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BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS AND A CONVEX PROBLEM OF BOLZA

A.R. AFTABIZADEH AND N.H. PAVEL
Department of Mathematics, Ohio University, Athens, Ohio 45701 USA

Abstract. In this paper we are concerned with three types of problems. (I) Existence and uniqueness of the solution u of the following boundary value problems:

$$p(t)u''(t) + r(t)u'(t) \in Au(t) + f(t), \quad \text{a.e. on } [0,T],\ T > 0 \tag{1}$$

$$u'(0) \in \alpha(u(0) - a), \quad u'(T) \in \beta(u(T) - b) \tag{2}$$

$$u''(t) \in Au(t) + f(t), \quad \text{a.e. on } [0,T] \tag{3}$$

$$u(0) = u(T), \quad u'(0) - u'(T) \in \gamma(u(0)) \tag{4}$$

$$u''(t) \in Au(t) + f(t), \quad \text{a.e. on } [0,T] \tag{3}$$

$$u'(0) = u'(T), \quad u(0) - u(T) \in \delta(u'(0)). \tag{5}$$

Here, A, α, $-\beta$, γ, δ are maximal monotone (possibly multivalued) operators acting in a real Hilbert space H, $a, b \in D(A)$, $T > 0$ arbitrary, $f \in L^2([0,T]; H)$ (L^2-with the weight function $\tilde{r}/p$, where $\tilde{r}(t) = \exp\big(\int_0^t (r(s)/p(s))\,ds\big)$, p, $r : [0,T] \to \mathbb{R}$ continuous with $p(t) \geq c > 0\ \forall\, t \in [0,T]$.

(II) Continuous dependence of $u = u(t, a, b, f)$ on a, b and f.

(III) In the case in which A, α and $-\beta$ are subdifferentials of some lower semi-continuous convex (l.s.c.) proper functions, we prove the equivalence of (1)-(2) with a convex problem of Bolza (Theorem 3.3).

1. Introduction. This paper contains several results on existence, uniqueness, continuous dependence on initial data and some optimization problems and application to some elliptic equations. The idea to work in $\mathcal{L}^2_{\tilde{r}/p} = L^2$ with the weight function $\tilde{r}$ enables one to eliminate the differentiability assumption on p and r in Theorem 3.1, and to prove the equivalance of (1)-(2) with an optimization problem. In section 2, we present some preliminary results which help to carry out the proof of the main results. Some of these results (e.g. Lemma 2.1 and Proposition 2.1) seem to be new. Theorem 3.1 in section 3 contains Theorem 3.1 of Veron [11]. There, he assumes $p \in W^{2,\infty}[0,T]$, $r \in W^{1,\infty}[0,T]$, while in this paper we assume only the continuity of p and r and we obtain the same conclusion. The result given in Theorem 3.3 is different from those of Barbu [4, p. 301], see Remark 3.4. Boundary conditions of the form (2) have been considered by Brezis [5] in the case $\alpha = \partial j_1$

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and by Pavel [6, 7] in the general case of (2). Boundary conditions (4) and (5) are new and extend the classical periodic boundary conditions

$$u(0) = u(T), \qquad u'(0) = u'(T).$$

From Theorem 3.2 we observe that if one of $D(\alpha)$ or $D(\beta)$ is bounded, then the problem (3.1)-(3.2) has a solution, without the additional assumption of the form

$$\lim_{\|x\|\to\infty} \|\alpha^0(x)\| = +\infty \quad \text{or} \quad \lim_{\|x\|\to\infty} \|\beta^0(x)\| = +\infty \tag{*}$$

used in [11], where $\alpha^0(x)$ denotes the element of the least norm of $\alpha(x)$. Note that Theorem 3.2 includes a result of Barbu [4, p. 311, Corollary 2.2]. Other results similar to Theorem 3.2 will be given [1]. In [11, Theorem 3.2] it is stated that the condition $(*)$ and differentiability assumptions on p and r guarantee the existence of a solution to (3.1)-(3.2). However, this conjecture is not yet completely proven, see [1].

In section 4, we state some recent results which will be completed and proved in [3].

2. Preliminary results. Let A, α, $-\beta$, be maximal monotone (operators) sets of $H \times H$ with $[0,0] \in \alpha \cap \beta \cap A$, where H is a real Hilbert space with inner product $\langle \cdot \rangle$ and norm $\|\cdot\|$. Let also $T > 0$ and p, q be two continuous functions from $[0,T]$ into $\mathbb{R}$, with $p(t) \neq 0$ for all $t \in [0,T]$. $\tilde{r}$ is given by

$$\tilde{r}(t) = \exp \int_0^t \frac{r(s)}{p(s)}\, ds\,. \tag{2.1}$$

Denote by $\mathcal{L}^2_{\tilde{r}/p} \equiv \mathcal{L}^2([0,T]; H)$ the space $L^2([0,T]; H)$ with the weight $\tilde{r}/p$. Then, the inner product $\ll \cdot\,, \gg$ of $\mathcal{L}^2$ is given by

$$\ll u, v \gg = \int_0^T \frac{\tilde{r}(t)}{p(t)} \langle u(t), v(t) \rangle \, dt, \qquad u,\, v \in \mathcal{L}^2 \tag{2.2}$$

and thus, the corresponding norm is

$$|u|^2 = \int_0^T \frac{\tilde{r}(t)}{p(t)} \|u(t)\|^2 \, dt, \qquad u \in \mathcal{L}^2\,. \tag{2.3}$$

As usual, by "$\rightarrow$" and "$\rightharpoonup$" we mean the strong and weak convergence in all the infinite dimensional spaces involved.

Define

$$H^2([0,T]; H) \equiv H^2 = \{u \in \mathcal{L}^2,\ u',\, u'' \in \mathcal{L}^2\} \tag{2.4}$$

$$D(B) = \{u \in H^2,\ u'(0) \in \alpha(u(0) - a),\ u'(T) \in \beta(u(T) - b)\}, \quad a, b \in H \tag{2.5}$$

$$Bu = -pu'' - ru' = -\frac{p}{\tilde{r}}(u'\tilde{r})', \quad u \in D(B) \tag{2.6}$$

$$D(\mathcal{A}) = \{v \in L^2,\ v(t) \in D(A), \text{ a.e. on } [0,T]\} \tag{2.7}$$

and for $u \in D(\mathcal{A})$, set

$$\mathcal{A}u = \{u \in L^2,\ v(t) \in Au(t), \text{ a.e. on } [0,T]\}. \tag{2.8}$$

We note that, every C^2-function u satisfying $u(0) = a$, $u(T) = b$, $u'(0) = u'(T) = 0$ belongs to $D(B)$ (as $0 \in \alpha(0)$ and $0 \in \beta(0)$), so $D(B)$ is non-empty.

The (set) operator $\mathcal{A}$ is said to be the realization of A in L^2. If A is maximal monotone in H, so is $\mathcal{A}$ in $\mathcal{L}^2$. Clearly,

$$(\mathcal{A}_\lambda u)(t) = A_\lambda u(t), \qquad \text{for all } t \in [0,T],$$

where A_λ and $\mathcal{A}_\lambda$ are Yosida approximations of A and $\mathcal{A}$ respectively.

Lemma 2.1. *Let p, r, $q : [0,T] \to \mathbb{R}$ be continuous, $p(t) \neq 0$ for all $t \in [0,T]$, and $p(t)q(t) \geq 0$ for all $t \in [0,T]$. Then, there are two solutions ϕ and ψ of the equation*

$$p(t)v''(t) + r(t)v'(t) - q(t)v(t) = 0, \qquad \text{on } [0,T] \tag{2.9}$$

such that

i) $\phi(0) = 0,\ \phi(T) > 0, \phi'(0) > 0,\ \phi'(T) > 0,$

ii) $\psi(0) < 0,\ \psi(T) = 0,\ \psi'(0) > 0,\ \psi'(T) = \phi'(0).$

iii) *In addition, if $q \neq 0$, then*

$$D = \begin{vmatrix} \phi'(T) & \phi'(0) \\ \psi'(T) & \psi'(0) \end{vmatrix} > 0.$$

Proof: Let ϕ and ψ be two solutions of (1.9) satisfying

$$p(t)\phi''(t) + r(t)\phi'(t) - q(t)\phi(t) = 0, \quad \phi(0) = 0,\ \phi'(0) > 0 \tag{2.10}$$

$$p(t)\psi''(t) + r(t)\psi'(t) - q(t)\psi(t) = 0, \quad \psi(T) = 0,\ \psi'(T) = \phi'(0) \tag{2.11}$$

for all $t \in [0,T]$. Then, we have

$$\phi'(t) = \tilde{r}^{-1}(t)\phi'(0) + \tilde{r}^{-1}(t)\int_0^t \frac{\tilde{r}(s)q(s)}{p(s)}\phi(s)\,ds, \qquad 0 \leq t \leq T \tag{2.12}$$

$$\psi'(t) = \tilde{r}(T)\tilde{r}^{-1}(t)\psi'(T) - \tilde{r}^{-1}(t)\int_t^T \tilde{r}(s)\frac{q(s)}{p(s)}\psi(s)\,ds, \quad 0 \leq t \leq T \tag{2.13}$$

$$\psi'(t) = \tilde{r}^{-1}(t)\psi'(0) + \tilde{r}^{-1}(t)\int_0^t \tilde{r}(s)\frac{q(s)}{p(s)}\psi(s)\,ds, \qquad 0 \leq t \leq T. \tag{2.14}$$

A simple combination of (2.12) and (2.14) leads to

$$\begin{aligned} &\phi'(T)\psi'(0) - \psi'(T)\phi'(0) = \\ &\tilde{r}^{-1}(T)\psi'(0)\int_0^T \tilde{r}(s)\frac{q(s)}{p(s)}\phi(s) - \tilde{r}^{-1}(T)\phi'(0)\int_0^T \tilde{r}(s)\frac{q(s)}{p(s)}\psi(s)\,ds. \end{aligned} \tag{2.15}$$

From (2.12) and (2.13) we derive, in view of (2.10) and (2.11), that $\phi'(t) > 0$ and $\psi'(t) > 0$ for all $t \in [0,T]$, so $\phi(t) > 0$ and $\psi(t) < 0$ for $t \in (0,T)$. This and (2.15) implies (iii) and thus the proof is complete.

Remark 2.1. Actually, we have proven that $\phi(t) > 0$, $\phi'(t) > 0$, $\psi(t) < 0$, and $\psi'(t) > 0$ for all $t \in (0,T)$.

Lemma 2.2. *For every continuous functions p and r from $[0,T]$ into $\mathbb{R}$, with $p(t) \geq c > 0$, for all $t \in [0,T]$, $a, b \in H$ and $T > 0$, the operator B defined by (2.6) is maximal monotone in $\mathcal{L}^2_{\tilde{r}/p}([0,T];H)$.*

Proof: The monotonicity of B in $\mathcal{L}^2$ is immediate. Indeed, if u, $v \in D(B)$, then

$$\begin{aligned} \ll Bu - Bv, u - v \gg &= -\int_0^T \langle (\tilde{r}(t)(u'(t) - v'(t)))',\ u(t) - v(t)\rangle\,dt \\ &= -\langle u'(t) - v'(t), u(t) - v(t)\rangle \tilde{r}(t)\Big|_0^T + \int_0^T \tilde{r}(t)\|u'(t) - v'(t)\|^2\,dt \geq 0. \end{aligned} \tag{2.16}$$

Now we show the maximal monotonicity of B in $\mathcal{L}^2$; i.e., $R(I+B) = L^2$. To do this, we need to prove that for each $f \in L^2$, there is a $u \in H^2$ such that

$$p(t)u''(t) + r(t)u'(t) - u(t) = f(t), \quad \text{a.e. on } [0,T] \tag{2.17}$$

$$u'(0) \in \alpha(u(0) - a), \quad u'(T) \in \beta(u(T) - b). \tag{2.18}$$

Now, let w be the solution of the Cauchy problem

$$\begin{aligned} &p(t)w''(t) + r(t)w'(t) - w(t) = f(t), \quad \text{a.e. on } [0,T] \\ &w(0) = w'(0) = 0. \end{aligned} \tag{2.19}$$

We will show that there are $x, y \in H$ such that

$$u(t) = w(t) + \phi(t)x + \psi(t)y \tag{2.20}$$

satisfies (2.18). Here, ϕ and ψ are as in Lemma 2.1 with $q(t) \equiv 1$. Indeed, u is a solution of (2.17) for every x, $y \in H$, so we have to prove the existence of x, $y \in H$ such that u satisfies (2.18). Clearly, u given by (2.20) verifies (2.18) if and only if (x, y) is a solution of

$$\begin{aligned} &\phi'(0)x + \psi'(0)y - \alpha(\psi(0)y - a) \ni 0 \\ &\phi'(T)x + \psi'(T)y - \beta(\phi(T)x - b - w(T)) \ni -w'(T). \end{aligned} \tag{2.21}$$

Now, we observe that the operator

$$F_1(x,y) = (\phi'(T)x + \psi'(0)y, \phi'(0)x + \psi'(0)y) \tag{2.22}$$

is symmetric and positive definite on $H \times H$, so

$$\langle F_1(x,y), (x,y)\rangle \geq \lambda_1(\|x\|^2 + \|y\|^2), \quad \forall\, x, y \in H \tag{2.23}$$

where λ_1 is the smallest eigenvalue of the matrix

$$M = \begin{bmatrix} \phi'(T) & \psi'(0) \\ \phi'(0) & \psi'(0) \end{bmatrix}. \tag{2.24}$$

Finally, set $D(F_2) = D(\beta_1) \times D(\alpha_1)$, and let

$$F_2(x,y) = \beta_1(x) \times \alpha_1(y) = \big\{(z_1, z_2) \in H \times H,\ z_1 \in \beta_1(x),\ z_2 \in \alpha_1(y)\big\} \tag{2.25}$$

where $\alpha_1(y) = -\alpha(\psi(0)y - a)$, $\beta_1(x) = -\beta(\phi(T)x - b - w(T))$. It is easy to check that F_2 is maximal monotone in $H \times H$. Therefore, $F_1 + F_2$ is surjective (it is actually a bijection from $D(F_2)$ onto $H \times H$), so the system (2.21), i.e., $F_1(x,y) + F_2(x,y) \ni (-w'(T), 0)$ has a unique solution (x,y), which completes the proof.

Proposition 2.1. *Let j_1, $j_2 : H \to \mathbb{R}$ be lower semi-continuous convex proper functionals and p, r, $q : [0,T] \to \mathbb{R}$ continuous, $p(t) \neq 0$ for all $t \in [0,T]$, $q(t)p(t) \geq 0$ on $[0,T]$, $q \neq 0$. Then, for every a, $b \in H$ and $f \in L^2$, the problem*

$$p(t)u''(t) + r(t)u'(t) - q(t)u(t) = f(t), \quad \text{a.e. on } [0,T] \tag{2.26}$$

$$u'(0) \in \partial j_1(u(0) - a), \quad -u'(T) \in \partial j_2(u(T) - b) \tag{2.27}$$

has a unique solution $u \in H^2([0,T];H)$. *Moreover,* u *is a solution of* (2.26) – (2.27) *if and only if* u *minimizes the functional*

$$F(u) = \begin{cases} \frac{1}{2}\int_0^T \tilde{r}(t)\big(\|u'(t)\|^2 + \frac{q(t)}{p(t)}\|u(t)\|^2\big)\,dt + \int_0^T \frac{\tilde{r}(t)}{p(t)}\langle f(t), u(t)\rangle\,dt \\ \qquad +j_1(u(0)-a) + \tilde{r}(T)j_2(u(T)-b), \quad \text{if } u \in H^1 \\ +\infty, \qquad \text{otherwise,} \end{cases} \tag{2.28}$$

i.e., if and only if

$$\inf\{F(v),\ v \in \mathcal{L}^2_{\tilde{r}}\} = F(u), \tag{2.29}$$

where $\mathcal{L}^2_{\tilde{r}}$ *is the space* $L^2([0,T];H)$ *with the weight*

$$\tilde{r}(t) = \exp\Big(\int_0^t (q(s)/p(s))\,ds\Big)$$

and ∂j_1 *is the subdifferential of* j_1 (cf. [9, p. 234]).

Proof: The existence and uniqueness of the solution u of (2.26)-(2.27) can be easily derived from Lemma 2.2. Define $\tilde{B} : D(B) \to \mathcal{L}^2_{\tilde{r}}$ by

$$(\tilde{B}u) = -u''(t) - \frac{r(t)}{p(t)}u'(t) = -\frac{1}{\tilde{r}(t)}(\tilde{r}(t)u'(t))', \quad u \in D(\tilde{B}) = D(B) \tag{2.30}$$

with $D(B)$ given by (2.5). Clearly, $\tilde{B}$ is also maximal monotone in $\mathcal{L}^2_{\tilde{r}}$ and we have

$$\tilde{B}u = \partial\phi_1(u), \quad \forall\, u \in D(\tilde{B}) \tag{2.31}$$

where $\phi_1 : \mathcal{L}^2_{\tilde{r}} \to (-\infty, +\infty]$ is given by

$$\phi_1(u) = \begin{cases} \frac{1}{2}\int_0^T \tilde{r}(t)\|u'(t)\|^2\,dt + j_1(u(0)-a) + j_2(u(T)-b)\tilde{r}(T), \quad \text{if } u \in H^1 \\ +\infty, \qquad \text{otherwise.} \end{cases} \tag{2.32}$$

To prove (2.31) we observe that ϕ_1 is (l.s.c.) and that $D(\tilde{B}) \subset D(\partial\phi_1)$ as well as $\tilde{B}u \in \partial\phi_1(u)$, $\forall\, u \in D(\tilde{B})$. As $\tilde{B}$ and $\partial\phi_1$ are maximal monotone in $\mathcal{L}^2_{\tilde{r}}$, it follows that $\tilde{B} = \partial\phi_1$. Set

$$Du = \frac{q}{p}u + \frac{f}{p}, \quad u \in \mathcal{L}^2_{\tilde{r}} \tag{2.33}$$

$$\phi_2 : \mathcal{L}^2_{\tilde{r}} \to [0, +\infty), \quad \phi_2(u) = \frac{1}{2}\big|u\sqrt{q/p}\big|^2_{\tilde{r}} + \ll \frac{f}{p}, u \gg_{\tilde{r}}, \tag{2.34}$$

where

$$|f|^2_{\tilde{r}} = \int_0^T \tilde{r}(t)\|f(t)\|^2\,dt, \quad \ll f, g \gg_{\tilde{r}} = \int_0^T \tilde{r}(t)\langle f(t), g(t)\rangle\,dt. \tag{2.35}$$

Similarly, it follows that

$$Du = \partial\phi_2(u), \quad \forall\, u \in \mathcal{L}^2_{\tilde{r}} \ \ (\text{so } D(\partial\phi_2) = \mathcal{L}^2_{\tilde{r}}). \tag{2.36}$$

Finally, we have

$$F = \phi_1 + \phi_2 \quad \text{and} \quad \partial F = \partial(\phi_1 + \phi_2) = \partial\phi_1 + \partial\phi_2; \tag{2.37}$$

i.e.,

$$\partial F(u) = \tilde{B}u + \frac{q}{p}u + \frac{f}{p}, \quad \forall\, u \in D(\tilde{B}) = D(\partial F). \tag{2.38}$$

Now, u is a solution of (2.29) if and only if $0 \in \partial F(u)$; i.e., if and only if $u \in D(\tilde{B})$ and

$$\tilde{B}u + \frac{q}{p}u + \frac{f}{p} = 0. \tag{2.39}$$

Since (2.39) is the $\mathcal{L}^2_{\tilde{r}}$ form of (2.26)-(2.27), the proof is complete.

Remark 2.2. In the case $H = \mathbb{R}$, the boundary conditions (2.18) and (2.27) coincide; this is because every maximal monotone operator acting in $\mathbb{R}$ is a subdifferential. These boundary conditions contain many classical boundary conditions. For example, the boundary conditions in [10, p. 13]

$$\begin{aligned} -u'(0)\cos\theta + u(0)\sin\theta &= \gamma_1, \\ u'(T)\cos\phi + u(T)\sin\phi &= \gamma_2 \end{aligned} \tag{2.40}$$

(where γ_1, γ_2, θ and ϕ are prescribed constants with $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$) are of the form (2.19) with $\alpha(x) = (\sin\theta/\cos\theta)x$, $\beta(x) = -(\sin\phi/\cos\phi)x$, $a = \gamma_1/\sin\theta$, $b = \gamma_2/\sin\phi$, $0 < \theta < \frac{\pi}{2}$, $0 < \phi < \frac{\pi}{2}$. Of course, in particular, α and $-\beta$ in (2.19) can be any continuous and monotone functions from H in $\mathbb{R}$. If $H = \mathbb{R}$, $\alpha(x) = \operatorname{sign} x$ and if it happens that $u(0) = 0$ in (2.19), then $|u'(0)| \le 1$. This is because if $\alpha = \partial j$ with $j(x) = |x|$, then

$$\alpha(x) = \partial j(x) = \operatorname{sign} x = \begin{cases} 1, & x > 0 \\ [-1, 1], & x = 0 \\ -1, & x < 0. \end{cases} \tag{2.41}$$

3. Boundary value problems and optimization. In this section, we study the equivalence of the boundary value problem

$$p(t)u''(t) + r(t)u'(t) \in Au(t) + f(t), \quad \text{a.e. on } [0,T] \tag{3.1}$$

$$u'(0) \in \alpha(u(0) - a), \quad u'(T) \in \beta(u(T) - b), \quad a, b \in D(A) \tag{3.2}$$

to a minimization problem of Bolza type

$$\inf\left\{F(v),\ v \in \mathcal{L}^2_{\tilde{r}/p}([0,T]; H)\right\} = F(u) \tag{3.3}$$

where $F : \mathcal{L}^2 \to (-\infty, +\infty]$ is a lower semi-continuous (but not continuous) convex proper functional defined in terms of r, A, f, α, β, a and b. By a solution of the problem (3.1)-(3.2), with $f \in \mathcal{L}^2$, we mean a function $u \in H^2([0,T]; H)$ satisfying $u(t) \in D(A)$ almost everywhere on $[0,T]$. We present also a result on continuous dependence of the solution $u = u(t, a, b, f)$ on a, b and f. The basic assumptions on A are maximal ω-monotonicity ($\omega \ge 0$); i.e.,

$$\langle y_1 - y_2, x_1 - x_2\rangle \ge \omega\|x_1 - x_2\|^2, \quad \forall\, x_j \in D(A),\ y_j \in Ax_j,\ j = 1, 2 \tag{3.4}$$

and

$$R(I + A) = H. \tag{3.5}$$

In order to study the equivalence of (3.1)-(3.2) with (3.3), we need to discuss first the existence and uniqueness of the solution of (3.1)-(3.2).

Definition 3.1. *We say that $u = u(t, a, b, f)$ depends continuously (in the weak (or strong) topology) on a, b and f if $u_n = u(t, a_n, b_n, f_n)$ is weakly (strongly) convergent in L^2 to $u = u(t, a, b, f)$ as $n \to \infty$ whenever a_n, $b_n \in D(A)$, $f_n \in L^2$ with $a_n \to a$, $b_n \to b$ strongly in H and $f_n \to f$ in L^2.*

Definition 3.2. *A is said to be locally bounded at $a \in D(A)$ if A is bounded in a neighborhood V_a of a; i.e., the set $\cup_{x \in V_a} Ax$ is bounded in H.*

Theorem 3.1. *Assume that A is maximal ω-monotone with $\omega > 0$ (i.e., (3.4) and (3.5) holds), α and $-\beta$ are maximal monotone, $0 \in \alpha(0) \cap \beta(0) \cap A(0)$ and A_λ is α monotone and β dissipative; i.e.,*

$$\langle A_\lambda x - A_\lambda y,\, z\rangle \geq 0, \quad \langle A_\lambda x - A_\lambda y,\, v\rangle \leq 0 \tag{3.6}$$

for all $\lambda > 0$, $z \in \alpha(x - y)$, $v \in \beta(x - y)$ and x, $y \in H$ with $x - y \in D(\alpha) \cap D(\beta)$. Then, for every a, $b \in D(A)$, p, $r : [0, T] \to \mathbb{R}$ continuous, with $p(t) \geq c > 0$, $\forall\, t \in [0, T]$ and $f \in L^2$, the boundary value problem (3.1) – (3.2) has a unique solution $u = u(t, a, b, f)$. In addition, if A is locally bounded, then the solution depends strongly continuous on a, b and f.

Proof: We first prove the theorem for the case when $a = b = 0$. In this case, the problem (3.1)-(3.2) can be written in $\mathcal{L}^2_{\tilde{r}/p}$ as

$$Bu + \mathcal{A}u \ni -f, \quad u \in D(B) \cap D(\mathcal{A}), \quad f \in \mathcal{L}^2_{\tilde{r}/p}. \tag{3.7}$$

But, for every $\lambda > 0$ and $u \in D(B)$,

$$\begin{aligned} \ll Bu, \mathcal{A}_\lambda u \gg &= -\int_0^T \langle (\tilde{r}(t)u'(t))',\, A_\lambda u(t)\rangle\, dt \\ &= -\tilde{r}(t)\langle u'(t), A_\lambda u(t)\rangle\Big|_0^T + \int_0^T \tilde{r}(t)\langle u'(t),\, (A_\lambda u(t))'\rangle\, dt \geq 0. \end{aligned} \tag{3.8}$$

This is because, in view of (3.6),

$$\langle u'(T),\, A_\lambda u(T)\rangle \leq 0, \quad \langle u'(0),\, A_\lambda u(0)\rangle \geq 0 \tag{3.9}$$

as $u'(T) \in \beta(u(T))$ and $u'(0) \in \alpha(u(0))$. Moreover, $t \to A_\lambda u(t)$ is almost everywhere differentiable on $[0, T]$ and the monotonicity of A_λ yields

$$\langle u'(t), (A_\lambda u(t))'\rangle \geq 0, \quad \text{a.e. on } [0, T]. \tag{3.10}$$

It follows that $B + \mathcal{A}$ is maximal monotone (see Barbu [4, p. 82] or [9, p. 118]). But, $\mathcal{A}$ is ω-coercive; i.e.,

$$\ll \mathcal{A}u, u \gg = \int_0^T \frac{\tilde{r}(t)}{p(t)}\langle Au(t), u(t)\rangle\, dt \geq \omega|u|^2, \quad \forall\, u \in D(\mathcal{A}). \tag{3.11}$$

Therefore, $B + \mathcal{A}$ is surjective (it is actually a bijection from $D(B) \cap D(\mathcal{A})$ onto $\mathcal{L}^2$) so, for every $f \in \mathcal{L}^2$, (3.7) has a unique solution. This proves the case when $a = b = 0$.

In the general case, a, $b \in D(A)$ with a, $b \neq 0$, $a \neq b$, the proof can be carried out as follows. We first observe that $B + \mathcal{A}$ is maximal monotone and $\frac{\omega}{2}$-coercive for $0 < \lambda \leq \frac{1}{\omega}$. This is because

$$\langle A_\lambda x - A_\lambda y, x - y\rangle \geq \frac{\omega}{1 + \lambda\omega}\|x - y\|^2 . \tag{3.12}$$

Therefore, $B+\mathcal{A}_\lambda$ is a bijection from $D(B)$ onto $\mathcal{L}^2$; so for every $f\in\mathcal{L}^2$, there is a unique $u_\lambda\in D(B)$ such that

$$Bu_\lambda+\mathcal{A}_\lambda u_\lambda=-f \tag{3.13}$$

which can be written as

$$Bu_\lambda+\mathcal{A}_\lambda u_\lambda-\mathcal{A}_\lambda v=-f-\mathcal{A}_\lambda v, \tag{3.14}$$

where v is the solution of

$$v''\in\mathcal{A}v,\quad v(0)=a,\ v(T)=b \tag{3.15}$$

(whose existence is well-known [4, p. 300]).

Following an idea from [11], we multiply (3.14) by $u_\lambda-v$ and then observe that

$$|\mathcal{A}_\lambda v|\le|\mathcal{A}_0 v|\le|v''|,\quad \langle\mathcal{A}_\lambda u_\lambda-\mathcal{A}_\lambda v\,,\,u_\lambda-v\rangle\ge\frac{\omega}{2}|u_\lambda-v|^2\ \ 0<\lambda<\omega^{-1} \tag{3.16}$$

$$\begin{aligned}\ll Bu_\lambda,u_\lambda-v\gg\ &\ge\int_0^T\tilde r(t)\|u_\lambda'(t)-v'(t)\|^2\,dt+\int_0^T\tilde r(t)\langle v'(t),u_\lambda'(t)-v'(t)\rangle\,dt\\&\quad+\int_0^T\tilde r(t)\|u_\lambda'(t)-v'(t)\|^2\,dt\\&\ge c|u_\lambda'-v'|^2.\end{aligned} \tag{3.17}$$

In such a manner, we get for $0<\lambda\le\frac{1}{\omega}$,

$$\frac{\omega}{2}|u_\lambda-v|^2+c|u_\lambda'-v'|^2\le(|f|+|v''|)|u_\lambda-v|+k(v')|u_\lambda'-v'|, \tag{3.18}$$

where

$$k(v')=\left\{\int_0^T\tilde r(t)p(t)\|v'(t)\|^2\,dt\right\}^{1/2}=\|v'\sqrt{\tilde rp}\|_{L^2}$$

which proves the boundedness of $u_\lambda-v$ and $u_\lambda'-v'$ in $\mathcal{L}^2$ for $0<\lambda\le\omega^{-1}$. From (3.14) and (3.18), we can easily get an estimate (upper bound) for $|\mathcal{A}_\lambda u_\lambda|$, independent of $0<\lambda<\frac{1}{\omega}$. Indeed, let $\psi_\lambda(t)=\frac{T-t}{T}A_\lambda a+\frac{t}{T}A_\lambda b$. Multiplying (3.14) by $\mathcal{A}_\lambda u_\lambda-\psi_\lambda$, and observing that

$$\begin{aligned}&\ll Bu_\lambda,\mathcal{A}_\lambda u_\lambda-\psi_\lambda\gg\\&=-\tilde r(t)\langle u_\lambda'(t),A_\lambda u_\lambda(t)-\psi_\lambda(t)\rangle\Big|_0^T+\int_0^T\tilde r(t)\langle u_\lambda'(t),(A_\lambda u_\lambda(t))'-\psi_\lambda'(t)\rangle\,dt\\&\ge-\int_0^T\tilde r(t)\langle u_\lambda'(t),\psi_\lambda'(t)\rangle\,dt\end{aligned}$$

(as $\langle u_\lambda'(t),(A_\lambda u_\lambda(t))'\rangle\ge0$ almost everywhere on $[0,T]$) we obtain

$$|\mathcal{A}_\lambda u_\lambda|^2\le|\mathcal{A}_\lambda u_\lambda|(|\psi_\lambda|+|f|)+|f|\,|\psi_\lambda|+\frac{1}{T}(\|A_0a\|+\|A_0b\|)\|\sqrt{\tilde rp}\|_{L^2}\,|u_\lambda'| \tag{3.19}$$

which gives the desired upper bound for $|\mathcal{A}_\lambda u_\lambda|$, say $|\mathcal{A}_\lambda u_\lambda|\le K$, for $0<\lambda<\frac{1}{\omega}$. Now, (3.14) and the identity

$$\mathcal{J}_\lambda u_\lambda+\lambda\mathcal{A}_\lambda u_\lambda=u_\lambda \tag{3.20}$$

yield

$$\langle Bu_\lambda-Bu_\lambda,u_\lambda-u_\mu\rangle+\langle\mathcal{A}_\lambda u_\lambda-\mathcal{A}_\mu u_\mu,\mathcal{J}_\lambda u_\lambda-\mathcal{J}_\mu u_\mu+\lambda\mathcal{A}_\lambda u_\lambda-\mu\mathcal{A}_\mu u_\mu\rangle=0$$

which can be written as

$$\frac{\omega}{2}|\mathcal{J}_\lambda u_\lambda - \mathcal{J}_\mu u_\mu| + c|u'_\lambda - u'_\mu| \le (\lambda+\mu)K^2, \quad \lambda,\, \mu \in (0, \frac{1}{\omega}) \tag{3.21}$$

and therefore $\mathcal{J}_\lambda u_\lambda$ is strongly convergent in $\mathcal{L}^2$ as $\lambda \to 0$, say $\mathcal{J}_\lambda u_\lambda \to u$ in $\mathcal{L}^2$. This and $\mathcal{J}_\lambda u_\lambda - u_\lambda = \lambda \mathcal{A}_\lambda u_\lambda \to 0$ as $\lambda \downarrow 0$ shows that $u_\lambda \to u$ as $\lambda \downarrow 0$. Say $\mathcal{A}_\lambda u_\lambda \rightharpoonup w$ as $\lambda \downarrow 0$. Since $\mathcal{A}_\lambda u_\lambda \in \mathcal{A}\mathcal{J}_\lambda u_\lambda$ and $\mathcal{A}$ is maximal monotone in $\mathcal{L}^2$, it follows that $u \in D(\mathcal{A})$ and $w \in \mathcal{A}u$. Now, (3.14) can be written as $-f - \mathcal{A}_\lambda u_\lambda \in Bu_\lambda$, with $u_\lambda \to u$ and $-f - \mathcal{A}_\lambda u_\lambda \rightharpoonup -f - w$. This implies that $u \in D(B)$ and $-f - w \in Bu$; i.e., $-f \in w + Bu$, so $-f \in \mathcal{A}u + Bu$ which means that (3.1)-(3.2) has a unique solution $u = u(t, a, b, f)$.

Now we prove the continuous dependence of u on a, b, f. Let a_n, $b_n \in D(A)$ and $f_n \in \mathcal{L}^2$ with $a_n \to a$, $b_n \to b$ in H, $f_n \to f$ in $\mathcal{L}^2$ with a, $b \in D(A)$. Let $u_n = u_n(t, a_n, b_n, f_n)$ be the solution to (3.1)-(3.2) corresponding to f_n, a_n and b_n. Then, according to (3.7), this means that

$$Bu_n + \mathcal{A}u_n \ni -f_n; \quad \text{i.e.,} \quad Bu_n + w_n = -f_n, \quad \text{with} \quad w_n \in \mathcal{A}u_n. \tag{3.22}$$

First, let us observe that (3.22) yields

$$\omega|u_n - u_m| \le |f_n - f_m|, \quad m, n = 1, 2, \cdots, \tag{3.23}$$

so $u_n \to u$ in $\mathcal{L}^2$ as $n \to \infty$. From (3.19) with f_n, a_n, b_n in place of f, a and b, we conclude that $\mathcal{A}_\lambda u^n_\lambda$ (u^n_λ corresponding to f_n, a_n, b_n) is bounded independently of λ and n, and moreover, $\mathcal{A}_\lambda u^n_\lambda \rightharpoonup w_n$ as $\lambda \downarrow 0$, so w_n is bounded in $\mathcal{L}^2$ with respect to n. We may assume that $w_n \rightharpoonup \tilde{w}$ as $n \to \infty$ in $\mathcal{L}^2$. Since $u_n \to \tilde{u}$ and $w_n \in \mathcal{A}u_n$, it follows that $\tilde{u} \in D(\mathcal{A})$ and $\tilde{w} \in \mathcal{A}\tilde{u}$. Then, by (3.22), in view of $-f_n - w_n \in Bu_n$, with $-f_n - w_n \rightharpoonup -f - \tilde{w}$, we conclude that $\tilde{u} \in D(B)$ and $-f \in B\tilde{u} + \mathcal{A}\tilde{u}$ so, $u = \tilde{u}$ as $-f \in Bu + \mathcal{A}u$ and this inclusion has a unique solution.

Remark 3.2. Unfortunately, the upper bounds for both $|u_\lambda - v|$ and $|u'_\lambda - v'|$ which are independent of $\lambda \in (0, \frac{1}{\omega})$ depend badly on ω; i.e., they contain ω at the denominator, so in this method we cannot obtain estimates for u_λ and u'_λ independent of $\omega \to 0$. Note also that u_λ cannot be estimated in terms of u'_λ as we do not have estimates for $u_\lambda(0)$ independent of λ.

Theorem 3.2. 1) *If A is (merely) maximal monotone in H; i.e., A satisfies* (3.4) *and* (3.5) *with $\omega = 0$, and if in addition to* (3.6) *we assume that at least one of $D(\alpha)$ or $D(\beta)$ is bounded, then for every a, $b \in D(A)$ and $f \in L^2$, the problem* (3.1) – (3.2) *has at least one solution $u = u(t, a, b, f)$.*

2) *If at least one of α, β or A is one-to-one, then the solution $u = u(t, a, b, f)$ is unique and depends weakly continuous on $f \in \mathcal{L}^2$. If, in addition, A is locally bounded, then u depends weakly continuous on a, b and f. In particular, if A is continuous, monotone and one-to-one, then u depends weakly continuous on a, b, and f.*

Proof: Here, $\mathcal{A}_\lambda$ is only monotone ($\ll \mathcal{A}_\lambda x - \mathcal{A}_\lambda y, x - y \gg \ge 0$) and $\frac{2}{\lambda}$-Lipschitz continuous so, $B + \mathcal{A}_\lambda$ is maximal monotone; i.e.,

$$R(\omega I + B + \mathcal{A}_\lambda) = L^2, \quad \forall\, \omega > 0.$$

Therefore, (3.14) becomes

$$Bu_\lambda + \mathcal{A}_\lambda u_\lambda + \omega u_\lambda \ni f, \quad \forall\, \lambda > 0,\, \omega > 0 \tag{3.24}$$

and (3.18) holds with ω in place of $\omega/2$ and for all $\lambda > 0$. If, say $D(\alpha)$ is bounded, then $u'_\lambda(0) \in \alpha(u_\lambda(0) - a)$ (i.e., $u_\lambda(0) - a \in D(\alpha)$) implies $u_\lambda(0)$ is bounded, say $\|u_\lambda(0)\| \leq K_1$, $\forall\, \lambda > 0$. Therefore,

$$\|u_\lambda(t) - v(t)\| \leq K_1 + \|a\| + \int_0^T \|u'_\lambda(s) - v(s)\|\, ds \leq \tilde{K}_1 + K_2|u'_\lambda - v'|, \tag{3.25}$$

where $K_2 = \|\sqrt{p/\tilde{r}}\|_{L^2}$, so

$$|u_\lambda - v| \leq \tilde{K}_1 K - 2 + K_2^2 |u'_\lambda - v'|, \quad \forall\, \lambda > 0. \tag{3.26}$$

This and (3.18) gives an upper bound $K_3 \geq |u'_\lambda - v'|$, K_3 independent of $\omega > 0$ and $\lambda > 0$. Therefore, (3.19) will provide an upper bound K_4 for $\mathcal{A}_\lambda u_\lambda$ with K_4 independent of λ and $\omega > 0$ while (3.26) provides an upper bound K_5 independent of λ and ω for u_λ. Finally, (3.24) provides an upper bound K_6 independent of λ and ω for $|u''_\lambda|$.

Passing to the limit, for $\lambda \downarrow 0$, in (3.24), using the same arguments as in the case of (3.14), we conclude that $u_\lambda \to u_\omega$ (in $\mathcal{L}^2$) and satisfies

$$Bu_\omega + \mathcal{A}u_\omega + \omega u_\omega \ni -f, \quad \forall\, \omega > 0. \tag{3.27}$$

This means that $\mathcal{A}_\lambda u_\lambda \rightharpoonup w_\omega \in \mathcal{A}u_\omega$ ($u'_\lambda \to u'_\omega$, $u''_\lambda \rightharpoonup u''_\omega$ as $\lambda \downarrow 0$) and $Bu_\omega + w_\omega + \omega u_\omega = -f$, so

$$-f - Bu_\omega - \omega u_\omega \in \mathcal{A}u_\omega. \tag{3.28}$$

Moreover, $|w_\omega| \leq K_4$, $|u_\omega| \leq K_5$, $\forall\, \omega > 0$. Multiplying (3.28) by $u_\omega - u_\delta$, we derive

$$c|u'_\omega - u'_\delta|^2 \leq (\omega + \delta) K_5^2,$$

so $u'_\omega \to v$ in $\mathcal{L}^2$ as $\omega \downarrow 0$. This and the boundedness of u''_ω in $\mathcal{L}^2$ imply $u'_\omega \to v$ in $C([0,T]; H)$ as $\omega \downarrow 0$. Since $u_\omega \rightharpoonup u$ as $\omega \downarrow 0$ (relabeling if necessary), we have actually $v = u'$ and $u_\omega(t) \rightharpoonup u(t)$ as $\omega \downarrow 0$ for all $t \in [0,T]$. Indeed, if $u_\omega(0) \rightharpoonup \ell$ as $\omega \downarrow 0$, then

$$u_\omega(t) = u_\omega(0) + \int_0^t u'_\omega(s), ds$$

yields $u_\omega(t) \rightharpoonup u(t)$

$$u(t) = \ell + \int_0^t u'(s)\, ds$$

with $\ell = u(0)$.

It is now easy to see that $u \in D(B)$. Indeed, $u_\omega \in D(B)$, so

$$u'_\omega(0) \in \alpha(u_\omega(0) - a), \qquad u'_\omega(T) \in \beta(u_\omega(T) - b). \tag{3.29}$$

Letting $\omega \downarrow 0$, (3.29) implies (2.18), thus $u \in D(B)$. Finally, we have

$$\begin{aligned} \langle Bu_\omega, u_\omega \rangle &= -\int_0^T \langle (\tilde{r}(t) u'_\omega(t))',\, u_\omega(t) \rangle\, dt \\ &= -\langle \tilde{r}(t) u'_\omega(t),\, u_\omega(t) \rangle \Big|_0^T + \int_0^T \tilde{r}(t) \|u'_\omega(t)\|^2\, dt \to \langle Bu, u \rangle \quad \text{as } \omega \downarrow 0. \end{aligned} \tag{3.30}$$

This enables us to pass to the limit (3.28) as $\omega \downarrow 0$ to get $-f - Bu \in \mathcal{A}u$; i.e.,

$$-f \in Bu + \mathcal{A}u, \tag{3.31}$$

so u is a solution to (3.1) and (3.2). Of course, u may not be unique in this case, so let v be another solution of (3.31); i.e., $-f \in Bv + \mathcal{A}v$. Then, $Bu - Bv + \mathcal{A}u - \mathcal{A}v = 0$. Multiplying by $u - v$, we get immediately

$$c|u' - v'|^2 \leq 0;$$

i.e., $u'(t) = v'(t)$ on $[0, T]$, so $u(t) - v(t)$ =constant. Therefore, we have $\alpha(u(0) - a) \cap \alpha(v(0) - a) \ni u'(0)$. If α is one-to-one, this implies $u(0) = v(0)$ and so on.

The proof of continuous dependence of u on a, b, f follows similar lines. This completes the proof.

Definition 3.3. *A multivalued operator α is said to be one-to-one if*

$$\alpha(x) \cap \alpha(y) \neq \emptyset \quad \text{implies} \quad x = y.$$

We are now prepared to discuss the equivalence of the boundary value problem (3.1)-(3.2) to an optimization problem. Let

$$A = \partial\phi, \qquad \tilde{r}(t) = \exp\left(\int_0^t r(s)\, ds\right) \tag{3.32}$$

(we take $p = 1$, for simplicity). Let $\phi : H \to (-\infty, +\infty]$ be a lower semi-continuous convex proper function with $\phi(0) = 0$ and $A = \partial\phi$. Let also $j_1, j_2 : H \to \mathbb{R}$ be (l.s.c.) with $j_1(0) = 0$, $j_2(0) = 0$ such that A_λ is ∂j_1 and ∂j_2 monotone. This is true if, e.g.

$$g(\mathcal{J}_\lambda x - \mathcal{J}_\lambda y) \leq g(x - y), \quad \forall x, y \in H,\ \lambda > 0 \tag{3.33}$$

with $g = j_1$, j_2.

Theorem 3.3. *Let $A = \partial\phi$ be ∂j_1 and ∂j_2 monotone. Assume one of the following conditions is satisfied:*

(i) *Either $D(\partial j_1)$ or $D(\partial j_2)$ is bounded and $r : [0, T] \to \mathbb{R}$ is continuous;*

or

(ii) *for some $L > 0$ and $x \in D(\partial j_1)\, (x \in D(\partial j_2))$*

$$\|(\partial j_1)^0(x)\| \geq L\|x\| \quad \text{or} \quad \|(\partial j_2)^0(x)\| \geq L\|x\| \tag{3.34}$$

and either ∂j_1 or ∂j_2 is one-to-one. Moreover, r is differentiable with $r \in W^{1,\infty}$, $r(t) \geq r(0) > 0$, $t \in [0, T]$.

Then, for every $T > 0$, a, $b \in D(\partial\phi)$ and $f \in L^2$, the boundary value problem

$$u''(t) + r(t)u'(t) \in \partial\phi(u(t)) + f(t), \quad \text{a.e. on } [0, T] \tag{3.35}$$

$$u'(0) \in \partial j_1(u(0) - a), \quad -u'(T) \in \partial j_2(u(T) - b) \tag{3.36}$$

has a unique solution u. Moreover, u is solution of (3.35) – (3.36) if and only if u is the solution of the following minimization problem (convex problem of Bolza)

$$\inf\{F(v),\ v \in \mathcal{L}^2_{\tilde{r}}\} = F(u) \tag{3.37}$$

where $F : L^2_{\tilde{r}} \to (-\infty, +\infty]$ *is l.s.c., given by*

$$F(u) = \begin{cases} \int_0^T \tilde{r}(t)\big(\frac{1}{2}\|u'(t)\|^2 + \phi(u(t))\big)\,dt + \int_0^T \tilde{r}(t)\langle f(t), u(t)\rangle\,dt \\ +j_1(u(0)-a) + \tilde{r}(T)j_2(u(T)-b) \ \ \textit{if } u \in H^1 \textit{ and } t \to \phi(u(t)) \textit{ is in } L^1([0,T];H) \\ +\infty, \qquad \textit{otherwise.} \end{cases} \tag{3.38}$$

Proof: We will prove that

$$\partial F(u) = Bu + \partial\tilde{\phi}(u) + f, \quad u \in D(B) \cap D(\partial\tilde{\phi}), \tag{3.39}$$

where $(\partial\tilde{\phi}(u))(t) = \partial\phi(u(t))$ almost everywhere on $[0,T]$; i.e., $\partial\tilde{\phi}$ is the realization of $\partial\phi$ in $L^2_{\tilde{r}}$. Indeed, set

$$Eu = Bu + \partial\tilde{\phi}(u) + f, \quad u \in D(E) \equiv D(B) \cap D(\partial\tilde{\phi}). \tag{3.40}$$

We know that $B + \partial\tilde{\phi}$ is a bijection from $D(B) \cap D\partial\tilde{\phi}$ onto $\mathcal{L}^2_{\tilde{r}}$ (see (3.31)) and that $B + \partial\tilde{\phi}$ is monotone. We now prove that $B + \partial\tilde{\phi}$ is maximal monotone in $\mathcal{L}^2_{\tilde{r}}$; i.e.,

$$R(I + B + \partial\tilde{\phi}) = L^2_{\tilde{r}}. \tag{3.41}$$

To do this, set $C = \partial\phi + I$, so $\tilde{C} = \partial\tilde{\phi} + I$. Let us prove that C is also ∂j monotone $(j = j_1, j_2)$. In other words, we have to prove that if A is α monotone in the sense of (3.6), then $C = A + I$ is also α monotone. Indeed,

$$\mathcal{J}^C_\lambda = (I + \lambda C)^{-1} = ((\lambda+1)I + \lambda A)^{-1} = \frac{1}{1+\lambda}\mathcal{J}^A_{\lambda/(\lambda+1)} \tag{3.42}$$

so

$$C_\lambda = \frac{1}{\lambda}(I - \mathcal{J}^C_\lambda) = \frac{1}{1+\lambda}I + \frac{1}{(\lambda+1)^2}A_{\lambda/(\lambda+1)}\,. \tag{3.43}$$

This implies, obviously, that C is also α monotone. According to Theorem 3.1, $B + \tilde{C}$ is surjective, see (3.31); i.e., (3.41) holds. In order to prove (3.39), we first can prove that

$$D(B) \cap D(\partial\tilde{\phi}) \subset D(\partial F), \quad \text{and} \quad Bu + \partial\tilde{\phi}(u) + f \subset \partial F(u) \tag{3.44}$$

for all $u \in D(B) \cap D(\partial\tilde{\phi})$. This means that we have to prove the inequality

$$F(v) - F(u) \geq \ll Bu + z + f, v - u \gg, \quad \forall\, z \in \partial\phi(u),\ \forall\, v \in L^2_{\tilde{r}}\,. \tag{3.45}$$

The proof of (3.45) involves the definition of the subdifferential, an integration by parts in $\ll Bu, v-u \gg$ and the elementary inequality

$$|\langle x, y\rangle| \leq \frac{1}{2}\big(\|x\|^2 + \|y\|^2\big), \qquad x,\, y \in H.$$

We omit these details. Now, as $B + \partial\tilde{\phi} + f$ is maximal monotone, (3.44) yields (3.39). Or, u is a solution of (3.37) if and only if "$0 \in \partial F(u)$;" that is, if and only if $0 \in Bu + \partial\tilde{\phi}(u) + f$ which is the functional form of (3.35) and (3.36). The proof is complete. ∎

Applications to partial differential equations are obtained by choosing $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$ and replacing A by some partial differential operators of monotone type satisfying the hypotheses of Theorems 3.1, 3.3, or Theorem 3.3. Here, we give only an example.

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\Gamma = \partial\Omega$. Let $j : \mathbb{R} \to \mathbb{R}$ be a (l.s.c),

$$Au = -\Delta u = -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \tag{3.46}$$

where the derivatives are taken in the sense of distributions, and

$$D(A) = \left\{u \in H^2(\Omega);\ -\frac{\partial u}{\partial \eta}(x) \in \partial j(u(x)),\ \text{ a.e. on } \Gamma\right\} \tag{3.47}$$

where $(\partial u/\partial \eta)$ is the outward normal derivative to Γ at $x \in \Gamma$. It is known that $-\Delta$ is the subdifferential $\partial\phi$ of the (l.s.c) functional $\phi : L^2(\Omega) \to (-\infty, +\infty]$ of Brezis

$$\phi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2\,dx + \int_\Gamma j(u(x))\,d\delta, & \text{if } u \in H^1(\Omega) \text{ and } j(u) \in L^1(\Gamma) \\ +\infty \quad \text{otherwise} \end{cases} \tag{3.48}$$

where $H^1(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces. Let also j_1, $j_2 : L^2(\Omega) \to \mathbb{R}$ be two (l.s.c) satisfying (3.33) for all x, $y \in L^2(\Omega)$ and $\lambda > 0$, with $\mathcal{J}_1 = (I - \Delta)^{-1}$, $\Delta = -\partial\phi$. Then, $-\Delta$ is ∂j_1, and ∂j_2 monotone in $L^2(\Omega)$. Finally, let $\mathcal{L}^2_{\tilde{r}} \equiv L^2_{\tilde{r}}([0,T]; L^2(\Omega))$ be the Hilbert space $L^2([0,T]; L^2(\Omega))$ with the weight function $\tilde{r}$ given by (2.1). From Theorem 3.3 with $H = L^2(\Omega)$, we derive

Corollary 3.1. *Let $T > 0$ be arbitrary and $r : [0,T] \to \mathbb{R}$ as in Theorem 3.3. Then, for every $a, b \in H^2(\Omega)$ with $-\frac{\partial a}{\partial \eta}(x) \in \partial j(a(x))$, $-\frac{\partial b}{\partial \eta}(x) \in \partial j(b(x))$ almost everywhere on Γ and $f \in L^2([0,T]; L^2(\Omega)) \equiv \mathcal{L}^2$, there exists a unique L^2 solution $u \in H^2([0,T]; L^2(\Omega))$ of the boundary value problem of elliptic type*

$$\frac{\partial^2 u}{\partial t^2}(t,x) + r(t)\frac{\partial u}{\partial t}(t,x) + \Delta_x u(t,x) = f(t,x), \quad \text{a.e. on } (0,T) \times \Omega \tag{3.49}$$

$$-\frac{\partial u}{\partial t}(t,x) \in \partial j(u(t,x)), \quad \text{a.e. on } [0,T] \times \Gamma \tag{3.50}$$

$$\frac{\partial u}{\partial t}(0,\cdot) \in \partial j_1(u(0,\cdot) - a), \quad -\frac{\partial u}{\partial t}(T,\cdot) \in \partial j_2(u(T,\cdot) - b). \tag{3.51}$$

Moreover, u is the L^2-solution to (3.49) – (3.51) if and only if u is the solution to the minimization problem

$$\inf\left\{F(v),\ v \in \mathcal{L}^2_{\tilde{r}}([0,T]; L^2(\Omega))\right\} = F(u) \tag{3.52}$$

where F is (l.s.c) functional on $L^2_{\tilde{r}}([0,T]; L^2(\Omega))$ given by

$$F(u) = \begin{cases} \int_0^T \tilde{r}(t)\left[\int_\Omega \frac{1}{2}\left(|\frac{\partial u}{\partial t}(t,x)|^2 + |\operatorname{grad}_x u(t,x)|^2\right)dx + \int_\Gamma j(u(t,x))\,d\sigma\right]dt + \\ \int_0^T \tilde{r}(t)\int_\Omega f(t,x)u(t,x)\,dx\,dt + \tilde{r}(T)j_1(u(0,\cdot) - a) + j_2(u(T,\cdot) - b), \\ \qquad \text{if } u \in H^1[[0,T]; L^2(\Omega)] \text{ and } t \to \phi(u(t,\cdot)) \text{ is in } L^1([0,T]; L^2(\Omega)) \\ +\infty, \quad \text{otherwise.} \end{cases} \tag{3.53}$$

Remarks. (I) The case of the two point boundary problems $u(0,x) = a(x)$, $u(T,x) = b(x)$ almost everywhere in Ω, corresponds to

$$j_k(u) = \begin{cases} 0, & u = 0 \\ & \qquad\qquad k = 1,2 \\ +\infty, & \text{otherwise,} \end{cases}$$

in (3.53), so $j_1(u(0,\cdot) - a) = j_2(u(T,\cdot) - b) = 0$. In this case, r can be any continuous function.

(II) The case $\frac{\partial u}{\partial t}(0,x) = u(0,x) - a(x)$ almost everywhere in Ω corresponds to $j_1(u) = \frac{1}{2}\|u\|^2$; i.e., to $\partial j_1(u) = u$, so in (3.53)

$$j_1(u(0,\cdot) - a) = \frac{1}{2}\int_\Omega |u(0,x) - a(x)|^2\, dx.$$

Recall also that the Dirichlet condition $u(t,x) = 0$ almost everywhere on $[0,T]\times\Gamma$ is obtained from (3.50) with

$$j(x) = \begin{cases} 0, & \text{if } x = 0 \\ +\infty, & \text{if } x \in L^2 - \{0\} \end{cases}$$

as in this case $D(\partial j) = \{0\}$, and $\partial j(0) = L^2(\Omega)$. The Neumann condition corresponds to $j(x)$ =constant. $-\frac{\partial u}{\partial t}(t,x) = u(t,x)$, almost everywhere on $[0,T]\times\Gamma$ corresponds to $j(x) = \frac{1}{2}|x|^2$, $\forall x \in \mathbb{R}$.

Remark 3.4. The problem (3.37) is said to be a convex problem of Bolza. This minimization problem is equivalent (in our hypotheses) to the boundary value problem (3.35)-(3.36). We can think of the equation (3.35) as a generalized Euler-Lagrange equation corresponding to the minimization problem (3.37). This problem is different from those of Barbu [4, p. 300-312] where he assumes at least one of $D(\phi)$, $D(j_1)$ or $D(j_2)$ has nonempty interior. Moreover, the left hand side of (3.35) is non-autonomous, which in Barbu's case has the form $-(\partial\psi(u'(t)))' = g(u')$; i.e., is autonomous.

4. A generalization of periodic boundary condition. For the first order differential equations

$$u'(t) = A(t)u(t),$$

with accretive (and dissipative) right hand side, we have considered (see [2]) boundary conditions of the form $u(0) \in g(u(T))$ where g is an expansive (possibly multivalued) operator acting in a Banach space X. The periodic boundary condition $u(0) = u(T)$ is obtained in the particular case $g = I$.

Recently, we have observed [3] that in the case of some second order differential equations, the periodic boundary conditions

$$u(0) = u(T), \qquad u'(0) = u'(T) \tag{4.1}$$

can be extended in two distinct ways:

$$u(0) = u(T), \quad u'(0) - u'(T) \in \gamma(u(0)) \tag{4.2}$$

$$u'(0) = u'(T), \quad u(0) - u(T) \in \delta(u'(0)). \tag{4.3}$$

The *main* result is given by

Theorem 3.1. *Let A, γ, δ be maximal monotone (possibly multivalued) operators acting in H with $(0,0) \in A \cap \gamma \cap \delta$. Assume that A is strongly monotone; i.e., $A - \omega I$ is monotone for some $\omega > 0$.*

(I) If γ is A-monotone; i.e.,

$$\langle y_1 - y_2, A_\lambda(x_1 - x_2)\rangle \geq 0, \quad \forall x_i \in D(\gamma),\ y_i \in \gamma(x_i),\ \ i = 1,2,\ \lambda > 0, \tag{4.4}$$

then for every $f \in L^2([0,T]; H)$, the problem $(4.2)-(4.5)$ has a unique solution $u \in H^2([0,T]; H)$ where

$$u''(t) \in Au(t) + f(t), \quad \text{a.e. on } [0,T]. \tag{4.5}$$

(II) If A is δ^{-1} monotone; i.e.,

$$\langle A_\lambda(x_1) - A_\lambda(x_2), y\rangle \geq 0, \quad \forall x_1, x_2 \in H \ \text{ with } x_1 - x_2 \in D(\delta^{-1}), \ \forall y \in \delta^{-1}(x_1 - x_2) \tag{4.6}$$

then the problem $(4.3) - (4.5)$ has a unique solution.

Remark 4.1. The condition (4.1) is obtained from either (4.2) or (4.3) with $\alpha = \beta = I$. The proof of Theorem 4.1, and examples and applications to ordinary and partial differential equations will be given in [3]. Moreover, in [3] we will consider also the more general case of

$$p(t)u'' + r(t)u'(t) \in Au(t) + f(t)$$

in place of (4.5).

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