QUARTERLY OF APPLIED MATHEMATICS

VOLUME LVII MARCH · 1999 NUMBER 1

MARCH 1999, PAGES 1-17

BOUNDARY VALUE PROBLEMS FOR DEGENERATE VON KÁRMÁN EQUATIONS

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Abstract. This article presents regularity results that admit a weak formulation for degenerate von Kármán boundary value problems modeling the deformation of clamped plates that lose stiffness in one direction. These boundary value problems are derived in the companion article, *A Derivation of Degenerate von Kármán Equations for Strongly Anisotropic Plates*, by the author. The equations are a fourth-order elliptic-parabolic system of weakly coupled nonlinear equations. The article includes the weak formulation and a brief description of the appropriate existence results for the formulation.

1. Introduction. In this article we establish regularity results that admit a weak formulation for degenerate von Kármán boundary value problems that model the deformation of clamped plates that lose stiffness in one direction. These boundary value problems are derived in a companion article [23]. The equations are a fourth-order elliptic-parabolic system of weakly coupled nonlinear equations.

Regularity results for elliptic-parabolic equations are well-known, for instance, the work of Oleĭnik in [17], and the well-known papers by Kohn and Nirenberg [14] and [15]. Elliptic-parabolic systems have been treated carefully by Cosner [6], Bertiger and Cosner [3], Philips and Sarason [19], and Tartakoff [26]. Fourth and higher-order equations have been examined by Weinacht in [28], [29] and elsewhere, Esposito [7], [8], Canfora [5], Benevento, Bruno and Castellano [2], and the author [20], [21], and [22]. Ivanov treated second-order quasi-linear elliptic-parabolic equations; his work is assembled in [13], while nonlinear fourth-order equations with similar degeneracy have been treated by Warnecke in [27]. One notable aspect of the present work is that it presents regularity estimates in the neighborhood of an intersection between the characteristic and noncharacteristic boundaries. These are the first such estimates for the fourth-order case of which the author is aware, as other regularity results for elliptic-parabolic problems employ the restriction that the characteristic and noncharacteristic portions of the boundary do not intersect. Lin and Tso have obtained results in the second-order case in [16].

Received July 12, 1994.

1991 *Mathematics Subject Classification.* Primary 35J70; Secondary 73C30, 35G05.

Although the present work does not present existence results, the techniques employed by Berger and Fife in [1] and [4] seem to be appropriate to the present problem; the difficulty in applying the standard tools of pseudomonotone operator theory (as presented by Zeidler in [30], for instance) lies in the bounded domain of the operator in the weak formulation presented in the conclusion of this paper, Sec. 4.

In an effort to alleviate the tedium of some of the estimates that follow, this formulation omits some coefficients included in the derivation found in [23], but the results obtained here apply to the more general model with only straightforward modifications.

2. Notation and statement of results. The boundary value problem under consideration is

$$u_{yyyy} + u_{xxyy} - [u, \overline{w}] = f \qquad \text{in } \Omega \subset \mathbf{R}^2, \tag{1}$$

$$\Delta^2 \overline{w} := \overline{w}_{xxxx} + \overline{w}_{yyyy} + 2\overline{w}_{xxyy} = -(u_x u_{yy})_x \quad \text{in } \Omega, \tag{2}$$

with boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma^*, \tag{3}$$

$$-\frac{d}{ds}\overline{w}_x = \tau n_y + \rho h_2, \quad \text{and} \quad \frac{d}{ds}\overline{w}_y = \tau n_x + \rho h_1 \quad \text{on } \partial\Omega. \tag{4}$$

The bilinear form in Eq. (1) is defined by

$$[u, w] := (w_{yy} u_x)_x + (w_{xx} u_y)_y - (w_{xy} u_x)_y - (w_{xy} u_y)_x. \tag{5}$$

We use Σ^* to refer to that part of the boundary $\partial\Omega$ on which the unit normal is not parallel to the x-axis. The vector $\mathbf{n} = (n_x, n_y)$ is the unit outer normal to the boundary, and d/ds refers to the tangential directional derivative in the counter-clockwise direction. The quantities τ and ρ are parameters convenient for stating our results.

The function $\overline{w}$ can be decomposed into three components: $\overline{w} = \frac{\tau}{2}(x^2 + y^2) + \rho w^0 + \hat{w}$, where w^0 solves $\Delta^2 w^0 = 0$ in Ω with boundary conditions

$$-\frac{dw_x^0}{ds} = h_2, \quad \frac{dw_y^0}{ds} = h_1 \quad \text{on } \partial\Omega, \tag{6}$$

and $\hat{w}$ solves $\Delta^2 \hat{w} = -(u_x u_{yy})_x$ with homogeneous boundary conditions. We will refer to $\rho w^0 + \hat{w}$ as w. Using this decomposition, we can rewrite Eq. (1) as

$$D[u] - [u, w] := \Delta(u_{yy} - \tau u) - [u, w] = f \quad \text{in } \Omega. \tag{7}$$

This defines the elliptic-parabolic operator $D[\cdot]$, which depends upon τ.

We use standard notation: $L^p(A)$ is the Banach space of Lebesgue p-integrable functions over the set $A \subset \mathbf{R}^2$, with norm $\|\cdot\|_{p;A}$. The Hilbert space $L^2(\Omega)$ has norm $\|\cdot\|$. The symbol $W^{(k,p)}(A)$ refers to the Banach space of functions that, together with their weak partial derivatives up to order k, are Lebesgue p-integrable on A; the norm will be denoted $\|\cdot\|_{k,p;A}$. If the domain is omitted it is understood to be Ω. The space $W_0^{(k,p)}(A)$ is the closure of $C_0^\infty(A)$ in $W^{(k,p)}(A)$. For an introduction to these spaces, see [10].

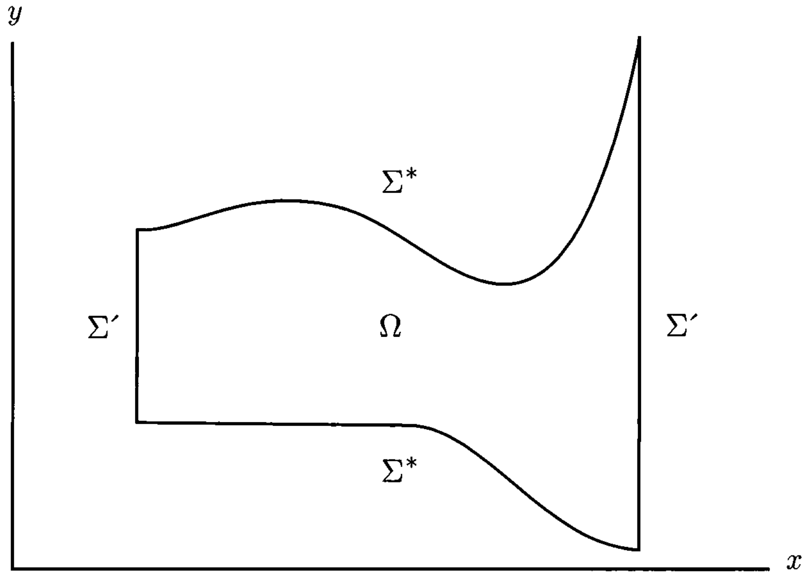

FIG. 1. An acceptable domain for the regularity results on u

To derive a weak formulation of Eq. (7) with boundary conditions (3) for fixed w, define the Hilbert space $\mathcal{H}$ by completing $C_0^\infty(\Omega)$ under

$$(u,v)_{\mathcal{H}} := \int_\Omega u_{xy}v_{xy} + u_{yy}v_{yy}dA + \int_\Omega \nabla u \cdot \nabla v\, dA.$$

Notice that if we integrate the operators in Eq. (7) against a function $v \in \mathcal{H}$ over Ω, integration by parts yields

$$\int_\Omega \Delta(u_{yy} - \tau u)v\, dA = \int_\Omega u_{yy}u_{yy} + u_{xy}v_{xy} + \tau\nabla u \cdot \nabla v\, dA =: B(u,v),$$

$$-\int_\Omega [u,w]v\, dA = \int_\Omega w_{yy}u_xv_x + w_{xx}u_yv_y - w_{xy}(u_xv_y + u_yv_x)\, dA =: c(w,u,v)$$

and these forms define a weak formulation of Eqs. (3) and (7):

$$\mathcal{B}_w(u,v) := B(u,v) + c(w,u,v) = \int_\Omega fv\, dA. \tag{8}$$

We call a u in $\mathcal{H}$ satisfying Eq. (8) for any v in $\mathcal{H}$ and fixed w and τ an $\mathcal{H}$-weak solution of the homogeneous Dirichlet boundary value problem for Eq. (7) for w and τ.

Put the following assumptions on the boundary of Ω. The portion denoted Σ', where n_y is zero, consists of two line segments. The complement of Σ' in $\partial\Omega, \Sigma^*$, is composed of two C^3 curves that intersect the line segments that comprise Σ' with bounded derivatives up to to order three. The regularity estimates apply so long as at points where Σ' intersects Σ^* the interior angle is right or acute. Figure 1 illustrates an acceptable geometry for the domain.

Finally, the Hilbert space in which the regularity result places the solution u:

$$\overline{\mathcal{H}} = \{u \in \mathcal{H} \mid u_{xx}, u_{xxy}, u_{xyy}, u_{yyy} \in L^2(\Omega)\}.$$

This paper proves the following result, and outlines its utility:

THEOREM. Assume f belongs to $L^2(\Omega)$, w belongs to $W^{(4,\frac{4}{3})}(\Omega)$, and Ω satisfies the conditions outlined above. When τ is sufficiently large and ρ, $\|w\|_{4,\frac{4}{3}}$, and $\|f\|$ are sufficiently small, the boundary value problem Eq. (8) has a unique $\mathcal{H}$-weak solution u that belongs to $\overline{\mathcal{H}}$.

3. Regularity of solutions to the degenerate equation. The inner product defined on $\mathcal{H}$ has positive definite quadratic form on $\mathcal{H}$ by the Poincaré-Friedrichs inequality since $\mathcal{H}$ is a subspace of $W_0^{(1,2)}(\Omega)$.

In order to obtain estimates on u near the curved portion of the boundary, Σ^*, we perform a local change of coordinates. The boundary arc Σ^* is the graph of $y = \Sigma(x)$, with $\frac{d}{dx}\Sigma(x) =: \sigma(x)$. We define $\xi = x$ and $\eta = y - \Sigma(x)$. In the new variables the bilinear form defined by Eq. (8) has a representation

$$\begin{aligned}\mathcal{B}_w(u,v) = \int_\Omega & Au_{\eta\eta}v_{\eta\eta} + \sigma(u_{\eta\eta}v_{\xi\eta} + u_{\xi\eta}v_{\eta\eta}) + u_{\xi\eta}v_{\xi\eta} \\ & + \tau(u_\xi v_\xi + Au_\eta v_\eta + \sigma(u_\xi v_\eta + u_\eta v_\xi)) \\ & + w_{\eta\eta}u_\xi v_\xi + (w_{\xi\xi} + \sigma' w_\eta)u_\eta v_\eta - w_{\xi\eta}(u_\xi v_\eta + u_\eta v_\xi)\, dA,\end{aligned}$$

where the coefficient $A = 1 + \sigma^2$ is always at least 1. In the course of the proof we use the operator and bilinear form expressed in these coordinates:

$$\begin{aligned} D[u] &= Au_{\eta\eta\eta\eta} + \sigma u_{\xi\eta\eta\eta} + (\sigma u_{\eta\eta\eta})_\xi + u_{\xi\xi\eta\eta} - \tau(u_{\xi\xi} + Au_{\eta\eta} + \sigma u_{\xi\eta} + (\sigma u_\eta)_\xi) \\ [u,w] &= (w_{\eta\eta}u_\xi)_\xi + ((w_{\xi\xi} + \sigma' w_\eta)u_\eta)_\eta - (w_{\xi\eta}u_\xi)_\eta - (w_{\xi\eta}u_\eta)_\xi. \end{aligned}$$

The first two lemmas give estimates on tangential derivatives of u near the boundary, and all derivatives in the interior of Ω.

LEMMA 1. Assume $u \in W^{(4,2)}(\Omega) \cap \mathcal{H}$ satisfies the conditions (3), $w \in W^{(3,4)}(\Omega)$ and there are functions ϕ, ψ, and χ belonging to $C_0^\infty(\mathbf{R}^2)$ with support that intersects $\partial\Omega$ only on a portion of Σ^* where a change of variables like that discussed above can be constructed. Assume further that $\psi \geq c > 0$ on $\operatorname{supp}\phi$ and $\chi \geq c$ on $\operatorname{supp}\psi$. Define $f_{(u,w)} := D[u] - [u,w]$. Then there are constants C and K depending upon the value c, the area of Ω, the norm $\|w\|_{3,4;\Omega}$, the maximum values of σ and ϕ and their derivatives to order two, and ψ and its first derivatives, with K depending on τ and C independent of τ such that

$$\|\phi u_{\xi xy}\|^2 + \|\phi u_{\xi yy}\|^2 + \tau\|\phi\nabla u_\xi\|^2 \leq C(\|\chi D^2 u\|^2 + \|\chi f_{(u,w)}\|^2) + K\|\chi\nabla u\|^2. \qquad (9)_\xi$$

An outline of the proof of this lemma can be found in the appendix 5. Lemmas of this type are frequently used in regularity results for degenerate elliptic equations, and similar lemmas together with their proofs can be found in [7], [14], [18], [20], [21], and [26]. The following lemma can be proved in a similar fashion.

LEMMA 2. Assume u and w are as in the previous lemma and ϕ, ψ, and χ are as above except that their supports intersect $\partial\Omega$, if at all, then only on Σ'. Then there are constants C and K depending upon the value c, the area of Ω, the norm $\|w\|_{3,4;\Omega}$, and the maximum values of ϕ and its derivatives to order two and ψ and its first derivatives, with K depending on τ and C independent of τ such that the inequality $(9)_y$ holds. Further, if $\operatorname{supp}\phi$ has no intersection with $\partial\Omega$, then $(9)_x$ holds as well, with constants having the dependencies given above.

By inequality $(9)_y$ we mean $(9)_\xi$ with y substituted for ξ.

Normal derivatives at the boundary must be estimated next. We will need the following trivial lemma, which is derived essentially the same way as the prior estimates.

LEMMA 3. Assume u, w, and $f_{(u,w)}$ are as in Lemma 1, and ϕ, ψ, and χ have the relationship described in Lemma 1 with support intersecting any portion of $\partial\Omega$. Then there are constants C and K depending upon the value c, the area of Ω, the norm $\|w\|_{3,4;\Omega}$, the maximum values of σ and its first derivative, ϕ and its derivatives to order three, and ψ and its first derivatives with K depending on τ and C independent of τ such that the following inequality holds:

$$\|\phi\nabla u_\eta\|^2 + \tau\|\phi\nabla u\|^2 \leq C(\|\chi\nabla u\|^2 + \|\chi f_{(u,w)}\|^2) + K\|\chi u\|^2. \tag{10}$$

The proof of this lemma depends upon the coercivity of the characteristic form of the operator $D[u]$ in local coordinates. If the maximum value of $|\sigma|$ is $\overline{\sigma} \geq 0$, then the minimum eigenvalue for the quadratic form $(1+\sigma^2)\eta^2 + 2\sigma\xi\eta + \xi^2$ is bounded below by

$$\frac{2 - \overline{\sigma}(\sqrt{\overline{\sigma}^2 + 4} - \overline{\sigma})}{2} > 0.$$

If we call this bound λ, then the bilinear form associated with $D[u]$ satisfies the following coercivity inequality

$$B(u,u) = Au_{\eta\eta}^2 + 2\sigma u_{\xi\eta}u_{\eta\eta} + u_{\xi\eta}^2 + \tau(u_\xi^2 + Au_\eta^2 + 2\sigma u_\xi u_\eta) \geq \lambda(|\nabla u_\eta|^2 + \tau|\nabla u|^2),$$

and this is all that is needed to obtain the result. Details are in the appendix.

We next turn to estimates of higher-order normal derivatives near Σ', where the operator is not coercive in the normal direction.

LEMMA 4. Assume we have two functions $u \in W^{(4,2)}(\Omega) \cap \mathcal{H}$ and $w \in W^{(3,4)}(\Omega)$. Assume further that the function $\phi \in C_0^\infty(\mathbf{R}^2)$ has support that intersects Σ', with $\phi_x = 0$ there, and possibly a contiguous component of Σ^*, in which case ϕ is a monotonic function of y on Σ'. If ψ and χ are $C_0^\infty(\mathbf{R}^2)$ functions related to ϕ as in the previous lemmas and a change of variables can be constructed on $\operatorname{supp}\chi$, then there exist constants C and K depending upon σ and its first and second derivatives, ϕ and its derivatives to order three, ψ and its first derivatives, and $\|w\|_{3,4;\Omega}$, with K depending on τ and C independent of τ such that the following inequality holds:

$$\|\phi u_{\xi xy}\|^2 + \|\phi u_{\xi yy}\|^2 + \tau\|\phi\nabla u_\xi\|^2 \leq C\|\chi D^2 u\|^2 + K(\|\chi\nabla u\|^2 + \|\phi f_{(u,w)}\|^2). \tag{11}$$

Proof. Multiplying the identity $f_{(u,w)} = D[u] - [w,u]$ by $(\phi^2 u_\xi)_\xi$ and integrating by parts over Ω gives

$$\begin{aligned}\int_\Omega \phi^2 f_{(u,w)\xi} u_\xi \, dA = \int_\Omega & (Au_{\eta\eta})_\xi(\phi^2 u_\xi)_{\eta\eta} + u_{\xi\xi\eta}(\phi^2 u_\xi)_{\xi\eta} \\ & + (\sigma u_{\xi\eta})_\xi(\phi^2 u_\xi)_{\eta\eta} + (\sigma u_{\eta\eta})_\xi(\phi^2 u_\xi)_{\xi\eta} + \tau(u_{\xi\xi}(\phi^2 u_\xi)_\xi + (Au_\eta)_\xi(\phi^2 u_\xi)_\eta \\ & + (\sigma u_\xi)_\xi(\phi^2 u_\xi)_\eta + (\sigma u_\eta)_\xi(\phi^2 u_\xi)_\xi) + (w_{\eta\eta}u_\xi)_\xi(\phi^2 u_\xi)_\xi \\ & - (w_{\xi\eta}u_\xi)_\xi(\phi^2 u_\xi)_\eta - (w_{\xi\eta}u_\eta)_\xi(\phi^2 u_\xi)_\xi + ((w_{\xi\xi} + \sigma' w_\eta)u_\eta)_\xi(\phi^2 u_\xi)_\eta \, dA \\ & + \int_{\partial\Omega} \phi^2[u_{\xi\xi\eta\eta} + (\sigma u_{\eta\eta})_{\xi\eta} - \tau(u_{\xi\xi} + (\sigma u_\eta)_\xi) - (w_{\eta\eta}u_\xi)_\xi + (w_{\xi\eta}u_\eta)_\xi]u_\xi n_\xi \, ds.\end{aligned} \tag{12}$$

Note that the boundary integral can be expressed as an area integral that can be rewritten as

$$\begin{aligned}-\int_\Omega \{ & (Au_{\eta\eta} + \sigma u_{\xi\eta})(\phi^2 u_\xi)_{\eta\eta} + [r(Au_\eta + \sigma u_\xi) \\ & - w_{\xi\eta}u_\xi + (w_{\xi\xi} + \sigma' w_\eta)u_\eta](\phi^2 u_\xi)_\eta - \phi^2 f_{(u,w)} u_\xi\}_\xi \, dA\end{aligned}$$

using the degenerate equation just given and integration by parts. Applying integration by parts repeatedly to Eq. (12) with the area integral replacing the boundary integral yields

$$\begin{aligned}\int_\Omega & f_{(u,w)}(\phi^2 u_\xi)_\xi \, dA \\ & = \int_\Omega \phi^2[Au_{\xi\eta\eta}^2 + 2\sigma u_{\xi\xi\eta}u_{\xi\eta\eta} + u_{\xi\xi\eta}^2 + \tau(Au_{\xi\eta}^2 + 2\sigma u_{\xi\xi}u_{\xi\eta} + u_{\xi\xi}^2)] \, dA + \mathcal{C} \\ & \quad + \int_{\Sigma'} \phi(2u_{\xi\eta}^2 + \tau u_\xi^2)(\phi_\xi - \sigma\phi_\eta) \, d\eta,\end{aligned} \tag{13}$$

where (by application of the Hölder and arithmetic-geometric mean inequalities) the quantity $\mathcal{C}$ can be bounded

$$|\mathcal{C}| \le C_1\|\chi D^2 u\| + K_1\|\chi \nabla u\|^2$$

with C_1 depending upon σ and its first and second derivatives, ϕ and its derivatives to order three, ψ and its first derivatives, and $\|w\|_{3,4;\Omega}$, and K_1 depending on all these and τ as well. Note that if $\operatorname{supp}\phi$ does not include Σ^*, then the boundary integral in Eq. (13) vanishes, since by hypothesis $\phi_\xi = 0$, and $\sigma = 0$ when the change of coordinates is trivial: $y = \eta$. In the case that $\operatorname{supp}\phi$ includes a portion of Σ^*, the boundary integral is positive, since $\sigma\phi_\eta \le 0$ by our hypothesis on the domain (stated in Sec. 2), and the hypotheses of this lemma on ϕ. From this the inequality (11) follows. □

In order to obtain an estimate on $u_{\eta\eta\eta}$ near Σ^* we apply the following lemma. It provides an estimate on the η derivative of a function in a neighborhood U of Σ^* under the following assumptions. Assume U is a domain in the (ξ,η)-plane whose boundary intersects $\partial\Omega$ on a connected portion of Σ^* and possibly on a contiguous connected portion of Σ'. Assume that the intersection of the boundary of U with Ω, denoted γ, is a convex C^3 curve that is normal to $\partial\Omega$ where they meet. If $\overline{U}$ intersects Σ', then denote by $\widehat{U}$ the union of U with its reflection about Σ' (including the portion of Σ').

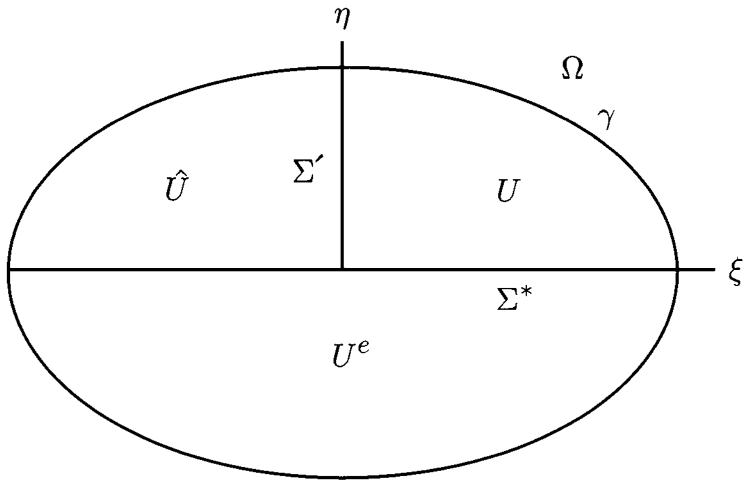

FIG. 2. Geometry of the region used in the proof of Lemma 5

Otherwise, take $\widehat{U}$ to be U. The union of $\widehat{U}$ with its reflection about the ξ-axis will be denoted U^e. We will choose γ so that ∂U^e is C^3.

LEMMA 5. Suppose U and γ are as described above, and for some $v \in W^{(1,2)}(\Omega)$ with $v = 0$ on Σ' in the sense of trace, the inequality

$$\left| \int_U v z_{\eta\eta} dA \right| \leq P \|z\|_{1,2;U} \tag{14}$$

holds for any $z \in C_0^\infty(U^e)$ with $z = z_\eta = 0$ on Σ^*, with P independent of z. Then, on any open set U' with $\overline{U'} \subset U$ that has $\gamma' = \partial U' \cap \Omega$ satisfying the conditions on γ, the following estimate holds:

$$\|v_\eta\|_{2;U'} \leq C(P + \|v_\xi\| + \|v\|) \tag{15}$$

for some constant C depending only on U and U'.

Proof. For convenience, we assume that η increases moving from Σ^* into U, and if $\overline{U}$ includes a corner point, then it is the origin of the coordinate system, and ξ increases moving from Σ' into U. The geometry of the region is shown in Figure 2. Since γ' satisfies the conditions on γ, U' can be extended to a region with C^3 boundary in the same way U is; call this set U'^e. Any function v defined in U can be extended into all of U^e in such a way that continuity of the derivatives controlled by the $\mathcal{H}$ norm is preserved, and $\|v\|_{1,2;U^e} \leq C\|v\|_{1,2;U}$.

To see this, extend v into $\widehat{U}$ (if necessary) by defining

$$v(\xi,\eta) = -3v(-\xi,\eta) + 4v(-\xi/2,\eta) \quad \text{if } (\xi,\eta) \in \widehat{U} \text{ and } \xi < 0.$$

This extension assures that formally

$$v|_{\xi=0^+} = v|_{\xi=0^-} \text{ and } v_\xi|_{\xi=0^+} = v_\xi|_{\xi=0^-},$$

and that $\|v\|_{1,2;\widehat{U}} \leq c\|v\|_{1,2;U}$.

Extend v from $\widehat{U}$ to U^e in the same way, taking $v(\xi,\eta) = -3v(\xi,-\eta)+4v(\xi,-\eta/2)$ if $(\xi,\eta)\in\widehat{U}$ and $\eta<0$; so a comparable statement about continuity of derivatives at $\eta=0$ and boundedness of the $W^{(1,2)}$ norm can be made. To make the following equations more readable we use the assignments $\lambda_1=-3$ and $\lambda_2=4$.

Notice that for a function v defined in this way, for any $\phi\in C_0^\infty(U^e)$, it follows that

$$\int_{U^e}\phi v\,dA=\int_U\phi^* v\,dA,$$

where

$$\phi^*(\xi,\eta)=\phi(\xi,\eta)+\sum_{J=1}^{2} j\lambda_j\phi(-j\xi,\eta)+\sum_{k=1}^{2} k\lambda_k\left(\phi(\xi,-k\eta)+\sum_{j=1}^{2} j\lambda_j\phi(-j\xi,-k\eta)\right).$$

If we consider a test function of the form $\phi=\psi_{\eta\eta}$, then $\phi^*=(\psi_{\eta\eta})^*=z_{\eta\eta}$, where

$$z(\xi,\eta)=\psi(\xi,\eta)+\sum_{j=1}^{2} j\lambda_j\psi(-j\xi,\eta)+\sum_{k=1}^{2}\frac{\lambda_k}{k}\left(\psi(\xi,-k\eta)+\sum_{j=1}^{2} j\lambda_j\psi(-j\xi,-k\eta)\right).$$

Note that z and z_η are zero on Σ^*. Further, since $\psi\in C_0^\infty(U^e)$, z is in $C_0^\infty(U^e)$. Thus the bound (14) holds for the test function z. From $(\psi_{\eta\eta})^*=z_{\eta\eta}$ it follows that

$$\left|\int_{U^e} v\psi_{\eta\eta}\,dA\right|\le P\|z\|_{1,2;U}\le C_1P\|\psi\|_{1,2;U^e}.$$

Integrating by parts gives

$$\left|\int_{U^e} v\psi_{\xi\xi}dA\right|\le\|v_\xi\|_{2;U^e}\|\psi\|_{1,2;U^e},$$

which we can combine with the preceding equation to get

$$\left|\int_{U^e} v\Delta\psi\,dA\right|\le C_2(P+\|v_\xi\|)\|\psi\|_{1,2;U^e}. \tag{16}$$

Here we have used that $\|v_\xi\|_{2;U^e}\le c\|v_\xi\|$.

Now, define h as the unique solution to $\Delta h=v$ in U'^e belonging to $W_0^{(1,2)}(U'^e)$. We know by classical results that $h\in W^{(3,2)}(U^e)$ (see for instance [9] or [12]). So, h defined on U' can be continuously extended to a function belonging to $W_0^{(3,2)}(U^e)$ (see [24]). In particular h satisfies $\|h\|_{1,2;U^e}\le c\|h\|_{1,2;U'^e}$. Integrating by parts and applying Eq. (16) we have for any $\psi\in C_0^\infty(U^e)$,

$$\left|\int_{U^e}\Delta h_\eta\Delta\psi\,dA\right|\le C_2(P+\|v_\xi\|)\|\psi_\eta\|_{1,2;U^e}\le C_2(P+\|v_\xi\|)\|\psi\|_{2,2;U^e}.$$

If we consider a sequence of ψ's that approach h_η in $W_0^{(2,2)}(U^e)$, then the inequality becomes

$$\|\Delta h_\eta\|_{2;U^e}^2\le C_2(P+\|v_\xi\|)\|h_\eta\|_{2,2;U^e}.$$

Now we can apply G̊arding's inequality for the biharmonic operator (see [9], e.g.) to get

$$\|h_\eta\|_{2,2;U^e}^2\le C_3[(P+\|v_\xi\|)\|h_\eta\|_{2,2;U^e}+\|h_\eta\|_{2;U^e}^2],$$

and since $\|h_\eta\|_{2;U^e} \le \|h\|_{1,2;U^e}$ and $\|h_\eta\|_{2;U^e} \le \|h_\eta\|_{2,2;U^e}$, we have

$$\|h_\eta\|_{2,2;U^e} \le C_3(P + \|v_\xi\| + \|h\|_{1,2;U^e}).$$

Finally, we can use $\|v_\eta\|_{2;U'} \le \|h_\eta\|_{2,2;U^e}$, and (by the continuous extension of h) $\|h\|_{1,2;U^e} \le c\|v\|$ to arrive at the desired estimate (15). □

LEMMA 6. Suppose $u \in W^{(4,2)}(\Omega) \cap \mathcal{H}$, $w \in W^{(3,4)}(\Omega)$, and $\phi, \psi_1, \psi_2, \psi_3$, and χ are $C_0^\infty(\mathbf{R}^2)$ functions with $\psi_1 \ge c$ on $\operatorname{supp}\phi$, $\psi_{i+1} \ge c$ on $\operatorname{supp}\psi_i$, $\chi \ge c$ on $\operatorname{supp}\psi_3$, and for $i = 1, 3$ the $\operatorname{int}\operatorname{supp}\psi_i \cap \Omega$ suitable U's, as described before Lemma 5. Then there are constants C and K so that

$$\|\phi u_{\eta\eta\eta}\|^2 \le C(\|\chi D^2 u\|^2 + \|\phi f_{(u,w)}\|^2) + K(\|\chi \nabla u\|^2 + \|\chi u\|^2). \tag{17}$$

Proof. Suppose z is a $C_0^2(\operatorname{int}\operatorname{supp}\psi_2)$ function with $z = z_\eta = 0$ on Σ^*. We know

$$\int_\Omega A(\phi u)_{\eta\eta} z_{\eta\eta}\, dA = \mathcal{B}_w(\phi u, z) - \mathcal{C}$$

where $\mathcal{C}$ includes all the bilinear forms except the term appearing on the left. By our assumption on z we may perform integration by parts and bounding to show

$$|\mathcal{C}| \le \{C_1(\|\psi_2 u_{\xi\xi\eta}\| + \|\psi_2 u_{\xi\eta\eta}\| + \|\psi_2 D^2 u\|) + K_1(\|\psi_2 \nabla u\| + \|\psi_2 u\|)\}\|\nabla z\|,$$

where C_1 depends upon the constant c, the maximum value of σ and its derivative, and the maximum values of ϕ and its derivatives up to order three and ψ_1, and K_1 depends upon these, the norm $\|w\|_{3,4;\Omega}$, the maximum values of ψ_1's first derivatives, and the parameter τ. Similarly we can show

$$|\mathcal{B}_w(\phi u, z) - \mathcal{B}_w(u, \phi z)| \le \{C_2\|\psi_2 D^2 u\| + K_2(\|\psi_2 \nabla u\| + \|\psi_2 u\|)\}\|z\|_{1,2;\Omega},$$

where C_2 and K_2 have the same dependencies as C_1 and K_1, respectively. Applying the identity

$$\int_\Omega A(\phi u)_{\eta\eta} z_{\eta\eta}\, dA = \int_\Omega z f_{(u,w)}\, dA + \mathcal{B}_w(\phi u, z) - \mathcal{B}_w(u, \phi z) - \mathcal{C},$$

it follows that

$$\begin{aligned} |(A(\phi u)_{\eta\eta}, z_{\eta\eta})| \le \{&C(\|\psi_2 u_{\xi\xi\eta}\| + \|\psi_2 u_{\xi\eta\eta}\| + \|\psi_2 D^2 u\|) \\ &+ K(\|\psi_2 \nabla u\| + \|\psi_2 u\|) + \|\phi f_{(u,w)}\|\}\|z\|_{1,2;\Omega}. \end{aligned}$$

Notice that this inequality has the form of the condition in Lemma 5; so its application using $\operatorname{int}\operatorname{supp}\psi_3$ as U and $\operatorname{int}\operatorname{supp}\psi_1$ as U' yields

$$\begin{aligned} \|\phi u_{\eta\eta\eta}\|^2 \le\ &C(\|\psi_2 u_{\xi\xi\eta}\|^2 + \|\psi_2 u_{\xi\eta\eta}\|^2 + \|\psi_2 D^2 u\|^2 + \|\phi f_{(u,w)}\|^2) \\ &+ K(\|\psi_2 \nabla u\|^2 + \|\psi_2 u\|^2). \end{aligned}$$

Eliminate the third-order derivatives on the right side by applying the estimate (9) if $\overline{U}$ does not include a corner point, and estimate (11) if it does, using ψ_2 as ϕ and ψ_3 as ψ to arrive at (17). This proves the lemma. □

REMARK. In the last step of the proof just completed we used the identities $\partial/\partial\xi = \partial/\partial x + \sigma\partial/\partial y$ and $\partial/\partial\eta = \partial/\partial y$ to obtain

$$|u_{\xi\xi\eta}| \le \max\{1, |\sigma|\}(|u_{\xi xy}| + |u_{\xi yy}|) \text{ and } u_{\xi\eta\eta} = u_{\xi yy}.$$

We will use this bound and the "inverse",

$$\|v_x\| \le \max\{1, |\sigma|\}(|v_\xi| + |v_\eta|),$$

to obtain the principal result, stated in Sec. 2:

Proof of theorem. First we show existence of the unique solution to the boundary value problem (8) in $\mathcal{H}$. Since w belongs to the space $W^{(3,4)}(\Omega)$, the Sobolev embedding theorem implies that the coefficients w_{xx}, w_{xy}, and w_{yy} are continuous on $\overline{\Omega}$, with maximum values bounded by a multiple of $\|w\|_{3,4;\Omega}$ (see [12], for instance). Thus for any w from a collection with uniformly bounded norm in $W^{(3,4)}(\Omega)$, given sufficiently large τ (depending on the bound), the bilinear form $\mathcal{B}_w(u,v)$ is both bounded and coercive on $\mathcal{H}$. Since the right-hand side of Eq. (8) is clearly a continuous linear functional on $\mathcal{H}$, the Lax-Milgram lemma implies existence of a unique in $u = u_w \in \mathcal{H}$ that solves (8) for every $v \in \mathcal{H}$ under the mentioned conditions on w and τ.

The proof of regularity consists of two parts. Using the estimates derived in the lemmas, we can establish the fundamental relation for any $u \in W^{(5,2)}(\Omega) \cap \mathcal{H}$ and satisfying the boundary conditions (3): if τ is sufficiently large then the basic estimate

$$\|u\|^2_{\overline{\mathcal{H}}} \le K(\|u\|^2_{\mathcal{H}} + \|f_{(u,w)}\|^2) \tag{18}$$

holds. Accepting this for the moment, we can show the completion of the proof. The solution u for the specified f can be the strong limit in $\mathcal{H}$ for a sequence of u^k for which the estimate Eq. (18) holds. Thus the corresponding $f_{(u^k,w)} =: f^k$ must converge weakly to f in $L^2(\Omega)$ since

$$(f^k, v) = \mathcal{B}_w(u^k, v) \to \mathcal{B}_w(u, v) = (f, v) \text{ for all } v \in \mathcal{H},$$

and $\mathcal{H}$ is dense in $L^2(\Omega)$. As the norms of a weakly convergent sequence we know that $\|f^k\|$ must be uniformly bounded. Thus taking the limit of

$$\|u^k\|^2_{\overline{\mathcal{H}}} \le K(\|u^k\|^2_{\mathcal{H}} + \|f^k\|^2)$$

we have

$$\|u\|^2_{\overline{\mathcal{H}}} \le K\|u\|^2_{\mathcal{H}} + P$$

with $P = K(\liminf \|f^k\|)^2 < \infty$. This is sufficient to show that u is in $\overline{\mathcal{H}}$.

In order to prove the basic estimate (18), we impose a pair of partitions of unity on the domain Ω, $\{\chi_l\}_{l=1}^N$, and $\{\phi_l\}_{l=1}^N$. The partitions must be chosen so that $\operatorname{supp}\phi_l \subset \operatorname{int}\operatorname{supp}\chi_l$, and each pair ϕ_l and χ_l satisfies one of the conditions:

1) $\operatorname{supp}\chi_l \subset \Omega$.
2) $\operatorname{supp}\phi_l$ intersects $\partial\Omega$ only on one component of Σ^* and satisfies the conditions on the set U in Lemma 5.
3) $\operatorname{supp}\phi_l$ intersects $\partial\Omega$ only on one component of Σ'. In this case ϕ_l must satisfy the conditions of Lemma 4.
4) $\operatorname{supp}\phi_l$ contains a corner point, and satisfies the conditions on the set U in Lemma 5. In this case ϕ_l must satisfy the conditions of Lemma 4.

The construction of a ϕ partition "inside" the χ partition is outlined in [20].

For a ϕ_l from a pair satisfying condition 1), we have by Lemma 2, estimate $(9)_x$ that

$$\tau\|\phi_l u_{xx}\|^2 \le C(\|\chi_l D^2 u\|^2 + \|\chi_l f_{(u,w)}\|^2) + K\|\chi_l \nabla u\|^2.$$

Under condition 2), by Lemma 1, estimate $(9)_\xi$ holds, and by Lemma 3, estimate (17) holds, and applying the bound in the Remark preceding this proof we have

$$\tau\|\phi_l u_{xx}\|^2 \le C\|\chi_l D^2 u\|^2 + K(\|\chi_l \nabla u\|^2 + \|\chi_l u\|^2 + \|\chi_l f_{(u,w)}\|^2).$$

Under either condition 3) or condition 4), by Lemma 3, estimate (17) holds, and by Lemma 4, estimate (11) holds, and we get the bound given for condition 2). Recall that in each of these bounds the constant C is independent of τ, so, for $\tau > C$ we have, under each condition

$$\|\phi_l u_{xx}\|^2 \le K(\|\chi_l u_{xy}\|^2 + \|\chi_l u_{yy}\|^2 + \|\chi_l \nabla u\|^2 + \|\chi_l u\|^2 + \|\chi_l f_{(u,w)}\|^2).$$

Summing over l we have

$$\int_\Omega \sum_{l=1}^N \phi_l^2 u_{xx}^2 dA \le K(\|u\|_{\mathcal{H}}^2 + \|f_{(u,w)}\|^2).$$

Now since $\sum_{l=1}^N \phi_l^2 \ge \frac{1}{N}(\sum_{l=1}^n \phi_l)^2 = \frac{1}{N}$, this inequality implies that $\|u_{xx}\|^2 \le K(\|u\|_{\mathcal{H}}^2 + \|f_{(u,w)}\|^2)$, from which it follows that

$$\|D^2 u\|^2 \le K(\|u\|_{\mathcal{H}}^2 + \|f_{(u,w)}\|^2) \tag{19}$$

as well.

The same technique yields estimates on the third-order derivatives with even less work. Under condition 1), by Lemma 2, estimates $(9)_x$ and $(9)_y$ hold, and we have

$$\|\phi_l u_{yyy}\|^2 + \|\phi_l u_{xyy}\|^2 + \|\phi_l u_{xxy}\|^2 \le K(\|\chi_l D^2 u\|^2 + \|\chi_l \nabla u\|^2 + \|\chi_l f_{(u,w)}\|^2).$$

Under condition 2), by Lemma 1, estimate $(9)_\xi$ holds, and by Lemma 6, estimate (17) holds, yielding

$$\|\phi_l u_{yyy}\|^2 + \|\phi_l u_{xyy}\|^2 + \|\phi_l u_{xxy}\|^2 \le K(\|\chi_l D^2 u\|^2 + \|\chi_l \nabla u\|^2 + \|\chi_l u\|^2 + \|\chi_l f_{(u,w)}\|^2).$$

Under condition 3), by Lemma 2, estimate $(9)_y$ holds, and by Lemma 4, estimate (11) holds, giving the bound given for condition 1). Under condition 4), by Lemma 4, estimate (11) holds, and by Lemma 6, estimate (17) holds, and we get the bound given for condition 2). Summing these bounds over l, and applying the definition of $\|u\|_{\mathcal{H}}$ and (19) we have

$$\int_\Omega \sum_{l=1}^N \phi_l^2 (u_{yyy}^2 + u_{xyy}^2 + u_{xxy}^2) dA \le K(\|u\|_{\mathcal{H}}^2 + \|f_{(u,w)}\|^2).$$

From here we may proceed as above to arrive at bounds on the norms of the third derivatives, and then the basic estimate (18) follows easily, noting that

$$\|u\|_{\overline{\mathcal{H}}}^2 = \|u\|_{\mathcal{H}}^2 + \|u_{yyy}\|^2 + \|u_{xyy}\|^2 + \|u_{xxy}\|^2 + \|u_{xx}\|^2. \quad \square$$

REMARK. The boundary conditions (6) satisfied by w^0 are equivalent up to affine transformation to Dirichlet boundary conditions under the physically motivated conditions on the boundary data:

$$\int_{\partial\Omega} h_\alpha ds = 0 \text{ for } \alpha = 1, 2, \quad \text{and} \quad \int_{\partial\Omega} (h_1 y - h_2 x)\, ds = 0.$$

Since w's second derivatives are the lowest order appearing in (7), we may assume that w^0 is a biharmonic function satisfying Dirichlet conditions on $\partial\Omega$.

The existence of a unique $(2,2)$-weak solution to the homogeneous Dirichlet problem for Eq. (2), denoted $\hat{w}$, for any $u \in \mathcal{H}$ follows from the Lax-Milgram lemma. The $(2,2)$-weak solution to this problem will belong to $W^{(4,p)}(\Omega)$ for any $1 \leq p < 2$ provided that u is in $\overline{\mathcal{H}}$, and Ω is a rectangle. This follows immediately from Theorem (7.3.2.1) in [11], since $u \in \mathcal{H}$ implies that the inhomogeneous term in Eq. (2), $(u_x u_{yy})_x$, belongs to $L^p(\Omega)$ for every $1 \leq p < 2$, by the Sobolev embedding theorem.

4. Conclusion. The regularity result just proved, in conjunction with the result cited in the remark concluding the previous section allow a weak formulation of the boundary value problem on a rectangle. Define u^0 as the $\overline{\mathcal{H}}$-weak solution to

$$D[u^0] - \rho[u^0, w^0] = f$$

with boundary conditions (3), w^l (for $l > 0$) as the $(4, \frac{4}{3})$-weak solution to

$$\Delta^2 w^l = (u_x^{l-1} u_{yy}^{l-1})_x$$

with homogeneous Dirichlet boundary conditions, and u^l (for $l > 0$) as the $\overline{\mathcal{H}}$-weak solution to

$$D[u^l] - [u^l, \rho w^0 + w^l] = f$$

with boundary conditions (3). Then the existence of a weak solution to the boundary value problem (1)–(4) is equivalent to the convergence of the sequence of ordered pairs (u^l, w^l). For the classical von Kármán equations, the convergence of this sequence in $W^{(2,2)}(\Omega)$ is demonstrated by formulating the problem as a pseudomonotone operator on this space [1]. Unfortunately, this method is not applicable in this case, since the weak formulation requires the regularity demonstrated here. Because of the restrictions on τ, ρ, and f needed to obtain the result (there are simple examples showing the necessity of these conditions; see for instance [20], [28]), the operator formulation is not defined on the entire space $\overline{\mathcal{H}}$, and so demonstrating that it is pseudomonotone is not straightforward.

5. Appendix.

Proof of Lemma 1. We begin by showing that there are constants:

C_1 depending upon the value c, the maximum value of σ and its derivatives to order two, and the maximum values of ϕ and its derivatives to order two,

C_2 depending upon the value c, the maximum values of ϕ and its first derivatives, and the maximum values of σ and its derivatives to order two,

C_3 depending upon the area of Ω, the value c, the maximum values of σ and its derivatives to order two, $\|w\|_{3,4;\Omega}$, and the maximum values of ϕ and its first derivatives,

such that

$$\begin{aligned} &|\mathcal{B}_w(\phi u_\xi, \phi u_\xi) + \mathcal{B}_w(u, (\phi^2 u_\xi)_\xi)| \\ &\quad \leq C_1(\|\psi u_{\xi\xi}\|^2 + \|\psi u_{\xi\eta}\|^2 + \|\psi u_{\eta\eta}\|^2 + \|\psi \nabla u\|^2) + \tau C_2 \|\psi \nabla u\|^2 \\ &\quad\quad + C_3(\|\psi \nabla u\|_4^2 + \|\psi u_{\xi\xi}\|^2 + \|\psi u_{\xi\eta}\|^2 + \|\psi \nabla u\|^2). \end{aligned} \tag{20$_{(\xi,\eta)}$}$$

The proof of this inequality consists essentially of integrations by parts to arrive at terms that can be bounded by the right-hand side of (20) by means of the Schwarz inequality and the arithmetic-geometric mean inequality. Notice that the normal to $\partial\Omega$ has no ξ component in the support of ϕ. Further, u, u_η, and the ξ derivatives of these quantities are all zero on $\partial\Omega$ in the support of ϕ by the boundary conditions specified in the theorem, Eq. (3). These facts assure no boundary terms appear in any of the integrations by parts performed in this proof. By applying integration by parts we arrive at

$$\begin{aligned}
&\mathcal{B}_w(u,(\phi^2u_\xi)_\xi)+\mathcal{B}_w(\phi u_\xi,\phi u_\xi)\\
&\quad=\int_\Omega \phi_\eta^2u_{\xi\xi}^2+[4\sigma\phi_\eta^2+2\phi_\eta\phi_\xi]u_{\xi\xi}u_{\xi\eta}+[4A\phi_\eta^2+4\sigma\phi_\eta\phi_\xi+\phi_\xi^2-2\sigma'(\phi^2)_\eta]u_{\xi\eta}^2\\
&\qquad-\sigma'(\phi^2)_\eta u_{\xi\xi}u_{\eta\eta}-[\sigma'(\phi^2)_\xi+2A'(\phi^2)_\eta]u_{\xi\eta}u_{\eta\eta}+\tfrac12(A'\phi^2)_\xi u_{\eta\eta}^2\\
&\qquad+2(A\phi_\eta^2u_\xi+2\sigma\phi_\eta\phi_\xi u_\xi+(\phi^2\sigma')_\xi u_\eta)_\eta u_{\xi\eta}+2(\sigma\phi_\eta^2u_\xi+\phi_\eta\phi_\xi u_\xi)_\xi u_{\xi\eta}\\
&\qquad+\left[A(\phi_\eta^2)_\eta+\sigma(\phi_\eta\phi_\xi)_\eta+\tfrac12\sigma(\phi_\eta^2)_\xi+\tfrac12(\phi_\xi^2)_\eta-\sigma'\phi_\eta^2-\sigma'\phi\phi_{\eta\eta}\right](u_\xi^2)_\eta\\
&\qquad-2[\sigma'\phi_\xi\phi_\eta+\sigma'\phi\phi_{\xi\eta}+A'(\phi_\eta^2+\phi\phi_{\eta\eta})](u_\eta u_\xi)_\eta\\
&\qquad+[A'\phi_\eta^2+A'\phi\phi_{\eta\eta}+\sigma'\phi_\eta\phi_\xi+\sigma'\phi\phi_{\xi\eta}](u_\eta^2)_\xi+\tfrac12[\sigma(\phi_\eta^2)_\eta+(\phi_\eta^2)_\xi](u_\xi^2)_\xi\\
&\qquad+[A\phi_{\eta\eta}^2+2\sigma\phi_{\eta\eta}\phi_{\xi\eta}+\phi_{\xi\eta}^2]u_\xi^2+(\phi_\xi^2w_{\eta\eta}+\phi_\eta^2w_\eta\sigma'-2\phi_\xi\phi_\eta w_{\xi\eta}+\phi_\eta^2w_{\xi\xi})u_\xi^2\\
&\qquad-(\phi^2)_\eta(\sigma'w_\eta)_\eta u_\xi u_\eta+\tfrac12[(\sigma'\phi^2)_\xi w_{\xi\eta}+(\sigma''\phi^2w_\eta)_\xi]u_\eta^2-(\phi^2u_\xi)_\eta u_\eta w_{\xi\xi\xi}\\
&\qquad+\left[(\phi^2)_\eta u_\xi^2+\tfrac12\sigma'\phi^2u_\eta^2+(\phi^2(u_\eta u_\xi))_\xi\right]w_{\xi\xi\eta}-(\phi^2u_\xi^2)_\xi w_{\xi\eta\eta}\\
&\qquad+\tau([\phi_\xi^2+A\phi_\eta^2+2\sigma\phi_\eta\phi_\xi-(\phi^2)_\eta\sigma']u_\xi^2\\
&\qquad+[\sigma_{\xi\xi}\phi^2-A'(\phi^2)_\eta]u_\xi u_\eta+\tfrac12(\phi^2A')_\xi u_\eta^2)\,dA.
\end{aligned}\tag{21}$$

We bound representative terms of the integral above. The second term of the integral is representative of those that are bounded by the C_1 term on the right side of (20). It is bounded by

$$\begin{aligned}
\left|\int_\Omega[4\sigma\phi_\eta^2+2\phi_\eta\phi_\xi]u_{\xi\xi}u_{\xi\eta}dA\right|&\le\frac{\|4\sigma\phi_\eta^2+2\phi_\eta\phi_\xi\|_\infty}{c^2}\left|\int_\Omega\psi^2u_{\xi\xi}u_{\xi\eta}\,dA\right|\\
&\le C(\|\psi u_{\xi\xi}\|^2+\|\psi u_{\xi\eta}\|^2).
\end{aligned}$$

The constant C is half the constant in the line above. The C_2 term in (20) arises from the terms containing τ in the integral in Eq. (21) in the same way. The C_3 term in (20) bounds all the terms in the integral in Eq. (21) that contain a derivative of w. The terms including derivatives up to order two of w can be easily bounded as above, since the Sobolev embedding theorem implies $w\in W^{(3,4)}$ has continuous derivatives at these orders. Further, the maximum values of these derivatives are bounded by the Sobolev inequality $\|g\|_4\le\frac{|\Omega|^{1/4}}{\pi}\|\nabla g\|$ (see [25] for the constant). A little more care must be shown in bounding the terms involving third derivatives of w. For example

$$\begin{aligned}&\left|\int_\Omega (\phi^2 u_\xi)_\eta u_\eta w_{\xi\xi\xi} dA\right| \\ &\quad \leq \left[\left(\int_\Omega (2\phi_\eta u_\xi)^2\, dA\right)^{1/2} + \left(\int_\Omega (\phi u_{\xi\eta})^2\, dA\right)^{1/2}\right]\left(\int_\Omega (\phi u_\eta w_{\xi\xi\xi})^2\, dA\right)^{1/2} \\ &\quad \leq \frac{\max\{\|4\phi_\eta^2\|_\infty, \|\phi^2\|_\infty\}^{1/2}}{c}(\|\psi u_\xi\| + \|\psi u_{\xi\eta}\|) \times \frac{\|\phi^2\|_\infty^{1/2}}{c}\|w_{\xi\xi\xi}\|_4\|\psi u_\eta\|_4 \\ &\quad \leq C\|w_{\xi\xi\xi}\|_4(\|\psi u_\eta\|_4^2 + \|\psi u_\eta\|^2 + \|\psi u_{\xi\eta}\|^2).\end{aligned}$$

All the remaining terms in the integral in Eq. (21) can be bounded in fashions analogous to those presented here; notice that the bounds derived apply as well to the absolute value of the integral of the term being bounded. This completes the proof of inequality (20).

We use this inequality to prove the lemma as follows. By the reasoning leading to Eq. (8), for u smooth enough to satisfy the conditions of the lemma of the following holds:

$$\mathcal{B}_w(u, (\phi^2 u_\xi)_\xi) = \int_\Omega f_{(u,w)}(\phi^2 u_\xi)_\xi\, dA.$$

Thus there is a constant C depending on the maximum values of ϕ and ϕ_ξ so that

$$|\mathcal{B}_w(u, (\phi^2 u_\xi)_\xi)| \leq C(\|\psi f_{(u,w)}\|^2 + \|\psi u_\xi\|^2 + \|\psi u_{\xi\xi}\|^2).$$

Further, by expanding the quadratic form on the left-hand side of (20) we have

$$\begin{aligned}\mathcal{B}_w(\phi u_\xi, \phi u_\xi) = \int_\Omega \phi^2 (A u_{\xi\eta\eta}^2 + u_{\xi\xi\eta}^2 + 2\sigma u_{\xi\eta\eta} u_{\xi\xi\eta} \\ + \tau(u_\xi^2 + A u_\eta^2 + 2\sigma u_\xi u_\eta))\, dA + \mathcal{C},\end{aligned} \tag{22}$$

where $|\mathcal{C}|$ has the form

$$\begin{aligned}\mathcal{C} = \int_\Omega &\left[\phi_\eta^2 - \frac{1}{2}(\phi^2)_{\eta\eta}\right] u_{\xi\xi}^2 + [4\sigma\phi_\eta^2 - \sigma(\phi^2)_{\eta\eta} + 2\phi_\eta\phi_\xi] u_{\xi\xi} u_{\xi\eta} \\ &+ \left[4A\phi_\eta^2 - A(\phi^2)_{\eta\eta} - \frac{1}{2}\sigma(\phi^2)_{\xi\eta} + 4\sigma\phi_\eta\phi_\xi + \phi_\xi^2 - \frac{1}{2}(\sigma(\phi^2)_\eta)_\xi - \frac{1}{2}(\phi^2)_{\xi\xi}\right] u_{\xi\eta}^2 \\ &- 2[(A\phi\phi_{\eta\eta}u_\xi + \sigma\phi\phi_{\xi\eta}u_\xi)_\eta + (\sigma\phi\phi_{\eta\eta}u_\xi + \phi\phi_{\xi\eta}u_\xi)_\xi] u_{\xi\eta} \\ &+ \left[\frac{1}{2}\sigma(\phi_\eta^2)_\eta + \frac{1}{2}(\phi_\eta^2)_\xi + A(\phi_\eta^2)_\eta\right](u_\xi^2)_\xi \\ &+ \left[\sigma(\phi_\eta\phi_\xi)_\eta + \frac{1}{2}(\phi_\xi^2)_\eta + \frac{1}{2}\sigma(\phi_\eta^2)_\xi\right](u_\xi^2)_\eta \\ &+ [A\phi_{\eta\eta}^2 + 2\sigma\phi_{\eta\eta}\phi_{\xi\eta} + \phi_{\xi\eta}^2] u_\xi^2 + \tau\Bigg[A\phi_\eta^2 - \frac{1}{2}A(\phi^2)_{\eta\eta} \\ &\qquad - \frac{1}{2}\sigma(\phi^2)_{\xi\eta} + 2\sigma\phi_\eta\phi_\xi + \phi_\xi^2 - \frac{1}{2}(\sigma(\phi^2)_\eta)_\xi - \frac{1}{2}(\phi^2)_{\xi\xi}\Bigg] u_\xi^2\, dA \\ &+ c(w, \phi u_\xi, \phi u_\xi).\end{aligned}$$

In the same fashion used to obtain (20), $\mathcal{C}$ can be bounded by an expression of the form of the right-hand side of that inequality. The differences are: the constants C_1 and C_2 depend upon ϕ and its derivatives up to order two and σ and its first derivative, while C_3 depends on ϕ and σ and their first derivatives, and the $\|\psi\nabla u\|_4$ term may be omitted. Since the integral on the right in Eq. (22) is ϕ^2 times a weak formulation of $D[u_\xi]u_\xi$, we can change variables to obtain

$$\begin{aligned}\|\phi u_{\xi xy}\|^2 + \|\phi u_{\xi yy}\|^2 + \tau\|\phi\nabla u_\xi\|^2 &\leq \mathcal{B}_w(\phi u_\xi, \phi u_\xi) + |\mathcal{C}| \\ &\leq |\mathcal{B}_w(\phi u_\xi, \phi u_\xi) + \mathcal{B}_w(u, (\phi^2 u_\xi)_\xi)| \\ &\qquad + |\mathcal{B}_w(u, (\phi^2 u_\xi)_\xi)| + |\mathcal{C}| \\ &\leq |\mathcal{D}| + C(\|\psi f_{(u,w)}\|^2 + \|\psi u_\xi\|^2 + \|\psi u_{\xi\xi}\|^2),\end{aligned}$$

where $|\mathcal{D}|$ is the sum of the right-hand side of inequality $(20)_{(\xi,\eta)}$ and the bound just derived for $\mathcal{C}$. Thus, by applying the Sobolev inequality to the quantity $\psi\nabla u$ in $|\mathcal{D}|$ we can show

$$|\mathcal{D}| \leq C(\|\chi u_{\xi\xi}\|^2 + \|\chi u_{\xi\eta}\|^2 + \|\chi u_{\eta\eta}\|^2) + K\|\chi\nabla u\|^2.$$

Substituting this bound into the previous inequality gives the bound of (9); so the proof is complete.

Proof of Lemma 3. By reasoning similar to that used in Lemma 1, since

$$\begin{aligned}\mathcal{B}_w(\phi u, \phi u) - \mathcal{B}_w(u, \phi^2 u) = \int_\Omega & u u_{\eta\eta}(-2A\phi_\eta^2 - 2\sigma\phi_\eta\phi_\xi) \\ &+ u u_{\xi\eta}(-2\sigma\phi_\eta^2 - 2\phi_\eta\phi_\xi) + u_\xi u_\eta(4\sigma\phi_\eta^2 + 2\phi_\eta\phi_\xi) \\ &+ u_\eta^2(4A\phi_\eta^2 + 4\sigma\phi_\eta\phi_\xi + \phi_\xi^2) + u_\xi^2\phi_\eta^2 + u u_\xi(2\sigma\phi_\eta\phi_{\eta\eta} + 2\phi_\eta\phi_{\xi\eta}) \\ &+ u u_\eta(4A\phi_\eta\phi_{\eta\eta} + 2\sigma\phi_{\eta\eta}\phi_\xi + 4\sigma\phi_\eta\phi_{\xi\eta} + 2\phi_\xi\phi_{\xi\eta}) \\ &+ u^2(A\phi_{\eta\eta}^2 + \phi_{\xi\eta}^2 + 2\sigma\phi_{\eta\eta}\phi_{\xi\eta} + \tau(A\phi_\eta^2 + 2\sigma\phi_\eta\phi_\xi + \phi_\xi^2) \\ &+ w_{\eta\eta}\phi_\xi^2 - 2w_{\xi\tau}\phi_\eta\phi_\xi + w_{\xi\xi}\phi_\eta^2 + \sigma'\phi_\eta^2 w_\eta)\, dA,\end{aligned} \tag{23}$$

we can show that

$$|\mathcal{B}_w(\phi u, \phi u) - \mathcal{B}_w(u, \phi^2 u)| \leq C_1\|\chi\nabla u\|^2 + K_1\|\chi u\|^2, \tag{24}$$

where C_1 and K_1 depend upon the value c, the area of Ω, the norm $\|w\|_{3,4;\Omega}$, the maximum values of σ and its first derivative, and the maximum values of ϕ and its derivatives to order two. As usual, C_1 is independent of and K_1 is dependent upon the parameter τ. Similarly, the identity

$$\begin{aligned}
&\mathcal{B}_w(\phi u, \phi u) - \int_\Omega \phi^2(Au_{\eta\eta}^2 + 2\sigma u_{\xi\eta}u_{\eta\eta} + u_{\xi\eta}^2 + \tau(u_\xi^2 + Au_\eta^2 + 2\sigma u_\xi u_\eta))\,dA \\
&\quad = \int_\Omega (4\sigma\phi_\eta^2 - \sigma(\phi^2)_{\eta\eta} + 2\phi_\eta\phi_\xi)u_\xi u_\eta + \left(\phi_\eta^2 - \tfrac{1}{2}(\phi^2)_{\eta\eta}\right) u_\xi^2 \\
&\quad + \left(4A\phi_\eta^2 - A(\phi^2)_{\eta\eta} + 4\sigma\phi_\eta\phi_\xi + \phi_\xi^2 - \tfrac{1}{2}(\sigma(\phi^2)_\eta)_\xi - \tfrac{1}{2}\sigma(\phi^2)_{\xi\eta} - \tfrac{1}{2}(\phi^2)_{\xi\xi}\right) u_\eta^2 \\
&\quad + \left(\tfrac{1}{2}\sigma(\phi_\eta^2)_\eta + \tfrac{1}{2}(\phi_\eta^2)_\xi\right)(u^2)_\xi + \left(A(\phi_\eta^2)_\eta + \sigma(\phi_\eta\phi_\xi)_\eta + \tfrac{1}{2}(\phi_\xi^2)_\eta + \tfrac{1}{2}\sigma(\phi_\eta^2)_\xi\right)(u^2)_\eta \\
&\quad - 2([A\phi_{\eta\eta} + \sigma\phi_{\xi\eta})\phi u]_\eta + [(\sigma\phi_{\eta\eta} + \phi_{\xi\eta})\phi u]_\xi)u_\eta + (A\phi_{\eta\eta}^2 + 2\sigma\phi_{\eta\eta}\phi_{\xi\eta} + \phi_{\xi\eta}^2)u^2 \\
&\quad + \tau\left(-A\phi\phi_{\eta\eta} + 2\sigma\phi_\eta\phi_\xi - \tfrac{1}{2}(\sigma(\phi^2)_\eta)_\xi - \tfrac{1}{2}\sigma(\phi^2)_{\xi\eta} - A\phi\phi_{\xi\xi}\right)u^2\,dA + c(w, \phi u, \phi u)
\end{aligned}$$

leads to the assertion

$$\begin{aligned}
&\left|\mathcal{B}_w(\phi u, \phi u) - \int_\Omega \phi^2(Au_{\eta\eta}^2 + 2\sigma u_{\xi\eta}u_{\eta\eta} + u_{\xi\eta}^2 + \tau(u_\xi^2 + Au_\eta^2 + 2\sigma u_\xi u_\eta))\,dA\right| \\
&\qquad \le C_2\|\chi\nabla u\|^2 + K_2\|\chi u\|^2,
\end{aligned} \tag{25}$$

where C_2 and K_2 have the same dependencies as C_1 and K_1, except that they depend upon the derivatives of ϕ up to order three. Using the coercivity of the integral in (25) mentioned in Sec. 3 (in the paragraph following the statement of the lemma), we arrive at the inequality (17), and the lemma is proved. □

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