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BOUNDARY REPRESENTATIONS FOR FAMILIES OF REPRESENTATIONS OF OPERATOR ALGEBRAS AND SPACES

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ABSTRACT. In analogy with the peak points of the Shilov boundary of a uniform algebra, Arveson defined the notion of boundary representations among the completely contractive representations of a unital operator algebra. However, he was unable to show that such representations always exist. Dropping his original condition that such representations should be irreducible, we show that a family of representations (in Agler's sense) of either an operator algebra or an operator space has boundary representations. This leads to a direct proof of Hamana's result that all unital operator algebras have enough such boundary representations to generate the C^*-envelope.

KEYWORDS: *Operator algebras, representations, Agler families, boundary representations, C^*-envelopes, liftings, dilations, Shilov boundary.*

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1. INTRODUCTION

Concretely, an operator algebra $\mathcal{A}$ is a subalgebra of $\mathcal{B}(K)$, the bounded linear operators on some Hilbert space K. It is unital if it contains the identity operator. The algebra $M_\ell(\mathcal{A})$ of $\ell \times \ell$ matrices with entries from $\mathcal{A}$ inherits a norm as a subspace of $M_\ell(\mathcal{B}(K))$ identified canonically with $\mathcal{B}\Big(\bigoplus\limits_1^\ell K\Big)$. The Blecher, Ruan and Sinclair Theorem ([5]) characterizes unital operator algebras in terms of a matrix norm structure, while a theorem of Blecher ([4]) does the same for non-unital algebras assuming the algebra multiplication is completely bounded. Consequently it is possible to speak abstractly of an operator algebra without reference to an ambient $\mathcal{B}(K)$.

Likewise operator spaces are concretely viewed as vector subspaces of some $\mathcal{B}(K)$. In a manner akin to that for operator algebras they too may be characterized in terms of a matrix norm structure, as Ruan showed ([11]).

A linear mapping $\phi : \mathcal{A} \to \mathcal{B}(H)$ ($\mathcal{A}$ is either an operator algebra or an operator space) induces a linear mapping $\phi_\ell : M_\ell(\mathcal{A}) \to \mathcal{B}\Big(\bigoplus_1^\ell H\Big)$ by applying ϕ entry-wise, so that $\phi_\ell(a_{jm}) = (\phi(a_{jm}))$. The map ϕ is completely bounded if ϕ is bounded and there exists C, independent of ℓ, such that $\|\phi_\ell\| \leqslant C$, it is completely contractive if it is completely bounded with $C \leqslant 1$, and it is completely isometric if ϕ_ℓ is an isometry for each ℓ. Finally, a representation of $\mathcal{A}$ on the Hilbert space H is an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}(H)$ (or simply a bounded linear map in the case that $\mathcal{A}$ is an operator space). If $\mathcal{A}$ is a unital operator algebra, it is assumed that any representation of $\mathcal{A}$ takes the unit to the identity operator.

A *boundary representation* ([3], [2]) of the unital operator algebra $\mathcal{A}$ consists of a completely isometric homomorphism $\phi : \mathcal{A} \to \mathcal{C}$, where $\mathcal{C}$ is a C^*-algebra and $C^*(\phi(\mathcal{A})) = \mathcal{C}$, together with a representation $\pi : \mathcal{C} \to \mathcal{B}(H)$ such that the only completely positive map on $\mathcal{C}$ agreeing with π on $\phi(\mathcal{A})$ is π itself. In originally defining boundary representations, Arveson also required that they be irreducible. We do not impose this condition.

The C^*-*envelope* of $\mathcal{A}$ (either an operator algebra or an operator space), denoted by $C^*_{\mathrm{e}}(\mathcal{A})$, is the essentially unique smallest C^*-algebra amongst those C^*-algebras $\mathcal{C}$ for which there is a completely isometric homomorphism $\phi : \mathcal{A} \to \mathcal{C}$. For instance, if $\mathcal{A}$ is a uniform algebra, then $C^*_{\mathrm{e}}(\mathcal{A})$ is the C^*-algebra of continuous functions on the Shilov boundary of $\mathcal{A}$. In fact, in this case the irreducible boundary representations correspond to peak points of $\mathcal{A}$. Arveson proved that $C^*_{\mathrm{e}}(\mathcal{A})$ exists provided there are enough boundary representations for $\mathcal{A}$. However, the existence of $C^*_{\mathrm{e}}(\mathcal{A})$ does not imply the existence of boundary representations and Hamana ([7]) established the existence of $C^*_{\mathrm{e}}(\mathcal{A})$ for operator algebras in general without recourse to boundary representations.

In this note we show, by elaborating on a construction of Agler essential to his approach to model theory ([1]) and using a characterization of boundary representations due to Muhly and Solel ([9]), that boundary representations exist, and then following an argument similar to Arveson's, we also derive the existence of $C^*_{\mathrm{e}}(\mathcal{A})$.

Agler's approach is to consider a *family* $\mathcal{F}_\mathcal{A}$ of representations of an associative algebra $\mathcal{A}$ (which is not necessarily an operator algebra). This is a collection of representations which is

(1) Closed with respect to direct sums (so if $\{\pi_\alpha\}$ is an arbitrary set of representations in the family, then $\bigoplus_\alpha \pi_\alpha$ is also a representation in the family);
(2) Hereditary (that is, if ϕ is a representation in the family and L is a subspace which is invariant for all $\phi(a)$, $a \in \mathcal{A}$, then $\phi|L$, the restriction of ϕ to L, is also in the family);
(3) Closed with respect to unital $*$-representations (so if $\phi : \mathcal{A} \to \mathcal{B}(H)$, and $\nu : \mathcal{B}(H) \to \mathcal{B}(K)$ is a unital $*$-representation, then $\nu \circ \phi$ is in the family).

If $\mathcal{A}$ is non-unital, then we also require

(4) $\mathcal{A}$ is closed with respect to spanning representations with respect to the partial ordering on dilations (defined below).

When $\mathcal{A}$ is unital, (1)–(3) can be shown to imply (4), using an argument similar to that used to prove (1) of Theorem 1.1 in [6].

Agler's definition of a family may readily be taken over to subspaces of associative algebras without alteration, and we do so here.

A consequence of (1) is that for each $a \in \mathcal{A}$, there is a constant C_a such that $\sup_{\pi \in \mathcal{F}_\mathcal{A}} \|\pi(a)\| \leqslant C_a$, and indeed, we may define a norm $\|\cdot\|_{\mathcal{F}_\mathcal{A}}$ on $\mathcal{A}$ such that $\|a\|_{\mathcal{F}_\mathcal{A}}$ is this supremum. Endowing $\mathcal{A}$ with this norm (which we do whenever $\mathcal{A}$ is not already a normed algebra or a subspace thereof), it happens that all representations in $\mathcal{F}_\mathcal{A}$ become contractive representations. On the other hand, if $\mathcal{A}$ is already (a subspace of) a normed algebra, then from (1) we likewise deduce that all representations are uniformly bounded in norm.

Special examples of families include the collection of all completely contractive representations of an operator space $\mathcal{A}$ and the collection of all representations π of the disc algebra such that $\pi(z)$ is an isometry.

A representation ϕ *lifts* to a representation ψ if ϕ is the restriction of ψ to an invariant subspace. Following Agler ([1]) we will say that the representation $\phi : \mathcal{A} \to \mathcal{B}(H)$ is *extremal*, if whenever K is a Hilbert space containing H and $\psi : \mathcal{A} \to B(K)$ is a representation such that H is invariant for $\psi(\mathcal{A})$ and $\phi = \psi|H$, then H reduces $\psi(\mathcal{A})$. Further, if $\rho : \mathcal{A} \to \mathcal{B}(L)$ is a representation, then ρ lifts to an extremal representation; i.e., there exists a Hilbert space H containing L and an extremal representation ϕ of $\mathcal{A}$ such that L is invariant for $\phi(\mathcal{A})$ and $\rho = \phi|L$ ([1], Proposition 5.10).

Lifting induces a partial ordering on representations, with $\phi_\alpha \leqslant \phi_\beta$ being equivalent to ϕ_α lifting to ϕ_β. If $\mathcal{S}$ is a totally ordered set of liftings (with respect to this partial ordering), then we define the *spanning representation* $\phi_\mathrm{s} : \mathcal{A} \to \mathcal{B}(H_\mathrm{s})$ by setting H_s to be the closed span of the H_α's over all $\alpha \in \mathcal{S}$, and then densely defining ϕ_s to be ϕ_α on H_α and extending to all of H_s by boundedness of the representations ϕ_α. It is readily verified that ϕ_s is a representation which lifts each ϕ_α.

The representation $\phi : \mathcal{A} \to \mathcal{B}(H)$ dilates to the representation $\psi : \mathcal{A} \to \mathcal{B}(K)$ if K contains H and $\phi(a) = P_H\psi(a)|H$ for all $a \in \mathcal{A}$. A fundamental result of Sarason ([12]) says that a representation ϕ dilates to a representation ψ if and only if H is semi-invariant for ψ. Thus, there exists subspaces $L \subset N \subseteq K$ invariant for ψ such that $H = N \ominus L$. Alternatively, $K = L \oplus H \oplus M$ with L and $L \oplus H$ invariant for ϕ. Just as in the case of liftings, dilating induces a partial ordering on representations in the obvious manner. We can also similarly define spanning representations of totally ordered sets of representations, and this is what is used in item (4) above. Note that liftings are also dilations (with $L = \{0\}$). Hence the partial ordering on dilations subsumes that of liftings, and in particular, any spanning representation of liftings is one in terms of dilations as well.

As was noted above, families of representations over unital algebras contain all spanning representations formed from chains of representations in the family, though it appears that this assumption needs to be added in the non-unital setting and for operator spaces. On the other hand, there are interesting collections of representations which are closed with respect to (1) and (4) of a family, but not necessarily (2) and (3). For example, take the algebra complex polynomials in a variable z and consider the collection of all representations obtained by mapping z to a contractive co-hyponormal operator. Since the theorems we prove below only depend on existence of spanning representations and the uniform boundedness of representations, we define the *extended family* of an associative algebra $\mathcal{A}$ to be a collection of representations of $\mathcal{A}$ which is closed under the formation of direct sums and spanning representations. We likewise define the notion of an extended family for subspaces of an associative algebra.

For dilations, the equivalent of an extremal will be referred to as a ∂-*representation*. The representation $\phi : \mathcal{A} \to \mathcal{B}(H)$ is a ∂-representation if whenever $\psi : \mathcal{A} \to B(K)$ dilates ϕ, then H reduces $\psi(\mathcal{A})$.

Muhly and Solel ([9]) show, in the language of Hilbert modules rather than representations, that for unital operator algebras ∂-representations coincide with boundary representations (forgetting the irreducibility requirement).

THEOREM 1.1. *Let $\mathcal{A}$ be a unital operator algebra. Then $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a ∂-representation if, and only if, given any completely isometric map $\phi : \mathcal{A} \to \mathcal{C}$ where $\mathcal{C}$ is a C^* algebra with $\mathcal{C} = C^*(\phi(\mathcal{A}))$, there exists a boundary representation $\pi : C^*(\phi(\mathcal{A})) \to \mathcal{B}(H)$ such that $\pi \circ \phi = \rho$.*

The proof of Muhly and Solel of this equivalence uses the existence of the C^*-envelope. Our main result and proof of the existence of the C^*-envelope do not depend on their work. However, it should be noted that a proof of the equivalence which does not already assume the existence of the C^*-envelope is possible, and we sketch a proof below based along a line of reasoning in Theorem 1.2 of [8].

Sketch of the proof of Theorem 1.1. Suppose $\phi : \mathcal{A} \to \mathcal{C} = C^*(\phi(\mathcal{A}))$ is completely isometric and $\pi : C^*(\phi(\mathcal{A})) \to \mathcal{B}(H)$ is a boundary representation. Set $\rho = \pi \circ \phi$, and note that it is completely contractive. Suppose $\nu : \mathcal{A} \to \mathcal{B}(K)$ dilates ρ. The goal is to show that H reduces ν.

To this end, define a map $\gamma : \phi(\mathcal{A}) \to \mathcal{B}(K)$ by $\gamma(\phi(a)) = \nu(a)$, $a \in \mathcal{A}$. This map is completely contractive, and so by the Arveson extension theorem extends to a completely positive unital map $\gamma : C^*(\phi(\mathcal{A})) \to \mathcal{B}(K)$ with $\gamma \circ \phi = \nu$. Observe that the map which takes $b \mapsto P_H\gamma(\phi(b))|H$, $b \in C^*(\phi(\mathcal{A}))$ is completely positive, and by definition, $P_H\gamma(\phi(a))|H = \rho(a) = \pi(\phi(a))$ for all $a \in \mathcal{A}$. We have assumed that π is a boundary representation, so in fact $P_H\gamma(\phi(b))|H = \pi(b)$ for all $b \in C^*(\phi(\mathcal{A}))$. From this we have for all $a \in \mathcal{A}$,

$$\rho(a)\rho(a)^* = \pi(\phi(a))\pi(\phi(a))^*$$

$$
\begin{aligned}
&= \pi(\phi(a)\phi(a)^*) \\
&= P_H \gamma(\phi(a)\phi(a)^*)|H \\
&\geqslant P_H \gamma(\phi(a))\gamma(\phi(a)^*)|H = P_H \nu(a)\nu(a)^*|H \\
&\geqslant P_H \gamma(\phi(a)) P_H \gamma(\phi(a)^*)|H = P_H \nu(a) P_H \nu(a)^*|H \\
&= \rho(a)\rho(a)^*,
\end{aligned}
$$

where the first inequality is the Cauchy-Schwarz inequality for completely positive maps ([10]). From this we see that $\nu(a)^* H \subseteq H$. An identical argument gives $\nu(a)H \subseteq H$, proving that H reduces ν.

The converse is a straightforward exercise and is left to the reader. ∎

Unless otherwise stated, we shall henceforth assume that $\mathcal{A}$ is either an associative algebra or a subspace of such an algebra. We will prove the following:

THEOREM 1.2. *If $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a representation in an extended family $\mathcal{F}_{\mathcal{A}}$, then there exists a Hilbert space K containing H and a ∂-representation $\phi : \mathcal{A} \to \mathcal{B}(K)$ also in $\mathcal{F}_{\mathcal{A}}$ such that ρ dilates to ϕ.*

As mentioned above, Arveson's original definition of boundary representation required π to be irreducible. Note that Theorem 1.2 does not imply the existence of irreducible boundary representations.

The authors would like to thank Jim Agler for sharing a draft of the first several chapters of a recent manuscript on his abstract model theory and Vern Paulsen for his valuable assistance and an advance look at a draft of his book, *Completely Bounded Maps and Operator Algebras* ([10]), particularly the chapters on the abstract characterization of operator algebras and the C^*-envelope.

The remainder of the paper is organized as follows. Section 2 establishes Theorem 2.1 giving the existence of extremals, which form the core of any model in Agler's approach to model theory ([1]). Although versions of this result are quite old, no proofs yet appear in the literature. In Section 3 we prove Theorem 1.2 and in Section 4 we explain how to obtain the existence of $C^*_e(\mathcal{A})$ from Theorem 1.2.

2. LIFTINGS AND EXTREMALS

In Agler's approach to model theory a family is a collection of representations of a unital algebra satisfying the first three canonical axioms listed in the last section. A key result in his model theory is that an arbitrary member of a family $\mathcal{F}_{\mathcal{A}}$ lifts to an extremal member of the family $\mathcal{F}_{\mathcal{A}}$ ([1], Proposition 5.10). We establish this result for extended families of associative algebras and their subspaces.

THEOREM 2.1. *If $\rho : \mathcal{A} \to \mathcal{B}(H)$ is a representation in an extended family $\mathcal{F}_{\mathcal{A}}$, then ρ lifts to an extremal representation in $\mathcal{F}_{\mathcal{A}}$.*

We establish a preliminary version of Theorem 2.1 in Lemma 2.2 below, and then indicate a proof of the theorem based on the lemma.

Suppose the representation $\phi_\alpha : \mathcal{A} \to \mathcal{B}(H_\alpha)$ lifts to the representation $\phi_\beta : \mathcal{A} \to \mathcal{B}(H_\beta)$. Then the lifting is *trivial* if H_α is reducing, not just invariant, for $\phi_\beta(\mathcal{A})$. If the only liftings of ϕ_α are trivial, then ϕ_α is extremal.

If ϕ_β lifts ϕ_α, then we define a lifting $\phi_\delta : \mathcal{A} \to \mathcal{B}(H_\delta)$ of ϕ_β to be *strongly non-trivial* with respect to ϕ_α if there exists an $a \in \mathcal{A}$ such that

$$P_{H_\delta \ominus H_\beta} \phi_\delta(a)^* | H_\alpha \neq 0.$$

Otherwise, the lifting is *weakly trivial* relative to ϕ_α. Finally, ϕ_β is *weakly extremal* relative to ϕ_α if every lifting of ϕ_β is weakly trivial relative to ϕ_α.

LEMMA 2.2. *Each representation $\phi_0 : \mathcal{A} \to \mathcal{B}(H_0)$ in an extended family $\mathcal{F}_\mathcal{A}$ lifts to a representation in $\mathcal{F}_\mathcal{A}$ which is weakly extremal relative to ϕ_0.*

Proof. The proof is by contradiction. Accordingly, suppose ϕ_0 does not lift to a weakly extremal representation relative to ϕ_0.

Let κ_0 be the cardinality of the set of points in the unit sphere of H_0, κ_1 the cardinality of the set of elements in the unit ball of $\mathcal{A}$. Set $\kappa = 2^{\aleph_0 \cdot \kappa_0 \cdot \kappa_1} > \kappa_0 \cdot \kappa_1$. Let λ be the smallest ordinal greater than or equal to κ. Note that for each a, there is a $C_a > 0$ so that for $\pi \in \mathcal{F}_\mathcal{A}$, $\|\pi(a)h\| \leqslant C_a \|h\|$ for all $h \in H_0$ and $a \in \mathcal{A}$.

Construct a chain of liftings in $\mathcal{F}_\mathcal{A}$ by transfinite recursion on the ordinal λ as follows: if $\alpha \leqslant \lambda$, and α has a predecessor, let ϕ_α denote a strong (with respect to ϕ_0) nontrivial lifting of $\phi_{\alpha-1}$. Such an lifting exists by the assumption that ϕ_0 does not lift to a weak extremal. If α is a limit ordinal, set ϕ_α to the spanning representation of $\{\phi_\delta\}_{\delta<\alpha}$. For any h in the unit sphere of H_0 and a in the unit ball of $\mathcal{A}$, there are at most countably many α's with predecessors where $P_{H_\alpha \ominus H_{\alpha-1}} \phi_\alpha(a)^* h \neq 0$. Since the cardinality of the set of ordinal numbers less than or equal to λ and having a predecessor is κ, there must be an ordinal $\beta < \lambda$ with predecessor where $P_{H_\beta \ominus H_{\beta-1}} \phi_\beta(a)^* h = 0$ for all h in the unit sphere of H_0 and a in the unit ball of $\mathcal{A}$, so that ϕ_β is a lifting of $\phi_{\beta-1}$ which is weakly trivial with respect to ϕ_0; a contradiction, ending the proof. ∎

Proof of Theorem 2.1. We use Theorem 2.2 to prove Theorem 2.1. Let $\phi_0 : \mathcal{A} \to \mathcal{B}(H_0)$ denote a given representation. Lift ϕ_0 to a representation $\phi_1 : \mathcal{A} \to \mathcal{B}(H_1)$ which is weakly extremal relative to ϕ_0. Lift ϕ_1 to a representation ϕ_2 which is weakly extremal relative to ϕ_1. Continuing in this manner, constructs a chain ϕ_j, $j \in \mathbb{N}$, with respect to the partial order on liftings with the property that ϕ_j is weakly extremal relative to ϕ_{j-1}. The resultant spanning representation $\phi_\infty : \mathcal{A} \to \mathcal{B}(H_\infty)$ lifts ϕ_0 and it is easily checked to be extremal, since it is weakly extremal relative to ϕ_j for all $j \in \mathbb{N}$. ∎

It is not difficult to see that the restriction of an extremal to a reducing subspace is an extremal. Also, in Theorem 2.1 if we were to take the intersection of all reducing subspaces of ϕ_∞ containing H_0, we end up with the smallest reducing subspace for ϕ_∞ containing H_0. Restricting to this gives a minimal extremal ϕ_e lifting ϕ_0, in the sense that if ψ lifts ϕ_0 and $\psi \leqslant \phi_e$, then $\psi = \phi_e$. Of course ϕ_e may still be reducible even if ϕ_0 is irreducible. In addition, there may be non-isomorphic minimal extremal liftings of ϕ_0.

3. DILATIONS AND BOUNDARY REPRESENTATIONS

Let $\phi_\alpha : \mathcal{A} \to \mathcal{B}(H_\alpha)$ be a representation. In parallel with the theory of liftings, a dilation $\phi_\beta : \mathcal{A} \to \mathcal{B}(H_\beta)$ is termed *trivial* if H_α is reducing for $\phi_\beta(\mathcal{A})$. If the only dilations of ϕ_α are trivial ones, then ϕ_α is a ∂-representation.

Likewise, suppose $\phi_\delta \geqslant \phi_\beta \geqslant \phi_\alpha$ in the partial ordering for dilations, with the representations mapping into the operators on H_δ, H_β and H_α, respectively. By assumption, we can write $H_\delta = L_\delta \oplus H_\beta \oplus M_\delta$, where L_δ and $L_\delta \oplus H_\beta$ are invariant for ϕ_δ. We say that ϕ_α is *strongly non-trivial* with respect to ϕ_α if there exists an $a \in \mathcal{A}$ such that either

$$P_{L_\delta}\pi_\beta(a)|H_\alpha \neq 0 \quad \text{or} \quad P_{M_\delta}\pi_\beta(a)^*|H_\alpha \neq 0.$$

Otherwise, the dilation is said to be *weakly trivial* relative to ϕ_α. Finally, ϕ_β is a *weak ∂-representation* relative to ϕ_α if every lifting of ϕ_β is weakly trivial relative to ϕ_α.

LEMMA 3.1. *Each representation $\phi_0 : \mathcal{A} \to \mathcal{B}(H_0)$ in an extended family $\mathcal{F}_\mathcal{A}$ dilates to a weak ∂-representation relative to ϕ_0 which is also in $\mathcal{F}_\mathcal{A}$.*

Proof. The proof closely follows that of the existence of weak extremals, and is by contradiction. Hence we suppose ϕ_0 does not lift to a weak ∂-representation relative to ϕ_0. We define the ordinal λ as in the proof of Lemma 2.2.

Construct a chain of dilations in $\mathcal{F}_\mathcal{A}$ where each of the representations by transfinite recursion on the ordinal λ as in Lemma 2.2: if $\alpha \leqslant \lambda$ and α is a limit ordinal, set ϕ_α to the spanning representation of $\{\phi_\delta\}_{\delta<\alpha}$ and if α has a predecessor, let ϕ_α be a dilation to a strong (with respect to ϕ_0) nontrivial dilation of $\phi_{\alpha-1}$, which exists by the assumption that ϕ_0 does not lift to a weak ∂-representation. Then for any h in the unit sphere of H_0 and a in the unit ball of $\mathcal{A}$, there are at most countably many α's with predecessors where $P_{L_\delta}\pi_\beta(a)h \neq 0$ or $P_{M_\delta}\pi_\beta(a)^*h \neq 0$. The same reasoning then gives a representation ϕ_β in our chain dilating $\phi_{\beta-1}$ which is weakly trivial with respect to ϕ_0, a contradiction. ∎

Proof of Theorem 1.2. This now follows the proof of Theorem 2.1. Construct a countably infinite chain of representations $\{\phi_i\}$ into the bounded operators on Hilbert spaces H_i, where ϕ_i is a weak ∂-representation with respect to ϕ_{i-1} for each $i \in \mathbb{N}$. Let ϕ_∞ denote the spanning representation on the Hilbert space H_∞. Since a dilation of a weak ∂-representation with respect to a representation ϕ is also a weak ∂-representation with respect to ϕ, ϕ_∞ is a weak ∂-representation with respect to ϕ_i for all i. It easily follows that ϕ_∞ is a ∂-representation. ∎

Minimal ∂-representations dilating a given representation can be defined in the manner of minimal extremals.

4. THE C^*-ENVELOPE AND THE SHILOV IDEAL

The C^*-envelope of the operator algebra $\mathcal{A}$, denoted $C^*_e(\mathcal{A})$, is a C^*-algebra which is determined by the property: there exists a completely isometric representation $\gamma : \mathcal{A} \to C^*_e(\mathcal{A})$ such that $C^*(\gamma(\mathcal{A})) = C^*_e(\mathcal{A})$ and if $\rho : \mathcal{A} \to \mathcal{B}(H)$ is any other completely contractive representation, then there exists an onto representation $\pi : C^*(\rho(\mathcal{A})) \to C^*(\gamma(\mathcal{A}))$ such that $\pi(\rho(a)) = \gamma(a)$ for all $a \in \mathcal{A}$.

It is not hard to see that $C^*_e(\mathcal{A})$ is essentially unique, for if ρ also has the properties of γ, then there exists an onto representation $\sigma : C^*(\gamma(\mathcal{A})) \to C^*(\rho(\mathcal{A}))$ with $\sigma(\gamma(a)) = \rho(a)$ for all $a \in \mathcal{A}$. It follows that σ is the inverse of π and thus, as C^*-algebras, $C^*(\gamma(\mathcal{A}))$ equals $C^*(\rho(\mathcal{A}))$.

THEOREM 4.1 ([7]). *Every unital operator algebra has a C^*-envelope.*

Proof. A proof follows directly from Theorem 1.2. Viewing the operator algebra $\mathcal{A}$ as a subspace of $\mathcal{B}(K)$, the inclusion mapping $\iota : \mathcal{A} \to \mathcal{B}(K)$ is a completely isometric representation and thus, according to this proposition, it dilates to a completely isometric representation $\gamma : \mathcal{A} \to \mathcal{B}(H)$ which is a ∂-representation.

To see that $C^*(\gamma(\mathcal{A}))$ is the C^*-envelope, suppose $\psi : \mathcal{A} \to \mathcal{B}(H_\psi)$ is also completely isometric. In this case $\sigma : \psi(\mathcal{A}) \to \mathcal{B}(H)$ given by $\sigma(\psi(a)) = \gamma(a)$ is completely contractive (and thus well-defined). By a theorem of Arveson, there exists a Hilbert space K containing H and a representation $\pi : C^*(\psi(\mathcal{A})) \to \mathcal{B}(K)$ such that $\gamma(a) = \sigma(\psi(a)) = P_H \pi(\psi(a))|H$ ([10], Corollary 6.7). Since $a \mapsto P_H \pi(\psi(a))|H$ is a representation of $\mathcal{A}$ and γ is a ∂-representation, H reduces $\pi(\psi(\mathcal{A}))$. Thus, σ extends to an onto representation $C^*(\psi(\mathcal{A})) \to C^*(\gamma(\mathcal{A}))$. ∎

Arveson says that $\mathcal{J}$ is the *Shilov boundary* of the concrete operator algebra $\mathcal{A} \subset \mathcal{B}(K)$ if $\mathcal{J}$ contains every ideal $\mathcal{I}$ with the property that the restriction of the quotient $q : C^*(\mathcal{A}) \to C^*(\mathcal{A})/\mathcal{I}$ to $\mathcal{A}$ is completely isometric. Since the inclusion of $\mathcal{A}$ into $\mathcal{B}(K)$ is completely isometric, there exists an onto representation $\pi : C^*(\mathcal{A}) \to C^*_e(\mathcal{A}) = C^*(\gamma(a))$ such that $\pi(a) = \gamma(a)$, where γ is a representation as in Theorem 4.1 which generates the C^*-envelope of $\mathcal{A}$. It is left to the interested reader to verify that the kernel of π is the Shilov ideal of $\mathcal{A}$.

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