right\}.$$

From (2.2) we have

$$C^{-1} \le v_{\xi\xi} \le C \quad \text{on } \partial G \cap \left\{ y_n < \frac{7}{8} \right\}.$$

The mixed derivative estimate

$$|v_{\gamma\xi}| \le C \quad \text{on } \partial G \cap \left\{ y_n < \frac{3}{4} \right\},$$

where $v_{\xi\gamma} = \sum \xi_i \gamma_j v_{y_i y_j}$, is found for example in [8] and [13]. For the mixed derivative estimate we need $f_t \in C^{0,1}$, with

$$|Df_t| \le C\tau^{\alpha-1} t^{1/2} \le C.$$

From (2.2) and equation (2.9) we have also

$$v_{\gamma\gamma} \le C \quad \text{on } \partial G \cap \left\{ y_n < \frac{3}{4} \right\}.$$

Next we derive an interior estimate for v.

Lemma 2.1. *Let v be as above. Then*

$$|D^2 v| \le C(1+M) \quad \text{in } G \cap \left\{ y_n < \frac{1}{2} \right\}, \tag{2.10}$$

where $M = \sup_{\{y_n<7/8\}} |Dv|^2$, $C > 0$ is independent of M.

Proof. First we show $v_{ii} \le C$ for $i = 1, \cdots, n-1$. Let

$$w(y) = \rho^4 \eta \left(\frac{1}{2} v_1^2 \right) v_{11},$$

where $v_1 = v_{y_1}$, $v_{11} = v_{y_1 y_1}$, and $\rho(y) = 2 - 3y_n$ is a cut-off function, $\eta(t) = (1 - \frac{t}{M})^{-1/8}$. If w attains its maximum at a boundary point, by the above boundary estimates we have $w \le C$. If w attains its maximum at an interior point y_0, by the linear transformation

$$\widetilde{y}_i = y_i, \quad i = 2, \cdots, n,$$
$$\widetilde{y}_1 = y_1 - \frac{v_{1i}(y_0)}{v_{11}(y_0)} y_i,$$

which leaves w unchanged, one may suppose $D^2 v(y_0)$ is diagonal. Then at y_0 we have

$$0 = (\log w)_i = 4\frac{\rho_i}{\rho} + \frac{\eta_i}{\eta} + \frac{v_{11i}}{u_{11}}, \tag{2.11}$$

$$0 \ge (\log w)_{ii} = 4\left(\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2} \right) + \left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} \right) + \left(\frac{v_{11ii}}{v_{11}} - \frac{v_{11i}^2}{v_{11}^2} \right). \tag{2.12}$$

Inserting (2.11) into (2.12) in the form $\frac{\rho_i}{\rho} = -\frac{1}{4}\left(\frac{\eta_i}{\eta} + \frac{v_{11i}}{v_{11}} \right)$ for $i = 2, \cdots, n$ and $\frac{v_{11i}}{v_{11}} = -(4\frac{\rho_i}{\rho} + \frac{\eta_i}{\eta})$ for $i = 1$, we obtain

$$\begin{aligned} 0 &\ge v^{ii} (\log w)_{ii} \\ &\ge v^{ii} \left(\frac{\eta_{ii}}{\eta} - 3\frac{\eta_i^2}{\eta^2} \right) - 36 v^{11} \frac{\rho_1^2}{\rho^2} + v^{ii} \frac{v_{11ii}}{v_{11}} - \frac{3}{2} \sum_{i=2}^{n} v^{ii} \frac{v_{11i}^2}{v_{11}^2}, \end{aligned} \tag{2.13}$$

where (v^{ij}) is the inverse matrix of (v_{ij}).

It is easy to verify that

$$v^{ii}\left(\frac{\eta_{ii}}{\eta}-3\frac{\eta_i^2}{\eta^2}\right)\geq\frac{C}{M}v_{11}-\frac{C}{M},$$

where $C>0$ is independent of M. Differentiating the equation

$$\log\det D^2v=\log(tf_t)$$

twice with respect to y_1, and observing that $|\partial_1 f_t|\leq C\tau^{\alpha-1}t^{1/2}\leq C$ and $|\partial_1^2 f_t|\leq C\tau^{\alpha-2}t\leq C$ after the transformation (2.8), we see the last two terms in (2.13) satisfy

$$v^{ii}\frac{v_{11ii}}{v_{11}}-\frac{3}{2}\sum_{i=2}^{n}v^{ii}\frac{v_{11i}^2}{v_{11}^2}\geq-\frac{1}{v_{11}}(\log f_t)_{11}\geq -C.$$

We obtain

$$\rho^4 v_{11}\leq C(1+M).$$

Hence $v_{ii}\leq C$ for $i=1,\cdots,n-1$ in $G\cap\{y_n<\frac{1}{2}\}$.

Next we show that $v_{nn}\leq C$. Let $w(y)=\rho^4\eta\left(\frac{1}{2}v_n^2\right)v_{nn}$ with the same ρ and η as above. If w attains its maximum at a boundary point, we have $v_{nn}\leq C$ by the boundary estimates. Suppose w attains its maximum at an interior point y_0. As above we introduce a linear transformation

$$\begin{aligned}\widetilde{y}_i&=y_i,\quad i=1,\cdots,n-1,\\ \widetilde{y}_n&=y_n-\frac{v_{in}(y_0)}{v_{nn}(y_0)}y_i,\end{aligned}$$

which leaves w unchanged. Then

$$w(y)=(2-\alpha_i y_i)^4\eta\left(\frac{1}{2}v_n^2\right)v_{nn}$$

and $D^2v(y_0)$ is diagonal. By the estimates for v_{ii}, $i=1,\cdots,n-1$, the constants α_i are uniformly bounded. Therefore the above argument applies. □

Scaling back to the coordinates x, we therefore obtain

$$\partial_\xi^2 u_t(x)\leq C\quad\text{in } D_{t/2}, \tag{2.14a}$$

$$|\partial_\xi\partial_\gamma u_t(x)|\leq C/\sqrt{t}\quad\text{in } D_{t/2}, \tag{2.14b}$$

$$\partial_\gamma^2 u_t(x)\leq C/t\quad\text{in } D_{t/2}, \tag{2.14c}$$

where C is independent of t, ξ is any unit tangential vector to $\partial\Omega_\delta$ and γ is the unit normal to $\partial\Omega_\delta$ ($\delta=\text{dist}(x,\partial\Omega)$), and $\partial_\xi\partial_\gamma u=\sum\xi_i\gamma_j u_{x_ix_j}$.

The proof of Lemma 2.1 is essentially due to Pogorelov [18]. Here we used a different auxiliary function, from which we obtain a linear dependence of $\sup|D^2v|$ on M, which will be used in the next section. The linear dependence can also be derived from Pogorelov's estimate by proper coordinate changes. Taking $\rho = -u$ in the auxiliary function w, we have the following estimate.

Corollary 2.1. *Let u be a convex solution of $\det D^2u = f$ in Ω. Suppose $\inf_\Omega u = -1$, and either $u = 0$ or $|D^2u| \le C_0(1+M)$ on $\partial\Omega$. Then*

$$|D^2u|(x) \le C(1+M), \quad \forall\ x \in \{u < -\frac{1}{2}\}, \tag{2.15}$$

where $M = \sup_{\{u<0\}} |Du|^2$, and C is independent of M.

Next we derive some estimates on the level sets of the solution u to (1.1), (1.2). Denote

$$S^0_{h,u}(y) = \{x \in \overline{\Omega} \mid u(x) < u(y) + Du(y)(x-y) + h\},$$
$$S_{h,u}(y) = \{x \in \overline{\Omega} \mid u(x) = u(y) + Du(y)(x-y) + h\}.$$

We will write $S_{h,u} = S_{h,u}(y)$ and $S^0_{h,u} = S^0_{h,u}(y)$ if no confusion arises. The set $S^0_{h,u}(y)$ is the section of u at center y and height h [4].

Lemma 2.2. *There exist positive constants $C_2 > C_1$ independent of h such that*

$$C_1 h^{n/2} \le |S^0_{h,u}(y)| \le C_2 h^{n/2} \tag{2.16}$$

for any $y \in \partial\Omega$, where $|\mathcal{K}|$ denotes the Lebesgue measure of a set $\mathcal{K}$.

Proof. It is known that for any bounded convex set $\mathcal{K} \subset \mathbf{R}^n$, there is a unique ellipsoid E containing $\mathcal{K}$ which achieves the minimum volume among all ellipsoids containing $\mathcal{K}$ [3]. E is called the minimum ellipsoid of $\mathcal{K}$. It satisfies $\frac{1}{n}(E - x_0) \subset \mathcal{K} - x_0 \subset E - x_0$, where x_0 is the center of E.

Suppose the origin is a boundary point of Ω, $\Omega \subset \{x_n > 0\}$, and locally $\partial\Omega$ is given by (2.6). By subtracting a linear function we also suppose u satisfies (2.7). Let E be the minimum ellipsoid of $S^0_{h,u}(0)$. Let v be the solution to $\det D^2u = \inf_\Omega f_t$ in $S^0_{h,u}$, $v = h$ on $\partial S^0_{h,u}$. If $|E| > Ch^{n/2}$ for some large $C > 1$, we have $\inf v < 0$. By the comparison principle, we obtain $\inf u \le \inf v < 0$, which is a contradiction to (2.7). Hence the second inequality of (2.16) holds.

Next we prove the first inequality. Denote

$$a_h = \sup\{|x'| \mid x \in S_{h,u}(0)\}, \tag{2.17}$$
$$b_h = \sup\{x_n \mid x \in S_{h,u}(0)\}. \tag{2.18}$$

If the first inequality is not true, $|S^0_{h,u}| = o(h^{n/2})$ for a sequence $h \to 0$. By (2.2), we have $S^0_{h,u} \supset \{x \in \partial\Omega \mid |x| < Ch^{1/2}\}$ for some $C > 0$. Hence $b_h = o(h^{1/2})$. By (2.2) we also have $u(x) \ge C_0|x|^2$ for $x \in \partial\Omega$. Hence if $a_h \le Ch^{1/2}$ for some $C > 0$, the function

$$v = \delta_0(|x'|^2 + \left(\frac{h^{1/2}}{b_h} x_n)^2\right) + \varepsilon x_n$$

for some small $\delta_0 > 0$, is a sub-solution to the equation $\det D^2 u = f$ in $S^0_{h,u}$ satisfying $v \le u$ on $\partial S^0_{h,u}$, where $\varepsilon > 0$ can be arbitrarily small. It follows by the comparison principle that $v_n(0) \le u_n(0) = 0$, which contradicts $v_n(0) = \varepsilon > 0$.

Hence, $a_h/h^{1/2} \to \infty$ as $h \to 0$. Let $x_0 = (x_{0,1}, 0, \cdots, 0, x_{0,n})$ (after a rotation of the coordinates x') be the center of E, where E is the minimum ellipsoid of $S^0_{h,u}$. Make the linear transformation

$$y_1 = x_1 - (x_{0,1}/x_{0,n})x_n, \qquad y_i = x_i \quad i = 2, \cdots, n$$

such that the center of E is moved to the x_n-axis. Let $E' = \{\sum_{i=1}^{n-1}(x_i/a_i)^2 < 1\}$ be the projection of E on $\{x_n = 0\}$. Since the origin $0 \in S^0_{h,u}$ and the center of E is located on the x_n-axis, one easily verifies that $a_1 \cdots a_n \le C|S^0_{h,u}| = o(h^{n/2})$, where $a_n = x_{0,n}$. Note that $x_{0,1} \le a_h$ and $x_{0,n} \le b_h \le 2n x_{0,n}$. By the uniform convexity of $\partial\Omega$,

$$\frac{x_{0,n}}{x_{0,1}} \ge C\frac{b_h}{a_h} \ge C a_h \gg h^{1/2}.$$

Hence after the above transformation, the boundary part $\partial\Omega \cap S^0_{h,u}$ is still uniformly convex. Also, as above, the function $v = \delta_0 \sum_{i=1}^{n}(\frac{h^{1/2}}{a_i} y_i)^2 + \varepsilon y_n$ is a sub-solution, and we reach a contradiction. $\square$

Next we show that the shape of the level set $S_{h,u}$ can be controlled by the mixed derivatives $u_{\xi\gamma}$ on $\partial\Omega$.

LEMMA 2.3. *Let u be the solution of* (1.1), (1.2). *Suppose as above that $\partial\Omega$ is given by* (2.6) *and u satisfies* (2.7). *If*

$$|\partial_{\xi\gamma} u(x)| \le K \quad \text{on} \quad \partial\Omega \tag{2.19}$$

for some $K \ge 1$, then

$$a_h \le CKh^{1/2}, \tag{2.20}$$

$$b_h \ge Ch^{1/2}/K \tag{2.21}$$

for some $C > 0$ independent of u, K and h.

Proof. We need only to prove (2.20) and (2.21) for small $h > 0$. Suppose the supremum a_h is attained at $x_h = (a_h, 0, \cdots, 0, c_h) \in S_{h,u}(0)$. Let $\ell = S_{h,u} \cap \{x_2 = \cdots = x_{n-1} = 0\}$. Then $\ell \subset \overline{\Omega}$ and it has an endpoint $\hat{x} = (\hat{x}_1, 0, \cdots, 0, \hat{x}_n) \in \partial\Omega$ with $\hat{x}_1 > 0$ such that $u(\hat{x}) = h$. If $a_h = \hat{x}_1$, by (2.2) we have $\hat{x}_1 \le Ch^{1/2}$, and by the upper bound in (2.16), $b_h \ge Ch^{1/2}$. Hence (2.20) and (2.21) hold.

When $a_h > \hat{x}_1$, let $\xi = (\xi_1, 0, \cdots, 0, \xi_n)$ be the unit tangential vector of $\partial\Omega$ at $\hat{x}$ in the x_1x_n-plane, and $\zeta = (\zeta_1, 0, \cdots, 0, \zeta_n)$ be the unit tangential vector of the curve ℓ at $\hat{x}$. Then all ξ_1, ξ_n, ζ_1, and $\zeta_n > 0$. Let θ_1 denote the angle between ξ and ζ at $\hat{x}$, and θ_2 the angle between ξ and the x_1-axis. By (2.2) and (2.19),

$$|\partial_\gamma u(\hat{x})| \le CK|\hat{x}|, \qquad |\partial_\xi u(\hat{x})| \ge C|\hat{x}|.$$

Hence

$$\frac{C}{K} \le \theta_1 < \pi - \frac{C}{K}. \tag{2.22}$$

But since all ξ_1, ξ_n, ζ_1, and $\zeta_n > 0$, we have $\theta_1 + \theta_2 < \frac{\pi}{2}$. Note that by (2.2) and (2.16), $a_h \ge Ch^{1/2}$ and $b_h \le Ch^{1/2}$. We obtain

$$a_h \le \hat{x}_1 + b_h/\mathrm{tg}\,(\theta_1+\theta_2) \le CKh^{1/2}, \;\; b_h \ge a_h \mathrm{tg}\,(\theta_1+\theta_2) \ge Ch^{1/2}/K. \tag{2.23}$$

Lemma 2.3 is proved. □

Lemma 2.3 shows that the shape of the sections $S^0_{h,u}(y)$ at boundary points y can be controlled by the mixed second order derivatives of u. If $S^0_{h,u}$ has a *good shape* for small $h > 0$, namely if the inscribed radius r is comparable to the circumscribed radius R,

$$R \le C_0 r \tag{2.24}$$

for some constant C_0 under control, the perturbation argument [4] applies and one infers that $|D^2 u(0)|$ is bounded. See Section 6. It follows that $u \in C^{2,\alpha}(\overline{\Omega})$ by [2], [19]. Estimation of the mixed second order derivatives on the boundary will be the key issue in the rest of the paper.

3. Mixed derivative estimates at the boundary

For $t > 0$ small let u_t be a solution of (2.5) and assume (2.6) (2.7) hold. As in Section 2 we use ξ and γ to denote tangential (parallel to $\partial\Omega$) and normal (vertical to $\partial\Omega$) vectors.

Lemma 3.1. *Suppose*

$$|\partial_\xi \partial_\gamma u_t| \le K \quad \text{on} \;\; \partial\Omega \tag{3.1}$$

for some $1 \le K \le Ct^{-1/2}$. *Then*

$$\partial_i^2 u_t \le C \qquad \text{in } D_t \cap \{x_n < t/8\}, \quad i = 1, \cdots, n-1, \tag{3.2a}$$

$$|\partial_i \partial_n u_t| \le CK \qquad \text{in } D_t \cap \{x_n < t/8\}, \tag{3.2b}$$

$$\partial_n^2 u_t \le CK^2 \qquad \text{in } D_t \cap \{x_n < t/8\}, \tag{3.2c}$$

where $C > 0$ *is a constant independent of* K *and* t.

Proof. By (2.14c), estimate (3.2a) is equivalent to (2.14a). The estimate (3.2b) follows from (3.2a) and (3.2c) by the convexity of u_t. By (2.2), (3.1), and equation (2.5), we obtain (3.2c) on the boundary $\partial\Omega$. By (2.15), the interior part of (3.2c) will follow if we have an appropriate gradient estimate for u_t in the set $S^0_{h,u_t}(0)$.

Let $h > 0$ be the largest constant such that $S^0_{h,u_t}(0) \subset D_{t/2}$ and u_t satisfies (2.14) in $\{u_t < h\}$. By the Lipschitz continuity of u, we have $h \leq Ct$. Let $v(y) = u_t(x)/h$, where $y = x/\sqrt{h}$. Then v satisfies the equation

$$\det D^2 v = f_t \quad \text{in} \quad \widetilde{\Omega} = \{x/\sqrt{h} \mid x \in \Omega\}. \tag{3.3}$$

By (2.16),

$$C_1 \leq |\{v < 1\}| \leq C_2. \tag{3.4}$$

We claim

$$|\partial_n v(y)| \leq CK \quad \forall \;\; y \in \left\{v < \frac{1}{2}\right\}. \tag{3.5}$$

If (3.5) holds, by Corollary 2.1 (with the auxiliary function $w(y) = (\frac{1}{2} - v)^4 \cdot \eta(\frac{1}{2}v_n^2)v_{nn}$ in the proof of Lemma 2.1), we obtain

$$\partial^2_{y_n} v \leq CK^2 \quad \text{in} \;\; \{v < 1/4\}.$$

In the above estimate we have used

$$\partial^2_{y_n} \log f_t(y) = h\, \partial^2_{x_n} \log f_t(x) \leq C \quad \text{in } \{x_n < t\}$$

by our definition of f_t in (2.3). Changing back to the x-coordinates we obtain (3.2c).

By convexity it suffices to prove (3.5) for $y \in \partial\{v < \frac{1}{2}\}$. Let $\overline{a}_h = h^{-1/2}a_h$, where a_h is as defined in (2.17). If $\overline{a}_h \leq C$, by (2.16), the set $\{v < 1\}$ has a good shape. By (2.1) and (2.2), the gradient estimate in $\{v < \frac{1}{2}\}$ is obvious.

If $\overline{a}_h \gg 1$ ($\overline{a}_h \leq CK$ by (2.20)), we divide $\partial\{v < \frac{1}{2}\}$ into two parts. Let $\partial_1\{v < \frac{1}{2}\}$ denote the set $y \in \{v = \frac{1}{2}\} \cap \widetilde{\Omega}$ such that the outer normal line of $\{v < \frac{1}{2}\}$ at y intersects $\{v = 1\} = \{y \in \widetilde{\Omega} \mid v(y) = 1\}$, and $\partial_2\{v < \frac{1}{2}\}$ denote the rest of $\partial\{v < \frac{1}{2}\}$, which consists of the boundary part $\{v < \frac{1}{2}\} \cap \partial\widetilde{\Omega}$ and the points $y \in \{v = \frac{1}{2}\}$ at which the outer normal line of $\{v < \frac{1}{2}\}$ intersects a boundary point in $\{v < 1\} \cap \partial\widetilde{\Omega}$.

Observe that for any $y \in \{v < 1\} \cap \partial\widetilde{\Omega}$, (3.5) holds by (3.1) since $Dv(0) = 0$. By convexity we obtain (3.5) on the part $\partial_2\{v < \frac{1}{2}\}$.

To verify (3.5) on $\partial_1\{v < \frac{1}{2}\}$, it suffices to show that

$$\text{dist}\left(\{v = 1\}, \left\{v < \frac{1}{2}\right\}\right) > \frac{C}{K}. \tag{3.6}$$

By the convexity of v we then have $|Dv| < CK$ on $\partial_1\{v < \frac{1}{2}\}$. From the last paragraph, $\text{dist}(\{v = 1\} \cap \partial\Omega, \{v < \frac{1}{2}\}) > C/K$.

We will construct appropriate sub-barriers to prove (3.6). Our sub-barrier will be a function defined on a cylinder $U = E \times (-a_n, a_n) \subset \mathbf{R}^n$ (after a rotation of axes), where $E = \sum_{i=1}^{n-1} x_i^2/a_i^2 < 1$ is an ellipsoid in $\mathbf{R}^{n-1}$.

First we derive a gradient estimate for such a sub-barrier. Suppose $a_1 \cdots a_n = 1$. Let w be the convex solution to $\det D^2 w = 1$ in U with $w = 0$ on ∂U. By making the linear transformation $\widetilde{y}_i = y_i/a_i$ for $i = 1, \cdots, n$ such that $U = \{|\widetilde{y}'| < 1\} \times (-1, 1)$, where $\widetilde{y}' = (\widetilde{y}_1, \cdots, \widetilde{y}_{n-1})$, we have the estimate $C_1 \le -\inf_U w \le C_2$ for two constants $C_2 > C_1 > 0$ depending only on n. By constructing proper sub-barriers [4], we see that w is Hölder continuous in $\widetilde{y}$. Hence for any $C_0 > 0$, by the convexity of w, the gradient estimate $C_1 < |D_{\widetilde{y}} w| < C_2$ on $\{w < -C_0\}$, for different $C_2 > C_1 > 0$ depends only on n and C_0. Changing back to the variable y, we obtain

$$C_1 a_n^{-1} \le |D_{y_n} w| \le C_2 a_n^{-1} \tag{3.7}$$

at any point $y \in \{w = -C_0\}$ such that $y' \in \frac{1}{2}E$. If $a := a_1 \cdots a_n \ne 1$, then by a dilation one sees that (3.7) holds with a_n replaced by a_n/a.

In order to use (3.7) to verify (3.5) on the part $\partial_1\{v < \frac{1}{2}\}$, we first show that

$$\inf_{|\nu|=1} \sup_{y,z \in \{v<1\}} \nu \cdot (y - z) \ge C/K, \tag{3.8}$$

namely the in-radius of the convex set $\{v < 1\}$ is greater than C/K, where $\nu \cdot y$ denotes the inner product in $\mathbf{R}^n$. To prove (3.8) we first observe that by (2.2),

$$B_{r_1}(0) \cap \partial\widetilde{\Omega} \subset \{v < 1\} \cap \partial\widetilde{\Omega} \subset B_{r_2}(0) \cap \partial\widetilde{\Omega}$$

for some $r_1, r_2 > 0$ independent of t. Let $\widetilde{y} = (0, \cdots, 0, \widetilde{y}_n)$ be a point on the positive x_n-axis such that $v(\widetilde{y}) = 1$. To prove (3.8), it suffices to show that

$$\widetilde{y}_n \ge C/K. \tag{3.9}$$

Let $\overline{y} = (\overline{a}, 0, \cdots, 0, \overline{c}) \in \partial\widetilde{\Omega}$ be an arbitrary point such that $v(\overline{y}) = 1$. Then similarly to (2.22), the angle at $\overline{y}$ of the triangle with vertices $\overline{y}, \widetilde{y}$ and the origin is larger than C/K. Hence $\widetilde{y}_n \ge Cr_1/K \ge C/K$. Hence (3.9) holds.

With (3.9), we can now prove (3.6). For any given point $\hat{y} \in \{v = 1\} \cap \partial\widetilde{\Omega}$, let P denote the tangent plane of $\{v = 1\}$ at $\hat{y}$. Choose a new coordinate system z such that $\hat{y}$ is the origin, $P = \{z_n = 0\}$ and the inner normal of $\{v < 1\}$ is the positive z_n-axis. Let S' denote the projection $\{v < 1\}$ on P. By (3.4) and (3.8) we have the volume estimate

$$|S'| \le CK. \tag{3.10}$$

Let $E \subset P$ be the minimum ellipsoid of S' with center z_0, and $E_0 \subset P$ be the translation of E such that its center is located at the origin $z = 0$ (the point $\hat{y}$). Then we have $S' \subset E \subset 4nE_0$. The latter inclusion is true when E is a ball and it is also invariant under linear transformations.

Let $U = \beta E_0 \times (0, 2/K)$ and $U_{1/2} = \beta E_0 \times (0, 1/K)$. Let w be the solution of $\det D^2 w = \sup_\Omega f_t$ in U such that $w = 1$ on ∂U. We may choose the constant $\beta \geq 8n$ such that $2E \subset \beta E_0$ and $\inf_U w \leq -1$ (note that since $|U| = 2\beta^{n-1}|E_0|/K$, β can be very large if $|E_0| \ll K$). Then by convexity we see that $w \leq 0 \leq v$ on $\{z_n = 1/K\} \cap \{v < 1\}$.

To verify that $w < v$ on $\partial\widetilde{\Omega} \cap \{v < 1\}$, we observe that either the distance from the plane $P = \{z_n = 0\}$ to the set $\{v < 1\} \cap \partial\widetilde{\Omega}$ is larger than C/K, or the angle θ_1 between the plane P and the plane $\{y_n = 0\}$ satisfies (2.22). In the former case, by (3.7) (with $a_n = 1/K$) we have $w \leq v$ on $\partial\widetilde{\Omega} \cap U_{1/2}$ if β is chosen large, independent of K. In the latter case, noting that the boundary part $\partial\widetilde{\Omega} \cap \{v < 1\}$ is very flat and that $|\partial_\xi v| \leq C$, where ξ is tangential to $\partial\widetilde{\Omega}$, by (3.7), we also have $w \leq v$ on $\partial\widetilde{\Omega} \cap U_{1/2}$. Therefore in both cases, $w \leq v$ on the boundary of the set $\{v < 1\} \cap U_{1/2}$.

By the comparison principle, it follows that $w \leq v$ in $\{v < 1\} \cap U_{1/2}$. By the gradient estimate (3.7) for w, it follows that the distance from $\{v < \frac{1}{2}\}$ to $\{v = 1\}$ is greater than C/K. This completes the proof. □

LEMMA 3.2. *Suppose* $|D^2 u_t| \leq K^2$ *in* $D_{t/8}$. *Then*

$$|D^2 u_t| \leq CK^2 \quad \text{in } D_{2t} \tag{3.11}$$

where $C > 0$ *is a constant independent of* K *and* t.

Proof. Fix a point $x_0 \in D_{2t} - D_{t/8}$. For any small $h > 0$, there exists a linear function $x_{n+1} = a \cdot x + b$ such that $a \cdot x_0 + b = u(x_0) + h$ and x_0 is the center of the minimum ellipsoid E of the section $\hat{S}_h := \{x \in \Omega \mid u(x) < a \cdot x + b\}$ [5], where a and b depend on h. Let h be the largest constant such that $\hat{S}_{h-\varepsilon} \subset\subset \Omega$ for any $\varepsilon > 0$.

Make a linear transformation $y = Tx$ such that $T(E)$ is a unit ball. Let $v = |T|^{2/n}(u - a \cdot x - b)$. Then v satisfies the equation $\det D^2 v = f_t(T^{-1}(y))$ in $T(\hat{S}_h)$ and $v = 0$ on the boundary $\partial T(\hat{S}_h)$. We have $C_1 \leq -\inf v \leq C_2$ for two constants $C_2 > C_1 > 0$ depending only on n, the upper and lower bounds of f_t. Let us assume simply that $\inf v = -1$.

Since f_t is Hölder continuous with exponent $\alpha' = \varepsilon_0 \alpha$, both before and after the transformation, by the Schauder-type estimate [4], we have $u \in C^{2,\alpha'}(T(\hat{S}_h))$. That is for any $\delta > 0$, there exist $C_2 > C_1 > 0$ depending on n, δ, $\alpha' \in (0, 1)$, the upper and lower bounds of f_t, and $\|f_t\|_{C^{\alpha'}(\overline{\Omega})}$, but independent of h, such that

$$C_1 I \leq \{D_y^2 v(y)\} \leq C_2 I \tag{3.12}$$

for any $y \in \{v < -\delta\}$, where I is the unit matrix. Note that (3.12) implies that the largest eigenvalue of $\{D_y^2 v\}$ is controlled by the smallest one.

Let $\delta = 1/64$. Since $\inf v = -1$, by convexity, $v(y_0) \leq -\frac{1}{2}$, where $y_0 = T(x_0)$. Since $\operatorname{dist}(x_0, \partial\Omega) \leq 2t$, by convexity, there exists a point $x^* \in D_{t/8}$

such that $v(y^*) \le -1/64$, where $y^* = T(x^*)$. From (3.12) we have

$$|D_y^2 v(y_0)| \le C|D_y^2 v(y^*)|.$$

Changing back to the x-variables, we obtain (3.11). □

The next lemma is simple but is important for our proof.

Lemma 3.3. *Suppose*

$$|D^2 u_t| \le C_0 t^{\beta-1} \quad \text{in} \quad D_{2t}, \tag{3.13}$$

where $\beta \in [0,1]$ *is a constant. Then in* $D_{t/2}$,

$$|u_t - u|(x) \le Ct^{\beta+\alpha'} \mathrm{dist}(x, \partial\Omega), \tag{3.14}$$

where $\alpha' = \varepsilon_0 \alpha$, C *is independent of* t.

Proof. By our construction we have $f_t \le f$ in Ω. Hence $u_t \ge u$ in Ω. Let

$$z = \begin{cases} -4t^{\beta+\alpha'} d_x + t^{\beta+\alpha'-1} d_x^2 & \text{if } d_x < 2t, \\ -4t^{\beta+\alpha'+1} & \text{if } d_x \ge 2t, \end{cases} \tag{3.15}$$

where $d_x = \mathrm{dist}(x, \partial\Omega)$. For any point $x \in D_{2t}$, choose the coordinates properly such that $D^2 z$ is diagonal with $z_{11} \le \cdots \le z_{nn}$. Then

$$\det D^2(u_t + C'z) \ge \det D^2 u_t + C'(\det \widetilde{D}^2 u_t) z_{nn},$$

where $\widetilde{D}^2 u = (u_{ij})_{i,j=1}^{n-1}$. From (3.15) we have $z_{nn} \ge Ct^{\beta+\alpha'-1}$. By (3.13), $\det \widetilde{D}^2 u_t \ge Ct^{1-\beta}$. Hence

$$\det D^2(u_t + C'z) \ge f_t + C't^{\alpha'} \ge f$$

if C' is chosen large. By the comparison principle, we obtain (3.14). □

In Lemma 3.2 we assume that $f \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. This condition is not satisfied in the proof of Theorem 1.2. For that proof, the trace of f on $\partial\Omega$ is smooth and we use f itself, rather than the mollification f_τ, in (2.3). We will need the following alternative of Lemma 3.3 in this case.

Lemma 3.3′. *Suppose* f *satisfies*

$$|f(x) - f(y)| \le C|x - y| \quad \forall \quad x \in \Omega, y \in \partial\Omega. \tag{3.16}$$

Then

$$|u - u_t|(x) \le Ct^{1+1/n} \mathrm{dist}(x, \partial\Omega) \tag{3.17}$$

for some constant $C > 0$ *independent of* t.

Proof. Let

$$z = \begin{cases} -4t^{1+1/n}d_x + t^{1/n}d_x^2 & \text{if } d_x < 2t, \\ -4t^{2+1/n} & \text{if } d_x \geq 2t. \end{cases} \tag{3.18}$$

Now,

$$\det D^2 z \geq Ct^n \quad \text{in} \quad D_{2t}$$

for some $C > 0$. Under assumption (3.16), we have $|f_t - f| \leq Ct$. Hence

$$\det D^2(u_t + Cz) \geq \det D^2 u_t + C(\det D^2 u_t)^{(n-1)/n}(\det D^2 z)^{1/n} \geq f \quad \text{in} \quad \Omega.$$

Similarly, $\det D^2(u + Cz) \geq \det D^2 u_t$ in Ω. It follows that

$$|u - u_t|(x) \leq C|z(x)|.$$

Hence (3.17) holds. □

Let $\theta = \alpha/16n$ if $f \in C^\alpha$, or $\theta = 1/16n$ if f satisfies (3.16), and $t' = t^{1+\theta}$. Let $u_{t'}$ be the corresponding solution of (2.5). By our construction of f_t, we may assume that $f_{t'} \geq f_t$ so that $u_{t'} \leq u_t$. Obviously Lemma 3.3 holds with u replaced by $u_{t'}$.

LEMMA 3.4. *Suppose u_t satisfies* (3.1). *Then*

$$|\partial_\xi \partial_\gamma u_{t'}| \leq CK \quad \text{on} \quad \partial\Omega, \tag{3.19}$$

where C is independent of K and t.

Proof. Suppose the origin is a boundary point and (2.6), (2.7) hold. For any $(x', s) \in \Omega$, where $s = t'/8$,

$$\begin{aligned} \partial_i u_t(x', s) &= \partial_i u_t(x', \rho(x')) + \partial_n \partial_i u_t(x', s_1)(s - \rho(x')), \quad i < n, \\ \partial_i \varphi(x', s) &= \partial_i \varphi(x', \rho(x')) + \partial_n \partial_i \varphi(x', s_2)(s - \rho(x')), \end{aligned} \tag{3.20}$$

for some $s_1, s_2 \in (\rho(x'), s)$. Since $Du_t(0) = 0$, by (3.1) we have $|\partial_\gamma u_t(x', \rho(x'))| \leq CK|x'|$. Hence

$$|\partial_n u_t(x', \rho(x'))| \leq CK|x'|.$$

Since $\partial_\xi(u_t - \varphi) = 0$,

$$|\partial_i(u_t - \varphi)|(x', \rho(x')) \leq C|x'|\,|\partial_n(u_t - \varphi)| \leq CK|x'|^2 \leq CKs.$$

By (3.2b) and (3.20),

$$|\partial_i(u_t - \varphi)(x', s)| \leq CKs. \tag{3.21}$$

Let $\beta \in [0, 1]$ such that $K = t^{(\beta-1)/2}$ (by (2.14c) we may assume $\beta \leq 1$). Then by (3.1) and Lemmas 3.1 and 3.2, $|D^2 u_t| \leq Ct^{\beta-1}$ in D_{2t}. Hence by Lemma 3.3,

$$|u_t - u_{t'}| \leq Ct^{\beta+\alpha'} s \quad \text{on} \quad \Omega \cap \{x_n = s\}.$$

By (3.2a),

$$\partial_i^2 u_t \le C \quad \text{and} \quad \partial_i^2 u_{t'} \le C \quad \text{on} \;\; \Omega \cap \{x_n = s\}.$$

Hence

$$|\partial_i(u_t - u_{t'})| \le C \sup_{\{x_n=s\}} |u_t - u_{t'}|^{1/2} \le C(t^{\beta+\alpha'} s)^{1/2} \qquad \text{on} \;\; \Omega \cap \{x_n = s\}.$$

Recalling that $s = t'/8 = t^{(1+\theta)}/8$, we obtain

$$\tag{3.22} \begin{aligned} |\partial_i(u_t - u_{t'})| &\le C t^{(\beta+\alpha'-1-\theta)/2} s \\ &\le C t^{(\beta-1)/2} s = CKs \qquad \text{on} \;\; \Omega \cap \{x_n = s\}. \end{aligned}$$

From (3.21) and (3.22),

$$\tag{3.23} |\partial_i(\varphi - u_{t'})| \le CKs \quad \text{on} \;\; \{x_n = s\}.$$

Next we estimate $\partial_n u_{t'}$ on $\{x_n = s\}$, first considering the point $(0, s)$. By convexity and (3.14),

$$\begin{aligned} \partial_n u_{t'}(0,s) &\le \frac{1}{s}[u_{t'}(0,2s) - u_{t'}(0,s)] \\ &\le \frac{1}{s}[u_t(0,2s) - u_t(0,s)] + Cs^{\beta+\alpha'} \\ &\le \partial_n u_t(0,2s) + Cs^{\beta+\alpha'}. \end{aligned}$$

By Lemma 3.1, $\partial_n^2 u_t \le CK^2$. Hence $\partial_n u_t(0,2s) \le \partial_n u_t(0) + CK^2 s = CK^2 s$. Note that $Ks^{1/2} < Kt^{(1+\theta)/2} \le t^{\theta/2}$. Now,

$$\begin{aligned} \partial_n u_{t'}(0,s) &\le CK^2 s + Cs^{\beta+\alpha'} \\ &\le Ct^{\theta/2} K s^{1/2} + Cs^{\beta+\alpha'} \le CKs^{1/2}. \end{aligned}$$

For any point $x = (x', s) \in \Omega$, note that $|\partial_n u_t(x', \rho(x'))| \le CK|x'|$, where $|x'| \le Cs^{1/2}$ by the uniform convexity of $\partial\Omega$. Hence, similarly, we have $|\partial_n u_{t'}(x', s)| \le CKs^{1/2}$. It follows that

$$\tag{3.24} |\partial_n u_{t'}(x)| \le CKs^{1/2} \quad \text{on} \;\; \{x_n = s\}.$$

Denote $T_i = \partial_i + \sum_{j<n} \rho_{x_i x_j}(0)(x_j \partial_n - x_n \partial_j)$ and let

$$z(x) = \pm T_i(u_{t'} - \varphi) + B(|x'|^2 + s^{-1} x_n^2) - \widetilde{C} K x_n.$$

By differentiating equation (1.1) with respect to T_i, one has, by [8],

$$\tag{3.25} \mathcal{L} z = \pm[T_i(\log f_{t'}) - \mathcal{L}(T_i \varphi)] + 2B \left(\sum_{i<n-1} u_{t'}^{ii} + s^{-1} u_{t'}^{nn} \right),$$

where $\mathcal{L} = u_{t'}^{ij} \partial_i \partial_j$ is the linearized operator of the equation $\log \det D^2 u_{t'} = \log f_{t'}$, and $\{u_{t'}^{ij}\}$ is the inverse of the Hessian matrix $\{D^2 u_{t'}\}$.

Let $G = \Omega \cap \{x_n < s\}$. First we verify $z \le 0$ on ∂G. By subtracting a smooth function we may assume that $D\varphi(0) = 0$. By the boundary condition we have $|T_i(u_{t'}-\varphi)| \le C|x|^2$ on $\partial\Omega\cap\partial G$. Hence for any given $B>0$, we may choose $\widetilde{C}$ large such that $z\le 0$ on $\partial G\cap\partial\Omega$. On the part $\partial G\cap\{x_n=s\}$, by (3.23) and (3.24),

$$|T_i(u_{t'}-\varphi)|(x) \le CKs + |x'|\,|\partial_n u_{t'}| \le CKs.$$

Hence, $z\le 0$ on ∂G.

Next we verify that $\mathcal{L}z \ge 0$ in G, computing

$$|D\log f_{t'}| \le C{\tau'}^{\alpha-1} \le C{t'}^{\varepsilon_0(\alpha-1)}, \tag{3.26}$$

where $\tau' = {t'}^{\varepsilon_0}$ ($\varepsilon_0 = 1/4n$) as in (2.3). Observe that

$$\sum_{i<n} u_{t'}^{ii} + s^{-1}u_{t'}^{nn} \ge ns^{-1/n}[\det D^2 u_{t'}]^{-1/n} \ge Cs^{-1/n}.$$

Hence we may choose the constant B large, independent of K, t, t', such that $\mathcal{L}z\ge 0$ in G. Now by the maximum principle we see that z attains its maximum at the origin. It follows that $z_n\le 0$; namely, $|\partial_i\partial_n u_{t'}(0)| \le CK$. □

Now we choose a fixed small constant $t_0>0$, and for $k=1,2,\cdots$, let

$$t_k = t_{k-1}^{1+\theta} = \cdots = t_0^{(1+\theta)^k}, \quad \theta = \frac{\alpha}{16n}, \tag{3.27}$$

and let $u_k = u_{t_k}$ be the solution of (2.5) with $t=t_k$. Then we have the estimates

$$\partial_\xi^2 u_k \le C \qquad \text{in } D_{t_k/8}, \tag{3.28a}$$

$$|\partial_\xi\partial_\gamma u_k| \le C^k/\sqrt{t_0} \qquad \text{in } D_{t_k/8}, \tag{3.28b}$$

$$\partial_\gamma^2 u_k \le C^k/t_0 \qquad \text{in } D_{t_k/8}. \tag{3.28c}$$

where the constant C is independent of k and t_0. Note that

$$C^k = O(|\log t_k|^m) \tag{3.29}$$

for some $m>0$ depending only on C. Hence for sufficiently large k, (3.13) holds with $\beta<1$ sufficiently close to 1. Hence in both Lemmas 3.3 and 3.4, we have

$$|u-u_t|(x) \le Ct^{1+\alpha'/2}\mathrm{dist}(x,\partial\Omega) \tag{3.30}$$

if $t>0$ is sufficiently small. In particular (3.30) holds for $u_t = u_{t_k}$ and $u = u_{t_{k+1}}$. From (3.28) and (3.29) we also have an improvement of (2.20) and (2.21), namely for any small $\delta>0$,

$$a_h \le Ch^{(1-\delta)/2}, \tag{3.31}$$

$$b_h \ge Ch^{(1+\delta)/2}, \tag{3.32}$$

provided h is sufficiently small, where C is independent of h.

With estimate (3.30), we may introduce the notion of *affine invariant neighborhood* (with respect to the origin). Let Γ_i, $(i = 1, 2)$, be two convex hypersurfaces which can be represented as radial graphs. That is $\Gamma_i = \rho_i(x)$ for $x \in S^n$, the unit sphere (or a subset of S^n). We say Γ_2 is in the affine invariant δ-neighborhood of Γ_1, denoted by $\Gamma_2 \subset A_\delta(\Gamma_1)$, if

$$(1-\delta)\rho_2 \le \rho_1 \le (1+\delta)\rho_2. \tag{3.33}$$

If $\Gamma_2 \subset A_\delta(\Gamma_1)$, then $T(\Gamma) \subset A_\delta(T(\partial\Omega))$ for any affine transformation T which leaves the origin invariant, namely $T(x) = T \cdot x$ for some matrix T.

Estimate (3.30) gives a control of the shape of the level set $S_{h,u_k}(0)$ for sufficiently large k. When $h = t_{k+1}^2$, by convexity and (3.30),

$$\begin{aligned} |u_k - u|(x) &\le C t_k^{1+\alpha'/2} \operatorname{dist}(x, \partial\Omega), \\ |Du_k|(x) &\ge h/|x| \quad \text{for } x \in S_{h,u_k}(0), \end{aligned}$$

where we assume that $u_k(0) = 0$, $Du_k(0) = 0$. It follows that

$$S_{h,u}(0) \subset A_\delta(S_{h,u_k}(0)) \tag{3.34}$$

with

$$\begin{aligned} \delta &\le \frac{t_k^{1+\alpha'/2} d_x}{h} = t_k^{\alpha'/2-1-2\theta} d_x \\ &\le t_k^{\alpha'/2-1-2\theta} t_{k+1} = t_k^{\alpha'/2-\theta} \le t_k^{\alpha'/4} \end{aligned} \tag{3.35}$$

up to a constant C. Note that $|x|$ does not appear in (3.35), and (3.34) also holds with u replaced by u_{k+1}.

As a consequence we have an estimate for the shape of the level set $S_{h,u}(y)$ for any $y \in \partial\Omega$. By subtracting a linear function (which depends on k), we assume $u_k(0) = 0$ and $Du_k(0) = 0$. By the second inequality of (2.16) we have $S_{h,u_k}(0) \subset D_{t_k/2}$ for $h = C_0 t_k^2$. For simplicity we assume that $C_0 = 1$. We define $a_{h,k}$ and $b_{h,k}$ as in (2.17) and (2.18) with $u = u_k$. Let

$$\overline{b}_{h,k} = \sup\{t \mid \ (0, \cdots, 0, t) \in S_{h,u_k}(0)\}.$$

By Lemma 2.3 and convexity,

$$\overline{b}_{h,k} \ge \frac{h^{1/2}}{a_{h,k}} b_{h,k} \ge \frac{t_0}{C^{2k}} h^{1/2}.$$

Note that $h^{1/2} = t_k = t_{k-1}^{1+\theta} = \cdots = t_0^{(1+\theta)^k}$. Consequently for any given $\delta > 0$,

$$\overline{b}_{h,k} \ge C h^{(1+\delta)/2}$$

provided k is sufficiently large, where $C = C(\delta, \theta, t_0)$. Let

$$\overline{b}_h = \sup\{t \mid \ (0, \cdots, 0, t) \in S_{h,u}(0)\}.$$

By (3.30), $\bar{b}_h \geq Ch^{(1+\delta)/2}$. Hence

$$u(0, x_n) \leq C x_n^{2/(1+\delta)} \tag{3.36}$$

for $x_n = h^{(1+\delta)/2}$ ($h = t_k^2$). As $k > 1$ can be chosen arbitrarily, the above estimate holds for all $x_n > 0$ small. By convexity and the boundary estimates (2.2), we then obtain

$$u(x) \leq C|x|^{2/(1+\delta)} \tag{3.37}$$

for $x \in \Omega$ near the origin. Therefore we have the following $C^{1,\alpha}$ estimate at the boundary.

THEOREM 3.1. *Let u be a solution of* (1.1), (1.2). *Suppose $\partial\Omega, \varphi$ and f satisfy the conditions in Theorem* 1.1. *Then for any $\hat{\alpha} \in (0,1)$, we have the estimate*

$$|u(x) - u(x_0) - Du(x_0)(x - x_0)| \leq C|x - x_0|^{1+\hat{\alpha}} \tag{3.38}$$

for any $x \in \Omega$ and $x_0 \in \partial\Omega$, where C depends on $\hat{\alpha}$.

Obviously Theorem 3.1 also holds for u_t with any $t > 0$, and the constant C in (3.38) is independent of t. In the next section we use a different form of (3.38). That is,

LEMMA 3.5. *Let u satisfy* (3.38). *Then*

$$|Du(y_0) - Du(y)| \leq C|y_0 - y|^{\hat{\alpha}} \tag{3.39}$$

for any $y_0 \in \partial\Omega$ where $y \in \Omega$.

Proof. Assume $u(0) = 0$, $Du(0) = 0$, and y is on the x_n-axis. By convexity we have $\partial_\nu u(y) \leq \frac{1}{t}[u(y+t\nu) - u(y)]$ for any unit vector ν such that $y + t\nu \in \Omega$, where $t = \frac{1}{2}|y|$. By (3.38), $u(y + t\nu), u(y) \leq Ct^{1+\hat{\alpha}}$. Hence $\partial_\nu u(y) \leq Ct^{\hat{\alpha}}$. It follows that $|Du(y) - Du(0)| \leq C|y|^{\hat{\alpha}}$. Similarly, $|\partial_n u(y_0) - \partial_n u(0)| \leq C|y_0|^\alpha$ for $y_0 \in \partial\Omega$ near the origin. From the boundary condition, we then infer that $|Du(y_0) - Du(0)| \leq C|y_0|^\alpha$. Hence (3.39) holds. □

4. Continuity estimates for second derivatives

Our passage to C^2 estimates at the boundary uses a modulus of continuity estimate for second derivatives proved by Caffarelli, Nirenberg, and Spruck in their treatment of the Dirichlet problem for the Monge-Ampère equation [8], [13].

Let u_t be the solution of (2.5). As before we always suppose the origin is a boundary point and near the origin $\partial\Omega$ is given by (2.6), and u_t satisfies (2.7).

Lemma 4.1. *Suppose* u_t *satisfies* (3.1). *Then*,

$$|\partial_\xi\partial_\gamma u_t(x) - \partial_\xi\partial_\gamma u_t(0)| \le \frac{CK^m}{|\log|x| - \log t|}, \tag{4.1}$$

where $m = 50$, $x \in \partial\Omega$, $|x| \le t/2$.

Proof. Although Lemma 4.1 is proved in [8], [13], we provide an outline here in order to display the polynomial dependence on the eigenvalue bounds of the coefficients.

Let $v = u_t/t^2$, $y = x/t$. Then v is defined on the set $\{\widetilde{\rho}(y') < y_n < 1\}$, where $\widetilde{\rho}(y') = \frac{1}{t}\rho(ty')$. By (2.2), $u_{\xi\xi} \ge C > 0$. By the upper bound in (2.16), $u_t(0, x_n) \ge Cx_n^2$. Hence we have

$$v \ge C \quad \text{on} \quad \{y_n = 1\} \tag{4.2}$$

for some positive constant C. By (3.1) and Lemma 3.1,

$$C^{-1}K^{-2} \le D^2 v \le CK^2 \quad \text{in} \quad G = B_{1/2}(0) \cap \{y_n > \widetilde{\rho}(y')\}, \tag{4.3}$$

where the constant C is independent of K.

Let $T = \partial_i + (\partial_i\widetilde{\rho})\partial_n$. Then $T(v - \psi) = T^2(v - \psi) = 0$ on $\partial G \cap B_{1/2}(0)$, where $\psi(y) = \varphi(ty)/t^2$ and φ is the boundary value in (1.2). By subtracting a smooth function we may suppose that $D\varphi(0) = 0$. Computation as in §4 in [8] shows that

$$\mathcal{L}(T^2(v - \psi)) \ge -CK^8, \tag{4.4}$$

where $\mathcal{L} = v^{ij}\partial_i\partial_j$. Note that the Hölder continuity of f_t suffices for (4.4), as in the proof of Lemma 2.1. By (4.3), the least eigenvalue λ and the largest eigenvalue Λ of $D^2 v$ satisfy $C^{-1}K^{-2} \le \lambda \le \Lambda \le CK^2$. Hence

$$z = a|y'|^2 - by_n^2 + cy_n$$

is an upper barrier of $T^2(v - \psi)$ (in a neighborhood of the origin) if we choose $a = C_1K^2$, $b = C_2K^{10}$, $c = C_3K^{10}$ such that $C_3 \gg C_2 \gg C_1 > 0$. It follows that

$$v_{iin}(0) \le CK^{10}. \tag{4.5}$$

Let $h = \widetilde{C}K^{10}|y|^2 - v_n$. Then

$$|\mathcal{L}h| \le CK^{12} \quad \text{in} \quad G. \tag{4.6}$$

Making the transformation $z' = y'$, $z_n = y_n - \widetilde{\rho}(y')$ to straighten the boundary $\partial\Omega$ near the origin, we may suppose $G = B^+_{1/2} = B_{1/2} \cap \{y_n > 0\}$. By (4.5), h is convex on $B_{1/2}(0) \cap \{x_n = 0\}$ if $\widetilde{C}$ is chosen large. Hence by the following Lemma 4.2, we obtain

$$|\partial_\xi\partial_\gamma v(y) - \partial_\xi\partial_\gamma v(0)| \le \frac{CK^m}{|\log|y||} \tag{4.7}$$

with $m = 50$. Scaling back, we obtain (4.1).

The following Lemma 4.2 is equivalent to Lemma 5.1 in [8].

LEMMA 4.2. *Let $h \in C^2(B^+_{1/2}) \cap C^0(B^+_{1/2} \cup T)$ satisfy*

$$\mathcal{L}h = a^{ij}\partial_i\partial_j h \le \overline{f} \tag{4.8}$$

in $B^+_{1/2}$, where $T = \partial B^+_{1/2} \cap \{x_n = 0\}$. Let λ and Λ be the least and the largest eigenvalues of the matrix $\{a^{ij}\}$. Suppose $h_{|T}$ is convex. Then for $x, y \in T$ near the origin,

$$|\partial_i h(x) - \partial_i h(y)| \le \frac{C}{|\log|x-y||}\frac{\Lambda}{\lambda}\Big(\frac{\overline{f}+\Lambda}{\lambda}\Big)^3 \sup(|h| + |Dh|), \quad i < n. \tag{4.9}$$

The main feature of Lemma 4.2 used in this paper is the polynomial dependence of the modulus of the logarithm continuity of $\partial_i h$ on the eigenvalues of the matrix $\{a_{ij}\}$. Alternatively we could have used the boundary Hölder estimate of Krylov [16], which would imply (4.1) with some modulus of continuity.

5. Mixed derivative estimates at the boundary, continued

To prove the $C^{2,\alpha}$ estimates at the boundary, we need a refinement of Lemma 3.4. Let t_k be as in (3.27) and u_k be the solution of (2.5) with $t = t_k$.

LEMMA 5.1. *For any given small $\sigma > 0$, there exists $K > 1$ sufficiently large such that if*

$$|\partial_\xi \partial_\gamma u_k| \le K \quad \text{on } \partial\Omega, \tag{5.1}$$

then

$$|\partial_\xi \partial_\gamma u_{k+1}| \le (1+\sigma)K \quad \text{on } \partial\Omega, \tag{5.2}$$

where ξ is any unit tangential vector on $\partial\Omega$, and γ is the unit outward normal to $\partial\Omega$.

The constant $\sigma > 0$ will be chosen small enough so that

$$(1+10\sigma)^m \le 1 + \frac{1}{2}\theta, \tag{5.3}$$

where $m = 50$ as in (4.1) and $\theta = \alpha/16n$ as defined before Lemma 3.4. We also assume K is sufficiently large and t_k sufficiently small such that

$$K\sigma^2 > 1, \tag{5.4}$$

$$K^{20} t_k \le \sigma^2. \tag{5.5}$$

Note that (5.5) is satisfied when k is large; see (3.29). Therefore we can also choose t_0 sufficiently small such that (5.5) holds for all k.

Proof of Lemma 5.1. The proof is also a refinement of that of Lemma 3.4. As before we suppose the origin is a boundary point, and near the origin $\partial\Omega$ is given by (2.6), and u_k satisfies (2.7). Then by (3.30),

$$|Du_{k+1}|(0) = O(t_k^{1+\alpha'/2}) = o(t_{k+1}). \tag{5.6}$$

By subtracting a smooth function we assume that $\varphi(0) = 0$, $D\varphi(0) = 0$.

Let $\mathcal{L} = u_{k+1}^{ij}\partial_i\partial_j$ be the linearized operator of the equation $\log \det D^2 u_{k+1} = \log f_{t_{k+1}}$. Let $G = D_{t_{k+1}/8} \cap \{x_n < s\}$, where $s = t_{k+1}^{1/4}$. Let

$$T = T_i = \partial_i + \sum_{j<n} \rho_{x_i x_j}(0)(x_j\partial_n - x_n\partial_j),$$
$$z(x) = \pm T_i(u_{k+1} - \varphi) + \frac{1}{2}(|x'|^2 + s^{-1}x_n^2) - (1 + 8\sigma)Kx_n.$$

If $\mathcal{L}z \ge 0$ in G and $z \le 0$ on ∂G, then by the maximum principle, z attains its maximum at the origin. Hence $z_n \le 0$ and so $|\partial_i\partial_n u_{k+1}(0)| \le (1 + 10\sigma)K$ if σK is large enough to control $|D^2\varphi|$. Hence Lemma 5.1 holds. In the following we verify that $\mathcal{L}z \ge 0$ in G and $z \le 0$ on ∂G.

The verification of $\mathcal{L}z \ge 0$ in G is similar to that in the proof of Lemma 3.4. We have

$$\mathcal{L}z = \pm[T(\log f_{t_{k+1}}) - \mathcal{L}(T\varphi)] + \Big(\sum_{i<n-1} u_{k+1}^{ii} + s^{-1}u_{k+1}^{nn}\Big). \tag{5.7}$$

Similar to (3.26),

$$|T(\log f_{t_{k+1}}) - \mathcal{L}(T\varphi)| \le Ct_{k+1}^{\varepsilon_0(\alpha-1)},$$
$$\sum_{i<n} u_{k+1}^{ii} + s^{-1}u_{k+1}^{nn} \ge ns^{-1/n}[\det D^2 u_{k+1}]^{-1/n} \ge Cs^{-1/n},$$

where $\varepsilon_0 = 1/4n$. Hence $\mathcal{L}z \ge 0$ as $s = t_{k+1}^{1/4}$ is very small.

To verify $z \le 0$ on ∂G, we divide the boundary ∂G into three parts; that is, $\partial_1 G = \partial G \cap \partial\Omega$, $\partial_2 G = \partial G \cap \{x_n = s\}$, and $\partial_3 G = \partial G \cap \partial\Omega_t$ ($t = t_{k+1}/8$).

First we consider the boundary part $\partial_1 G$. For any boundary point $x \in \partial\Omega$ near the origin, let $\xi = \xi_T$ be the projection of the vector $T = \partial_i + \rho_{ij}(0)(x_j\partial_n - x_n\partial_i)$ on the tangent plane of $\partial\Omega$ at x. We have

$$|(T - \xi)|(x) \le C|x|^2. \tag{5.8}$$

Hence for $x \in \partial\Omega$ near the origin, we have, by (3.39) and (5.6), noting that $\partial_\xi(u_{k+1} - \varphi) = 0$,

$$\begin{aligned} |T(u_{k+1} - \varphi)(x)| &\le C|x|^2|\partial_\gamma(u_{k+1} - \varphi)(x)| \\ &\le C|x|^2(|x|^{\hat\alpha} + |\partial_\gamma(u_{k+1} - \varphi)(0)|) \\ &\le C|x|^2(|x|^{\hat\alpha} + t_{k+1}), \end{aligned} \tag{5.9}$$

where $t_{k+1} = s^4$. Hence $z \le 0$ on $\partial_1 G$.

Next we consider the part $\partial_2 G$. For any given point $x=(x',s)\in\partial_2 G$, let $\hat{x}=(x',\rho(x'))\in\partial\Omega$. As above let ξ be the projection of $T(\hat{x})$ on $\partial\Omega$. Then

$$\partial_\xi(u_{k+1}-\varphi)(x)=\partial_\xi(u_{k+1}-\varphi)(\hat{x})+\partial_n\partial_\xi(u_{k+1}-\varphi)(x',s')(s-\rho(x'))$$

for some $s'\in(\rho(x'),s)$. By Lemma 3.4,

$$|\partial_n\partial_\xi u_{k+1}|\le|\partial_\gamma\partial_\xi u_{k+1}|+|\partial_\xi^2 u_{k+1}|\le CK.$$

Note that $\partial_\xi(u_{k+1}-\varphi)(\hat{x})=0$ and $|s-\rho(x')|\le(1+C|x'|^2)t_{k+1}=2s^4$. Hence by (5.8),

$$\begin{aligned}(5.10)\quad |T(u_{k+1}-\varphi)(x)|&\le|\partial_\xi(u_{k+1}-\varphi)(x)|+|T-\xi|\,|\partial_\gamma(u_{k+1}-\varphi)(x)|\\&\le Cs^4K+C|x|^{2+\hat{\alpha}},\end{aligned}$$

where we have used that $|T(x)-\xi|\le|T(x)-T(\hat{x})|+|T(\hat{x})-\xi|$ and

$$|T(x)-T(\hat{x})|=|\sum_j\rho_{ij}(0)(x_n-\hat{x}_n)\partial_j|\le Ct_{k+1}=Cs^4.$$

Hence $z\le 0$ on $\partial_2 G$.

Finally we consider the part $\partial_3 G$. We introduce a mapping $\eta=\eta_k$ from $\partial\Omega$ to $\partial\Omega_t$ for $t=t_{k+1}/8$. For any boundary point $y\in\partial\Omega$, by the strict convexity of u_k, the infimum

$$\inf\{u_k(x)-u_k(y)-Du_k(y)(x-y)\mid\ x\in\partial\Omega_t\}$$

is attained at a (unique) point $z\in\partial\Omega_t$. We define $\eta(y)=z$. In other words, z is the unique point in $\partial\Omega_t\cap S_{h,u_k}(y)$ with $h>0$ the largest constant such that $S^0_{h,u_k}(y)\subset D_t$. The mapping η is continuous and one-to-one by the strict convexity and smoothness of $\partial\Omega_t$. The purpose of introducing the mapping η is to give a more accurate estimate for $|T(u_k-\varphi)|(p)$ for $p\in\partial\Omega_t$.

First we consider the point $p=(p_1,\cdots,p_n)\in\partial\Omega_t$ such that $\eta^{-1}(p)$ is the origin. Suppose as before that locally near the origin, $\partial\Omega$ is given by (2.6) and $u_k(0)=0$, $Du_k(0)=0$. Then $h=\inf_{\partial\Omega_t}u_k$. By a rotation of the coordinates x', we suppose that $\{\partial_{ij}u_k(0)\}_{i,j=1}^{n-1}$ is diagonal. We want to prove that

$$(5.11)\qquad |p_i|\le\frac{1+4\sigma}{\partial_i^2 u_k(0)}Kt\quad\forall\ i=1,\cdots,n-1,$$

$$(5.12)\qquad p_n\le t+o(t).$$

By (2.2), $\partial_i^2 u_k(0)$ has positive upper and lower bounds for $1\le i\le n-1$. By (3.39), the tangential second derivatives of u_k are Hölder continuous. Indeed, by the boundary condition $u_k=\varphi$ on $\partial\Omega$, we have

$$(5.13)\qquad \partial^2_{\xi\zeta}u_k+\rho_{\xi\zeta}\partial_\gamma u_k=\partial^2_{\xi\zeta}\varphi+\rho_{\xi\zeta}\partial_\gamma\varphi,$$

where ξ and ζ are unit tangential vectors, and γ is the unit outer normal. By (3.39), $\partial_\gamma u_k$ is Hölder continuous. Hence

$$(5.14)\qquad |\partial^2_{\xi\zeta}u_k(x)-\partial^2_{\xi\zeta}u(0)|\le\sigma^2$$

for any $x\in\partial\Omega$ near the origin and any unit tangential vectors ξ and ζ.

We will prove (5.11) for $i = 1$. By restricting to the 2-plane determined by the x_1-axis and x_n-axis, without loss of generality we may assume that $n = 2$. Denote

$$a_h = \sup\{|x_1| \mid x \in S_{h,u_k}(0)\},$$
$$b_h = \sup\{x_n \mid x \in S_{h,u_k}(0)\},$$

where $h = \inf_{\partial\Omega_t} u_k$. Then it suffices to prove

$$a_h \le \frac{1+4\sigma}{\partial_1^2 u_k(0)} Kt, \tag{5.11$'$}$$

$$b_h \le t + o(t). \tag{5.12$'$}$$

Note that we have now $x = (x_1, x_n)$, and the domains D_t, Ω_t denote the restriction on the 2-plane.

Assume the supremum a_h is achieved at $x_h = (a_h, c_h)$. In the two dimensional case, the level set $\ell := S_{h,u_k}$ is a curve in $\overline{\Omega}$, which has an endpoint $\hat{x} = (\hat{x}_1, \hat{x}_n) \in \partial\Omega$ with $\hat{x}_1 > 0$.

If $a_h \le Ch^{1/2}$ for some $C > 0$ under control, by (2.16) we have $b_h \ge C_1 h^{1/2}$. In this case we have $t \ge C_2 h^{1/2}$. Hence (5.11$'$) holds for sufficiently large K.

If $a_h \ge Ch^{1/2}$ (let us choose $C = \sigma^{-2}$), let $\xi, \zeta, \theta_1, \theta_2$ be as in the proof of Lemma 2.3. Then $\theta_1 + \theta_2 < \pi/2$. By (5.1) and (5.14),

$$|\partial_\gamma u_k(\hat{x})| \le (1+\sigma)K|\hat{x}|, \tag{5.15}$$

$$|\partial_\xi u_k(\hat{x})| \ge (1-\sigma)\partial_1^2 u_k(0)\,|\hat{x}|. \tag{5.16}$$

Hence $\mathrm{tg}\theta_1 \ge \frac{(1-\sigma)\partial_1^2 u_k(0)}{(1+\sigma)K}$. Note that $\mathrm{tg}(\theta_1+\theta_2) \le c_h/(a_h - \hat{x}_1)$ by the convexity of ℓ. We obtain

$$a_h \le \hat{x}_1 + \frac{1+2\sigma}{\partial_1^2 u_k(0)} K c_h.$$

Recall that $h^{1/2} \le \sigma^2 a_h$ by assumption, and $\hat{x}_1 \le Ch^{1/2}$ by (2.2). Hence we obtain

$$a_h \le \frac{1+3\sigma}{\partial_1^2 u_k(0)} K c_h. \tag{5.17}$$

Suppose $\partial\Omega_t$ is locally given by

$$x_n = \rho_t(x'). \tag{5.18}$$

Then ρ_t is smooth and uniformly convex. It is easy to see that $\rho_t(0) = t$ and $|D\rho_t|(0) = o(t)$. Hence we have

$$c_h \le t + C_1 a_h^2 + o(t) a_h. \tag{5.19}$$

By (3.31), $a_h \le Ch^{(1-\delta)/2}$. By (3.36), $h \le Ct^{2/(1+\delta)}$, where $\delta > 0$ can be arbitrarily small as long as t is sufficiently small. Hence we have $c_h \le t + o(t)$. Therefore (5.11) holds.

To prove (5.12), assume that the supremum b_h is attained at $\hat{x}_h = (d_h, b_h)$. Then $b_h \le \rho_t(d_h)$. Hence

$$b_h \le t + C_1 d_h^2 + o(t) d_h \le t + o(t). \tag{5.20}$$

Recall that $d_h \le a_h \le C h^{(1-\delta)/2}$, and by our definition of h, $b_h \ge t$. Hence (5.12) holds.

Now we prove

$$|T(u_k - \varphi)|(p) \le (1 + 6\sigma) K p_n \tag{5.21}$$

at $p = \eta(0)$. Let ξ be the projection of $T(p)$ on the tangent plane of $\partial\Omega_t$ at p. We have

$$|T(p)| \le 1 + C(p_n + |p|^2), \tag{5.22}$$

$$|(T - \xi)(p)| \le C(p_n + |p|^2). \tag{5.23}$$

Hence

$$|T(u_k - \varphi)(p)| \le |\partial_\xi (u_k - \varphi)(p)| + C(p_n + |p|^2)|D(u_k - \varphi)(p)|. \tag{5.24}$$

By (3.39),

$$|D(u_k - \varphi)(p)| \le C|p|^{\hat{\alpha}}.$$

Hence the second term in (5.24) is small. By (5.13), we have $\partial_{ij}^2 \varphi(0) = \partial_{ij}^2 u_k(0)$ for $i, j = 1, \cdots, n-1$ (recall that we assume $D\varphi(0) = 0$ at the beginning). Hence near the origin we have, by the Taylor expansion and (5.11),

$$\begin{aligned} |\partial_i \varphi(p)| &\le (1+\sigma)|p_j\, \partial_i \partial_j u_k(0)| \\ &\le (1 + 5\sigma) K p_n. \end{aligned} \tag{5.25}$$

By our definition of the mapping η, $\partial_\xi u_k = 0$ at p. (This is the purpose of introducing the mapping η.) Hence

$$|\partial_\xi (u_k - \varphi)(p)| \le (1 + 6\sigma) K p_n. \tag{5.26}$$

By (5.24) we therefore obtain (5.21).

Next we prove (5.21) for any given $p \in \partial_3 G$. Let $y = \eta^{-1}(p)$, where η is the mapping introduced above. Then by (5.14) we have, similarly to (5.11),

$$|p_i - y_i| \le \frac{1 + 5\sigma}{\partial_i^2 u_k(0)} K t. \tag{5.27}$$

Choose a new coordinate system such that y is the origin and the positive x_n-axis is the inner normal at y. Subtract a linear function from both u_k and φ (which does not change the value of $T(u_k - \varphi)$) such that $Du_k(y) = 0$. As above let ξ be the projection of $T(p)$ on the tangent plane of $\partial\Omega_t$ at p. By (3.39), $|Du_k|, |D\varphi| \le \sigma^2$ in G. Hence

$$\begin{aligned} |Tu_k(p)| &\le |\partial_\xi u_k(p)| + |T(p) - \xi|\, |Du_k(p)| \le C p_n, \\ |T\varphi(p)| &\le |\partial_\xi \varphi(p)| + C p_n. \end{aligned}$$

By (5.13) and noting that $|D\varphi| \le \sigma^2$, we have, similar to (5.14),

$$|\partial^2_{\xi\zeta}\varphi(x) - \partial^2_{\xi\zeta}u_k(0)| \le \sigma^2.$$

Hence as (5.25),

$$|\partial_\xi\varphi(p)| \le (1+6\sigma)Kp_n.$$

Hence (5.21) holds at any point $p \in \partial_3 G$.

With (5.21) we are now in position to prove $z \le 0$ on $\partial_3 G$. By (3.30),

$$|u_{k+1} - u_k|(x) \le Ct_k^{1+\alpha'/2}t, \quad x \in \partial\Omega_t.$$

Hence by (3.28a),

$$|\partial_\xi(u_{k+1} - u_k)(x)| \le C(t_k^{1+\alpha'/2}t)^{1/2} \le Ct_k^{\alpha'/8}t, \quad x \in \partial\Omega_t,$$

where ξ is any unit tangential vector to $\partial\Omega_t$. Hence

$$\begin{aligned}|T(u_{k+1} - u_k)(x)| &\le |\partial_\xi(u_{k+1} - u_k)| + C(t + |x|^2)|D(u_k - \varphi)| \\ &\le Ct_k^{\alpha'/8}t + Cx_n.\end{aligned}$$

In view of (5.21), it follows that

$$|T(u_{k+1} - \varphi)(x)| \le (1+7\sigma)Kx_n,\ x \in \partial\Omega_t. \tag{5.28}$$

From (5.28) and noting that $\sigma K \gg 1$, we obtain $z \le 0$ on $\partial_3 G$. This completes the proof. □

By Lemma 5.1, we improve (3.28) to

$$\partial^2_\xi u_k \le C \qquad \text{in } D_{t_k/8}, \tag{5.29a}$$

$$|\partial_\xi\partial_\gamma u_k| \le C(1+\sigma)^k \qquad \text{in } D_{t_k/8}, \tag{5.29b}$$

$$\partial^2_\gamma u_k \le C(1+\sigma)^{2k} \qquad \text{in } D_{t_k/8}, \tag{5.29c}$$

where C depends only on $n, \partial\Omega, f, t_0$, and φ.

Now we apply the estimate (4.1) to the section $S^0_{h,u_k}(0)$, where

$$h = t_{k+1}^2 = t_k^{2(1+\theta)}, \quad \theta = \alpha/16n.$$

For any $x \in \partial\Omega \cap S^0_{h,u_k}$, we have by (2.2),

$$|x| \le Ch^{1/2} \le Ct_{k+1}.$$

By (4.1),

$$|\partial_\xi\partial_\gamma u_k(x) - \partial_\xi\partial_\gamma u_k(0)| \le \frac{[C(1+\sigma)^k]^m}{|\log|x| - \log t_k|}.$$

By our definition, $t_k = t_{k-1}^{1+\theta} = \cdots = t_0^{(1+\theta)^k}$. We obtain, by the choice of σ in (5.3),

$$|\partial_\xi\partial_\gamma u_k(x) - \partial_\xi\partial_\gamma u_k(0)| \le \widetilde{C}\frac{(1+\theta/2)^k}{(1+\theta)^k}, \tag{5.30}$$

where $\widetilde{C}$ depends only on $n, \partial\Omega, f, \varphi$ and t_0, and is independent of k.

Proof of Theorem 1.1. We will first prove

$$\sup_{x\in\Omega} |D^2u(x)| \le C. \tag{5.31}$$

Suppose the origin is a boundary point such that $\Omega \subset \{x_n > 0\}$. We will prove D^2u is bounded at the origin. By making a linear transformation of the form

$$\begin{aligned} y_n &= x_n \\ y_i &= x_i - \alpha_i x_n, \quad i = 1, \cdots, n-1, \end{aligned} \tag{5.32}$$

we may suppose $\partial_i\partial_n u_k(0) = 0$, where, by (5.29b),

$$|\alpha_i| \le C(1+\sigma)^k \le C|\log h|.$$

Hence the boundary part $\{x \in \partial\Omega \mid u_k(x) < h\}$ is smooth and uniformly convex after the transformation (5.32). By (5.30) there is a sufficiently large k_0 such that when $k \ge k_0$,

$$|\partial_\xi\partial_\gamma u_k(x)| \le C \tag{5.33}$$

for $x \in \partial\Omega$ with $|x| < t_{k+1}$. Thus, from (2.20) and (2.21),

$$\begin{aligned} a_{h,k} &= \sup\{|x'| \mid x \in S_{h,u_k}(0)\} \le \widetilde{C}h^{1/2}, \\ b_{h,k} &= \sup\{x_n \mid x \in S_{h,u_k}(0)\} \ge h^{1/2}/\widetilde{C} \end{aligned} \tag{5.34}$$

for some $\widetilde{C} > 0$ depending only on n, f, φ and $\partial\Omega$, but independent of k. That is, the section S^0_{h,u_k} has a good shape, as defined in (2.24).

By (3.34), $S^0_{h,u}$ also has a good shape for $h \le t^2_{k+1}$. Now the perturbation argument [4, §6], implies that

$$C_1|x|^2 \le u(x) \le C_2|x|^2, \tag{5.35}$$

where we assume $u(0) = 0$, $Du(0) = 0$. Furthermore, $|D^2u(x)| \le C$, for $x \in \Omega$ near the origin. Making the inverse transformation of (5.32), we obtain (5.31) for x near the origin. The interior second order derivative estimate was established in [4]. Hence (5.31) holds.

Estimate (5.31) implies the Monge-Ampère equation is uniformly elliptic, and so the $C^{2,\alpha}$ estimate follows [2], [19]. □

Remark. Estimate (5.30) actually implies a continuity estimate for the mixed second derivatives of u on the boundary. By the $C^{1,\alpha}$ estimate (Lemma 3.5) and the equation itself, we can then infer a continuity estimate for D^2u on the boundary. However, unless the inhomogeneous term f is smoother, we shall need to use the perturbation argument of the next section to derive continuity estimates for D^2u near the boundary.

6. The perturbation argument

In this section we provide the perturbation argument [4] which enables us to proceed from a level set of good shape to second derivative estimates.

THEOREM 6.1. *Let u be a convex solution to* (1.1), (1.2). *Suppose there is an $h_0 > 0$ such that for any boundary point $y \in \partial\Omega$, $S^0_{h_0,u}(y)$ has a good shape. Then under the assumptions of Theorem* 1.1, *u is $C^{2,\alpha}$ smooth up to the boundary.*

Proof. Let the origin be a boundary point such that $\Omega \subset \{x_n > 0\}$. By subtracting a linear function we suppose

$$u(0) = 0, \quad Du(0) = 0. \tag{6.1}$$

By a rescaling $u \to u/h_0$, $x \to x/\sqrt{h_0}$, we may suppose $h_0 = 1$ and

$$|f(x) - f(0)| \le \varepsilon |x|^\alpha \tag{6.2}$$

for some $\varepsilon > 0$ sufficiently small. For simplicity we suppose $f(0) = 1$. By (2.2) we have

$$C^{-1} \le u_{\xi\xi} \le C \quad \text{on} \quad \partial\Omega \tag{6.3}$$

for any unit tangential vector ξ. First we need two lemmas.

LEMMA 6.1. *Let u_i, $i = 1, 2$, be two convex solutions of* $\det D^2 u = 1$ *such that $u_1 = u_2$ on $\partial\Omega$. Suppose $\|u_i\|_{C^{2,\alpha}} \le C_0$ in $S^0_{1,u_1}(0)$. Then if*

$$|u_1 - u_2| \le \delta \quad \textit{in} \quad S^0_{1,u_1} \tag{6.4}$$

for some sufficiently small $\delta > 0$,

$$|D^2(u_1 - u_2)| \le C\delta \quad \text{in} \quad S^0_{1/2,u_1}. \tag{6.5}$$

Proof. We have

$$\begin{aligned}\det D^2 u_2 - \det D^2 u_1 &= \int_0^1 \frac{d}{dt} \det[D^2 u_1 + t(D^2 u_2 - D^2 u_1)]dt \\ &= a_{ij}(x)\partial_i\partial_j(u_2 - u_1) = 0,\end{aligned} \tag{6.6}$$

where $L = a_{ij}(x)\partial_i\partial_j$ is a linear, uniformly elliptic operator with Hölder continuous coefficients. By the Schauder estimates for linear elliptic equations, we obtain (6.5). □

LEMMA 6.2. *Let u be as above such that $S^0_{1,u}$ has a good shape. Then for* $h \in (0, 1/4]$,

$$S_{h,u} \subset N_\delta(h^{1/2}E) \tag{6.7}$$

with

$$\delta \le C(h^{(1+\hat{\alpha})/2} + h^{-1/2}\varepsilon), \tag{6.8}$$

where $\hat{\alpha}$ *is any constant in* $(0,1)$, N_δ *denotes the* δ*-neighborhood,* E *is an ellipsoid of good shape.*

Proof. Let v be the solution of

$$\det D^2 v = f(0) = 1 \quad \text{in} \quad S^0_{1,u}$$

such that $v = u$ on $\partial S^0_{1,u}$. Since $u = \varphi \in C^3$ on $\partial\Omega$ and $\partial\Omega \in C^3$, from [22] we have $v \in C^{2,\hat{\alpha}}(S^0_{3/4,u})$ $\forall$ $\hat{\alpha} \in (0,1)$. By the Taylor expansion,

$$v(x) = v(0) + v_i(0)x_i + \frac{1}{2}v_{ij}(0)x_i x_j + O(|x|^{2+\hat{\alpha}}),$$

we have, on $S_{h,v}(0)$,

$$C^{-1}h^{1/2} \le |Dv| \le Ch^{1/2}. \tag{6.9}$$

Hence

$$S_{h,v}(0) \le N_{\hat{\delta}}(h^{1/2}E)$$

with $\hat{\delta} \le Ch^{(1+\hat{\alpha})/2}$, where E is the ellipsoid $\{x \in \mathbf{R}^n \mid \frac{1}{2}v_{ij}(0)x_i x_j = 1\}$.

By (6.2) it is easy to verify that $|u - v| \le C\varepsilon$, and by (6.3) we have $|Dv(0)| \le C\varepsilon$. Hence by (6.9),

$$S^0_{h-Ch^{-1/2}\varepsilon,v}(0) \le S^0_{h,u}(0) \le S^0_{h+Ch^{-1/2}\varepsilon,v}(0) \tag{6.10}$$

provided $\varepsilon \ll h^{1/2}$. Hence

$$S_{h,u} \subset N_{Ch^{-1/2}\varepsilon}(S_{h,v}) \subset N_{Ch^{(1+\hat{\alpha})/2}+Ch^{-1/2}\varepsilon}(h^{1/2}E). \qquad \square$$

Proof of Theorem 6.1 *continued.* Let u_k, $k = 0, 1, \cdots$, be the solution of

$$\det D^2 u_k = 1 \quad \text{in } S^0_{4^{-k},u},$$
$$u_k = u \quad \text{on } \partial S^0_{4^{-k},u}.$$

Since $S^0_{1,u}$ has a good shape, by the regularity of the Monge-Ampère equation, $\|u_0\|_{C^{2,\alpha}(S^0_{3/4,u})} \le C$. Denote

$$\omega_k = \sup\{|f(x) - 1| \mid x \in S^0_{4^{-k},u}\},$$

where $f(0) = 1$ by assumption. By the comparison principle, $|u - u_0| \le C\omega_0$. Hence if the constant ε in (6.2) is sufficiently small, $S^0_{1/4,u}$ has a good shape. It follows $\|u_1\|_{C^{2,\hat{\alpha}}(S^0_{3/16,u_1})} \le C$. Note that $|u_1 - u_0| \le C\omega_0$. By Lemma 6.1 we obtain

$$|D^2u_0(x) - D^2u_1(x)| \le C\omega_0 \quad \text{for} \quad x \in S^0_{4^{-2},u_1}. \tag{6.11}$$

It follows that $2^2 S^0_{4^{-2},u_1}$ has a good shape, where $t\Omega = \{x \in \mathbf{R}^n \mid \ tx \in \Omega\}$.

Let $R_k = \sup\{|x| \mid x \in S^0_{4^{-k},u}\}$, namely $B_{R_k}(0)$ is the smallest ball containing $S^0_{4^{-k},u}$. By (6.11) there is a constant $\beta > 0$ such that

$$R_1 < (1-\beta)R_0. \tag{6.12}$$

For $k = 1, 2, \cdots$, applying the same argument to $\hat{u}_0 := 4^k u_k(2^{-k}x)$ and $\hat{u}_1 := 4^k u_{k+1}(2^{-k}x)$, we obtain

$$|D^2u_k(x) - D^2u_{k+1}(x)| \le C\omega_k \quad \text{for} \quad x \in S^0_{4^{-k-2},u_{k+1}}. \tag{6.13}$$

From (6.2) and by induction we have

$$\begin{aligned} R_k &\le (1-\beta)R_{k-1} \le C(1-\beta)^k, \\ \omega_k &\le C\varepsilon(1-\beta)^{\alpha k}. \end{aligned}$$

Hence we obtain from (6.13),

$$|D^2u_0(x) - D^2u_{k+1}(x)| \le C\sum_{i=0}^{k}\omega_i \quad \text{for} \quad x \in S^0_{4^{-k-2},u_{k+1}}, \tag{6.14}$$

where the right-hand side $\le C\varepsilon$. Hence $S^0_{4^{-k},u} = S^0_{4^{-k},u_k}$ has a good shape. From (6.14) we see that $\{D^2u_{k+1}(0)\}$ is convergent. Hence u is twice differentiable at 0, and $D^2u(0) = \lim_{k\to\infty} D^2u_k(0)$. Moreover, (D^2u) is positive definite, so that the Monge-Ampère equation (1.1) is uniformly elliptic. The Hölder continuity of D^2u follows from [2], [19].

The Hölder continuity of D^2u also follows from (6.14) immediately. Indeed, let $\hat{x}$ be a point in $\overline{\Omega}$ near the origin. Choose k_0 such that $\hat{x} \in S_{4^{-k_0-1},u}(0)$. For $k \ge k_0$, let $\hat{u}_k$ be the solution of

$$\begin{aligned} \det D^2\hat{u}_k &= \hat{f}_k \quad &&\text{in } S^0_{4^{-k},u}(\hat{x}), \\ \hat{u}_k &= u \quad &&\text{on } \partial S^0_{4^{-k},u}(\hat{x}), \end{aligned}$$

where $\hat{f}_k = \inf\{f(x) \mid x \in S^0_{4^{-k},u}(\hat{x})\}$. Then, similarly,

$$|D^2\hat{u}_{k_0}(\hat{x}) - D^2\hat{u}_{k+1}(\hat{x})| \le C\sum_{i=k_0}^{k}\hat{\omega}_i, \tag{6.15}$$

where $\hat{\omega}_k \le \sup\{|f(x) - f(\hat{x})| \mid x \in S^0_{4^{-k},u}(\hat{x})\}$. Since f is Hölder continuous, $\sum_{i=k_0}^{\infty}\hat{\omega}_i \le Cd_0^{\alpha}$ and $\sum_{i=k_0}^{\infty}\omega_i \le Cd_0^{\alpha}$, where d_0 is the diameter of the set $S_{4^{-k_0-1},u}(0)$. From (6.14), (6.15), and the interior smoothness of u_{k_0}, and by choosing appropriate k_0, we obtain the Hölder continuity at the origin,

$$|D^2u(\hat{x}) - D^2u(0)| \le C|\hat{x}|^{\alpha'} \tag{6.16}$$

for some $\alpha' \in (0, \alpha)$. From (6.16) we obtain the global Hölder continuity for D^2u. Indeed, let $x, y \in \Omega$ and be close to $\partial\Omega$. If $|x - y| \ge \delta_0(\text{dist}(x, \partial\Omega) +$

$\mathrm{dist}(y,\partial\Omega))$ for some constant $\delta_0 > 0$, let $\hat{x}, \hat{y} \in \partial\Omega$ be the boundary points closest to x, y. Then by (6.16) (denote $\mathcal{A}(x,y) = |D^2u(x) - D^2u(y)|$ for short),

$$\mathcal{A}(x,y) \le \mathcal{A}(x,\hat{x}) + \mathcal{A}(\hat{x},\hat{y}) + \mathcal{A}(\hat{y},y) \le C|x-y|^{\alpha'}.$$

Otherwise the estimate for $\mathcal{A}(x,y)$ is equivalent to the interior one [4]. □

Remark 6.1. For the estimate (6.16), if $\hat{x}$ is also a boundary point, the proof uses only the Hölder continuity of f in the sets $S^0_{h,u}(x)$ for $x \in \partial\Omega$. Hence if f satisfies (3.16), D^2u is Hölder continuous on $\partial\Omega$. We do not require that f be Hölder in Ω.

Remark 6.2. We have actually proved that D^2u is continuous if f is Dini continuous, that is if

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty,$$

where $\omega(t) = \sup\{|f(x) - f(y)| \mid |x-y| < t\}$, so that the right-hand side of (6.14) is convergent.

Remark 6.3. For the interior $C^{2,\alpha}$ estimate, the condition that $S^0_{h_0,u}$ has a good shape is automatically satisfied if u is a strictly convex solution, since the convex set $S^0_{h_0,u}$ can be normalized by a linear transformation. However for the $C^{2,\alpha}$ estimate at the boundary, we can only do a linear transformation of the form (5.32) with relatively small α_i, and must prove (5.34) for u so that the level set has a good shape. Other linear transformations may worsen the boundary condition.

7. Application to the affine mean curvature equation

In this section we prove Theorem 1.2. First we prove the uniqueness of solutions.

LEMMA 7.1. *There is at most one uniformly convex solution $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ of the second boundary value problem* (1.4)–(1.6).

Proof. Suppose both u_1 and u_2 are solutions. We have, by the concavity of the affine area functional A,

$$\begin{aligned} A(u_1) - A(u_2) &= \int_\Omega \Big((\det D^2u_1)^{1/(n+2)} - (\det D^2u_2)^{1/(n+2)} \Big) \\ &\le \frac{1}{n+2} \int_\Omega w_2 U_2^{ij} D_{ij}(u_1 - u_2) \\ &= \frac{1}{n+2} \big[\int_{\partial\Omega} \gamma_i D_j(u_1-u_2) w_2 U_2^{ij} + \int_\Omega (u_1-u_2) f(x) \big]. \end{aligned}$$

where we have used the divergence-free relation $\sum_i \partial_i U^{ij} = 0 \ \forall \ j$. Similarly we have

$$A(u_2) - A(u_1) \leq \frac{1}{n+2}\left[\int_{\partial\Omega} \gamma_i D_j(u_2 - u_1) w_1 U_1^{ij} - \int_\Omega (u_1 - u_2) f(x)\right].$$

Note that $w_1 = w_2$ on $\partial\Omega$. Hence

$$0 \leq \int_{\partial\Omega} w_1 \gamma_i D_j(u_1 - u_2)(U_2^{ij} - U_1^{ij}) = -\int_{\partial\Omega} w_1 \gamma_i D_j(u_1 - u_2)(U_1^{ij} - U_2^{ij}).$$

For any given boundary point, suppose $e_n = (0, \cdots, 0, 1)$ is the inner normal there. Then $\gamma = -e_n$, and the right-hand side of the above inequality is equal to

$$-\int_{\partial\Omega} w_1 D_n(u_1 - u_2)(U_1^{nn} - U_2^{nn}),$$

where $U^{nn} = \det(u_{x_i x_j})|_{i,j=1}^{n-1}$. Since $u_1 = u_2$ on $\partial\Omega$,

$$U_1^{nn} - U_2^{nn} > 0 \quad \text{if} \quad \frac{\partial u_1}{\partial x_n} < \frac{\partial u_2}{\partial x_n}.$$

Hence we obtain

$$0 \leq \int_{\partial\Omega} w_1 D_n(u_1 - u_2)(U_1^{nn} - U_2^{nn}) < 0,$$

which implies $Du_1 = Du_2$ on $\partial\Omega$. Hence $u_1 = u_2$ by the concavity of the affine area functional. This completes the proof. □

In the following we always assume that $u \in C^4(\overline{\Omega})$ is a uniformly convex solution of (1.4)–(1.6) and the conditions of Theorem 1.2 hold. By Aleksandrov's maximum principle [13], $u \in W^{4,p}(\Omega)$ $(p \geq n)$ suffices for the estimates below. Note that $u \in W^{4,1}_{\text{loc}}(\Omega) \cap C^2(\overline{\Omega})$ suffices for Lemma 7.1. The following lemma is taken from [21]

LEMMA 7.2. *There exists a constant $C > 0$ such that any solution u of* (1.4) *satisfies*

$$C^{-1} \leq w \leq C \quad \text{in} \ \ \Omega, \tag{7.1}$$

$$|w(x) - w(x_0)| \leq C|x - x_0| \quad \forall \ \ x \in \Omega, x_0 \in \partial\Omega, \tag{7.2}$$

where C depends only on n, $\text{diam}(\Omega)$, $\sup_\Omega |f|$, *and* $\sup_\Omega |u|$.

Proof. Let $z = \log w - u$. If z attains its minimum at a boundary point, by the boundary condition (1.6) we have $w \geq C$ in Ω. Let us suppose z attains its minimum at an interior point $x_0 \in \Omega$. At this point we have

$$0 = z_i = \frac{w_i}{w} - u_i,$$
$$0 \leq z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - u_{ij}$$

as a matrix. Hence

$$0 \le u^{ij} z_{ij} \le \frac{f}{d^\theta} - n$$

where $d = \det D^2 u$, $\theta = 1/(n+2)$. We obtain $d(x_0) \le C$. Since $z(x) \ge z(x_0)$, we obtain

$$w(x) \ge w(x_0) \exp(u(x) - u(x_0)). \tag{7.3}$$

The first inequality in (7.1) follows.

Next let $z = \log w + A|x|^2$. If z attains its maximum at a boundary point, by (1.6) we have $w \le C$ and so (7.1) holds. If z attains its maximum at an interior point x_0, we have, at x_0,

$$0 = z_i = \frac{w_i}{w} + 2Ax_i,$$
$$0 \ge z_{ii} = \frac{w_{ii}}{w} - \frac{w_i^2}{w^2} + 2A.$$

Suppose $(D^2 u)$ is diagonal at x_0. Then

$$0 \ge u^{ij} z_{ij} = \frac{f}{d^\theta} - 4A^2 x_i^2 u^{ii} + 2Au^{ii} \ge \frac{f}{d^\theta} + Au^{ii} \tag{7.4}$$

if A is small. Observe that

$$d^\theta \sum u^{ii} \ge C \left(\sum u^{ii} \right)^{2/(n+2)}$$

We obtain $\sum u^{ii} \le C$, and hence (7.1) is proved.

Let v be a smooth, uniformly convex function in Ω such that $v = \psi$ on $\partial\Omega$ and $D^2 v \ge K$. Then

$$U^{ij} v_{ij} \ge K \sum U^{ii} \ge CK[\det D^2 v]^{(n-1)/n} \ge CK.$$

Hence if K is large enough, v is a lower barrier of w (where (1.4) is a second order elliptic equation of w). We thus obtain

$$w(x) - w(x_0) \ge -C|x - x_0| \quad \forall \ \ x \in \Omega, x_0 \in \partial\Omega. \tag{7.5}$$

Similarly one can construct an upper barrier for w. Hence (7.2) holds. □

In (7.3) the lower bound for w depends on the uniform estimate for u which we obtain in turn need to find the lower bound for w, namely the upper bound for $\det D^2 u$. To avoid the mutual dependence we assume $f \le 0$, so that w attains its minimum on the boundary by the maximum principle. This condition can be relaxed to $f \le \varepsilon$ for some $\varepsilon > 0$ small but cannot be removed completely, as is easily seen by solving equation (1.4) in the one-dimensional case.

Lemma 7.3. *Let $u \in C^4(\overline{\Omega})$ be a solution of the boundary value problem* (1.4)–(1.6). *Then we have the estimate*

$$\sup_{\Omega} |D^2 u| \leq C, \tag{7.6}$$

where C depends only on $n, \partial\Omega$, $\|f\|_{L^\infty}$, $\|\varphi\|_{C^4(\overline{\Omega})}$, $\|\psi\|_{C^4(\overline{\Omega})}$, and $\inf \psi$.

Proof. Consider the Monge-Ampère equation

$$\det D^2 u = w^{-(n+2)/(n+1)} \quad \text{in } \ \Omega. \tag{7.7}$$

By Lemma 7.2, the right-hand side of (7.7) is positive and satisfies condition (3.16). Hence by the argument in the preceding sections, $D^2 u$ is bounded and Hölder continuous on the boundary; see Remark 6.1. For any $\delta > 0$, by (7.1) the solution of the linearized Monge-Ampère equation

$$U^{ij} w_{ij} = f \quad \text{in } \ \Omega \tag{7.8}$$

is Hölder continuous [7]; namely, $\det D^2 u \in C^\alpha(\Omega_\delta)$ for some $\alpha \in (0,1)$. Hence $u \in C^{2,\alpha}(\Omega_\delta)$ [4]. So we are left to consider a point $\hat{x} \in \Omega$ near the boundary. Choosing an appropriate coordinate system, we assume that $\hat{x}$ is on the positive x_n-axis, the origin is a boundary point, and $\Omega \subset \{x_n > 0\}$. Suppose $u(0) = 0$, $Du(0) = 0$. Then the arguments of the preceding sections apply, with $\theta = \frac{1}{16n}$, and we conclude as before the quadratic growth estimate (5.35). Let $\hat{h}$ be the largest constant such that $S^0_{\hat{h},u}(\hat{x}) \subset \Omega$. By (5.35), the section $S^0_{\hat{h},u}(\hat{x})$ has a good shape. Hence the argument in [7] applies, and we also conclude that w is bounded and Hölder continuous near $\hat{x}$. Hence (7.6) holds. □

Lemma 7.4. *If $f \in L^\infty(\Omega)$, then for any $p > 1$,*

$$\|u\|_{W^{4,p}(\Omega)} \leq C, \tag{7.9}$$

where C depends only on $n, p, \partial\Omega$, $\|f\|_{L^\infty}$, $\|\varphi\|_{C^4(\overline{\Omega})}$, $\|\psi\|_{C^4(\overline{\Omega})}$, and $\inf \psi$. If $f \in C^\alpha(\overline{\Omega})$, $\varphi \in C^{4,\alpha}(\overline{\Omega})$, $\psi \in C^{4,\alpha}(\overline{\Omega})$, and $\partial\Omega \in C^{4,\alpha}$ for some $\alpha \in (0,1)$, then

$$u \in C^{4,\alpha}(\overline{\Omega}) \leq C \tag{7.10}$$

where C depends, in addition, on α.

Proof. Regard the fourth order equation (1.4) as a system of two second order partial differential equations (7.7) (7.8). By estimate (7.6), both (7.7) and (7.8) are uniformly elliptic. It follows that w is Hölder continuous up to the boundary and so $u \in C^{2,\alpha}(\overline{\Omega})$ [2], [19]. Hence (7.8) is a linear, uniformly elliptic equation with Hölder coefficients and $w \in W^{2,p}(\Omega)$ for any $p < \infty$. From (7.7) we also conclude the global $C^{4,\alpha}$ *a priori* estimate for u. □

Proof of Theorem 1.2. We have proved the uniqueness and established the *a priori* estimate for solutions of (1.4)–(1.6). To prove the existence of solutions we use the degree theory as follows.

For any positive $w \in C^{0,1}(\overline{\Omega})$, let $u = u_w \in C^{2,\alpha}(\overline{\Omega})$ be the solution of

$$\det D^2 u = w^{-(n+2)/(n+1)} \quad \text{in} \quad \Omega, \qquad u = \varphi \quad \text{on} \quad \partial\Omega. \tag{7.11}$$

Next let w_t, $t \in [0,1]$, be the solution of

$$U^{ij} w_{ij} = t f(x) \quad \text{in} \quad \Omega, \qquad w_t = t\psi + (1-t) \quad \text{on} \quad \partial\Omega. \tag{7.12}$$

We have thus defined a compact mapping $T_t : \ w \in C^{0,1}(\overline{\Omega}) \to w_t \in C^{0,1}(\overline{\Omega})$. By the *a priori* estimate (7.9), the degree $\deg(T_t, B_R, 0)$ is well defined, where B_R is the set of all positive functions satisfying $\|u\|_{C^{0,1}(\overline{\Omega})} \le R$. When $t = 0$, from (7.12) we have, obviously, $w \equiv 1$. Namely, T_0 has a unique fixed point $w \equiv 1$. Hence the degree $\deg(T_t, B_R, 0) = 1$ for all $t \in [0,1]$. This completes the proof. □

Remark. Theorem 1.2 extends to more general equations (1.4) where

$$w = [\det D^2 u]^{\theta - 1}, \quad 0 < \theta \le \frac{1}{n}.$$

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References

[1] W. Blaschke, *Vorlesungen úber Differential Geometrie*, Berlin, 1923.

[2] L. A. Caffarelli, Interior *a priori* estimates for solutions of fully nonlinear equations, *Ann. of Math.* **130** (1989), 189–213.

[3] ———, A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity, *Ann. of Math.* **131** (1990), 129–134.

[4] ———, Interior $W^{2,p}$ estimates for solutions of Monge-Ampère equations, *Ann. of Math.* **131** (1990), 135–150.

[5] ———, Boundary regularity of maps with convex potentials, *Comm. Pure Appl. Math.* **45** (1992), 1141–1151.

[6] ———, Boundary regularity of maps with convex potentials II, *Ann. of Math.* **144** (1996), 453–496.

[7] L. A. Caffarelli and C. E. Gutiérrez, Properties of the solutions of the linearized Monge-Ampère equations, *Amer. J. Math.* **119** (1997), 423–465.

[8] L. A. Caffarelli, L. Nirenberg, and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations I. Monge-Ampère equation, *Comm. Pure Appl. Math.* **37** (1984), 369–402.

[9] E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, *Michigan Math. J.* **5** (1958), 105–126.

[10] ———, Hypersurfaces with maximal affinely invariant area, *Amer. J. Math.* **104** (1982), 91–126.

[11] S.-Y. Cheng and S.-T. Yau, Complete affine hypersurfaces, I. The completeness of affine metrics, *Comm. Pure Appl. Math.* **39** (1986), 839–866.

[12] S. S. Chern, Affine minimal hypersurfaces, in *Minimal Submanifolds and Geodesics*, *Proc. Japan-United States Sem.* (Tokyo, 1977), 17–30.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.

[14] P. Guan, N. S. Trudinger, and X.-J. Wang, On the Dirichlet problem for degenerate Monge-Ampère equations, *Acta Math.* **182** (1999), 87–104.

[15] N. Ivochkina, *A priori* estimate of $\|u\|_{C^2(\overline{\Omega})}$ of convex solutions of the Dirichlet problem for the Monge-Ampère equation, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (LOMI) **96** (1980), 69–79 (Russian); English translation in *J. Soviet Math.* **21** (1983), 689–697.

[16] N. V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, Reidel, Dordrecht-Boston, 1987.

[17] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, Cambridge, 1994.

[18] A. V. Pogorelov, *The Multidimensional Minkowski Problems*, J. Wiley & Sons, New York, 1978.

[19] M. V. Safonov, Classical solution of second-order nonlinear elliptic equations, *Izv. Akad. Nauk SSSR Ser. Mat.* **52** (1988), 1272–1287 (Russian); English translation in *Math. USSR-Izv.* **33** (1989), 597–612.

[20] N. S. Trudinger and X.-J. Wang, The Bernstein problem for affine maximal hypersurfaces, *Invent. Math.* **140** (2000), 399–422.

[21] ———, The affine Plateau problem, *J. Amer. Math. Soc.* **18** (2005), 253–289.

[22] X.-J. Wang, Regularity for Monge-Ampère equation near the boundary, *Analysis* **16** (1996), 101–107.

[23] ———, Affine maximal hypersurfaces, *Proc. Internat. Congress of Mathematicians* Vol. III, 2002, 221–231.

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