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BOUNDARY VALUE PROBLEMS FOR GENERAL SECOND ORDER EQUATIONS AND SIMILARITY SOLUTIONS TO THE RAYLEIGH PROBLEM

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Abstract. Several fairly general existence theorems are established for two-point boundary value problems associated with a nonlinear differential equation which occurs in the study of the p-Laplace equation, generalized diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in a porous medium. One of them implies the existence of similarity solutions to the Rayleigh problem for a power-law fluid.

1. Introduction. The method of upper and lower solutions has become a standard tool for studying the solvability of boundary value problems associated with the second-order nonlinear differential equation

$$y''=f(x, y, y')\,; \tag{1.1}$$

see, for instance, [1]–[5] and the references therein. Basically, α and β are called lower and upper solutions relative to the equation (1.1), respectively, if

$$\alpha''(x)\geq f(x, \alpha(x), \alpha'(x)) \quad \text{and} \quad \beta''(x)\leq f(x, \beta(x), \beta'(x))$$

for x in some interval. Here and henceforth the prime denotes differentiation with respect to the independent variable x.

The main purpose of this paper is to establish several fairly general existence theorems for boundary value problems associated with the nonlinear differential equation

$$[\phi(y')]'=k(x)f(x, y, y') \tag{1.2}$$

by employing the method of lower and upper solutions.

Concerning the equation (1.2), the following hypotheses on ϕ, k and f are adopted:

(H_1) ϕ: $\boldsymbol{R}\to\boldsymbol{R}$ is continuous and strictly increasing and $\phi(\boldsymbol{R})=\boldsymbol{R}$, where $\boldsymbol{R}=(-\infty,+\infty)$.

(H_2) k: $(a, b)\to\boldsymbol{R}_+$ is continuous and $\int_a^b k(x)dx<+\infty$, where $\boldsymbol{R}_+=(0, +\infty)$.

($\tilde{H}_2$) k: $(a, +\infty)\to\boldsymbol{R}_+$ is continuous and $\int_a^b k(x)dx<+\infty$ for each fixed $b>a$.

(H_3) f: $[a, b]\times\boldsymbol{R}^2\to\boldsymbol{R}$ is continuous and satisfies a Nagumo condition on the set

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$$(1.3)\qquad \Omega := \{(x, y, z);\ a \le x \le b,\ \alpha(x) \le y \le \beta(x),\ z \in \boldsymbol{R}\}\,,$$

where $\alpha, \beta \in C[a, b]$ with $\alpha(x) \le \beta(x)$ on $[a, b]$, and $C[a, b]$ denotes the set of all continuous real-valued functions defined on $[a, b]$.

($\tilde{\mathrm{H}}_3$) $f:[a, +\infty) \times \boldsymbol{R}^2 \to \boldsymbol{R}$ is continuous and for fixed $b > a$ satisfies a Nagumo condition on the set Ω, where $\alpha, \beta \in C[a, +\infty)$ with $\alpha(x) \le \beta(x)$ on $[a, +\infty)$.

A continuous function $f: [a, b] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ is said to satisfy a Nagumo condition on the set Ω if there exists a continuous function $\theta: \bar{\boldsymbol{R}}_+ \to \boldsymbol{R}_+$ and a real number σ, $1 \le \sigma \le +\infty$, such that

$$(1.4)\qquad |f(x, y, z)| \le \theta(|z|) \quad \text{on} \quad \Omega$$

and

$$(1.5)\qquad \int_{-\infty}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{(\sigma-1)/\sigma} du}{\theta(|\phi^{-1}(u)|)}\,, \quad \int_{\phi(\nu)}^{+\infty} \frac{|\phi^{-1}(u)|^{(\sigma-1)/\sigma} du}{\theta(|\phi^{-1}(u)|)} > \mu^{(\sigma-1)/\sigma} \|k\|_\sigma\,,$$

where $\bar{\boldsymbol{R}}_+ = [0, +\infty)$, $\phi^{-1}(u)$ is the function inverse to $\phi(t)$,

$$(1.6)\qquad \mu := \max_{a \le x \le b} \beta(x) - \min_{a \le x \le b} \alpha(x)\,,$$

$$(1.7)\qquad \nu := \max\{|\alpha(a) - \beta(b)|,\ |\alpha(b) - \beta(a)|\}/(b - a)\,,$$

and

$$(1.8)\qquad \|k\|_\sigma := \begin{cases} \sup_{a<x<b} k(x)\,, & \text{if} \quad \sigma = +\infty\,, \\ [\int_a^b (k(x))^\sigma dx]^{1/\sigma}\,, & \text{if} \quad 1 \le \sigma < +\infty\,. \end{cases}$$

We denote $\mu^0 := 1$, $|\phi^{-1}(u)|^0 := 1$.

Moreover, most of our existence theorems are stated in terms of upper and lower solutions and a Nagumo condition.

Functions $\alpha, \beta \in C^1[a, b]$ (resp. $C^1[a, +\infty)$) are called lower and upper solutions of the equation (1.2) on $[a, b]$ (resp. $[a, +\infty)$), respectively, if $[\phi(\alpha')]'$, $[\phi(\beta')]' \in C(a, b)$ (resp. $C(a, +\infty)$) and

$$(1.9)\qquad \begin{aligned} &[\phi(\alpha'(x))]' \ge k(x) f(x, \alpha(x), \alpha'(x)) && \text{in} \quad (a, b)\ (\text{resp. } (a, +\infty))\,, \\ &[\phi(\beta'(x))]' \le k(x) f(x, \beta(x), \beta'(x)) && \text{in} \quad (a, b)\ (\text{resp. } (a, +\infty))\,; \end{aligned}$$

A function $y \in C^1[a, b]$ (resp. $C^1[a, +\infty)$) is said to be a solution of the equation (1.2) on $[a, b]$ (resp. $[a, +\infty)$), if it is both a lower and an upper solution of (1.2) on $[a, b]$ (resp. $[a, +\infty)$), i.e.,

$$[\phi(y'(x))]' = k(x) f(x, y(x), y'(x)) \quad \text{in} \quad (a, b)\ (\text{resp. } (a, +\infty))\,.$$

Equations of the form (1.2), in particular $\phi(t) = |t|^{N-1} t$, $N > 0$, occur in the study of the $(N+1)$-Laplace equation [6], generalized diffusion theory [7]–[9], non-Newtonian fluid theory [10], [11], and the turbulent flow of a gas in a porous medium [12], [13].

The existence results in this paper will improve, extend, and complement the existing theory in [1]–[15]. Actually, some boundary value problems for the equation (1.2), including the case $\phi(t)=|t|^{N-1}t$, $N>0$, have been studied by O'Regan [15], Bobisud [13], Phan-Thien [10], Kaper et al. [6], and Piao et al. [14]. Their arguments are all different from ours. To our knowledge, the method of upper and lower solutions for the equation (1.2) with $\phi(t)$ being genuinely nonlinear has not yet been developed.

Besides the above, we also study the Rayleigh problem for a power-law fluid

$$\begin{cases} \dfrac{\partial}{\partial\eta}\left(\left|\dfrac{\partial u}{\partial\eta}\right|^{N-1}\dfrac{\partial u}{\partial\eta}\right)=\dfrac{\partial u}{\partial t}, & \eta>0, \quad t>0; \quad N>0, \\ u(\eta, 0)=0, & \eta>0, \\ u(0, t)=U_0 t^k, & t>0; \quad U_0>0, \\ u(+\infty, t)=0, & t>0 \end{cases}$$

under suitable restrictions on the parameters N and k.

The plan of this paper is as follows. In Section 2, we study the boundary value problems for (1.2) with y satisfying some linear boundary conditions, with a view to extending Theorems 1.5.1 and 1.7.1 in the monograph [1] and improving some existence principles and results in [13], [15]. In Section 3 we demonstrate that the Rayleigh problem has similarity solutions under suitable restrictions on N and k, by employing one of the existence theorems established in Section 2. Finally, in the appendix we prove that the Rayleigh problem has no similarity solutions when $N=1$ and $k=-1, -2, -3, \ldots$.

2. Linear boundary conditions. The present section is the cream of this paper. We study the following three boundary value problems

$$\begin{cases} [\phi(y')]'=k(x)f(x, y, y'), & a<x<b, \\ y(a)=A, \quad y(b)=B, \end{cases} \tag{2.1}$$

$$\begin{cases} [\phi(y')]'=k(x)f(x, y, y'), & a<x<b, \\ y'(a)=A, \quad y(b)=B, \end{cases} \tag{2.2}$$

and

$$\begin{cases} [\phi(y')]'=k(x)f(x, y, y'), & a<x<+\infty, \\ y(a)=A, \end{cases} \tag{2.3}$$

where A and B are prescribed real numbers.

A few and significant modifications of the proof of Theorems 2.3 and 2.1 in [15] yield the following existence result.

THEOREM 2.1. *Let* (H_1), (H_2) *hold and let* $f:[a,b]\times \mathbf{R}^2 \to \mathbf{R}$ *be continuous and bounded. Then* (2.1) *and* (2.2), *respectively, have at least one solution.*

PROOF. First we prove that (2.1) has a solution. Let us define a mapping $T_1: C[a,b]\to C[a,b]$ by

$$(T_1z)(x):=\phi^{-1}\left(\tau+\int_a^x k(s)f(s,(Jz)(s),z(s))ds\right), \qquad a\le x\le b, \quad \forall z\in C[a,b],$$

where $\varphi^{-1}: \mathbf{R}\to\mathbf{R}$ is the function inverse to ϕ,

$$(Jz)(x):=B-\int_x^b z(s)ds, \qquad a\le x\le b, \tag{2.4}$$

and τ is determined by the equation

$$\int_a^b \phi^{-1}\left(\tau+\int_a^x k(s)f(s,(Jz)(s),z(s))ds\right)dx=:w(\tau)=B-A. \tag{2.5}$$

Concerning the mapping T_1, we claim that

(i) for each fixed $z\in C[a,b]$ the equation (2.5) has a unique solution $\tau\in\mathbf{R}$,

(ii) the image of $C[a,b]$ under the mapping T_1 is uniformly bounded and equicontinuous on $[a,b]$, and

(iii) the mapping T_1 is continuous on $C[a,b]$.

Let $z\in C[a,b]$ be fixed. Then we have, by the definition of $w(\tau)$,

$$(b-a)\phi^{-1}(\tau-M\|k\|_1)\le w(\tau)\le(b-a)\phi^{-1}(\tau+M\|k\|_1), \qquad \forall\tau\in\mathbf{R}, \tag{2.6}$$

where $\|k\|_1$ is defined by (1.8) and M a positive number such that

$$|f(x,y,z)|\le M \qquad \text{on} \quad [a,b]\times\mathbf{R}^2.$$

Because ϕ^{-1} is a continuous, strictly increasing function on $\mathbf{R}$ with $\phi^{-1}(-\infty)=-\infty$ and $\phi^{-1}(+\infty)=+\infty$, so is w (for each fixed $z\in C[a,b]$), by the definition of w and (2.6). Thus, there exists a unique $\tau\in\mathbf{R}$ satisfying the equation (2.5), the the claim (i) is true.

By the first mean value theorem for integrals, it follows from (2.5) that for each fixed $z\in C[a,b]$ there exists a $\xi\in(a,b)$ such that

$$\phi^{-1}\left(\tau+\int_a^\xi k(s)f(s,(Jz)(s),z(s))ds\right)=\frac{B-A}{b-a},$$

i.e.,

$$\tau=\phi\left(\frac{B-A}{b-a}\right)-\int_a^\xi k(s)f(s,(Jz)(s),z(s))ds,$$

where τ is the unique solution of (2.5) corresponding to the function $z\in C[a,b]$.

Consequently, we have

$$(2.7)\quad |\tau|, \left|\tau+\int_a^x k(s)f(s,(Jz)(s),z(s))ds\right| \le \left|\phi\left(\frac{B-A}{b-a}\right)\right| + M\|k\|_1 =: N\,, \qquad a\le x\le b\,.$$

From the definition of T_1 and (2.7), we conclude that

$$|(T_1 z)(x)| \le \max_{-N\le\mu\le N} |\phi^{-1}(u)|\,, \qquad a\le x\le b,\quad \forall z\in C[a,b]\,.$$

This shows that the image of $C[a, b]$ is uniformly bounded on $[a, b]$.

From the uniform continuity of $\phi^{-1}(u)$ on the closed interval $[-N, N]$, it follows that for any $\varepsilon>0$ there exists a $\rho>0$ such that

$$|\phi^{-1}(u_1)-\phi^{-1}(u_2)|<\varepsilon\,, \quad \text{whenever} \quad u_1, u_2\in[-N, N] \quad \text{and} \quad |u_1-u_2|<\rho\,.$$

Put

$$u_j := \tau + \int_a^{x_j} k(s)f(s,(Jz)(s),z(s))ds\,, \qquad j=1,2\,,$$

and

$$K(x) := \int_a^x k(s)ds\,, \qquad a\le x\le b\,.$$

Then $K(s)$ is absolutely continuous on $[a, b]$. We assert that there exists a $\delta>0$ such that for any $z\in[a, b]$,

$$|u_1-u_2|\le M|K(x_1)-K(x_2)|\le\rho\,, \quad \text{whenever} \quad x_1, x_2\in[a,b] \quad \text{and} \quad |x_1-x_2|<\delta\,.$$

Hence, there exists a $\delta>0$, independent of $\varepsilon>0$ and $z\in C[a, b]$, such that

$$|(T_1z)(x_1)-(T_1z)(x_2)|<\varepsilon\,, \quad \text{whenever} \quad x_1, x_2\in[a,b] \quad \text{and} \quad |x_1-x_2|<\delta\,.$$

This shows that the image of $C[a, b]$ is equicontinuous on $[a, b]$. Consequently, the claim (ii) holds.

Now assume that $z_n\in C[a, b]$, $n=0, 1, 2, \ldots,$ and z_n converges to z_0 uniformly on $[a, b]$ as $n\to\infty$. By the definition of T_1, we have

$$(T_1 z_n)(x) = \phi^{-1}\left(\tau_n + \int_a^x k(s)f(s,(Jz_n)(s),z_n(s))ds\right), \qquad n=0,1,2,\ldots,$$

where τ_n satisfies the condition

$$(2.5)_n \qquad \int_a^b \phi^{-1}\left(\tau_n + \int_a^x k(s)f(s,(Jz_n)(s),z_n(s))ds\right)dx = B-A\,.$$

Applying the first mean value theorem for integrals to the difference between $(2.5)_n$, $n=1, 2, 3, \ldots,$ and $(2.5)_0$, we get

$$\phi^{-1}\left(\tau_n+\int_a^{\xi_n} k(s)f(s,(Jz_n)(s),z_n(s))ds\right)-\phi^{-1}\left(\tau_0+\int_a^{\xi_n} k(s)f(s,(Jz_0)(s),z_0(s))ds\right)=0\,,$$

where $\xi_n \in (a, b)$, $n=1, 2, 3, \ldots$, i.e.,

$$\tau_n-\tau_0=\int_a^{\xi_n} k(s)[f(s,(Jz_0)(s),z_0(s))-f(s,(Jz_n)(s),z_n(s))]ds\,, \qquad n=1,2,3,\ldots\,.$$

Thus from the assumption that $z_n(x)$ converges to $z_0(x)$ uniformly on $[a, b]$, it follows that

$$\lim_{n\to\infty} \tau_n=\tau_0$$

and hence

$$\lim_{n\to\infty} (T_1z_n)(x)=(T_1z_0)(x) \quad \text{uniformly on} \quad [a,b]\,.$$

This proves the claim (iii).

From the claims (ii) and (iii), we conclude that the mapping $T_1: C[a, b]\to C[a, b]$ is bounded and completely continuous, by the Arzela-Ascoli theorem. The Schauder fixed point theorem then yields a fixed point of T_1 in $C[a, b]$.

We denote the fixed point of T_1 by $z(x)$. Then we get

$$z(x)=\phi^{-1}\left(\tau+\int_a^x k(s)f(s,(Jz)(s),z(s))ds\right), \qquad a\le x\le b\,,$$

where τ satisfies the constraint condition (2.5). Utilizing the fixed point $z(x)$, we define a function $y(x)$ by

(2.8)
$$\begin{aligned} y(x):&=(Jz)(x)=B-\int_x^b z(s)ds \\ &=B-\int_x^b \phi^{-1}\left(\tau+\int_a^t k(s)f(s,y(s),y'(s))ds\right)dt\,, \qquad a\le x\le b\,. \end{aligned}$$

Consequently, we have

$$y\in C^1[a,b]\,, \quad y(a)=A\,, \quad y(b)=B\,,$$

$$\phi(y'(x))=\tau+\int_a^x k(s)f(s,y(s),y'(s))ds\,, \qquad a\le x\le b\,,$$

and

$$[\phi(y'(x))]'=k(x)f(x,y(x),y'(x))\,, \qquad a<x<b\,.$$

Therefore, the function $y(x)$ is a solution to (2.1).

We now prove that (2.2) has a solution. Define a mapping $T_2: C[a,b]\to C[a,b]$ by

$$(T_2z)(x):=\phi^{-1}\left(\phi(A)+\int_a^x k(s)f(s,(Jz)(s),z(s))ds\right),\qquad a\leq x\leq b,\quad \forall z\in C[a,b].$$

In the same way as above, we deduce that the mapping T_2 has at least one fixed point in $C[a,b]$. Thus, the fixed point of T_2, denoted by $z(x)$, satisfies the equation

$$z(x)=\phi^{-1}\left(\phi(A)+\int_a^x k(s)f(s,(Jz)(s),z(s))ds\right),\qquad a\leq x\leq b.$$

Putting

$$\begin{aligned} y(x):&=(Jz)(x)=B-\int_x^b z(t)dt \\ &=B-\int_x^b \phi^{-1}\left(\phi(A)+\int_a^t k(s)f(s,y(s),y'(s))ds\right)dt,\qquad a\leq x\leq b, \end{aligned} \tag{2.9}$$

we obtain

$$y\in C^1[a,b],\quad y'(a)=A,\quad y(b)=B.$$

$$\phi(y'(x))=\phi(A)+\int_a^x k(s)f(s,y(s),y'(s))ds,\qquad a\leq x\leq b.$$

$$[\phi(y'(x))]'=k(x)f(x,y(x),y'(x)),\qquad a<x<b.$$

This shows that the function $y(x)$ defined by (2.9) is a solution to the boundary value problem (2.2).

From the above proof, we see that the uniform continuity of ϕ^{-1} and f on compact sets implies the complete continuity of the mapping T_j, $j=1,2$. Thereupon, we can conclude that none of the existence results established in [15] requires the assumption that ϕ^{-1} is continuously differentiable on $\boldsymbol{R}$. In particular. Theorems 2.3 and 2.1 in [15] can be recast as follows.

THEOREM 2.2. *Let* (H_1), (H_2) *hold and* $f:[a,b]\times\boldsymbol{R}^2\to\boldsymbol{R}$ *be continuous. Then the boundary value problems* (2.1) *has at least one solution if there is a real number* M, *independent of* λ, *such that* $|y'(x)|\leq M$ *on* $[a,b]$ *for each* $\lambda\in(0,1)$ *and for any solution* $y(x)$ *to the boundary value problems*

$$\begin{cases} [\phi(y')]'=\lambda k(x)f(x,y,y'), & a<x<b, \\ y(a)=A,\quad y(b)=B. \end{cases} \tag{2.1$_\lambda$}$$

Similarly, the boundary value problems (2.2) *has at least one solution if there is a real number* M, *independent of* λ, *such that* $|y'(x)|\leq M$ *on* $[a,b]$ *for each* $\lambda\in(0,1)$ *and* *for any solution $y(x)$ to the boundary value problems*

$$(2.2)_\lambda \qquad \begin{cases} [\phi(y')]' = \lambda k(x) f(x, y, y'), & a < x < b, \\ y'(a) = A, \quad y(b) = B. \end{cases}$$

We omit the proof of this theorem here, because there are only a few (but significant) modifications of the original and the modified parts are similar to those in the proof of the preceding theorem.

A generalization of Theorem 1.5.1 in the monograph [1] is the following.

THEOREM 2.3. *Let $\alpha, \beta \in C^1[a, b]$, respectively, be lower and upper solutions of* (1.2) *on $[a, b]$ with $\alpha(x) \le \beta(x)$ on $[a, b]$ and let* (H_1), (H_2) and (H_3) *hold. Then, for any $\alpha(a) \le A \le \beta(a)$, $\alpha(b) \le B \le \beta(b)$, the boundary value problem* (2.1) *has at least one solution $y \in C^1[a, b]$ with $[\phi(y')]' \in C(a, b)$,*

$$\alpha(x) \le y(x) \le \beta(x) \quad \textit{and} \quad |y'(x)| \le M \qquad \textit{on} \quad [a, b],$$

where the constant M depends only on α, β, ϕ, θ and k.

PROOF. Let us consider the modified boundary value problem of the form

$$(2.10) \qquad \begin{cases} [\phi(y')]' = k(x) F(x, y, y'), & a < x < b, \\ y(a) = A, \quad y(b) = B, \end{cases}$$

where the function $F(x, y, z)$ is what is called the modification of $f(x, y, z)$ associated with the triple $\alpha(x)$, $\beta(x)$, M, i.e.,

$$F(x, y, z) := \begin{cases} f^*(x, \beta(x), z) + \dfrac{y - \beta(x)}{1 + y^2}, & \text{for} \quad y \ge \beta(x), \\ f^*(x, y, z), & \text{for} \quad \alpha(x) \le y \le \beta(x), \\ f^*(x, \alpha(x), z) + \dfrac{y - \alpha(x)}{1 + y^2}, & \text{for} \quad y < \alpha(x), \end{cases}$$

$$f^*(x, y, z) := \begin{cases} f(x, y, M), & \text{for} \quad z > M, \\ f(x, y, z), & \text{for} \quad -M \le z \le M, \\ f(x, y, -M), & \text{for} \quad z < -M, \end{cases}$$

while M is a positive constant satisfying

$$M > v, \quad M > |\alpha'(x)|, \ |\beta'(x)| \qquad \text{on} \quad [a, b]$$

and

$$(2.11) \qquad \int_{\phi(-M)}^{\phi(-v)} \frac{|\phi^{-1}(u)|^{(\sigma-1)/\sigma} du}{\theta(|\phi^{-1}(u)|)}, \quad \int_{\phi(v)}^{\phi(M)} \frac{|\phi^{-1}(u)|^{(\sigma-1)/\sigma} du}{\theta(|\phi^{-1}(u)|)} > \mu^{(\sigma-1)/\sigma} \|k\|_\sigma, \qquad 1 \le \sigma \le +\infty.$$

By (1.5), such a constant can be chosen. It follows from the definition that $F(x, y, z)$ is continuous and bounded on $[a, b]\times \boldsymbol{R}^2$.

By Theorem 2.1, the modified boundary value problem (2.10) has a solution $y\in C^1[a, b]$ with $[\phi(y')]'\in C(a, b)$. Concerning the solution $y(x)$, we claim that

(I) $\alpha(x)\le y(x)\le \beta(x)$ on $[a, b]$;

(II) $|y'(x)|\le M$ on $[a, b]$.

We shall prove only that $y(x)\le\beta(x)$ on $[a, b]$. The arguments are essentially the same for the case $\alpha(x)\le y(x)$ on $[a, b]$.

Assume that $y(x)>\beta(x)$ for some $x\in(a, b)$, since $y(a)=A\le\beta(a)$, $y(b)=B\le\beta(b)$. Then the function $w(x):=y(x)-\beta(x)$ has a positive maximum at a point $x_0\in(a, b)$. Hence it follows that $y'(x_0)=\beta'(x_0)$, $|y'(x_0)|<M$, and

$$\begin{aligned}[\phi(y'(x_0))]' &= k(x_0)F(x_0, y(x_0), y'(x_0)) \\ &= k(x_0)\left[f(x_0, \beta(x_0), \beta'(x_0))+\frac{y(x_0)-\beta(x_0)}{1+y^2(x_0)}\right] \\ &> k(x_0)f(x_0, \beta(x_0), \beta'(x_0))\ge[\phi(\beta'(x_0))]' .\end{aligned}$$

From the continuity of $[\phi(y'(x))]'$ and $[\phi(\beta'(x))]'$ in (a, b), we conclude that there exists a $\delta>0$ such that $(x_0-\delta, x_0+\delta)\subset(a, b)$ and

$$[\phi(y'(t))]'>[\phi(\beta'(t))]' \qquad \text{in} \quad (x_0-\delta, x_0+\delta) .$$

Integrating both sides of the above inequality with respect to t from x_0 to $x\in(x_0, x_0+\delta)$, we get

$$\phi(y'(x))-\phi(y'(x_0))>\phi(\beta'(x))-\phi(\beta'(x_0)) \qquad \text{in} \quad (x_0, x_0+\delta) ,$$

i.e.,

$$w'(x)=y'(x)-\beta'(x)>0 \qquad \text{in} \quad (x_0, x_0+\delta) ,$$

which is impossible at a maximum of $w(x)$. We conclude that $y(x)\le\beta(x)$ on $[a, b]$. Thus, the claim (I) is true.

The mean value theorem asserts that there exists a point $x_0\in(a, b)$ such that

$$v_0:=|y'(x_0)|=|B-A|/|b-a| .$$

Then by (1.7), we have $v_0\le v$. Assume that the claim (II) is not true. Then there exists an interval $[x_1, x_2]\subset[a, b]$ such that one of the following cases holds:

(i) $y'(x_1)=v_0$, $y'(x_2)=M$ and $v_0<y'(x)<M$ on (x_1, x_2),

(ii) $y'(x_1)=M$, $y'(x_2)=v_0$ and $v_0<y'(x)<M$ on (x_1, x_2),

(iii) $y'(x_1)=-v_0$, $y'(x_2)=-M$ and $-M<y'(x)<-v_0$ in (x_1, x_2),

(iv) $y'(x_1)=-M$, $y'(x_2)=-v_0$ and $-M<y'(x)<-v_0$ on (x_1, x_2).

Let us consider the case (i). By (1.4), we obtain

$$|[\phi(y'(x))]'| = k(x)|f(x, y(x), y'(x))| \leq k(x)\theta(|y'(x)|) \quad \text{on} \quad [x_1, x_2]$$

and as a result

$$\int_{\phi(v_0)}^{\phi(M)} \frac{|\phi^{-1}(u)|^{(\sigma-1)/\sigma} du}{\theta(|\phi^{-1}(u)|)} = \int_{x_1}^{x_2} \frac{|y'(x)|^{(\sigma-1)/\sigma}[\phi(y'(x))]' dx}{\theta(|y'(x)|)}$$

$$\leq \int_{x_1}^{x_2} \frac{|y'(x)|^{(\sigma-1)/\sigma}|[\phi(y'(x))]'| dx}{\theta(|y'(x)|)} \leq \int_{x_1}^{x_2} k(x)|y'(x)|^{(\sigma-1)/\sigma} dx$$

$$\leq \left(\int_{x_1}^{x_2} |k(x)|^{\sigma} dx\right)^{1/\sigma} (y(x_2) - y(x_1))^{(\sigma-1)/\sigma} \leq \|k\|_{\sigma} \mu^{(\sigma-1)/\sigma}$$

if $1 < \sigma \leq +\infty$ and

$$\int_{\phi(v_0)}^{\phi(M)} \frac{du}{\theta(|\phi^{-1}(u)|)} = \int_{x_1}^{x_2} \frac{[\phi(y'(x))]' dx}{\theta(|y'(x)|)} \leq \int_{x_1}^{x_2} k(x) dx \leq \|k\|_1$$

if $\sigma = 1$. This contradicts (2.11). We can deal with the remaining possibilities in a similar way and therefore we conclude that the claim (II) is valid.

The claims (I) and (II) imply that the solution $y(x)$ to (2.10) is also a solution to the boundary value problem (2.1). The proof is complete.

Concerning the boundary value problem (2.2), the following statement holds.

THEOREM 2.4. *Let $\alpha, \beta \in C^1[a, b]$, respectively, be lower and upper solutions of* (1.2) *on $[a, b]$ with $\alpha'(a) = \beta'(a)$ and $\alpha(x) \leq \beta(x)$ on $[a, b]$ and let* (H_1), (H_2) and (H_3) *hold. Then, for $A = \alpha'(a)$ and any $\alpha(b) \leq B \leq \beta(b)$, the boundary value problem* (2.2) *has a solution $y \in C^1[a, b]$ with $[\phi(y')]' \in C(a, b)$,*

$$\alpha(x) \leq y(x) \leq \beta(x) \quad and \quad |y'(x)| \leq M \quad on \quad [a, b],$$

where the constant M depends only on α, β, ϕ, θ and k.

The proof is omitted here, because it is very similar to and simpler than that of Theorem 2.3.

The following two statements are consequences of Theorems 2.3 and 2.4, which, respectively, improve Theorem 4.2 in [15] and Theorem 1 in [12].

THEOREM 2.5. *Suppose that $f: [a, b] \times \mathbf{R}^2 \to \mathbf{R}^2$ is continuous and there is a constant $M \geq 0$ such that $yf(x, y, 0) \geq 0$ for $|y| \geq M$ and for all $x \in [a, b]$. Let $\alpha = -N$, $\beta = N$, where $N = \max\{|A|, |B|, M\}$ if we are examining* (2.1), *whereas $N = \max\{|B|, M\}$, if we are interested in* (2.2) *with $A = 0$. In addition, suppose that* (H_1), (H_2) *and* (H_3) *hold. Here $v_0 = |B - A|/|b - a|$ can be chosen as v if we are examining* (2.1), *whereas $v_0 = 0$ can be taken as v if we are interested in* (2.2) *with $A = 0$. Then the boundary value problems* (2.1) *and* (2.2), *respectively, have a solution, where $A = 0$ in* (2.2).

PROOF. Clearly, $\alpha=-N$ and $\beta=N$ are lower and upper solution of (1.2) on $[a, b]$, respectively. Theorems 2.3 and 2.4, respectively, assert the existence of solutions to (2.1) and (2.2).

THEOREM 2.6. *Suppose that* $f: [a, b]\times \bar{\boldsymbol{R}}_+\times \boldsymbol{R}\to \boldsymbol{R}$ *is continuous, and* $f(x, y, z)>0$ *for* $y>0$, $f(x, 0, z)=0$ *for* $(x, z)\in[a, b]\times \boldsymbol{R}$. *Let* $B\geq A\geq 0$, $\alpha=0$, *and* $\beta=B$. *Suppose that* (H_1), (H_2) *and* (H_3) *are satisfied, where* $v=v_0=(B-A)/(b-a)$. *Then the boundary value problem* (2.1) *has a solution.*

PROOF. We may assume that f is extended to $\boldsymbol{R}$ as an odd function of y. The f remains continuous. We shall denote the extension by f also.

Clearly, $\alpha=0$ and $\beta=B$ are lower and upper solutions of (1.2) on $[a, b]$, respectively. Theorem 2.3 tells us that the boundary value problems (2.1) has a solution.

Concerning the boundary value problem (2.3), the following statement is valid, which is a generalization of Theorem 1.7.1 stated in the monograph [1].

THEOREM 2.7. *Let* $\alpha, \beta\in C^1[a, b]$, *respectively, be lower and upper solutions of* (1.2) *on* $[a, +\infty)$ *such that* $\alpha(x)\leq\beta(x)$ *on* $[a, \infty)$ *and let* (H_1), $(\tilde{\mathrm{H}}_2)$ *and* $(\tilde{\mathrm{H}}_3)$ *be satisfied. Then for any* $\alpha(a)\leq A\leq\beta(a)$ *the boundary value problem* (2.3) *has a solution* $y\in C^1[a, \infty)$ *with* $[\phi(y')]'\in C(a, \infty)$ *and* $\alpha(x)\leq y(x)\leq\beta(x)$ *on* $[a, \infty)$.

PROOF. By Theorem 2.3, it follows that for each $n\geq 1$ there is a solution $y_n\in C^1[a, a+n]$ of (2.2) such that $y_n(a)=A$, $y_n(a+n)=\beta(a+n)$, $\alpha(x)\leq y(x)\leq\beta(x)$ on $[a, a+n]$, and

$$\phi(y_n'(x))-\phi(y_n'(a))=\int_a^x k(s)f(s, y_n(s), y_n'(s))ds\,, \qquad a\leq x\leq a+n\,,$$

$$(2.12) \qquad y_n(x)=A+\int_a^x y_n'(s)ds\,, \qquad a\leq x\leq a+n\,.$$

Furthermore, there is an $M_n>0$ such that $|y'(x)|\leq M_n$ on $[a, a+n]$ for any solutions of (1.2) satisfying $\alpha(x)\leq y(x)\leq\beta(x)$ on $[a, a+n]$. By carefully examining the proof of Theorem 2.3, we know that, for any $n\geq 1$, $y_m(x)$ is a solution on $[a, a+n]$ verifying $\alpha(x)\leq y_m(x)\leq\beta(x)$ and $|y_m'(x)|\leq M_n$ on $[a, a+n]$ for all $m\geq n$. Consequently, for $m\geq n$ the sequences $\{y_m(x)\}$, $\{y_m'(x)\}$ are both uniformly bounded and equicontinuous on $[a, a+n]$. Thus, employing the standard diagonalization arguments, we obtain subsequences $\{y_{m(k)}(x)\}$, $\{y'_{m(k)}(x)\}$ which converge uniformly on all compact subintervals of $[a, \infty)$ to functions $y(x)$ and $z(x)$, respectively. Substituting $\{y_{m(k)}(x)\}$ and $\{y'_{m(k)}(x)\}$ into (2.12) and then letting $m(k)\to+\infty$, we get

$$\phi(z(x))-\phi(z(a))=\int_a^x k(s)f(s, y(s), z(s))ds\,, \qquad x\geq a\,,$$

$$y(x)=A+\int_a^x z(s)ds\,, \qquad x\geq a\,.$$

This shows that the function $y(x)$ is a solution to the boundary value problem (2.3).

It is worth pointing out that our results can be extended to the study of the equation (1.2) with some nonlinear boundary value conditions. Especially, Erbe in [3] established several fairly general existence results to the particular case where $\phi(t)=t$, $k(x)\equiv 1$ in (1.2) with some nonlinear boundary value conditions. All the arguments there are independent of ϕ and k. Consequently, one can re-establish all the existence results therein. Since the proofs are exactly the same, we will not repeat the results and the proofs here and leave them to the interested readers.

3. The Rayleigh problem. The initial-boundary value problem

$$\text{(3.1)}\qquad \begin{cases} \dfrac{\partial}{\partial\eta}\left(\left|\dfrac{\partial u}{\partial\eta}\right|^{N-1}\dfrac{\partial u}{\partial\eta}\right)=\dfrac{\partial u}{\partial t}\,, & \eta>0\,,\quad t>0\,;\quad N>0\,,\\ u(\eta,0)=0\,, & \eta>0\,,\\ u(0,t)=U_0t^k\,, & t>0\,;\quad U_0>0\,,\\ u(+\infty,t)=0\,, & t>0 \end{cases}$$

is usually called the Rayleigh problem, which has been suggested as a model for a power-law fluid near the suddenly accelerated plane wall. For details see [16] and [10].

If we write

$$\text{(3.2)}\qquad u=U_0t^ky(x)\,,\quad x=X\eta t^{-\rho}\,,\quad \rho:=\frac{1+(N-1)k}{N+1}\,,\quad X^{N+1}:=\rho U_0^{1-N}\,,$$

then we arrive at the boundary value problem

$$\text{(3.3)}\qquad (|y'(x)|^{N-1}y'(x))'=\lambda y(x)-xy'(x)\,,\qquad x>0\,,$$

$$\text{(3.4)}\qquad y(0)=1\,,$$

$$\text{(3.5)}\qquad \lim_{x\to+\infty} x^{[-\lambda]_+}y(x)=0\,,$$

where

$$\lambda:=k/\rho\,,\quad [-\lambda]_+:=\max\{-\lambda,0\}\,,$$

and ρ is required to be positive.

Conversely, if $y(x)$ is a solution to (3.3)–(3.5), then the function

$$u(\eta,t)=U_0t^ky(X\eta t^{-\rho})\,,$$

is a similarity solution to the Rayleigh problem (3.1), where the similarity variable $x = X\eta t^{-\rho}$ and ρ, X are given by (3.2). That is to say, similarity solutions to the Rayleigh problem (3.1) exist if and only if solutions to the boundary value problem (3.3)–(3.5) exist.

The Rayleigh problem (3.1) was considered by Phan-Thien [10], but there were two errors:

(i) The boundary condition (3.5) was not presented, and hence

(ii) his conclusion that similarity solutions to the Rayleigh problem (3.1) exist for any real number k is not true.

In the appendix we shall prove that the boundary value problem (3.3)–(3.5) has no solutions when $N=1$ and $\lambda = -2, -4, -6, \ldots$.

Concerning the boundary value problem (3.3)–(3.5), the following statement holds.

THEOREM 3.1. *When* $N>0$, $\lambda \geq -1$, *the boundary value problem* (3.3)–(3.5) *has a solution* $y(x)$. *If* $N>1$, *then* $y \in C^1[0, +\infty)$ *and*

$$0 \leq y(x) \leq \left[1 - \frac{N-1}{N+1} x^{(N+1)/N}\right]_+^{N/(N-1)} \quad on \quad [0, +\infty), \tag{3.6}$$

where $[w]_+ := \max\{w, 0\}$, *if* $N=1$, *then* $y \in C^2[0, +\infty)$ *and*

$$0 \leq y(x) \leq e^{-x^2/2} \quad on \quad [0, +\infty) \tag{3.7}$$

while if $0<N<1$, *then* $y \in C^2[0, +\infty)$ *and*

$$0 \leq y(x) \leq \left(1 + \frac{1-N}{1+N} x^{(N+1)/N}\right)^{-N/(1-N)} \quad on \quad [0, +\infty). \tag{3.8}$$

From (3.6), we conclude that when $N>1$ and $\lambda \geq -1$, the similarity solution $y(x)$ has compact support, i.e., there exists a point $0 < x_0 \leq ((N+1)/(N-1))^{N/(N+1)}$ such that $y(x) \equiv 0$ for $x \geq x_0$.

PROOF OF THEOREM 3.1. Put

$$\alpha(x) = 0, \quad \beta(x) := \begin{cases} \left[1 - \frac{N-1}{N+1} x^{(N+1)/N}\right]_+^{N/(N-1)}, & x \geq 0, \text{ if } N>1, \\ e^{-x^2/2}, & x \geq 0, \text{ if } N=1, \\ \left(1 + \frac{1-N}{1+N} x^{(N+1)/N}\right)^{-N/(1-N)}, & x \geq 0, \text{ if } 0<N<1. \end{cases}$$

Clearly, $\alpha(x)$ and $\beta(x)$ are lower and upper solutions of (3.3) on $[0, +\infty)$, respectively.

When $N>1$, we define

$$\phi(t) = |t|^{N-1} t, \quad k(x) \equiv 1, \quad f(x, y, z) = \lambda y - xz.$$

Then, for each $b>0, f(x, y, z)$ satisfies a Nagumo condition on the set

$$\Omega := \{(x, y);\ 0 \le x \le b,\ 0 \le y \le 1\}$$

provided

$$\theta(|z|) = |\lambda| + b|z| \quad \text{and} \quad \sigma = +\infty .$$

When $0 < N \le 1$, the equation (3.3) can be rewritten as

$$y''(x) = (\lambda y(x) - xy'(x))|y'(x)|^{(1-N)}/N .$$

We define

$$\phi(t) = t , \quad k(x) \equiv 1 , \quad f(x, y, z) = (\lambda y - xz)|z|^{1-N}/N .$$

Then, for each $b>0, f(x, y, z)$ satisfies a Nagumo condition on Ω provided

$$\theta(|z|) = (|\lambda| + b|z|)|z|^{1-N}/N , \quad \sigma = +\infty .$$

Theorem 2.7 tells us that the boundary value problem (3.3)–(3.5) has a solution y and

$$\alpha(x) \le y(x) \le \beta(x) \qquad \text{on} \quad [0, +\infty) .$$

In particular, when $N \in (0, 1]$ the solution $y(x)$ is in $C^2[0, +\infty)$. This proves the theorem.

By Theorem 3.1, it follows from (3.2) that the Rayleigh problem (3.1) has a similarity solution when

$$N \ge 1 \quad \text{and} \quad k \ge -1/2N$$

or when

$$0 < N < 1 \quad \text{and} \quad -1/2N \le k < 1/(1-N) .$$

Appendix. In the appendix we study the boundary value problem

$$(1)_n \qquad y'' + xy' + (n+1)y = 0 , \qquad x > 0 ,$$

$$(2) \qquad y(0) = 1 ,$$

$$(3)_n \qquad \lim_{x \to +\infty} x^{n+1} y(x) = 0 ,$$

where $n = 0, 1, 2, \ldots$, which is a special case of (3.3)–(3.5) when $N=1$, $\lambda = -(n+1)$. Our aim is to demonstrate the following:

THEOREM. *If $n = 0, 2, 4, \ldots$, then the boundary value problem $(1)_n$–$(3)_n$ has a unique* *solution; if* $n=1, 3, 5, \ldots$, *then* $(1)_n$–$(3)_n$ *has no solution.*

From the theorem, we conclude that the Rayleigh problem (3.1) has no similarity solutions when $N=1$ and $k=-1, -2, -3, \ldots$.

We begin by introducing the following formulas relative to the Hermite polynomials [17, pp. 837, 1033–1034]:

$$He_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}, \qquad n=0, 1, 2, \ldots, \tag{4}$$

$$\frac{d}{dx}He_n(x)=nHe_{n-1}(x), \qquad n=1, 2, \ldots, \tag{5}$$

$$He_{n+1}(x)=xHe_n(x)-nHe_{n-1}(x), \qquad n=1, 2, \ldots, \tag{6}$$

$$\int_{-\infty}^{+\infty} e^{-x^2/2}He_m(x)He_n(x)dx=\begin{cases}\sqrt{2\pi}n!, & m=n,\\ 0, & m\neq n.\end{cases} \tag{7}$$

The first eleven Hermite polynomials are

$$He_0=1, \quad He_1=x, \quad He_2=x^2-1, \quad He_3=x^3-3x, \quad He_4=x^4-6x^2+3,$$
$$He_5=x^5-10x^3+15x, \quad He_6=x^6-15x^4+45x^2-15,$$
$$He_7=x^7-21x^5+105x^3-105x, \quad He_8=x^8-28x^6+210x^4-420x^2+105,$$
$$He_9=x^9-36x^7+378x^5-1260x^3+945x,$$
$$He_{10}=x^{10}-45x^8+630x^6-3150x^4+4725x^2-945.$$

Employing the first eleven Hermite polynomials, we have already checked up the identity

$$He_n(x)\sum_{m\geq 0}(-1)^m\frac{d^m}{dx^m}He_{n-m}(x)+He_{n+1}(x)\sum_{m\geq 1}(-1)^m\frac{d^{m-1}}{dx^{m-1}}He_{n-m}(x)=n!, \qquad n=1, 2, \ldots, 9. \tag{8$_n$}$$

Thus, we conjecture that it is still true when $n=10, 11, 12, \ldots$. However, we have not yet been able to present its proof.

Clearly, $y=e^{-x^2/2}$ is a solution of the equation $(1)_0$. Using the method of reduction of order, we can find the general solution of $(1)_0$

$$y_0(x)=Ae^{-x^2/2}\int_0^x e^{t^2/2}dt+Be^{-x^2/2}, \tag{9$_0$}$$

where A and B are arbitrary constants.

From the structure of the equation $(1)_n$, we conclude that the function

$$(9)_n \qquad y_n(x) := (-1)^n \frac{d^n}{dx^n} y_0(x) = A\left(He_n(x) e^{-x^2/2} \int_0^x e^{t^2/2} dt + \sum_{m \geq 1} (-1)^m \frac{d^{m-1}}{dx^{m-1}} He_{n-m}(x) \right) + B He_n(x) e^{-x^2/2}, \qquad n = 1, 2, 3, \ldots$$

is the general solutions of $(1)_n$.

From the representation of $y_n(x)$, we assert that $A=0$ if and only if the solution $y_n(x)$ satisfies the boundary condition $(3)_n$.

To prove the assertion, it is enough to prove that when $A=1$, $\lim_{x\to+\infty} x^{n+1} y_n(x)$ does not exist or exists but is equal to a nonzero constant. Indeed, applying the l'Hospital rule, we obtain

$$(10)_0 \qquad \lim_{x\to+\infty} x y_0(x) = \lim_{x\to+\infty} \frac{\int_0^x e^{t^2/2} dt}{x^{-1} e^{x^2/2}} = \lim_{x\to+\infty} \frac{1}{1 - x^{-2}} = 1,$$

$$\lim_{x\to+\infty} x^{n+1} y_n(x) = \lim_{x\to+\infty} \frac{\int_0^x e^{t^2/2} dt + e^{x^2/2} He_n^{-1}(x) \sum_{m\geq 1} (-1)^m \frac{d^{m-1}}{dx^{m-1}} He_{n-m}(x)}{He_n^{-1}(x) x^{-(n+1)} e^{x^2/2}}$$

$$= \lim_{x\to+\infty} \frac{M_1 + M_2}{x^{-n} He_n(x) - (n+1) x^{-(n+2)} He_n(x) - x^{-(n+1)} \frac{d}{dx} He_n(x)},$$

where

$$M_1 = He_n(x) \sum_{m\geq 0} (-1)^m \frac{d^m}{dx^m} He_{n-m}(x),$$

$$M_2 = \left(x He_n(x) - \frac{d}{dx} He_n(x) \right) \sum_{m\geq 1} (-1)^m \frac{d^{m-1}}{dx^{m-1}} He_{n-m}(x),$$

i.e.,

$$(10)_n \qquad \lim_{x\to+\infty} x^{n+1} y_n(x) = \lim_{x\to+\infty} \left(He_n(x) \sum_{m\geq 0} (-1)^m \frac{d^m}{dx^m} He_{n-m}(x) + He_{n+1}(x) \sum_{m\geq 1} (-1)^m \frac{d^{m-1}}{dx^{m-1}} He_{n-m}(x) \right), \qquad n = 1, 2, ,3, \ldots.$$

Here we have used (5) and (6). The reason why the l'Hospital rule can be applied is that

$$F_n(x) = \int_0^x e^{t^2/2} dt + e^{x^2/2} He_n^{-1}(x) \sum_{m\geq 1} (-1)^m \frac{d^{m-1}}{dx^{m-1}} He_{n-m}(x), \qquad n = 1, 2, 3, \ldots$$

approaches $+\infty$ as x tends to $+\infty$.

The polynomial in the brackets on the right hand side of $(10)_n$ is exactly the left hand side of $(8)_n$. We conjecture that it is equal to $n!$. If it is a polynomial of degree nonzero, then the limit does not exist, if it is a polynomial of degree zero, then it is exactly $n!$ by (5) and (7). This shows that our assertion is valid. Therefore, when $n=0, 2, 4, \ldots,$

$$y_n(x)=e^{-x^2/2}He_n(x)/He_n(0)$$

is the unique solution of $(1)_n$. When $n=1, 3, \ldots,$ the equation $(1)_n$ has no solutions, because $He_n(0)=0$ in this case.

This proves the theorem.

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