SIAM J. CONTROL OPTIM.
Vol. 48, No. 6, pp. 4014–4031

BOUNDARY STABILIZATION OF A 1-D WAVE EQUATION WITH IN-DOMAIN ANTIDAMPING*

ANDREY SMYSHLYAEV†, EDUARDO CERPA‡, AND MIROSLAV KRSTIC†

Abstract. We consider the problem of boundary stabilization of a 1-D (one-dimensional) wave equation with an internal spatially varying antidamping term. This term puts all the eigenvalues of the open-loop system in the right half of the complex plane. We design a feedback law based on the backstepping method and prove exponential stability of the closed-loop system with a desired decay rate. For plants with constant parameters the control gains are found in closed form. Our design also produces a new Lyapunov function for the classical wave equation with passive boundary damping.

Key words. wave equation, stabilization, backstepping

AMS subject classifications. 35L05, 93D15

DOI. 10.1137/080742646

1. Introduction.

The asymptotic stability and stabilization by feedback of wave equations in bounded domains are topics which have been widely studied over the past 30 years. The wave equation being conservative, the main idea is to add some dissipation by means of boundary (see, e.g., [3], [11]) or distributed (see, e.g., [5]) damping terms. If the dissipation is large enough, then one expects that the energy of the system is uniformly decreasing. Thus, one expects the solutions to converge polynomially or exponentially to zero. In order to deal with this kind of problem, several tools have been applied. Among them are spectral methods [15], [9], the LQR (linear-quadratic regulator) approach [12], the multiplier technique [7], [13], the microlocal analysis [1], Lyapunov functionals [20], and the Gramian approach [8], [21].

In this paper we are concerned with the stabilization problem of a 1-D (one-dimensional) wave equation with an internal destabilizing term. Because of this term, the system is antistable in the sense that the eigenvalues of the open-loop system can all be in the right half of the complex plane, which produces an exponential growth of the norm of the solutions.

The objective in this paper is to design a boundary control law to stabilize the system. We consider a wave equation on a unit interval as its spatial domain and assume the availability of a Dirichlet actuator at the right end point of the unit interval. We assume that the string is pinned (homogeneous Dirichlet boundary condition) at the left end point. Our approach is based on the backstepping method which uses a Volterra transformation to map an unstable system into a stable "target" PDE (partial differential equation). This method allows us to achieve an arbitrarily large exponential decay rate for the closed-loop system.

In the framework of infinite-dimensional systems, the backstepping method has been mainly used for parabolic and first-order hyperbolic equations [14], [17], [18].

*Received by the editors December 4, 2008; accepted for publication (in revised form) February 27, 2010; published electronically May 5, 2010.
http://www.siam.org/journals/sicon/48-6/74264.html
†Department of Mechanical and Aerospace Engineering, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093 (asmyshly@ucsd.edu, krstic@ucsd.edu).
‡Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile (eduardo.cerpa@usm.cl).

Recently, in [10] the authors have extended the method in order to deal with second-order hyperbolic systems. In that paper, they deal with an unstable wave equation. The instability comes from a boundary term of antistiffness type which generates a finite number of eigenvalues for the open-loop system in the right half of the complex plane. A more challenging problem has been dealt with in [19], where an infinite number of unstable eigenvalues is generated by the boundary antistable term.

To eliminate internal antidamping and add an arbitrary amount of positive damping and stiffness, we develop a novel backstepping transformation. This transformation has a 2×2 structure and is invertible, and the kernels of its four Volterra operators are generated from two coupled second-order hyperbolic PDEs in Goursat form. For plants with constant coefficients, these PDEs can be solved explicitly, resulting in closed-form control gains.

Our design also produces a new Lyapunov function for the classical undamped wave equation with passive boundary damping. This Lyapunov function is "perfect" in the sense that it gives a decay rate that is exactly equal to the one determined by eigenvalues.

The paper is organized as follows. In section 2 we formulate the problem and state the main result. In section 3 we introduce the transformation and the boundary feedback which transform the plant into the "target" system. In section 4, we show that this target PDE is exponentially stable. In section 5, we prove exponential stability of the closed-loop system. In section 6 several explicit control designs are presented. Possible extensions of our approach are discussed in section 7.

2. Statement of the problem and main result.

Consider the wave equation

$$\left\{\begin{array}{l} u_{tt}(x,t) = u_{xx}(x,t) + 2\lambda(x)u_t(x,t) + \alpha(x)u_x(x,t) + \beta(x)u(x,t), \\ u(0,t) = 0, \quad u(1,t) = U(t), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{array}\right. \tag{2.1}$$

where for each time $t \geq 0$, $U(t) \in \mathbb{R}$ is the input and the functions $u(t,\cdot), u_t(t,\cdot) : [0,1] \to \mathbb{R}$ form the state of the system. The functions u_0, u_1 are the initial conditions and the functions λ, α, β are coefficients whose regularity will be defined later. The open-loop plant (i.e., with $U(t) = 0$) may be unstable depending on the function λ. For instance, for positive $\lambda(x)$ and $\beta(x) = \alpha(x) = 0$, all the eigenvalues of the system are located in the right half of the complex plane. Our objective is to design a feedback law which stabilizes (2.1) at the origin.

Without loss of generality, we set $\alpha(x) \equiv 0$. Indeed, if α is not identically zero, the following rescaling of the state variable,

$$v(x,t) = \mathrm{e}^{\frac{1}{2}\int_0^x \alpha(\tau)d\tau} u(x,t),$$

would transform the original wave equation into another one that does not have the first-order spatial derivative term.

Note that for constant λ, one can eliminate the antidamping term by introducing the new variable $v(x,t) = \mathrm{e}^{-\lambda t}u(x,t)$. Then one designs the controller for the v-system that achieves a decay rate larger than λ. However, this idea does not work for spatially varying $\lambda(x)$.

The main idea of this paper is to use the transformation

$$w(x,t) = h(x)u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^x s(x,y)u_t(y,t)dy \tag{2.2}$$

and the feedback

$$U(t) = \frac{1}{h(1)}\left\{\int_0^1 k(1,y)u(y,t)dy + \int_0^1 s(1,y)u_t(y,t)dy\right\}, \tag{2.3}$$

where the function $h = h(x)$ and kernels $k = k(x,y)$ and $s = s(x,y)$ are appropriately chosen, to convert the original system (2.1) into the following:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - 2d(x)w_t(x,t) - c(x)w(x,t), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases} \tag{2.4}$$

with appropriate functions $d = d(x)$ and $c = c(x)$ so that this new system is exponentially stable. The functions d and c can always be chosen to provide any desired decay rate.

Then, we use exponential stability of (2.4) and the invertibility of the transformation (2.2) to obtain stability of the closed-loop system (2.1), (2.3).

Introducing the space $H_L^1(0,1)$ defined by

$$H_L^1(0,1) := \{w \in H^1(0,1)\,;\, w(0) = 0\}$$

and endowed with the H^1-norm, and the domain

$$\mathcal{T} := \{(x,y) \in \mathbb{R}^2;\, 0 \le x \le 1,\, 0 \le y \le x\},$$

we can state our main result.

THEOREM 2.1. *Let $\lambda \in C^2([0,1])$ and $\alpha, \beta \in C^0([0,1])$. There exist functions $h \in C^2([0,1])$ and $k, s \in C^2(\mathcal{T})$ such that for any $(u_0, u_1) \in H_L^1(0,1) \times L^2(0,1)$ satisfying the compatibility condition*

$$u_0(1) = \frac{1}{h(1)}\left\{\int_0^1 k(1,y)u_0(y)dy + \int_0^1 s(1,y)u_1(y)dy\right\},$$

there exists a unique solution of the closed-loop system (2.1), (2.3) in the space $C([0,\infty); H_L^1(0,1)) \cap C^1([0,\infty); L^2(0,1))$. Moreover, for any $\omega > 0$, there exists a positive constant C independent of the initial data such that the solutions satisfy

$$\|(u(\cdot,t), u_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \le C\mathrm{e}^{-\omega t}\|(u_0,u_1)\|_{H^1(0,1)\times L^2(0,1)}. \tag{2.5}$$

3. Control design.

In this section we derive the equations for the functions $h(x)$, $k(x,y)$, and $s(x,y)$ and show that they have a unique twice continuously differentiable solution.

3.1. Derivation of the equations satisfied by the kernels. Using the transformation (2.2) and equation (2.1) with $\alpha = 0$, we get

$$\begin{aligned} w_{tt} - w_{xx} &+ 2d(x)w_t + c(x)w \\ &= \int_0^x u(y)\Big[k_{xx} - k_{yy} - (c(x)+\beta(y))k - 2(\lambda(y)+d(x))s_{yy} \\ &\qquad - 2(\lambda(y)\beta(y) + \lambda''(y) + d(x)\beta(y))s - 4\lambda'(y)s_y\Big]dy \\ &+ \int_0^x u_t(y)\Big[s_{xx} - s_{yy} - 2(\lambda(y)+d(x))k \\ &\qquad - \big(4\lambda^2(y) + 4d(x)\lambda(y) + c(x) + \beta(y)\big)s\Big]dy \\ &+ s(x,0)u_{tx}(0) + u(x)\Big[2\frac{d}{dx}k(x,x) + 2\big(\lambda(x)+d(x)\big)s_y(x,x) \\ &\qquad + 2\lambda'(x)s(x,x) + \big(c(x)+\beta(x)\big)h(x) - h''(x)\Big] \\ &+ u_x(0)\Big[k(x,0) + 2\big(\lambda(0)+d(x)\big)s(x,0)\Big] \\ &+ u_t(x)\Big[2\frac{d}{dx}s(x,x) + 2\big(\lambda(x)+d(x)\big)h(x)\Big] \\ &- u_x(x)\Big[2\big(\lambda(x)+d(x)\big)s(x,x) + 2h'(x)\Big]. \end{aligned}$$

In order to satisfy (2.4), we choose $k = k(x,y)$ and $s = s(x,y)$ as solutions of

$$\begin{aligned} k_{xx}(x,y) - k_{yy}(x,y) = {} & 2\big(\lambda(y)+d(x)\big)s_{yy}(x,y) + \big(c(x)+\beta(y)\big)k(x,y) \\ & + 2\big(\lambda(y)\beta(y) + \lambda''(y) + d(x)\beta(y)\big)s(x,y) \\ & + 4\lambda'(y)s_y(x,y), \end{aligned} \tag{3.1}$$

$$\begin{aligned} 2k'(x,x) = {} & -2\big(\lambda(x)+d(x)\big)s_y(x,x) - 2\lambda'(x)s(x,x) \\ & -\big(c(x)+\beta(x)\big)h(x) + h''(x), \end{aligned} \tag{3.2}$$

$$k(x,0) = 0 \tag{3.3}$$

and

$$\begin{aligned} s_{xx}(x,y) - s_{yy}(x,y) = {} & 2\big(\lambda(y)+d(x)\big)k(x,y) \\ & + \big(4\lambda^2(y) + 4d(x)\lambda(y) + c(x) + \beta(y)\big)s(x,y), \end{aligned} \tag{3.4}$$

$$s'(x,x) = -\big(\lambda(x)+d(x)\big)h(x), \tag{3.5}$$

$$(\lambda(x)+d(x))s(x,x) = -h'(x), \tag{3.6}$$

$$s(x,0) = 0. \tag{3.7}$$

Dividing (3.6) by (3.5), we get $h'(x)h(x) = s(x,x)s'(x,x)$, or, integrating, $h(x)^2 = s(x,x)^2 + A$. Let us choose $h(0) = 1$ so that when all the coefficients of the original and target systems are the same, we have the identity $w(x,t) = u(x,t)$. From (3.7) we have $s(0,0) = 0$, which gives $A = 1$. Using (3.6), we obtain

$$\frac{h'(x)}{\sqrt{h(x)^2 - 1}} = \lambda(x) + d(x),$$

which gives

$$h(x) = \cosh\left(\int_0^x a(\tau)d\tau\right), \tag{3.8}$$

where $a = a(x)$ is defined by

$$a(x) := \lambda(x) + d(x). \tag{3.9}$$

Thus, we can write

$$s(x,x) = -\frac{h'(x)}{a(x)} = -\sinh\left(\int_0^x a(\tau)d\tau\right).$$

Our next goal is to find $k(x,x)$ explicitly. Let us denote $f(x) = s_y(x,x)$. Integrating (3.2) and using (3.3), we obtain

$$2k(x,x) = h'(x) + \int_0^x \left[-2a(\tau)f(\tau) + 2\frac{\lambda'(\tau)h'(\tau)}{a(\tau)} - (\beta(\tau)+c(\tau))h(\tau)\right] d\tau\,. \tag{3.10}$$

We see that we have to find $f(x)$ in order to get $k(x,x)$. From (3.5) we get

$$s_x(x,x) = -a(x)h(x) - f(x).$$

Using (3.5), (3.9), and the previous equation, we get

$$s_{xx}(x,x) - s_{yy}(x,x) = [s_x(x,x) - s_y(x,x)]' = [-a(x)h(x) - 2f(x)]'\,,$$

which shows that f is the solution of the integrodifferential equation

$$\begin{cases} 2f'(x) - 2a(x)\displaystyle\int_0^x a(\tau)f(\tau)d\tau = L(x), \\ f(0) = -a(0), \end{cases} \tag{3.11}$$

where $L = L(x)$ is defined by

$$\begin{aligned} L(x) := {} & (4\lambda(x)a(x) - 2a^2(x) + c(x) + \beta(x))\sinh\left(\int_0^x a(\tau)\,d\tau\right) - a'(x)h(x) \\ & - 2a(x)\int_0^x \frac{\lambda'(\tau)h'(\tau)}{a(\tau)}\,d\tau + a(x)\int_0^x(\beta(\tau)+c(\tau))h(\tau)\,d\tau\,. \end{aligned} \tag{3.12}$$

From (3.11) we obtain the following second order ODE (ordinary differential equation) for $f(x)$:

$$\begin{cases} 2a(x)f''(x) - 2a'(x)f'(x) - 2a^3(x)f(x) = L'(x)a(x) - L(x)a'(x), \\ f(0) = -a(0), \quad f'(0) = -a'(0)/2\,. \end{cases} \tag{3.13}$$

The solution of (3.13) is

$$f(x) = -a(0)\cosh\left(\int_0^x a(\tau)\,d\tau\right) + \frac{1}{2}\int_0^x L(y)\cosh\left(\int_y^x a(\tau)\,d\tau\right)dy. \tag{3.14}$$

Using (3.14), (3.12), and (3.10), after tedious but straightforward calculations we obtain

$$\begin{aligned} k(x,x) = m(x) := {} & \frac{h'(x)}{2a(x)}(2\lambda(x) + a(x) + a(0)) \\ & + \frac{h(x)}{2}\int_0^x (d^2(y) - \lambda^2(y) - \beta(y) - c(y))\,dy. \end{aligned} \tag{3.15}$$

Let us define $\rho_i = \rho_i(x,y)$ with $i = 1,\dots,5$ by

$$\rho_1(x,y) = 2(\lambda(y) + d(x)), \quad \rho_2(x,y) = c(x) + \beta(y), \quad \rho_4(x,y) = 4\lambda'(y),$$

$$\rho_3(x,y) = 2(\lambda(y)\beta(y) + \lambda''(y) + d(x)\beta(y)), \quad \rho_5(x,y) = 4\lambda^2(y) + 4d(x)\lambda(y) + c(x) + \beta(y);$$

then one gets the following equations for the kernel functions:

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = \rho_1(x,y)s_{yy}(x,y) + \rho_2(x,y)k(x,y) \\ \qquad\qquad + \rho_3(x,y)s(x,y) + \rho_4(x,y)s_y(x,y), \\ k(x,x) = m(x), \\ k(x,0) = 0, \end{cases} \tag{3.16}$$

$$\begin{cases} s_{xx}(x,y) - s_{yy}(x,y) = \rho_1(x,y)k(x,y) + \rho_5(x,y)s(x,y), \\ s(x,x) = -\sinh\left(\int_0^x a(\tau)d\tau\right), \\ s(x,0) = 0. \end{cases} \tag{3.17}$$

3.2. Existence of the kernel functions. To prove the existence of solutions of (3.16), (3.17), we perform the following change of variable:

$$\xi = x + y, \quad \eta = x - y.$$

Let us define the functions $G = G(\xi,\eta)$ and $G^s = G^s(\xi,\eta)$ by

$$G(\xi,\eta) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right), \quad G^s(\xi,\eta) = s\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$$

and denote

$$g_1(\xi) := m(\xi/2), \quad g_2(\xi) := -\sinh\left(\int_0^{\xi/2} a(\tau)d\tau\right),$$

$$b_i(\xi,\eta) := \rho_i\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \quad \forall i = 1,\dots,4.$$

From (3.16) and (3.17), one obtains the PDEs

$$\begin{cases} G_{\xi\eta} = b_1(G^s_{\xi\xi} - 2G^s_{\xi\eta} + G^s_{\eta\eta}) + b_2 G + b_3 G^s + b_4(G^s_\xi - G^s_\eta), \\ G(\xi,0) = g_1(\xi), \\ G(\xi,\xi) = 0 \end{cases} \tag{3.18}$$

and

$$\begin{cases} G^s_{\xi\eta} = b_1 G + b_5 G^s, \\ G^s(\xi,0) = g_2(\xi), \\ G^s(\xi,\xi) = 0. \end{cases} \tag{3.19}$$

Integrating (3.18), first with respect to η between 0 and η, and then with respect to ξ between η and ξ, one gets

$$
\begin{aligned}
(3.20)\quad G(\xi,\eta) = g_1(\xi) - g_1(\eta) &+ \frac{1}{4}\int_\eta^\xi \int_0^\eta b_2(\tau,s)G(\tau,s)dsd\tau \\
&+ \frac{1}{4}\int_\eta^\xi \int_0^\eta b_1(\tau,s)\left(G^s_{\xi\xi}(\tau,s) - 2G^s_{\xi\eta}(\tau,s) + G^s_{\eta\eta}(\tau,s)\right)dsd\tau \\
&+ \frac{1}{4}\int_\eta^\xi \int_0^\eta \left(b_3(\tau,s)G^s(\tau,s) + b_4(\tau,s)(G^s_\xi(\tau,s) - G^s_\eta(\tau,s))\right)dsd\tau\,.
\end{aligned}
$$

In the same way, we integrate (3.19), first with respect to η between 0 and η, and then with respect to ξ between η and ξ. One gets

$$
(3.21)\quad G^s(\xi,\eta) = g_2(\xi) - g_2(\eta) + \frac{1}{4}\int_\eta^\xi \int_0^\eta \left(b_1(\tau,s)G(\tau,s) + b_5(\tau,s)G^s(\tau,s)\right)dsd\tau.
$$

We use next a classical iterative method in order to prove that the coupled equations (3.20) and (3.21) have a unique solution. Let us define the functions G^0 and $G^{s,0}$ as

$$
G^0(\xi,\eta) = g_1(\xi) - g_1(\eta)\,, \qquad G^{s,0}(\xi,\eta) = g_2(\xi) - g_2(\eta)
$$

and set up the following recursion for $n = 0, 1, 2, \ldots$:

$$
\begin{aligned}
G^{n+1}(\xi,\eta) = &\frac{1}{4}\int_\eta^\xi \int_0^\eta b_2(\tau,s)G^n(\tau,s)dsd\tau \\
&+ \frac{1}{4}\int_\eta^\xi \int_0^\eta b_1(\tau,s)\left(G^{s,n}_{\xi\xi}(\tau,s) - 2G^{s,n}_{\xi\eta}(\tau,s) + G^{s,n}_{\eta\eta}(\tau,s)\right)dsd\tau \\
&+ \frac{1}{4}\int_\eta^\xi \int_0^\eta \left(b_3(\tau,s)G^{s,n}(\tau,s) + b_4(\tau,s)(G^{s,n}_\xi(\tau,s) - G^{s,n}_\eta(\tau,s))\right)dsd\tau, \\
G^{s,n+1}(\xi,\eta) = &\frac{1}{4}\int_\eta^\xi \int_0^\eta \left(b_1(\tau,s)G^n(\tau,s) + b_5(\tau,s)G^{s,n}(\tau,s)\right)dsd\tau.
\end{aligned}
$$

By defining $M := \max\{2\|g_1'\|_{L^\infty(0,1)}, 2\|g_2'\|_{L^\infty(0,1)}, \|g_2''\|_{L^\infty(0,1)}\}$, we obtain

$$
\begin{aligned}
&|G^0(\xi,\eta)| = |g_1(\xi) - g_1(\eta)| \le \|g_1'\|_{L^\infty(0,1)}|\xi-\eta| \le 2\|g_1'\|_{L^\infty(0,1)} \le M, \\
&|G^{s,0}(\xi,\eta)| = |g_2(\xi) - g_2(\eta)| \le \|g_2'\|_{L^\infty(0,1)}|\xi-\eta| \le 2\|g_2'\|_{L^\infty(0,1)} \le M, \\
&|G^{s,0}_\xi(\xi,\eta)| = |g_2'(\xi)| \le \|g_2'\|_{L^\infty(0,1)} \le M, \\
&|G^{s,0}_\eta(\xi,\eta)| = |g_2'(\eta)| \le \|g_2'\|_{L^\infty(0,1)} \le M, \\
&|G^{s,0}_{\eta\xi}(\xi,\eta)| = 0\,, \\
&|G^{s,0}_{\xi\xi}(\xi,\eta)| = |g_2''(\xi)| \le \|g_2''\|_{L^\infty(0,1)} \le M, \\
&|G^{s,0}_{\eta\eta}(\xi,\eta)| = |g_2''(\eta)| \le \|g_2''\|_{L^\infty(0,1)} \le M.
\end{aligned}
$$

Let us now suppose that for some $n \in \mathbb{N}$ we have

$$
(3.22)\quad \begin{cases}
|G^n(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!}, \quad |G^{s,n}(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!}, \\
{}^{1.5pt}|G^{s,n}_\xi(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!}, \quad |G^{s,n}_\eta(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^n}{n!}, \\
{}^{1.5pt}|G^{s,n}_{\xi\xi}(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^{n-1}}{(n-1)!}, \quad |G^{s,n}_{\xi\eta}(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^{n-1}}{(n-1)!}, \\
{}^{1.5pt}|G^{s,n}_{\eta\eta}(\xi,\eta)| \le MK^n\frac{(\xi+\eta)^{n-1}}{(n-1)!}.
\end{cases}
$$

From (3.20) and (3.21) we obtain

$$
\begin{aligned}
|G^{s,n+1}(\xi,\eta)| &\leq \frac{1}{4}\|b_1\|_{L^\infty}\int_\eta^\xi\int_0^\eta |G^n(\tau,s)|dsd\tau + \frac{1}{4}\|b_5\|_{L^\infty}\int_\eta^\xi\int_0^\eta |G^{s,n}(\tau,s)|dsd\tau \\
&\leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{4}\right)\frac{MK^n}{n!}\int_\eta^\xi\int_0^\eta (\tau+s)^n dsd\tau \\
&\leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{4}\right)\frac{MK^n}{(n+1)!}\int_\eta^\xi ((\tau+\eta)^{n+1}-\tau^{n+1})d\tau \\
&\leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{4}\right)\frac{MK^n}{(n+1)!}|\xi-\eta|(\xi+\eta)^{n+1} \\
&\leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{2}\right)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
|G^{n+1}(\xi,\eta)| &\leq \frac{\|b_2\|_{L^\infty}}{4}\int_\eta^\xi\int_0^\eta |G^n(\tau,s)|dsd\tau + \frac{\|b_3\|_{L^\infty}}{4}\int_\eta^\xi\int_0^\eta |G^{s,n}(\tau,s)|dsd\tau \\
&\quad + \frac{\|b_1\|_{L^\infty}}{4}\int_\eta^\xi\int_0^\eta |G^{s,n}_{\xi\xi}(\xi,s)| + 2|G^{s,n}_{\eta\xi}(\eta,s)||G^{s,n}_{\eta\eta}(\xi,s)|dsd\tau \\
&\quad + \frac{\|b_4\|_{L^\infty}}{4}\int_\eta^\xi\int_0^\eta |G^{s,n}_{\xi}(\xi,s)| + |G^{s,n}_{\eta}(\eta,s)|dsd\tau \\
&\leq \left(\frac{\|b_2\|_{L^\infty}+\|b_3\|_{L^\infty}+2\|b_4\|_{L^\infty}}{4}\right)\frac{MK^n}{n!}\int_\eta^\xi\int_0^\eta (\tau+s)^n dsd\tau \\
&\quad + \|b_1\|_{L^\infty}\frac{MK^n}{(n-1)!}\int_\eta^\xi\int_0^\eta (\tau+s)^{(n-1)}dsd\tau \\
&\leq \left(\frac{\|b_2\|_{L^\infty}+\|b_3\|_{L^\infty}+2\|b_4\|_{L^\infty}+4\|b_1\|_{L^\infty}}{2}\right)\frac{MK^n}{(n+1)!}(\xi+\eta)^{n+1}.
\end{aligned}
$$

In a very similar way, we obtain

$$
\left|G^{s,n+1}_{\xi}(\xi,\eta)\right| \leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{4}\right)MK^n\frac{(\xi+\eta)^{n+1}}{(n+1)!},
$$

$$
\left|G^{s,n+1}_{\eta}(\xi,\eta)\right| \leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{4}\right)MK^n\frac{(\xi+\eta)^{n+1}}{(n+1)!},
$$

$$
\left|G^{s,n+1}_{\xi\eta}(\xi,\eta)\right| \leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}}{4}\right)MK^n\frac{(\xi+\eta)^{n}}{n!},
$$

$$
\left|G^{s,n+1}_{\xi\xi}(\xi,\eta)\right| \leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}+\|b_{1\xi}\|_{L^\infty}+\|b_{5\xi}\|_{L^\infty}}{4}\right)MK^n\frac{(\xi+\eta)^{n}}{n!},
$$

and

$$
\left|G^{s,n+1}_{\eta\eta}(\xi,\eta)\right| \leq \left(\frac{\|b_1\|_{L^\infty}+\|b_5\|_{L^\infty}+\|b_{1\eta}\|_{L^\infty}+\|b_{5\eta}\|_{L^\infty}}{4}\right)MK^n\frac{(\xi+\eta)^{n}}{n!}.
$$

Thus, by induction we have proved that (3.22) holds with a constant K given, for instance, by

$$K = \frac{1}{2} \max \big\{ \|b_1\|_{C^1} + \|b_5\|_{C^1}, 4\|b_1\|_{L^\infty} + \|b_2\|_{L^\infty} + \|b_3\|_{L^\infty} + 2\|b_4\|_{L^\infty} \big\}.$$

Once the estimates (3.22) are proved, it follows that the solutions of (3.20) and (3.21) are given by the series

$$G^s(\xi,\eta) = \sum_{n=0}^{\infty} G^{s,n}(\xi,\eta), \quad G(\xi,\eta) = \sum_{n=0}^{\infty} G^n(\xi,\eta), \tag{3.23}$$

which define two continuous functions. To see that these functions are indeed more regular, we use the equations which they satisfy. From (3.21), we see that G^s belongs to C^2 if b_1 and b_5 are continuous. Then, from (3.20), we see that if b_i with $i = 1, \dots, 4$ are continuous functions, then G belongs to C^2. Thus, we obtain the following result asserting the existence of the kernel functions k and s.

THEOREM 3.1. *Let $\lambda \in C^2([0,1])$, $\beta \in C^0([0,1])$, and $d, c \in C^0([0,1])$. Then (3.16) and (3.17) have a unique solution $k, s \in C^2(\mathcal{T})$.*

3.3. Transforming the plant into the "target" system. Let us define the map

$$\begin{aligned} \Pi : H^1_L(0,1) \times L^2(0,1) &\longrightarrow \quad H^1_L(0,1) \times L^2(0,1), \\ (q_1, q_2) &\longmapsto \quad \Pi(q_1,q_2) = (z_1, z_2), \end{aligned}$$

where z_1, z_2 are defined by

$$\begin{aligned} z_1(x) &:= h(x)q_1(x) - \int_0^x k(x,y)q_1(y)dy - \int_0^x s(x,y)q_2(y)dy, \\ z_2(x) &:= s_y(x,x)q_1(x) - s(x,x)q_1'(x) + h(x)q_2(x) \\ &\quad - \int_0^x \Big[\lambda(y)s(x,y) + k(x,y)\Big] q_2(y)dy - \int_0^x \Big[\beta(y)s(x,y) + s_{yy}(x,y)\Big] q_1(y)dy. \end{aligned}$$

This linear map is continuous, and hence there exists a positive constant D_1 such that

$$\|\Pi(q_1,q_2)\|_{H^1(0,1)\times L^2(0,1)} \le D_1 \|(q_1,q_2)\|_{H^1(0,1)\times L^2(0,1)}. \tag{3.24}$$

The importance of Π is that it maps solutions $(q(t), q_t(t))$ of

$$\begin{aligned} q_{tt}(x,t) &= q_{xx}(x,t) + 2\lambda(x)q_t(x,t) + \beta(x)q(x,t), \\ q(0,t) &= 0, \quad q(1,t) = \frac{1}{h(1)} \left\{ \int_0^1 k(1,y)q(y,t)dy + \int_0^1 s(1,y)q_t(y,t)dy \right\} \end{aligned}$$

into solutions $(z(t), z_t(t)) := \Pi(q(t), q_t(t))$ of

$$\begin{aligned} z_{tt}(x,t) &= z_{xx}(x,t) - 2d(x)z_t(x,t) - c(x)z(x,t), \\ z(0,t) &= z(1,t) = 0. \end{aligned}$$

The map Π, converting the original unstable system into the target system, is invertible. Indeed, to obtain the kernel functions $\hat{k} = \hat{k}(x,y)$ and $\hat{s} = \hat{s}(x,y)$ defining Π^{-1}, we simply replace the functions $d(x)$ by $-\lambda(x)$ and $\lambda(x)$ by $-d(x)$ in the previous analysis for the kernels $k = k(x,y)$ and $s = s(x,y)$. Thus, we get a map

$$\Pi^{-1} : H^1_L(0,1) \times L^2(0,1) \longrightarrow H^1_L(0,1) \times L^2(0,1)$$

and a positive constant D_2 such that

$$\|\Pi^{-1}(z_1,z_2)\|_{H^1(0,1)\times L^2(0,1)} \le D_2 \|(z_1,z_2)\|_{H^1(0,1)\times L^2(0,1)}. \tag{3.25}$$

4. Stability of the target system. Recall that the target system is

$$\text{(4.1)} \qquad \begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - 2d(x)w_t(x,t) - c(x)w(x,t), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases}$$

with $d = d(x)$ and $c = c(x)$ two continuous functions that can be chosen arbitrarily (as long as exponential stability is maintained). The exponential stability of (4.1) has been studied by Cox and Zuazua in [5] in the case $c = 0$ and by Shubov in [16] in the general case, considering even a nonconstant diffusive coefficient. Their approach is spectral: they prove that the eigenfunctions of the underlying nonself-adjoint operator form a Riesz basis of the space and that the best exponential decay rate is exactly given by $\sup_{k\in\mathbb{N}} \Re(\sigma_k)$, where the set $\{\sigma_k\}_{k\in\mathbb{N}}$ is the set of eigenvalues of the stationary operator and $\Re(z)$ stands for the real part of a complex number z. The result in [5] is the following.

THEOREM 4.1 (Cox and Zuazua [5]). *There exist two positive constant C, ω such that for any $(w_0, w_1) \in H_0^1(0,1) \times L^2(0,1)$, the solution of* (4.1) *satisfies*

$$\text{(4.2)} \quad \|(w(\cdot,t), w_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \le Ce^{-\omega t}\|(w_0,w_1)\|_{H^1(0,1)\times L^2(0,1)} \quad \forall t > 0.$$

In this paper, the functions d and c are part of the design of the feedback law, and hence we are able to consider (4.1) with constant coefficients. In this case, for any $\omega > 0$, we can find the parameters d and c so that (4.2) holds.

For the sake of completeness, we give a proof of (4.2) here. First, we study the constant coefficient case, where we can choose the parameters d and c so that the exponential decay rate ω is as large as desired. Then, in the nonconstant case, we apply the Lyapunov approach to quickly get the stability result. In this case, we do not get an arbitrarily large decay rate.

Remark 4.2. In [6] and [2] the exponential stability of (4.1) is proved in the case of an indefinite damping term, i.e., with a damping taking positive and negative values. It is done under a spectral hypothesis involving the damping and the spectral elements of the underlying operator. This hypothesis implies that the damping is "more positive than negative."

4.1. Constant coefficient case. Let us first consider the case where the design functions d and c are positive constants. It is well known that if we take $d > 0$ and $c = 0$, then the system is exponentially stable, but an arbitrary decay rate cannot be achieved due to overdamping. As can be seen below in the formula for the eigenvalues, the maximal decay rate in this case is π even for large values of the parameter d. That is the reason why we also consider a nonzero parameter c.

In order to apply a spectral approach, let us define the operator $A : D(A) \subset H_0^1(0,1) \times L^2(0,1) \to H_0^1(0,1) \times L^2(0,1)$ as follows:

$$D(A) := H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1),$$

$$A(v_1, v_2) = (v_2, v_1'' - 2dv_2 - cv_1).$$

By defining $v := w_t$, we write system (4.1) as

$$\begin{cases} \frac{d}{dt}(w(t), v(t)) = A(w(t), u(t)), \\ (w(0), v(0)) = (w_0, w_1). \end{cases}$$

The eigenvalues of the operator A are given by

$$\sigma_{-k} = -d - \sqrt{d^2 - c - k^2\pi^2}, \quad \sigma_k = -d + \sqrt{d^2 - c - k^2\pi^2} \quad \forall k \in \mathbb{N},$$

with the corresponding complex-valued eigenfunctions

$$\varphi_{-k} = \sqrt{2}\begin{pmatrix} \dfrac{1}{\sigma_{-k}}\sin(k\pi x) \\ \sin(k\pi x) \end{pmatrix}, \quad \varphi_k = \sqrt{2}\begin{pmatrix} \dfrac{1}{\sigma_k}\sin(k\pi x) \\ \sin(k\pi x) \end{pmatrix} \quad \forall k \in \mathbb{N}.$$

For any $d > 0$, we choose the parameter c such that $d^2 - c < \pi^2$. Thus for any $k \in \mathbb{Z}$, $\Re(\sigma_k) = -d$. As the parameter d can be chosen arbitrarily, we obtain the exponential decay to zero with any prescribed decay rate. Indeed, since the functions $\{\sqrt{2}\sin(k\pi x)\}_{k\in\mathbb{N}}$ form a basis of the space $L^2(0,1)$ and the functions $\{\frac{\sqrt{2}}{k\pi}\sin(k\pi x)\}_{k\in\mathbb{N}}$ form a basis of the spaces $H_0^1(0,1)$, we can write for any $w_0 \in H_0^1(0,1)$ and $w_1 \in L^2(0,1)$,

$$w_0(x) = \sum_{k\in\mathbb{N}} w_0^k \frac{\sqrt{2}\sin(k\pi x)}{k\pi}, \quad w_1(x) = \sum_{k\in\mathbb{N}} w_1^k \sqrt{2}\sin(k\pi x),$$

with $\{w_0^k\}_{k\in\mathbb{N}}, \{w_1^k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\|w_0\|_{H^1(0,1)} = \left(\sum_{k\in\mathbb{N}} |w_0^k|^2\right)^{1/2}, \quad \|w_1\|_{L^2(0,1)} = \left(\sum_{k\in\mathbb{N}} |w_1^k|^2\right)^{1/2}.$$

By using these spectral elements, it is not difficult to see that the solution of (4.1) can be written as

$$w(x,t) = \sum_{k\in\mathbb{N}} e^{-dt}\left\{w_0^k\cos(\alpha_k t) + \left(w_1^k + \frac{dw_0^k}{k\pi}\right)\sin(\alpha_k t)\right\}\frac{\sqrt{2}}{k\pi}\sin(k\pi x)$$

with $\alpha_k = \sqrt{k^2\pi^2 + c}$, and therefore one gets (4.2) with $\omega = d$.

4.2. General case. Even though one can always choose c and d as constants and achieve any prescribed decay rate, in some cases, it may be desirable to choose these parameters to be spatially varying, for example, to improve performance. Therefore, we do not limit ourselves to the constant parameter case and give here the stability proof for general $c(x) > 0$, $d(x) > 0$.

Given $w_0 \in H_0^1(0,1)$ and $w_1 \in L^2(0,1)$, we consider $w \in C([0,+\infty); H_0^1(0,1)) \cap C^1([0,+\infty); L^2(0,1))$ the solution of the target system (4.1). Along the trajectory defined by this solution, we define the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 \left(|w_x|^2 + |w_t|^2\right)dx + \delta\int_0^1 ww_t dx + \frac{1}{2}\int_0^1 c(x)|w|^2 dx,$$

where δ is a positive real number. The parameter δ has to satisfy the inequalities

$$\delta < 1, \quad \kappa + \delta < c(x), \quad \kappa + 2\delta < 2d(x) \ \ \forall x \in [0,1],$$

where κ is a positive real number.

The following two lemmas are easily proved by direct calculation.

Lemma 4.3. *The function V is nonincreasing. Moreover, we have*

$$\dot{V}(t) \leq -2\delta V(t).$$

Lemma 4.4. *The function V is positive. Moreover, we have*

$$V(t) \geq \min\left\{\frac{1-\delta}{2}, \frac{\kappa}{2}\right\} \|(w, w_t)\|^2_{H^1(0,1)\times L^2(0,1)}.$$

From Lemma 4.3, we obtain that $V(t) \leq V(0)e^{-2\delta t}$, and from Lemma 4.4 we obtain the exponential decay of the $H^1 \times L^2$-norm in Theorem 4.1 with $\omega = \delta$ and

$$C = \frac{\max\{\delta+1, \delta+\|c\|_{L^\infty}\}}{\min\{1-\delta, \kappa\}}.$$

5. Closed-loop system. From the previous analysis, it is easy to see the following:

- If $u_0 \in H^1_L(0,1)$ and $u_1 \in L^2(0,1)$ satisfy the compatibility condition

$$u_0(1) = \frac{1}{\cosh\left(\int_0^1 a(\tau)d\tau\right)} \left\{\int_0^1 k(1,y)u_0(y)dy + \int_0^1 s(1,y)u_1(y)dy\right\},$$

  then $(w_0, w_1) := \Pi(u_0, u_1)$ belongs to $H^1_0(0,1) \times L^2(0,1)$. Furthermore, we have (see (3.24))

$$\|(w_0, w_1)\|_{H^1(0,1)\times L^2(0,1)} \leq D_1 \|(u_0, u_1)\|_{H^1(0,1)\times L^2(0,1)}.$$

- For $(w_0, w_1) \in H^1_0(0,1) \times L^2(0,1)$ we have that $w \in C([0,\infty); H^1_0(0,1)) \cap C^1([0,\infty); L^2(0,1))$, the unique solution of (4.1), satisfies (see Theorem 4.1)

$$\|(w(\cdot,t), w_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \leq Ce^{-\omega t}\|(w_0, w_1)\|_{H^1(0,1)\times L^2(0,1)} \quad \forall t > 0.$$

- By defining for any $t > 0$, $(u(t), u_t(t)) := \Pi^{-1}(w(t), w_t(t))$, we get the solution of the closed-loop system

$$\begin{aligned}
&u_{tt}(x,t) = u_{xx}(x,t) + 2\lambda(x)u_t(x,t) + \beta(x)u(x,t),\\
&u(0,t) = 0, \quad u(1,t) = \frac{\int_0^1 k(1,y)u(y,t)dy + \int_0^1 s(1,y)u_t(y,t)dy}{\cosh\left(\int_0^1 a(\tau)d\tau\right)},\\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x).
\end{aligned}$$

  Moreover, we have (see (3.25))

$$\|(u, u_t)\|_{H^1(0,1)\times L^2(0,1)} \leq D_2 \|(w, w_t)\|_{H^1(0,1)\times L^2(0,1)}.$$

Thus, we obtain the exponential decay to zero of the solutions of the closed-loop system (2.1), (2.3), which ends the proof of Theorem 2.1.

6. Closed-form controllers. In this section we present several explicit control designs.

6.1. Undamped wave equation. Let $\lambda \equiv 0$, $\beta \equiv 0$ in (2.1):

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), \\ u(0,t) = 0, \\ u(1,t) = U(t)\,. \end{cases} \tag{6.1}$$

This wave equation has all of its eigenvalues on the imaginary axis. Let us move all of them to the left in the complex plane by the same distance, parallel to the real axis (in other words, only real parts of the eigenvalues are changed). This corresponds to selecting the "critically damped" target system, a special case of (2.4) with $c = d^2$:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - 2dw_t(x,t) - d^2 w(x,t), \\ w(0,t) = 0, \\ w(1,t) = 0. \end{cases} \tag{6.2}$$

All of the eigenvalues of the above system lie on the vertical line $\Re\{\sigma_k\} = -d$, which is easy to see by using the transformation $w = \mathrm{e}^{-dt}v$ and showing that v satisfies the undamped wave equation.

The PDEs (3.16) and (3.17) become

$$\begin{cases} k_{xx}(x,y) = k_{yy}(x,y) + 2ds_{yy}(x,y), \\ k(x,0) = 0, \\ k(x,x) = d\sinh(dx) \end{cases} \tag{6.3}$$

and

$$\begin{cases} s_{xx}(x,y) = s_{yy}(x,y) + 2dk(x,y), \\ s(x,0) = 0, \\ s(x,x) = -\sinh(dx)\,. \end{cases} \tag{6.4}$$

The form of the boundary conditions in the above PDEs suggests

$$k(x,y) = d\sinh(dy), \quad s(x,y) = -\sinh(dy) \tag{6.5}$$

as a guess for a solution. Substituting these functions into the PDEs (6.3), (6.4) we confirm that (6.5) is indeed a (unique) solution.

The transformation (2.2) can now be written as

$$w(x,t) = \cosh(dx)u(x,t) + \int_0^x \sinh(dy)(u_t(y,t) - du(y,t))\,dy \tag{6.6}$$

and the controller is

$$U(t) = -\int_0^1 \frac{\sinh(dy)}{\cosh(d)}(u_t(y,t) - du(y,t))\,dy\,. \tag{6.7}$$

Note that the gain of this controller is bounded by a linear function of d.

Thanks to the gain of the transformation (6.6) being only a function of y, we can write (6.6) in an algebraic form

$$\begin{bmatrix} w_t + dw \\ w_x \end{bmatrix} = \begin{bmatrix} \cosh(dx) & \sinh(dx) \\ \sinh(dx) & \cosh(dx) \end{bmatrix} \begin{bmatrix} u_t \\ u_x \end{bmatrix}. \tag{6.8}$$

The inverse transformation is

$$\begin{bmatrix} u_t \\ u_x \end{bmatrix} = \begin{bmatrix} \cosh(dx) & -\sinh(dx) \\ -\sinh(dx) & \cosh(dx) \end{bmatrix} \begin{bmatrix} w_t + dw \\ w_x \end{bmatrix}, \tag{6.9}$$

or, in the integral form,

$$u(x,t) = \cosh(dx)w(x,t) - \int_0^x \sinh(dy)(w_t(y,t) + 2dw(y,t))\,dy. \tag{6.10}$$

6.2. "Perfect" Lyapunov function for passively damped wave equation. Consider the plant

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), \\ u(0,t) = 0, \\ u_x(1,t) = U(t)\,. \end{cases} \tag{6.11}$$

It is well known that a so-called passive damper $U(t) = -c_1 u_t(1,t)$, $c_1 > 0$, $c_1 \neq 1$, exponentially stabilizes this system. Let us see what the backstepping design gives for this plant. We use the transformation (6.6) and the following target system:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - 2dw_t(x,t) - d^2 w(x,t), \\ w(0,t) = 0, \\ w_x(1,t) = 0\,. \end{cases} \tag{6.12}$$

From (6.8) it is easy to see that the controller is

$$U(t) = -\tanh(d)u_t(1,t), \tag{6.13}$$

so we recover the classical passive damper with $c_1 = \tanh(d)$. This is not surprising, because our design moves eigenvalues to the left parallel to the real axis (since (6.12) is critically damped) and that is also exactly what a passive damper is known to do. To put it another way, we found the similarity transformation (6.6) between the plant with boundary damping and the plant with internal damping (critically damped). The benefit of that similarity transformation is that for system (6.12) it is much easier to come up with the Lyapunov function that shows arbitrary decay rate. In fact, the simple Lyapunov function

$$V = \frac{1}{2}\int_0^1 (w_t + dw)^2\,dx + \frac{1}{2}\int_0^1 w_x^2\,dx \tag{6.14}$$

gives

$$\dot{V} = -2dV, \tag{6.15}$$

which is the exact decay rate given by the eigenvalues (hence one can call this Lyapunov function "perfect" in some sense).

Using the transformation (6.8), we rewrite the above Lyapunov function in the original variables. After simple calculations one gets

$$V = \frac{1}{2}\int_0^1 \cosh(2dx)(u_t^2 + u_x^2)\,dx + \int_0^1 \sinh(2dx)u_t u_x\,dx, \tag{6.16}$$

or, in another form,

$$V = \frac{1}{4}\int_0^1 \mathrm{e}^{2dx}(u_t + u_x)^2\,dx + \frac{1}{4}\int_0^1 \mathrm{e}^{-2dx}(u_t - u_x)^2\,dx. \tag{6.17}$$

To the best of our knowledge, such a Lyapunov function (which shows the precise decay rate given by eigenvalues) does not exist in the previous literature on this classical problem. In the form (6.17) our Lyapunov function resembles the one in [4] for the first-order hyperbolic equations. However, the control design in [4] is different (passive dampers on both ends for two transport PDEs interconnected through boundaries) and the best decay rate is not shown.

Given that $c_1 = \tanh(d) < 1$ for all $d > 0$, it may appear that the design above recovers a passive damper only for $0 < c_1 < 1$, while it is known that $c_1 > 1$ also works. However, simply modifying the boundary condition of the target system (6.12) at $x = 1$ to the dynamic boundary condition $w_t(1,t) + dw(1,t) = 0$ (which shifts eigenvalues vertically by $\pi/2$), and using the transformation (6.8), we get $U(t) = -\coth(d)u_t(1,t)$, $c_1 = \coth(d) > 1$. The Lyapunov functions (6.14), (6.16), (6.17) with $d = \coth^{-1}(c_1)$ give $\dot{V} = -2dV$.

6.3. Assignment of arbitrary damping and stiffness for critically anti-damped wave equation. Consider the plant

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + 2\lambda u_t(x,t) - \lambda^2 u(x,t), \\ u(0,t) = 0, \\ u(1,t) = U(t)\,. \end{cases} \tag{6.18}$$

All eigenvalues of this plant lie on the vertical line $\Re\{\sigma_k\} = \lambda$. We assign arbitrary damping and stiffness using a two-step design.

Step 1. Transform the plant into the critically damped system (6.2). This corresponds to moving all eigenvalues to the left by $(\lambda + d)$. The PDEs for k and s are

$$\begin{cases} k_{xx}(x,y) = k_{yy}(x,y) + 2(\lambda+d)s_{yy}(x,y) + (d^2-\lambda^2)k - 2\lambda^2(\lambda+d)s, \\ k(x,0) = 0, \\ k(x,x) = (2\lambda+d)\sinh((d+\lambda)x) \end{cases} \tag{6.19}$$

and

$$\begin{cases} s_{xx}(x,y) = s_{yy}(x,y) + 2(\lambda+d)k(x,y) + (3\lambda^2 + 4\lambda d + d^2)s, \\ s(x,0) = 0, \\ s(x,x) = -\sinh\left((\lambda+d)x\right). \end{cases} \tag{6.20}$$

As in section 6.1, based on the boundary conditions we make the following guess:

$$k(x,y) = (2\lambda+d)\sinh((\lambda+d)y), \qquad s(x,y) = -\sinh((\lambda+d)y)\,. \tag{6.21}$$

One can then verify that this pair of functions is indeed a solution of the PDEs (6.19), (6.20).

Step 2. Adjust the stiffness coefficient to the desired level. We use the transformation

$$\bar{w}(x,t) = w(x,t) - \int_0^x p(x,y)w(y,t)\,dy \tag{6.22}$$

to convert (6.2) into the system

$$\begin{cases} \bar{w}_{tt}(x,t) = \bar{w}_{xx}(x,t) - 2d\bar{w}_t(x,t) - c\bar{w}(x,t), \\ \bar{w}(0,t) = 0, \\ \bar{w}(1,t) = 0. \end{cases} \tag{6.23}$$

One can show that $p(x,y)$ satisfies

$$\begin{cases} p_{xx}(x,y) = p_{yy}(x,y) + (c-d^2)p(x,y), \\ p(x,0) = 0, \\ p(x,x) = -\frac{1}{2}(c-d^2)x. \end{cases} \tag{6.24}$$

The solution to this PDE is [17]

$$p(x,y) = -(c-d^2)y\frac{I_1\left(\sqrt{(c-d^2)(x^2-y^2)}\right)}{\sqrt{(c-d^2)(x^2-y^2)}}\,, \tag{6.25}$$

where I_1 is the modified Bessel function of order one.

To find the total transformation from u to $\bar{w}$, we combine the transformations (2.2) with (6.22):

$$\begin{aligned} \bar{w}(x,t) = &\left[\cosh((\lambda+d)x)u(x,t) - \int_0^x k(x,y)u(y,t)\,dy - \int_0^x s(x,y)u_t(y,t)\,dy\right] \\ &- \int_0^x p(x,y)\cosh((\lambda+d)y)u(y,t)\,dy \\ &+ \int_0^x p(x,y)\left[\int_0^y k(y,\xi)u(t,\xi)\,d\xi + \int_0^y s(y,\xi)u_t(t,\xi)\,d\xi\right]dy \\ = &\cosh((\lambda+d)x)u(x,t) \\ &- \int_0^x \left[k(x,y) + p(x,y)\cosh((\lambda+d)y) - \int_y^x p(x,\xi)k(\xi,y)\,d\xi\right]u(y,t)\,dy \\ &- \int_0^x \left[s(x,y) - \int_y^x p(x,\xi)s(\xi,y)\,d\xi\right]u_t(y,t)\,dy \\ = &\cosh((\lambda+d)x)u(x,t) - \int_0^x \bar{k}(x,y)u(y,t)\,dy - \int_0^x \bar{s}(x,y)u_t(y,t)\,dy\,. \end{aligned}$$

Using the expressions (6.21), (6.24) for the gains k, s, and p, we obtain

$$\begin{aligned} \bar{s}(x,y) &= -I_0\left(\sqrt{(c-d^2)(x^2-y^2)}\right)\sinh((\lambda+d)y)\,, \\ \bar{k}(x,y) &= (2\lambda+d)I_0\left(\sqrt{(c-d^2)(x^2-y^2)}\right)\sinh((\lambda+d)y) \\ &\quad - (c-d^2)y\frac{I_1\left(\sqrt{(c-d^2)(x^2-y^2)}\right)}{\sqrt{(c-d^2)(x^2-y^2)}}\cosh((\lambda+d)y). \end{aligned}$$

The feedback law is

$$\begin{aligned} U(t) = &\int_0^1 \frac{\sinh((\lambda+d)y)}{\cosh(\lambda+d)}I_0\left(\sqrt{(c-d^2)(1-y^2)}\right)[(2\lambda+d)u(y,t) - u_t(y,t)]\,dy \\ &- \int_0^1 (c-d^2)y\frac{\cosh((\lambda+d)y)}{\cosh(\lambda+d)}\frac{I_1\left(\sqrt{(c-d^2)(1-y^2)}\right)}{\sqrt{(c-d^2)(1-y^2)}}u(y,t)\,dy. \end{aligned}$$

6.4. Plant with "pure" antidamping. For the plant

$$u_{tt}(x,t) = u_{xx}(x,t) + 2\lambda u_t(x,t)$$

the two-step approach described above gives the following gains for controller (2.3):

$$s(x,y) = \sinh((\lambda+d)y) + \lambda y r(x,y),$$
$$k(x,y) = -\lambda y \cosh((\lambda+d)x)\frac{I_1(\lambda\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} - (2\lambda+d)\left[\sinh((\lambda+d)y) + \lambda y r(x,y)\right],$$

where

$$r(x,y) = \int_y^x \sinh((\lambda+d)\xi)\frac{I_1(\lambda\sqrt{\xi^2-y^2})}{\sqrt{\xi^2-y^2}}d\xi.$$

7. Extensions. The control design presented in this paper allows several straightforward extensions.

7.1. Neumann actuation. To extend the design to the plants with Neumann actuation we modify one of the boundary conditions of the target system (2.4) from $w(1,t) = 0$ to $w_x(1,t) = 0$. Using the exact same transformation (2.2) then gives the following feedback:

$$u_x(1,t) = \frac{1}{h(1)}\Bigg[(-h'(1) + k(1,1))u(1,t) + s(1,1)u_t(1,t) + \int_0^1 k_x(1,y)u(y,t)\,dy + \int_0^1 s_x(1,y)u_t(y,t)\,dy\Bigg]. \tag{7.1}$$

7.2. Robin boundary condition at the uncontrolled end. For plants with the boundary condition $u_x(0,t) = -qu(0,t)$ instead of the Dirichlet $u(1,t) = 0$ the transformation (2.2) leads to the PDEs (3.16), (3.17) with boundary conditions modified as follows:

$$k(x,x) = m(x) + q, \quad k_y(x,0) = -qk(x,0), \quad s_y(x,0) = -qs(x,0), \tag{7.2}$$

and the same boundary condition for $s(x,x)$. Using the method of successive approximations with very slight modifications compared to section 3.2, one proves existence and uniqueness of the solution of the control gain PDEs.

7.3. In-domain boundary and integral terms. The transformation (2.2) also works for the following class of plants:

$$u_{tt} = u_{xx} + 2\lambda(x)u_t + \beta(x)u + g_1(x)u(0,t) + g_2(x)u_x(0,t) + \int_0^x f(x,y)u(y,t)\,dy, \tag{7.3}$$

which may appear as a part of the design for more complex systems. The extra terms here are strict-feedback and therefore do not pose any difficulties for the backstepping design.

7.4. Observers and output feedback. In the designs in previous sections we assumed the measurements of u and u_t across the domain. Using the ideas presented in [18], it is possible to design dual observers which require only boundary measurements of u and u_t, either on the same or on the opposite boundary with actuation. These observers can then be combined with the backstepping controllers using the certainty equivalence principle.

REFERENCES

[1] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065.

[2] A. Benaddi and B. Rao, *Energy decay rate of wave equations with indefinite damping*, J. Differential Equations, 161 (2000), pp. 337–357.

[3] G. Chen, *Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain*, J. Math. Pures Appl., 58 (1979), pp. 249–273.

[4] J. M. Coron, B. d'Andréa-Novel, and G. Bastin, *A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws*, IEEE Trans. Automat. Control, 52 (2007), pp. 2–11.

[5] S. Cox and E. Zuazua, *The rate at which energy decays in a damped string*, Comm. Partial Differential Equations, 19 (1994), pp. 213–243.

[6] P. Freitas and E. Zuazua, *Stability results for the wave equation with indefinite damping*, J. Differential Equations, 132 (1996), pp. 338–352.

[7] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, RAM Res. Appl. Math., Masson, Paris; John Wiley & Sons, Ltd., Chichester, UK, 1994.

[8] V. Komornik, *Rapid boundary stabilization of linear distributed systems*, SIAM J. Control Optim., 35 (1997), pp. 1591–1613.

[9] V. Komornik and P. Loreti, *Fourier Series in Control Theory*, Springer Monogr. Math., Springer-Verlag, New York, 2005.

[10] M. Krstic, B. Z. Guo, A. Balogh, and A. Smyshlyaev, *Output-feedback stabilization of an unstable wave equation*, Automatica, 44 (2008), pp. 63–74.

[11] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Differential Equations, 50 (1983), pp. 163–182.

[12] I. Lasiecka and R. Triggiani, *Riccati equations for hyperbolic partial differential equations with* $L_2(0,T;L_2(\Gamma))$*—Dirichlet boundary terms*, SIAM J. Control Optim., 24 (1986), pp. 884–925.

[13] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Masson, Paris, 1988.

[14] W. Liu, *Boundary feedback stabilization of an unstable heat equation*, SIAM J. Control Optim., 42 (2003), pp. 1033–1043.

[15] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639–739.

[16] M. A. Shubov, *Nonselfadjoint operators generated by the equation of a nonhomogeneous damped string*, Trans. Amer. Math. Soc., 349 (1997), pp. 4481–4499.

[17] A. Smyshlyaev and M. Krstic, *Closed-form boundary state feedbacks for a class of* 1*-D partial integro-differential equations*, IEEE Trans. Automat. Control, 49 (2004), pp. 2185–2202.

[18] A. Smyshlyaev and M. Krstic, *Backstepping observers for a class of parabolic PDEs*, Systems Control Lett., 54 (2005), pp. 613–625.

[19] A. Smyshlyaev and M. Krstic, *Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary*, Systems Control Lett., 58 (2009), pp. 617–623.

[20] M. Tucsnak, *Boundary stabilization for the stretched string equation*, Differential Integral Equations, 6 (1993), pp. 925–935.

[21] J. M. Urquiza, *Rapid exponential feedback stabilization with unbounded control operators*, SIAM J. Control Optim., 43 (2005), pp. 2233–2244.