

RESEARCH

Open Access



# Boundary value conditions for linear differential equations with power degenerations

Aigerim Kalybay<sup>1\*</sup> 

\*Correspondence:  
kalybay@kimep.kz  
<sup>1</sup>KIMEP University, Almaty,  
Kazakhstan

## Abstract

On the interval  $[0, 1]$  we consider the  $n$ th order linear differential equation, the coefficient of the highest derivative of which is equivalent to the power function  $t^\mu$  when  $t \rightarrow 0$ . The main aim of the paper is to pose “generalized” Cauchy conditions for the given equation at the point of singularity  $t = 0$ , which would be correct for any  $\mu > 0$ .

**MSC:** Primary 34B05; secondary 46N20

**Keywords:** Linear differential equation; Boundary value problem; Weight function; Multiweighted derivative; Point of singularity

## 1 Introduction

Let us consider the following  $n$ th order linear differential equation:

$$\sum_{i=0}^n a_i(t)y^{(i)}(t) = f(t), \quad t \in [0, 1], \quad (1)$$

where the coefficients  $a_i(\cdot)$ ,  $i = 0, 1, \dots, n$ , and the right-hand side  $f(\cdot)$  are continuous functions on  $[0, 1]$ . Moreover,  $a_n(t) > 0$  when  $t \in (0, 1]$  and  $a_n(t)$  is equivalent to  $t^\mu$  when  $t \rightarrow 0$ , i.e., equation (1) has a singularity at the point  $t = 0$  of order  $\mu$ . It is known that if  $0 < \mu < 1$ , then all solutions of equation (1) belong to  $C^n[0, 1]$ . Hence, in this case we can pose the same boundary conditions for (1) as for a nonsingular equation. In particular, at the point of singularity the following Cauchy conditions can be posed:

$$y^{(i)}(0) = 0, \quad i = 0, 1, \dots, n - 1.$$

In the case  $\mu \geq 1$ , there are, in general, no finite limits  $\lim_{t \rightarrow 0} y^{(i)}(t)$  for all  $i = 0, 1, \dots, n - 1$ . Therefore, the Cauchy conditions have no meaning. The main aim of the paper is to pose “generalized” Cauchy conditions for (1) at  $t = 0$ , which would be correct for any  $\mu > 0$ .

The problem will be solved in the following way. Let  $\gamma_i < 1$ ,  $i = 0, 1, \dots, n - 1$ , be an arbitrary set of  $n$  real numbers and  $\gamma_n = 1$ . Suppose that the set of numbers  $\bar{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n)$

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

is such that

$$\alpha_i = \gamma_{i-1} - \gamma_i + 1, \quad i = 1, 2, \dots, n, \tag{2}$$

and  $\alpha_0$  is calculated from the equality

$$\mu = \gamma_0 + \alpha_0 + n - 1.$$

That gives

$$\sum_{i=0}^n \alpha_i = \mu, \quad \gamma_i = \alpha_n + \sum_{k=i+1}^{n-1} (\alpha_k - 1) < 1, \quad i = 0, 1, \dots, n - 2, \text{ and } \gamma_{n-1} = \alpha_n < 1. \tag{3}$$

Using this set of numbers  $\bar{\alpha}$  for  $y(t) \in C^n(0, 1]$ , we construct the following operations:

$$D_{\bar{\alpha}}^0 y(t) = t^{\alpha_0} y(t),$$

$$D_{\bar{\alpha}}^i y(t) = t^{\alpha_i} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \dots t^{\alpha_1} \frac{d}{dt} t^{\alpha_0} y(t), \quad i = 1, 2, \dots, n.$$

We call this differential operation (or operator)  $D_{\bar{\alpha}}^i$  the *multiweighted derivative of the function  $y$  of order  $i$* ,  $i = 0, 1, \dots, n$ . At the point of singularity  $t = 0$  of equation (1) we pose the boundary conditions

$$D_{\bar{\alpha}}^i y(0) = 0, \quad i = 0, 1, \dots, n - 1, \tag{4}$$

where each  $D_{\bar{\alpha}}^i y(0) = 0$ ,  $i = 0, 1, \dots, n - 1$ , is understood in the sense of the existence of the finite limit  $\lim_{t \rightarrow 0} D_{\bar{\alpha}}^i y(t) = D_{\bar{\alpha}}^i y(0)$ . Conditions (4) are the required “generalized” Cauchy conditions, and in the paper we prove that problem (1) and (4) with (3) has a unique solution.

Let us note that the operator  $D_{\bar{\alpha}}^n$  gives the basis for a space  $W_{p,\bar{\alpha}}^n = W_{p,\bar{\alpha}}^n(I)$  of functions  $y : I \rightarrow \mathbb{R}$  with the finite semi-norm

$$\|y\|_{W_{p,\bar{\alpha}}^n(I)} = \|D_{\bar{\alpha}}^n y\|_p,$$

where  $1 < p < \infty$ ,  $I = (0, 1)$  or  $I = (1, \infty)$ .

The idea to study function spaces with the purpose to apply them to different problems concerning differential equations appeared in works by Sobolev in the thirties. From this time the theory of Sobolev spaces has been developed to be a very powerful instrument for solving boundary value problems of differential equations. Moreover, such concept as a “weight function” was introduced to take care of different problems connected to singularities. Correspondingly, Kudryavtsev presented a fairly complete theory of one-dimensional Sobolev spaces with power weights (see, e.g., [8–15] and the references given there). He considered a space  $L_{p,\gamma}^n = L_{p,\gamma}^n(I)$  of functions  $y : I \rightarrow \mathbb{R}$ , which on the interval  $I$  have  $n$ th order derivative with the finite semi-norm

$$\|y\|_{L_{p,\gamma}^n} = \|t^\gamma y^{(n)}\|_p.$$

For the interval  $I = (0, 1)$  it was shown that if  $\gamma > n - \frac{1}{p}$ , then the function  $y$ , in general, does not have finite limit value when  $t \rightarrow 0$ . Thus, the introduction of an additional function as “weight” does not always solve singularity problems. This argumentation served as a motivation for constructing the operator  $D_{\alpha}^n$  and defining the space  $W_{p,\alpha}^n$  with  $n + 1$  weights. The theory of the function space  $W_{p,\alpha}^n$  generalizes the previous theory of the function space  $L_{p,\gamma}^n$ . In the series of works (see, e.g., [1–6]) there were considered similar problems for the  $W_{p,\alpha}^n$  as those considered by Kudryavtsev for the space  $L_{p,\gamma}^n$ . The results for the space  $W_{p,\alpha}^n$  cover such singularities that cannot be handled by one weight, but can be handled with many weights, and they give the basis for the main result of the present paper. More precisely, in the paper [5] it was shown that the properties of the space  $W_{p,\alpha}^n$  are dependent on the values  $\gamma_i, i = 0, 1, \dots, n - 1$ , in accordance with which we have the following three cases of the degeneration of the weight functions  $t^{\alpha_i}, i = 1, 2, \dots, n$ :

1.  $\gamma_{\max} = \max_{0 \leq i \leq n-1} \gamma_i < 1 - \frac{1}{p}$  (weak degeneration);
2.  $\gamma_{\min} = \min_{0 \leq i \leq n-1} \gamma_i > 1 - \frac{1}{p}$  (strong degeneration);
3.  $\gamma_{\min} < 1 - \frac{1}{p} < \gamma_{\max}$  (mixed degeneration).

Then in the paper [6] it was proved that, for any function  $y \in W_{p,\alpha}^n$ , the condition  $\gamma_{\max} < 1 - \frac{1}{p}$  (weak degeneration) is necessary and sufficient for the existence of limit values  $\lim_{t \rightarrow 0} D_{\alpha}^i y(t) = D_{\alpha}^i y(0)$  for all  $0 \leq i \leq n - 1$ . If we resolve the equalities in (3) with respect to  $\alpha_i, i = 1, 2, \dots, n$ , we get the equalities in (2). Moreover, by the assumption the values  $\gamma_i, i = 0, 1, \dots, n - 1$ , satisfy the condition  $\gamma_i < 1$ . Thus, we have a weak degeneration of the weight functions  $t^{\alpha_i}, i = 1, 2, \dots, n$ , which guarantees the existence of characteristics at the singular point  $t = 0$ .

The paper is organized as follows: In Sect. 2 we collect all the required notations, definitions, and statements; in Sect. 3 we state and prove our main result concerning the existence of a unique solution of problem (1) and (4).

## 2 Preliminaries

Let us introduce the following family of functions  $K_k(x, t), k = 0, 1, \dots, n$ , assuming that  $K_n(x, t) \equiv 1, K_{n-1}(x, t) = \int_x^t s^{-\alpha_{n-1}} ds, K_{n-2}(x, t) = \int_x^t y^{-\alpha_{n-2}} \int_x^y s^{-\alpha_{n-1}} ds dy$  and, in general,

$$K_k(x, t) = \int_x^t t_{k+1}^{-\alpha_{k+1}} \int_x^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \dots \int_x^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} dt_{n-1} dt_{n-2} \dots dt_{k+1}, \quad k = 0, 1, \dots, n - 1,$$

for  $t > x$ . Moreover, we assume that  $K_k(x, t) = 0, k = 0, 1, \dots, n - 1$ , for  $t \leq x$ .

Let  $n - 1 \geq k \geq 0$ . If in the integrals of  $K_k(x, t)$  we successively change the variables  $t_{n-1} = x\tau_{n-1}, t_{n-2} = x\tau_{n-2}, \dots, t_{k+1} = x\tau_{k+1}$ , we get

$$x^{-\alpha_n} K_k(x, t) = x^{-\gamma_k} K_k\left(1, \frac{t}{x}\right)$$

and

$$x^{-\alpha_n} K_k(x, t) \geq x^{-\gamma_k} K_k(1, 2) \quad \text{for } 0 < x \leq \frac{t}{2}, \tag{5}$$

i.e., the function  $x^{-\alpha_n} K_k(x, t)$  has a singularity at zero of order  $x^{-\gamma_k}$  for  $0 < \gamma_k < 1$ .

Let  $n - 1 \geq k \geq 0, i = k, k + 1, \dots, n - 1$ . Changing the order of integration and using (3), for  $t > 0$ , we get

$$\begin{aligned}
 & \int_0^t s^{-\alpha_n} K_k(s, t) ds \\
 &= \int_0^t s^{-\alpha_n} \int_s^t t_{k+1}^{-\alpha_{k+1}} \int_s^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \dots \int_s^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} dt_{n-1} dt_{n-2} \dots dt_{k+1} ds \\
 &= \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \dots \int_0^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} \int_0^{t_{n-1}} s^{-\alpha_n} ds dt_{n-1} \dots dt_{k+2} dt_{k+1} \\
 &= \frac{1}{1 - \gamma_{n-1}} \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \dots \int_0^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} dt_{n-1} \dots dt_{k+2} dt_{k+1} \\
 &= \frac{1}{(1 - \gamma_{n-1})(1 - \gamma_{n-2})} \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \dots \int_0^{t_{n-3}} t_{n-2}^{-\alpha_{n-2}} dt_{n-2} \dots dt_{k+2} dt_{k+1} \\
 &= \dots = \frac{1}{\prod_{i=k+1}^{n-1} (1 - \gamma_i)} \int_0^t t_{k+1}^{-\alpha_{k+1}} dt_{k+1} = d_k t^{1-\gamma_k}, \tag{6}
 \end{aligned}$$

where  $d_k = \frac{1}{\prod_{i=k}^{n-1} (1 - \gamma_i)}$ .

**Lemma 1** *Let  $\gamma_{n-1} < 1$ . Then, for  $y \in C^n(0, 1]$  satisfying the condition  $\sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| < \infty$ , there exists  $D_{\alpha}^{n-1} y(0)$  and the estimate*

$$t^{\gamma_{n-1}-1} |D_{\alpha}^{n-1} y(t) - D_{\alpha}^{n-1} y(0)| \leq d_{n-1} \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| \tag{7}$$

holds.

*Proof.* Indeed, by assumption it follows that the function  $t^{-\alpha_n} D_{\alpha}^n y(t)$  is absolutely summable on the interval  $(0, 1]$ . Therefore, by Newton–Leibniz formula for  $0 < t \leq 1$ , we have

$$\begin{aligned}
 \int_0^t s^{-\alpha_n} D_{\alpha}^n y(s) ds &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^t \frac{d}{ds} D_{\alpha}^{n-1} y(s) ds \\
 &= D_{\alpha}^{n-1} y(t) - \lim_{\varepsilon \rightarrow 0} D_{\alpha}^{n-1} y(\varepsilon) = D_{\alpha}^{n-1} y(t) - D_{\alpha}^{n-1} y(0). \tag{8}
 \end{aligned}$$

Since  $\alpha_n = \gamma_{n-1}$  and  $\gamma_{n-1} < 1$ , from (8) we get

$$|D_{\alpha}^{n-1} y(t) - D_{\alpha}^{n-1} y(0)| \leq \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| \int_0^t s^{-\alpha_n} ds = \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| \cdot \frac{t^{1-\gamma_{n-1}}}{1 - \gamma_{n-1}}.$$

The last gives (7). The proof of Lemma 1 is complete.

**Lemma 2** *Let  $n - 1 \geq k \geq 0$  and  $\gamma_i < 1$  for  $i = k, k + 1, \dots, n - 1$ . Let a function  $y \in C^n(0, 1]$  satisfy the conditions:  $\sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| < \infty$  and  $D_{\alpha}^i y(0) = 0, i = k + 1, \dots, n - 1$ . Then there exist  $D_{\alpha}^k y(0)$  and*

$$t^{\gamma_k-1} |D_{\alpha}^k y(t) - D_{\alpha}^k y(0)| \leq d_k \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)|, \tag{9}$$

$$D_{\alpha}^k y(t) = D_{\alpha}^k y(0) + \int_0^t K_k(s, t) s^{-\alpha_n} D_{\alpha}^n y(s) ds. \tag{10}$$

*Proof.* Since  $\sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| < \infty$  and  $\gamma_i < 1, i = k, k + 1, \dots, n - 1$ , from (6) we have that the function  $K_i(s, t)s^{-\alpha_n} D_{\alpha}^n y(s)$  is absolutely summable on  $(0, t)$ . Therefore, there exists

$$\begin{aligned} & \int_0^t K_i(s, t)s^{-\alpha_n} D_{\alpha}^n y(s) ds \\ &= \int_0^t t_{i+1}^{-\alpha_{i+1}} \int_0^{t_{i+1}} t_{i+2}^{-\alpha_{i+2}} \dots \int_0^{t_{n-2}} t_{n-1}^{-\alpha_{n-1}} \int_0^{t_{n-1}} t_n^{-\alpha_n} D_{\alpha}^n y(t_n) dt_n dt_{n-1} \dots dt_{i+1} \end{aligned} \tag{11}$$

for all  $i = k, k + 1, \dots, n - 1$ .

Since  $D_{\alpha}^{n-1} y(0) = 0$ , from (8) we have

$$D_{\alpha}^{n-1} y(t) = \int_0^t t_n^{-\alpha_n} D_{\alpha}^n y(t_n) dt_n, \quad t \in (0, 1].$$

The last, together with (11) for  $i = n - 2$ , gives

$$D_{\alpha}^{n-2} y(t) - D_{\alpha}^{n-2} y(0) = \int_0^t t_{n-1}^{-\alpha_{n-1}} \int_0^{t_{n-1}} t_n^{-\alpha_n} D_{\alpha}^n y(t_n) dt_n dt_{n-1}.$$

By the condition  $D_{\alpha}^{n-2} y(0) = 0$ , hence

$$D_{\alpha}^{n-2} y(t) = \int_0^t t_{n-1}^{-\alpha_{n-1}} \int_0^{t_{n-1}} t_n^{-\alpha_n} D_{\alpha}^n y(t_n) dt_n dt_{n-1}.$$

If we continue this process, using the fact that  $D_{\alpha}^i y(0) = 0, i = k + 1, \dots, n - 1$ , and the finiteness of (11), we get

$$D_{\alpha}^{k+1} y(t) = \int_0^t t_{k+2}^{-\alpha_{k+2}} \int_0^{t_{k+2}} t_{k+3}^{-\alpha_{k+3}} \dots \int_0^{t_{n-1}} t_n^{-\alpha_n} D_{\alpha}^n y(t_n) dt_n dt_{n-1} \dots dt_{k+2}.$$

That gives

$$\begin{aligned} D_{\alpha}^k y(t) - D_{\alpha}^k y(0) &= \int_0^t t_{k+1}^{-\alpha_{k+1}} \int_0^{t_{k+1}} t_{k+2}^{-\alpha_{k+2}} \dots \int_0^{t_{n-1}} t_n^{-\alpha_n} D_{\alpha}^n y(t_n) dt_n dt_{n-1} \dots dt_{k+1} \\ &= \int_0^t K_k(s, t)s^{-\alpha_n} D_{\alpha}^n y(s) ds, \end{aligned} \tag{12}$$

i.e., there exists  $D_{\alpha}^k y(0)$  and (10) holds. From (6) and (12) we have

$$\begin{aligned} |D_{\alpha}^k y(t) - D_{\alpha}^k y(0)| &\leq \sup_{0 \leq t \leq 1} |D_{\alpha}^k y(t)| \int_0^t K_k(s, t)s^{-\alpha_n} ds \\ &= \frac{1}{\prod_{i=k}^{n-1} (1 - \gamma_i)} t^{1-\gamma_k} \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)|. \end{aligned}$$

Thus, we get (9). The proof of Lemma 2 is complete.

**Corollary 1** *Let  $n - 1 \geq k \geq 0$ . Suppose that the conditions of Lemma 2 hold and  $D_{\alpha}^k y(0) = 0$ . Then*

$$D_{\alpha}^i y(t) = \int_0^t K_i(s, t)s^{-\alpha_n} D_{\alpha}^n y(s) ds \tag{13}$$

and

$$\sup_{0 \leq t \leq 1} t^{\gamma_i-1} |D_{\alpha}^i y(t)| \leq d_i \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| \tag{14}$$

for  $i = k, k + 1, \dots, n - 1$ .

**Lemma 3** *Let  $y : I \rightarrow \mathbb{R}$  be such that  $y \in C^n(0, 1]$ . Then*

$$y^{(k)}(t) = \sum_{i=0}^k b_{k,i} t^{\gamma_i-\gamma_0-\alpha_0-k} D_{\alpha}^i y(t), \quad k = 0, 1, \dots, n, \tag{15}$$

where  $b_{k,k} = 1, k = 0, 1, \dots, n$ , and coefficients  $b_{k,i}, i = 1, 2, \dots, k - 1, k = 1, 2, \dots, n$ , and  $b_{k,0}, k = 1, 2, \dots, n$ , are defined by the recurrent formulas:

$$b_{k,i} = b_{k-1,i-1} + b_{k-1,i}(\gamma_i - \gamma_0 - \alpha_0 - (k - 1)) \quad \text{and} \quad b_{k,0} = -b_{k-1,0}(\alpha_0 + k - 1).$$

*Proof.* For  $k = 0$ , we have

$$y(t) = t^{-\alpha_0} D_{\alpha}^0 y(t) = b_{0,0} t^{\gamma_0-\gamma_0-\alpha_0-0} D_{\alpha}^0 y(t),$$

where  $b_{0,0} = 1$ .

For  $k = 1$ , we have

$$D_{\alpha}^1 y(t) = t^{\alpha_1} (\alpha_0 t^{-1} D_{\alpha}^0 y(t) + t^{\alpha_0} y'(t)).$$

That, using  $\alpha_1 = \gamma_0 - \gamma_1 + 1$ , gives

$$\begin{aligned} y'(t) &= -\alpha_0 t^{-\alpha_0-1} D_{\alpha}^0 y(t) + t^{-\alpha_0-\alpha_1} D_{\alpha}^1 y(t) \\ &= b_{1,0} t^{\gamma_0-\gamma_0-\alpha_0-1} D_{\alpha}^0 y(t) + b_{1,1} t^{\gamma_1-\gamma_0-\alpha_0-1} D_{\alpha}^1 y(t), \end{aligned}$$

where  $b_{1,0} = -b_{0,0}(\alpha_0 + 1 - 1)$  and  $b_{1,1} = 1$ .

Now, we assume that

$$y^{(k-1)}(t) = \sum_{i=0}^{k-1} b_{k-1,i} t^{\gamma_i-\gamma_0-\alpha_0-(k-1)} D_{\alpha}^i y(t)$$

is true. Then, using (2), we get

$$\begin{aligned} y^{(k)}(t) &= \sum_{i=0}^{k-1} b_{k-1,i} t^{\gamma_i-\gamma_0-\alpha_0-(k-1)-\alpha_{i+1}} D_{\alpha}^{i+1} y(t) \\ &\quad + \sum_{i=0}^{k-1} b_{k-1,i} (\gamma_i - \gamma_0 - \alpha_0 - (k - 1)) t^{\gamma_i-\gamma_0-\alpha_0-k} D_{\alpha}^i y(t) \\ &= b_{k-1,k-1} t^{\gamma_{k-1}-\gamma_0-\alpha_0-(k-1)-\alpha_k} D_{\alpha}^k y(t) \\ &\quad + \sum_{i=1}^{k-1} b_{k-1,i-1} t^{\gamma_{i-1}-\gamma_0-\alpha_0-(k-1)-\alpha_i} D_{\alpha}^i y(t) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{k-1} b_{k-1,i} (\gamma_i - \gamma_0 - \alpha_0 - (k-1)) t^{\gamma_i - \gamma_0 - \alpha_0 - k} D_{\alpha}^i y(t) \\
 & + b_{k-1,0} (\gamma_0 - \gamma_0 - \alpha_0 - (k-1)) t^{\gamma_0 - \gamma_0 - \alpha_0 - k} D_{\alpha}^0 y(t) \\
 = & -b_{k-1,0} (\alpha_0 + k - 1) t^{\gamma_0 - \gamma_0 - \alpha_0 - k} D_{\alpha}^0 y(t) \\
 & + \sum_{i=1}^{k-1} (b_{k-1,i-1} + b_{k-1,i} (\gamma_i - \gamma_0 - \alpha_0 - (k-1))) t^{\gamma_i - \gamma_0 - \alpha_0 - k} D_{\alpha}^i y(t) \\
 & + b_{k-1,k-1} t^{\gamma_{k-1} - \gamma_0 - \alpha_0 - k} D_{\alpha}^{k-1} y(t).
 \end{aligned}$$

The last implies (15). The proof of Lemma 3 is complete.

### 3 Main result

**Theorem 1** *Let (3) hold. Let the coefficients  $a_i(t)$ ,  $i = 0, 1, \dots, n$ , of equation (1) be continuous functions on  $(0, 1]$  and satisfy the conditions*

$$c_1 t^\mu \leq a_n(t) \leq c_2 t^\mu, \quad \mu > 0, t \in (0, 1], \tag{16}$$

$$a_i(t) = o(t^{\mu-n+i}), \quad i = 0, 1, \dots, n-1, \text{ for } t \rightarrow 0, \tag{17}$$

where the constants  $c_1 > 0$  and  $c_2 > 0$  do not depend on  $t \in (0, 1]$ . Then, for any  $\mu > 0$  and for any function  $f(\cdot)$  continuous on  $[0, 1]$ , there exists a unique solution of problem (1) and (4), and the following estimate

$$\sum_{k=0}^n \sup_{0 \leq t \leq 1} t^{\mu-n+k} |y^{(k)}(t)| \leq c \max_{0 \leq t \leq 1} |f(t)| \tag{18}$$

holds, where  $c > 0$  does not depend on  $f$ .

*Proof.* If we substitute (15) in (1), then we have

$$\sum_{k=0}^n \left( \sum_{i=k}^n a_i(t) b_{i,k} t^{\gamma_k - \gamma_0 - \alpha_0 - i} \right) D_{\alpha}^k y(t) = f(t).$$

By introducing the notations  $\tilde{a}_k(t) = \sum_{i=k}^n a_i(t) b_{i,k} t^{\gamma_k - \gamma_0 - \alpha_0 - i}$ ,  $k = 0, 1, \dots, n$ , we get

$$\sum_{k=0}^n \tilde{a}_k(t) D_{\alpha}^k y(t) = f(t). \tag{19}$$

From conditions (16) and (17) we have

$$c_1 \leq \tilde{a}_n(t) \leq c_2, \quad t \in (0, 1], \tag{20}$$

$$\tilde{a}_k(t) = o(t^{\gamma_k - 1}), \quad k = 0, 1, \dots, n-1, \text{ for } t \rightarrow 0. \tag{21}$$

By condition (3) we have that  $\gamma_i < 1$ ,  $i = 0, 1, \dots, n-1$ , therefore (14) and (13) are valid for  $k = 0$ .

Let  $z(t) = \tilde{a}_n(t)D_{\alpha}^n y(t)$ , then from (4), (13), and (19) we obtain

$$z(t) + \int_0^t \sum_{k=0}^{n-1} \tilde{a}_k(t)K_k(s,t)s^{-\alpha_n}\tilde{a}_n^{-1}(s)z(s) ds = f(t). \tag{22}$$

Now, we prove that integral equation (22) has a unique solution continuous on  $[0, 1]$ , for which the estimate

$$\max_{0 \leq t \leq 1} |z(t)| \leq \bar{c} \max_{0 \leq t \leq 1} |f(t)| \tag{23}$$

holds, where  $\bar{c} > 0$  does not depend on  $f$ .

Despite the fact that integral equation (22) has the form of a Volterra equation, (5) implies that the kernel of the integral operator in (22) is unbounded if, for some  $0 \leq k \leq n - 1$ , we have  $0 < \gamma_k < 1$ . Therefore, we cannot apply the regular theory of Volterra integral equations in the space of continuous functions (see, e.g., [16]). Hence, let us first solve equation (22) on some interval  $[0, \delta]$ ,  $0 < \delta < 1$ , using the method of contraction mapping (see, e.g., [7]).

By (21) there exists  $1 > \delta > 0$  such that

$$c_1^{-1} \sup_{0 \leq t \leq \delta} \sum_{k=0}^{n-1} \frac{1}{\prod_{i=k}^{n-1} (1 - \gamma_i)} |\tilde{a}_k(t)| t^{1-\gamma_k} = q < 1. \tag{24}$$

In  $C[0, \delta]$  we consider the integral operator

$$Kz(t) = \sum_{k=0}^{n-1} \tilde{a}_k(t) \int_0^t K_k(s,t)s^{-\alpha_n}\tilde{a}_n^{-1}(s)z(s) ds, \quad t \in [0, \delta].$$

Due to (6), (21), and (24) we have

$$\begin{aligned} \sup_{0 \leq t \leq \delta} |Kz(t)| &\leq \sup_{0 \leq t \leq \delta} \sum_{k=0}^{n-1} |\tilde{a}_k(t)| \int_0^t K_k(s,t)s^{-\alpha_n} |\tilde{a}_n(s)|^{-1} |z(s)| ds \\ &\leq \sup_{0 \leq t \leq \delta} c_1^{-1} \sum_{k=0}^{n-1} |\tilde{a}_k(t)| \sup_{0 \leq t \leq \delta} |z(t)| \int_0^t K_k(s,t)s^{-\alpha_n} ds \\ &= c_1^{-1} \sup_{0 \leq t \leq \delta} \sum_{k=0}^{n-1} \frac{1}{\prod_{i=k}^{n-1} (1 - \gamma_i)} |\tilde{a}_k(t)| t^{1-\gamma_k} \sup_{0 \leq t \leq \delta} |z(t)| = q \sup_{0 \leq t \leq \delta} |z(t)|. \end{aligned}$$

Therefore,  $K$  is a contraction operator in  $C[0, \delta]$ . Applying the method of contraction mapping to integral equation (22) ([7], pp. 88–89), we have that equation (22) has a unique solution  $\bar{z}_0 \in C[0, \delta]$ ; in addition,  $\max_{0 \leq t \leq \delta} |\bar{z}_0(t)| \leq \bar{c}_0 \max_{0 \leq t \leq 1} |f(t)|$ . The successive approximations  $z_0, z_1, \dots, z_n, \dots$  to this solution are of the form

$$z_n(t) + \int_0^t \sum_{k=0}^{n-1} \tilde{a}_k(t)K_k(s,t)s^{-\alpha_n}\tilde{a}_n^{-1}(s)z_{n-1}(s) ds = f(t),$$

where any function from  $C[0, \delta]$  can be chosen as the first approximation  $z_0(t)$ .



Thus, we have found the solution  $\bar{z}_0 \in C[0, \delta]$  of equation (22) on the interval  $[0, \delta]$ . From equation (22) we have

$$\bar{z}_0(\delta) = f(\delta) - \sum_{k=0}^{n-1} \tilde{a}_k(\delta) \int_0^\delta K_k(s, \delta) s^{-\alpha_n} \tilde{a}_n^{-1}(s) \bar{z}_0(s) ds.$$

Let us now solve equation (22) on the interval  $[\delta, 1]$  with the condition

$$z(\delta) = \bar{z}_0(\delta). \tag{25}$$

On  $[\delta, 1]$  we present equation (22) in the form

$$\begin{aligned} z(t) + \sum_{k=0}^{n-1} \tilde{a}_k(t) \left( \int_0^\delta K_k(s, t) s^{-\alpha_n} \tilde{a}_n^{-1}(s) z(s) ds + \int_\delta^t K_k(s, t) s^{-\alpha_n} \tilde{a}_n^{-1}(s) z(s) ds \right) \\ = f(t). \end{aligned}$$

Since  $z = \bar{z}_0$  on the interval  $[0, \delta]$ , we have

$$\begin{aligned} z(t) + \sum_{k=0}^{n-1} \tilde{a}_k(t) \int_\delta^t K_k(s, t) s^{-\alpha_n} \tilde{a}_n^{-1}(s) z(s) ds \\ = f(t) - \sum_{k=0}^{n-1} \tilde{a}_k(t) \int_0^\delta K_k(s, t) s^{-\alpha_n} \tilde{a}_n^{-1}(s) \bar{z}_0(s) ds. \end{aligned} \tag{26}$$

It means that the kernel of the integral operator  $K$  is continuous on the bounded set  $\delta \leq s \leq t \leq 1$ . Hence, equation (22) is a regular Volterra integral equation on the interval  $[\delta, 1]$ . Therefore, it has a unique continuous solution  $\bar{z}_1$  on the interval  $[\delta, 1]$  (see, e.g., [16]), for which the estimate  $\max_{\delta \leq t \leq 1} |\bar{z}_1(t)| \leq \bar{c}_1 \max_{0 \leq t \leq 1} |f(t)|$  holds.

From (26) we have

$$\bar{z}_1(\delta) = f(\delta) - \sum_{k=0}^{n-1} \tilde{a}_k(\delta) \int_0^\delta K_k(s, \delta) s^{-\alpha_n} \tilde{a}_n^{-1}(s) \bar{z}_0(s) ds.$$

The last gives that  $\bar{z}_0(\delta) = \bar{z}_1(\delta)$ , i.e., (25) holds. Hence, the function

$$z(t) = \begin{cases} \bar{z}_0(t), & 0 \leq t \leq \delta, \\ \bar{z}_1(t), & \delta \leq t \leq 1, \end{cases}$$

belongs to  $C[0, 1]$  and it is a unique solution of (22). From the estimates for  $\bar{z}_0$  and  $\bar{z}_1$ , we get (23) with  $\bar{c} = \max\{\bar{c}_0, \bar{c}_1\}$ .

Thus, problem (1) and (4) is reduced to the problem

$$\begin{cases} D_{\tilde{\alpha}}^i y(t) = \tilde{a}_n^{-1}(t) z(t), \\ D_{\tilde{\alpha}}^i y(0) = 0, \quad i = 0, 1, \dots, n-1, \end{cases} \tag{27}$$

with condition (3), where  $z \in C[0, 1]$  is a unique solution of (22).

Since  $z \in C[0, 1]$ , the function  $\tilde{a}_n^{-1}(t) = (a_n(t)t^{-\mu})^{-1}$  is continuous on  $(0, 1]$  and by (20) we have that  $\frac{1}{c_2} \leq \tilde{a}_n^{-1}(t) \leq \frac{1}{c_1}$ ,  $t \in (0, 1]$ . Then from (27) we get  $D_{\alpha}^n y(t) \in C(0, 1]$  and  $\sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| < \infty$ . Moreover, by condition (3) we have  $\gamma_i < 1$ ,  $i = 0, 1, \dots, n - 1$ . Then, on the basis of Lemmas 1 and 2 and from (13) and (14), we get

$$\begin{aligned} D_{\alpha}^i y(t) &= \int_0^t K_i(s, t) s^{-\alpha_n} D_{\alpha}^n y(s) ds \\ &= \int_0^t K_i(s, t) s^{-\alpha_n} \tilde{a}_n^{-1}(s) z(s) ds, \quad i = 0, 1, \dots, n - 1, \end{aligned} \tag{28}$$

and the estimate

$$\sup_{0 \leq t \leq 1} t^{\gamma_i - 1} |D_{\alpha}^i y(t)| \leq d_i \sup_{0 \leq t \leq 1} |D_{\alpha}^n y(t)| \leq \frac{d_i}{c_1} \max_{0 \leq t \leq 1} |z(t)|, \quad i = 0, 1, \dots, n. \tag{29}$$

From (29) the uniqueness of problem (27) follows.

From (23) and (29) we obtain

$$\sup_{0 \leq t \leq 1} t^{\gamma_i - 1} |D_{\alpha}^i y(t)| \leq \frac{\bar{c} d_i}{c_1} \max_{0 \leq t \leq 1} |f(t)|, \quad i = 0, 1, \dots, n. \tag{30}$$

Since  $\mu = \gamma_0 + \alpha_0 + n - 1$ , from (15) we have

$$t^{\mu - n + k} y^{(k)}(t) = \sum_{i=0}^k b_{k,i} t^{\gamma_i - 1} D_{\alpha}^i y(t), \quad k = 0, 1, \dots, n.$$

The last, together with (30), gives the estimate

$$\sup_{0 \leq t \leq 1} |t^{\mu - n + k} y^{(k)}(t)| \leq \tilde{c} \max_{0 \leq t \leq 1} |f(t)|, \quad k = 0, 1, \dots, n,$$

where  $\tilde{c} > 0$  depends on  $\bar{c}$ ,  $c_1$ ,  $d_i$  and  $b_{k,i}$ ,  $k, i = 0, 1, \dots, n$ , and does not depend on  $f$ . The last implies the validity of (18). The proof of Theorem 1 is complete.

*Remark 1* Under the conditions of Theorem 1 problem (1) and (4) is solvable for any set  $\alpha_0, \alpha_1, \dots, \alpha_n$  satisfying condition (3).

**Acknowledgements**

The author would like to thank Professor Ryskul Oinarov and the unknown referees for their generous suggestions and remarks, which have improved this paper.

**Funding**

The paper was written with financial support of the Ministry of Education and Science of the Republic of Kazakhstan, grant no. AP05130975 in the area "Scientific research in the field of natural sciences".

**Availability of data and materials**

Not applicable.

**Competing interests**

The author declares that she has no competing interests.

**Authors' contributions**

All authors read and approved the final manuscript.

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 21 February 2020 Accepted: 9 June 2020 Published online: 18 June 2020

## References

1. Abdikalikova, Z., Baiyristanov, A., Oinarov, R.: Compactness of embedding between spaces with multiweighted derivatives—the case  $p \leq q$ . *Math. Inequal. Appl.* **14**(4), 793–810 (2011)
2. Abdikalikova, Z., Kalybay, A.: Summability of a Tchebysheff system of functions. *J. Funct. Spaces Appl.* **8**(1), 87–102 (2010). <https://doi.org/10.1155/2010/405313>
3. Abdikalikova, Z., Oinarov, R., Persson, L.-E.: Boundedness and compactness of the embedding between spaces with multiweighted derivatives when  $1 \leq q < p < \infty$ . *Czechoslov. Math. J.* **61** 7 (2011). <https://doi.org/10.1007/s10587-011-0014-1>
4. Baideldinov, B.L.: On stabilization to the generalized polynomial of functions with an integrable weight derivative. *Dokl. Math.* **354**(3), 295–297 (1997)
5. Baideldinov, B.L.: On equivalent norms in multiweighted spaces. *Izv. Nats. Acad. Nauk Resp. Kaz. Ser. Fiz.-Mat.* **3**, 8–14 (1998) (in Russian)
6. Kalybay, A.: A generalized multiparameter weighted Nikol'skii–Lizorkin inequality. *Dokl. Math.* **68**(1), 121–127 (2003)
7. Kolmogorov, A.N., Fomin, S.V.: *Elements of the Theory of Functions and Functional Analysis*. Nauka, Moscow (1989) (in Russian)
8. Kudryavtsev, L.D.: Direct and inverse embedding theorems. Applications to the solution of elliptic equations by variational method. *Trudy Mat. Inst. Akad. Nauk SSSR* **55**, 3–182 (1959) (in Russian)
9. Kudryavtsev, L.D.: Imbedding theorems for functions defined on unbounded regions. *Sov. Math. Dokl.* **4**, 530–532 (1963) English transl. in
10. Kudryavtsev, L.D.: An imbedding theorem for a class of functions defined in the whole space or in the half-space. I. *Mat. Sb.* **69**(111)(4), 616–639 (1966) (in Russian)
11. Kudryavtsev, L.D.: Imbedding theorems for classes of functions defined on the entire space or on a half space. II. *Mat. Sb.* **70**(112)(1), 3–35 (1966) (in Russian)
12. Kudryavtsev, L.D.: Equivalent norms in weighted spaces. *Proc. Steklov Inst. Math.* **170**, 185–218 (1987)
13. Kudryavtsev, L.D.: On variational problems for quadratic weight functionals on infinite intervals. *Proc. Steklov Inst. Math.* **172**, 223–235 (1987)
14. Kudryavtsev, L.D.: An analogue for an infinite interval of the Lizorkin–Nicol'skii inequality. *Proc. Steklov Inst. Math.* **173**, 151–160 (1987)
15. Kudryavtsev, L.D.: On some inequalities for functions and their derivatives on an infinite interval. *Proc. Steklov Inst. Math.* **204**, 143–147 (1994)
16. Smirnov, V.I.: *A Course of Higher Mathematics*. Nauka, Moscow (1974) (in Russian)

Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)

---