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BOUNDARY VALUE PROBLEMS FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS TREATED BY PSEUDO-DIFFERENTIAL METHODS.

GERD GRUBB and VSEVOLOD A. SOLONNIKOV

1. Introduction.

Let Ω be a bounded connected open set in R^n with smooth boundary $\partial\Omega = \Gamma$, and denote by $\vec{n} = (n_1, \ldots, n_n)$ the unit interior normal vector field defined near Γ. Let $I =]0, b[$, where $b \in]0, \infty]$, let $Q = \Omega \times I$, and let $S = \Gamma \times I$. Denote $f|_\Gamma = \gamma_0 f$ and $\gamma_0 \partial_\nu^k f = \gamma_k f$, where $\partial_\nu f = \Sigma_{j=1}^n \; n_j \partial_j f$, $\partial_j f = \partial f / \partial x_j$. We consider the Navier-Stokes problem

$$
(1.1)\qquad
\begin{array}{lrl}
\text{(i)}\ \partial_t u - \Delta u + \Sigma_{i=1}^n u_i \partial_i u + \operatorname{grad} p = f & & \text{for } (x,t) \in Q,\\
\text{(ii)} & \operatorname{div} u = 0 & \text{for } (x,t) \in Q,\\
\text{(iii)} & T_k \begin{pmatrix} u \\ p \end{pmatrix} = \varphi_k & \text{for } (x,t) \in S,\\
\text{(iv)} & u|_{t=0} = u_0 & \text{for } x \in \Omega,
\end{array}
$$

where u is the velocity vector $u = (u_1, \ldots, u_n)$ and p is the pressure. Here T_k is one of the boundary operators

$$
(1.2)\quad T_1 \begin{pmatrix} u \\ p \end{pmatrix} = \chi_1 u - \gamma_0 p \vec{n}, \quad T_0 \begin{pmatrix} u \\ p \end{pmatrix} = \gamma_0 u, \quad \text{or} \quad T_2 \begin{pmatrix} u \\ p \end{pmatrix} = (\chi_1 u)_\tau + \gamma_0 u_\nu \vec{n},
$$

$v_\nu \vec{n}$ resp. v_τ stand for the normal resp. tangential components of a vector field v defined near Γ, and χ_1 is the special first order boundary operator defined via the strain tensor $S(u) = (\partial_i u_j + \partial_j u_i)_{i,j=1,\ldots,n}$ as

$$
(1.3)\qquad \chi_1 u = \gamma_0 S(u) \vec{n} = \gamma_0 (\textstyle\sum_j (\partial_i u_j + \partial_j u_i) n_j)_{i=1,\ldots,n}.
$$

The case $k = 1$ (considered earlier e.g. in Solonnikov [S2,3]) is important for the study of non-stationary free boundary problems (cf. [S5,6]); it has been studied much less than the Dirichlet case $k = 0$ (that is the main subject of an abundant literature) or the intermediate case $k = 2$, that enters in stationary free boundary

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problems (cf. [S4]). Also the following boundary operators will be studied:

$$T_3\begin{pmatrix} u \\ p \end{pmatrix} = \gamma_1 u - \gamma_0 p\vec{n}, \quad T_4\begin{pmatrix} u \\ p \end{pmatrix} = \gamma_1 u_\tau + \gamma_0 u_\nu \vec{n}. \tag{1.4}$$

Intermediate conditions like the cases $k = 2$ or 4 have been studied e.g. by Miyakawa [Mi] and Giga [Gi2], see also Solonnikov and Ščadilov [S-Šš]). Mogilevski [Mo] made a systematic study of general boundary conditions in dimension 3.

Along with (1.1) we consider the Stokes problem (the linearized Navier-Stokes problem):

$$\begin{aligned} &\text{(i) } \partial_t u - \Delta u + \operatorname{grad} p = f \quad \text{for } (x,t) \in Q, \\ &\quad\text{with (ii), (iii) and (iv) as in (1.1)}, \end{aligned} \tag{1.5}$$

which is used in the treatment of (1.1). Let $J(\Omega) = \{u \in L_2(\Omega)^n \mid \operatorname{div} u = 0\}$ and $J_0(\Omega) = \{u \in L_2(\Omega)^n \mid \operatorname{div} u = 0, \gamma_0 u_\nu = 0\}$, and denote

$$\begin{aligned} &J_k = J \quad \text{for } k = 1,3, \quad J_k = J_0 \quad \text{for } k = 0,2,4; \\ &\operatorname{pr}_{J_k} = \text{the orthogonal projection of } L_2(\Omega)^n \text{ onto } J_k(\Omega). \end{aligned} \tag{1.6}$$

Then we assume that the data satisfying

$$u_0 \in J_k, \text{ for each } k, \quad \text{and } \varphi_{k,\nu} = 0 \text{ when } k = 0, 2 \text{ or } 4. \tag{1.7}$$

We shall present a pseudo-differential treatment of these problems that was developed primarily to handle the case $k = 1$, but works also for the other cases, when suitably modified.

The main difficulty with the linearized problem (1.5) is that the "interior" operator $\tilde{L}$:

$$\tilde{L}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \partial_t - \Delta & \operatorname{grad} \\ -\operatorname{div} & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} \tag{1.8}$$

is only *degenerate* parabolic, in the sense that its principal symbol

$$\tilde{l}(\xi,\tau) = \begin{pmatrix} i\tau + |\xi|^2 & \dots & 0 & i\xi_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & i\tau + |\xi|^2 & i\xi_n \\ -i\xi_1 & \dots & -i\xi_n & 0 \end{pmatrix} \tag{1.9}$$

(considered for $(\xi,\tau) \in \mathbb{R}^{n+1}$) has the determinant $-(i\tau + |\xi|^2)^{n-1}|\xi|^2$, that is zero not only when $(\xi,\tau) = (0,0)$ but also when $\xi = 0$, $\tau \neq 0$. Then the standard parabolic theory is not directly applicable, and various techniques have been invented to circumvent the problem. Most of the known methods involve working in the "solenoidal" spaces J_0 and J (cf. e.g. Fujita and Kato [F-K], Ladyzhenskaya [L], Temam [T1, 2], [Gi1,2], [Mi], . . .) and treating homogeneous boundary conditions. In the cases $k = 0$, 2 or 4, there is a well-known reduction of (1.5) (with $\varphi_k = 0$) to a problem for $-\mathrm{pr}_{J_0}\Delta$ in the space J_0, but the cases $k = 1$ and 3 do not give the analogous operator in J. Moreover, from a systematic point of view, working in J_0 or J is somewhat inconvenient, because the property $\operatorname{div} u = 0$ is easily violated under standard differential equation techniques such as change of variables, multiplication by cut off functions, etc.

What we propose here is a different method that eliminates the condition $\operatorname{div} u = 0$ altogether; in fact we "divide out" the degeneracy by use of pseudo-differential techniques. The price one pays for this is that one then has to deal with a system containing certain nonlocal operators (so-called singular Green operators); on the other hand, the problem one obtains is parabolic *in the nondegenerate sense* (and nonhomogeneous boundary conditions are naturally included). For parabolic pseudo-differential initial-boundary value problems, a general solvability theory was set up in Grubb [G4], and further elaborated in Grubb and Solonnikov [G-S4], where the mapping properties of the solution operators are studied in anisotropic L_2 Sobolev-Slobodetskiĭ spaces. We apply that theory here; and note the advantage that the delicate study of roots of polynomials associated with (1.9) and the boundary conditions (as in [Mo]), where the complexity increases with the dimension, is replaced by general limit considerations independent of the dimension (cf. Section 6).

The present method easily allows lower order terms in the first line of (1.1) and (1.5), i.e. allows $-\Delta u$ to be replaced by the (generally nonselfadjoint) operator

$$-\Delta u + Bu, \quad \text{where } Bu = \textstyle\sum_{i=1}^{n} B_j \partial_j u + B_0 u, \tag{1.10}$$

the $B_0(x), \dots, B_n(x)$ being $n \times n$-matrices; this includes the Oseen equation [O], where the B_j are constants times the identity matrix.

Pseudo-differential methods have been used before by Giga in [Gi1,2], treating the cases $k = 0, 2, 4$ (and other cases where the operator in (1.5) takes the form $-\mathrm{pr}_{J_0}\Delta$ on J_0) in an L_q framework ($1 < q < \infty$); and the main point there was to reduce the problem to the study of parameter-dependent pseudo-differential operators *in the boundary Γ*, which is itself a boundaryless $(n-1)$-dimensional manifold. Let us also recall the singular integral operator approach in Fabes, Lewis and Rivière [F-L-R], leading to L_{q_1,q_2} estimates. In those works, the pressure p does not enter in the boundary conditions. In the present study, where p does enter in the boundary condition (for $k = 1$ and 3), we draw on the full calculus of parameter-dependent pseudo-differential boundary operators presented in [G4] (associated with the manifold $\bar{\Omega}$ and its boundary Γ), generalizing the calculus of Boutet de Monvel [BM].

Part of the results have already been published: [G-S1] gives a short account of the reduction of the linear problems (1.5) to pseudo-differential problems and states the ensuing linear solvability theorem; [G-S2] gives a survey; and [G-S3] presents in detail the deduction of the nonlinear results from the linear results. However, the explanation of the parabolicity of the reduced problems is not shown in detail in those papers, and that is a main object of the present paper. Here one must show that the singular Green operators occurring in the reduced problem are *of class* $\leqq 2$. We account for this in detail, presenting several choices that shed light on the precise contributions from the various data in (1.1); this also allows us to establish finer estimates of p in anisotropic Sobolev spaces.

The present paper has been under way for a long time (in fact the writing up was started before that of [G-S1]), so the authors must apologize for the delay, which is partly due to a slowness in communication.

Here is an outline of the contents:

Section 2 contains definitions and introductory material. We first recall the definition of anisotropic Sobolev spaces, and then we recall some elements of the Boutet de Monvel calculus of pseudo-differential boundary operators, and list their mapping properties in the anisotropic spaces (taken from [G-S4]). Among other examples we study the projection operators pr_{J_k}, that are shown to belong to the calculus. Section 3 gives the appropriate Green's formulas linking the boundary conditions with sesquilinear forms, and shows an ellipticity property for each fixed value of a certain parameter. In Section 4, we explain the precise reduction of the Neumann type problems (the cases $k = 1$ and 3) to pseudo-differential boundary problems (notably containing singular Green operators), for the linear as well as nonlinear Navier-Stokes systems. In Section 5 we show how the method is modified to include the Dirichlet and intermediate cases ($k = 0, 2$, and 4), and sum up the results. In Section 6 we show the parabolicity of the linear reduced problems, that hinges on (i) the normality of the boundary conditions, (ii) the fact that the reductions have been made in such a way that the singular Green terms only contain γ_j with $j \leqq 1$ (i.e., are "of class $\leqq 2$"). A consequence (based on [G4]) is drawn concerning resolvent estimates for the stationary problems. In Sections 7, we first define the appropriate compatibility conditions, both for the linear and the nonlinear time-dependent problems, and show how they correspond to compatibility conditions for the reduced problems. Then we derive the general solvability results for the linear cases in $H^{r,r/2}(Q)$ spaces ($r \geqq 0$) on the basis of [G-S4] and discuss various improvements of the estimates of p. Finally, we connect this to the nonlinear treatment in [G-S3] by a brief outline (when $r + 2 \geqq n/2$); also here there are improved estimates of p, when $r + 2 > n/2$. Section 8 gives further comments: on the solvability with less regular initial data when f and φ_k equal zero; on the solvability when $I =]0, \infty[$; on the realizations of the stationary linear problems restricted to the solenoidal spaces $J_k(\Omega)$; and on the spectra of these realizations, where we obtain an eigenvalue estimate in the cases $k = 1, 2, 3, 4$, by perturbation from a result of Kozhevnikov [K] for the Dirichlet case.

There are two appendices, one giving the details concerning the decomposition of vectors and differential operators into a normal and a tangential part near the boundary, the other containing the proofs of the auxiliary estimates of the bilinear term $\mathsf{K}(u, v) = \Sigma_{i=1}^{n} u_i \partial_i v$ used in the present paper and in [G-S3].

In addition to the fact that the present results include the less frequently studied cases $k = 1$ and 3, and include B, let us point to the following features of interest: 1) The treatment of truly non-homogeneous initial-boundary value problems in high generality, allowing nonzero boundary values φ_k, and allowing $f \in L_2(Q)^n$ (in the nonlinear case, $f \in L_2(Q)^n$ is allowed for $n \leqq 4$). 2) The explicit compatibility conditions of all orders (formulated without reference to fractional powers of operators), leading e.g. to solutions that are as smooth as one may want on $\bar{Q}$, *including* the "corner" $\Gamma \times \{0\}$. 3) The fact that the half-integer cases of r are included, by integral compatibility conditions. 4) The study of smoothless of p in t.

The present paper and its precedents have all been concerned with L_2 Sobolev estimates. There is a recent extension of the parabolic pseudo-differential theory to L_q Sobolev spaces (for $1 < q < \infty$) that will be applicable in a similar way to these problems, cf. Grubb and Kokholm [G7], [G-K]. (It may be of interest that also spaces of negative order ($r < 0$) are included there to some extent.)

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2. Preliminaries.

2.1. *Sobolev spaces.* In the following, Ω denotes a bounded open connected subset of R^n with smooth boundary $\partial\Omega = \Gamma$; or in some cases $\Omega = \mathsf{R}^n_+ = \{x \in \mathsf{R}^n \mid x_n > 0\}$, with $\partial\Omega = \mathsf{R}^{n-1}$, whose points are denoted $x' = (x_1, \ldots, x_{n-1})$. ($\Omega$ can also be equal to R^n.) With $I = I_b =]0, b[$ for some $b \in]0, \infty]$, we set $Q = \Omega \times I$ and $S = \Gamma \times I$ (the notation Q_b resp. S_b can be used if the dependence on b must be stated explicitly). The vectors x and x' (and ξ, ξ', etc.) are usually understood as column vectors. In circumstances where they enter as row vectors, we may indicate this explicitly by writing ${}^t x$ resp. ${}^t x'$, etc. $\langle x \rangle$ stands for $(1 + |x|^2)^{1/2}$ (also if $x \in \mathsf{C}^n$), and $[s]$ denotes the largest integer $\leqq s$. We denote

$$u(x,t)|_{x \in \Gamma} = \gamma_0 u, \quad u(x,t)|_{t=t_0} = r_{t_0} u; \tag{2.1}$$

and we sometimes write $\Omega \times \{t_0\} = \Omega_{(t_0)}$ and $\Gamma \times \{t_0\} = \Gamma_{(t_0)}$ instead of just Ω or Γ. The higher order boundary operators are denoted $\gamma_j = \gamma_0 \partial_\nu^j$, see Appendix A for precise definitions of these and other particular operators entering in the theory. We generally work with complex valued functions, although the original problems are real.

The problems will be considered in Sobolev-Slobodetskiĭ spaces over Ω and Γ, and over Q and S (in the latter case they are anisotropic in (x, t)), for which we use a standard notation, that is explained in detail in [G-S4]. Let us here just recall a few elements. For $r \in \mathsf{R}$, $H^r(\mathsf{R}^n)$ denotes the space of distributions $v \in S'(\mathsf{R}^n)$ such that the following norm is finite: $\|v\|_{H^r(\mathsf{R}^n)} = (2\pi)^{-n/2}\|\langle\xi\rangle^r \mathscr{F}v\|_{L_2(\mathsf{R}^n)}$ (here $\mathscr{F}$ denotes the Fourier transform $v(x) \mapsto \hat{v}(\xi) = \int e^{-ix\cdot\xi}v(x)\,dx$). $H^r(\Omega)$ is the space of restrictions to Ω,

$$\text{(2.3)}\quad \begin{aligned} &\|u\|^2_{H^r(\Omega)} \simeq \|u\|^2_{H^{[r]}(\Omega)} + \textstyle\sum_{\alpha=[r]\neq r} \|D^\alpha u\|^2_{r-[r],*,\Omega}, \text{ where} \\ &\|f\|^2_{s,*,\Omega} = \int_\Omega\int_\Omega |f(x)-f(y)|^2 \frac{dx\,dy}{|x-y|^{n+2s}}, \text{ for } s \in]0,1[. \end{aligned}$$

The norm is also simply denoted $\|u\|_r$, when Ω is understood. For $r \in \mathsf{R}$, $H^r_0(\bar{\Omega})$ denotes the closed subspace of $H^r(\mathsf{R}^n)$ consisting of those elements that are supported in $\bar{\Omega}$. For $r \geqq 0$, the elements of $H^r_0(\bar{\Omega})$ are identified with the L_2 functions they define on Ω (extended by zero on $\mathsf{R}^n \setminus \Omega$), and

$$\text{(2.4)}\quad \|u\|^2_{H^r_0(\bar\Omega)} \simeq \|u\|^2_{H^r(\Omega)} + \sum_{|\alpha|=[r]} \int_\Omega |D^\alpha u(x)|^2 \frac{dx}{\operatorname{dist}(x,\Gamma)^{2(r-[r])}}, \text{ for all } r \geqq 0.$$

Here the $H^r_0(\bar{\Omega})$ norm is strictly stronger than the $H^r(\Omega)$ norm, when $r - 1/2 \in \mathsf{N}$, whereas when $r - 1/2 \in \mathsf{R}_+ \setminus \mathsf{N}$, $H^r_0(\bar{\Omega})$ equals the closure of $C_0^\infty(\Omega)$ in $H^r(\Omega)$ (it is precisely the set where the traces $\gamma_j u$ with $j < r - 1/2$ are 0).

For r and $s \geqq 0$, we define

$$\text{(2.5)}\quad \begin{aligned} &H^{r,s}(Q) = L_2(I; H^r(\Omega)) \cap H^s(I; L_2(\Omega)), \\ &\text{with norm } \|u\|_{H^{r,s}(Q)} = (\|u\|^2_{L_2(I;H^r(\Omega))} + \|u\|^2_{H^s(I;L_2(\Omega))})^{1/2}, \end{aligned}$$

also written $\|u\|_{r,s}$; and denote by $H^{r,s}_{(0)}(Q)$ the closed subspace of $H^{r,s}(\Omega \times]-\infty, b[)$ consisting of the functions supported in $\{t \geqq 0\}$. (An extension to negative certain orders is described in Appendix B.) The space $H^{r,s}_{(0)}(Q)$ identifies with a subspace of $H^{r,s}(Q)$, closed if $s + 1/2 \notin \mathsf{N}$ (then it is the subspace of functions u with $r_0\partial_t^j u = 0$ for $2j + 1 < s$), and dense but not closed if $s + 1/2 \in \mathsf{N}$; and the norm is equivalent with

$$\text{(2.6)}\quad \|u\|^2_{H^{r,s}_{(0)}(\bar\Omega)} \simeq \|u\|^2_{H^{r,s}(Q)} + \int_0^b \|D_t^{[s]} u\|^2_{L_2(\Omega)} \frac{dt}{t^{2(s-[s])}}, \text{ for all } r, s \geqq 0.$$

There is a continuous imbedding

$$\text{(2.7)}\quad H^{s'}(I; H^{r'}(\Omega)) \subset H^{r,s}(\Omega \times I), \text{ when } 0 \leqq r' \leqq r, 0 \leqq s' \leqq s(r-r')/r.$$

The definitions generalize easily to vector valued functions; and when E is a Hermitian C^∞ vector bundle over $\bar{\Omega}$, the corresponding Sobolev spaces of sections are denoted $H^r(\Omega, E)$, etc., possibly abbreviated to $H^r(E)$. When

$E = \bar{\Omega} \times \mathbf{C}^N$, we just write $H^r(\Omega)^N$. When E is a vector bundle over $\bar{\Omega}$, we denote by $\underline{E}$ its lifting to $\bar{Q}$ (or to $\bar{\Omega} \times \bar{I}'$ for other intervals I'), and use a similar notation for liftings from Γ to S.

2.2. *Pseudo-differential boundary operators.* We shall use the calculus of pseudo-differential boundary operators as defined in Boutet de Monvel [BM], and further elaborated in Grubb [G4] and (for parabolic problems) in [G-S4]. Referring to these works for details, we just recall some definitions.

A pseudo-differential operator (ps.d.o.) P on $\mathbf{R}^n$ with symbol $p(x,\xi)$, a Poisson operator K (going from $\mathbf{R}^{n-1}$ to $\mathbf{R}^n_+$) with symbol-kernel $\tilde{k}(x,\xi')$, a trace operator T' of class 0 (going from $\mathbf{R}^n_+$ to $\mathbf{R}^{n-1}$) with symbol-kernel $\tilde{t}'(x,\xi')$, resp. a singular Green operator (s.g.o.) G' of class 0 (going from $\mathbf{R}^n_+$ to $\mathbf{R}^n_+$) with symbol-kernel $\tilde{g}'(x', x_n, y_n, \xi')$, is defined by the formula, respectively,

$$
\begin{aligned}
Pu(x) &= \mathrm{OP}(p)u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\,d\xi,\\
Kv(x) &= \mathrm{OPK}(\tilde{k})v(x) = (2\pi)^{1-n} \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\,\tilde{k}(x',x_n,\xi')\,\hat{v}(\xi')\,d\xi',\\
T'u(x') &= \mathrm{OPT}(\tilde{t}')u(x')\\
&= (2\pi)^{1-n} \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{t}'(x',x_n,\xi')\,\acute{u}(\xi',x_n)\,dx_n\,d\xi',\\
G'u(x) &= \mathrm{OPG}(\tilde{g}')u(x)\\
&= (2\pi)^{1-n} \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{g}'(x',x_n,y_n,\xi')\acute{u}(\xi',y_n)\,dy_n\,d\xi';
\end{aligned}
\tag{2.8}
$$

here $\hat{u}(\xi) = \mathscr{F}_{x\to\xi}u(x), \hat{v}(\xi') = \mathscr{F}_{x'\to\xi'}v(x')$, and $\acute{u}(\xi',x_n) = \mathscr{F}_{x'\to\xi'}u(x',x_n)$. The symbol resp. symbol-kernels are C^∞ functions satisfying estimates:

$$
\begin{aligned}
&|D_x^\beta D_\xi^\alpha\, p(x,\xi)| \leqq c(x)\langle\xi\rangle^{d-|\alpha|},\\
&\|D_{x'}^\beta x_n^m D_{x_n}^{m'} D_{\xi'}^\alpha\, \tilde{k}(x',x_n,\xi')\|_{L_{2,x_n}(\mathbf{R}_+)} \leqq c(x')\langle\xi'\rangle^{d-1/2-|\alpha|-m+m'},\\
&\|D_{x'}^\beta x_n^m D_{x_n}^{m'} D_{\xi'}^\alpha\, \tilde{t}(x',x_n,\xi')\|_{L_{2,x_n}(\mathbf{R}_+)} \leqq c(x')\langle\xi'\rangle^{d+1/2-|\alpha|-m+m'},\\
&\|D_{x'}^\beta x_n^m D_{x_n}^{m'} y_n^k D_{y_n}^{k'} D_{\xi'}^\alpha\, \tilde{g}'(x',x_n,y_n,\xi')\|_{L_{2,x_n,y_n}(\mathbf{R}_+\times\mathbf{R}_+)}\\
&\qquad\qquad \leqq c(x')\langle\xi'\rangle^{d-|\alpha|-m+m'-k+k'},
\end{aligned}
\tag{2.9}
$$

for all indices (here $c(x)$ and $c(x')$ denote arbitrary continuous functions of x resp. x'), and the operators and symbol(-kernel)s are then said to be of order d. The *symbols* of K, T' resp. G' are $k(x',\xi) = \mathscr{F}_{x_n\to\xi_n}\tilde{k}(x,\xi')$, $t'(x',\xi) = \bar{\mathscr{F}}_{x_n\to\xi_n}\tilde{t}'(x,\xi')$, resp. $g'(x', \xi', \xi_n, \eta_n) = \mathscr{F}_{x_n \to \xi_n} \bar{\mathscr{F}}_{y_n \to \eta_n} \tilde{g}(x', x_n, y_n, \xi')$; here $\bar{\mathscr{F}}$ is the conjugate Fourier transform $f(x_n) \to \int e^{+ix_n\xi_n} f(x_n)dx_n$.

For integer $l > 0$, trace and singular Green operators *of class* l and order d are operators of the form, respectively,

$$Tu = \sum_{0 \leqq j \leqq l-1} S_j \gamma_j u + T'u,$$
$$Gu = \sum_{0 \leqq j \leqq l-1} K_j \gamma_j u + G'u, \tag{2.10}$$

where the γ_j are the standard trace operators (cf. (A.7)), the S_j are pseudo-differential operators in $\mathbb{R}^{n-1}$ of order $d-j$, the K_j are Poisson operators of order $d-j$, and T' resp. G' are trace resp. s.g.o.s of class 0 and order d as described above. They have the symbols, respectively,

$$t(x', \xi) = \sum_{0 \leqq j \leqq l-1} s_j(x', \xi')\xi_n^j + t'(x', \xi),$$
$$g(x', \xi, \eta_n) = \sum_{0 \leqq j \leqq l-1} k_j(x', \xi)\eta_n^j + g'(x', \xi, \eta_n). \tag{2.11}$$

In the present paper we mainly consider polyhomogeneous symbols, namely symbols that have expansions in series of terms homogeneous in ξ resp. (ξ, η_n) (the homogeneity is required for $|\xi| \geqq 1$ for p but only for $|\xi'| \geqq 1$ when k, t and g are considered). Then the principal symbol p^0 (resp. k^0, t^0, g^0) is defined as the term of highest degree in the expansion, the degree being d for P and T and $d-1$ for K and G.

When P is a ps.d.o on $\mathbb{R}^n$, the truncated operated P_Ω on Ω is defined by

$$P_\Omega = r_\Omega P e_\Omega, \tag{2.12}$$

where e_Ω denotes extension by zero on $\mathbb{R}^n \setminus \Omega$ and r_Ω denotes restriction from $\mathbb{R}^n$ to Ω. To assure that P_Ω has good continuity properties in the Sobolev spaces over $\bar{\Omega}$, P is assumed to have the so-called transmission property at Γ (two-sided, cf. [G-H]); this is in particular satisfied by operators composed of differential operators and inverses of elliptic differential operators, and it requires the order d of P to be integer (under the assumption of polyhomogeneity).

For each of the types of operators P_Ω, K, T and G, the *principal boundary symbol operator* denotes the operator defined for functions on $\mathbb{R}_+$ by applying the operator definition to the principal symbol (at $x_n = 0$ for P) with respect to the x_n-variable alone (so that x' and ξ' are kept as parameters); it is also called the "model operator". The operator classes are preserved under coordinate changes. In particular, the principal boundary symbol operators are preserved, when a coordinate change $x' \mapsto \kappa(x')$ is accompanied by the change $\xi' \mapsto (\partial\kappa/\partial x')^{-1}\xi'$, where $\partial\kappa/\partial x'$ is the functional matrix $(\partial\kappa_j/\partial x_k)_{j,k=1,\dots,n-1}$.

The definition of K, T and G is carried over from the case $\Omega = \mathbb{R}^n_+$ to the general case of $\Omega \subset \mathbb{R}^n$ by the help of local coordinates. One can moreover define the operators in smooth vector bundles over $\bar{\Omega}$ resp. Γ.

In addition to the abovementioned operators we also need to consider ps.d.o.s S over Γ, of any order.

Singular Green operators arise typically when Poisson and trace operators are composed (as in $G = KT$); another source is the composition of truncated ps.d.o.s., where

$$L(P, Q) \equiv (PQ)_\Omega - P_\Omega Q_\Omega \tag{2.13}$$

is a singular Green operator.

The five types of operators P_Ω, S, K, T and G intoduced above have the property that compositions among them again lead to operators belonging to these types (together, they form an "algebra"). They are usually considered together in systems, generally of the form

$$\mathscr{A} = \begin{pmatrix} P_\Omega + G & K \\ & \\ T & S \end{pmatrix} : \begin{matrix} H^s(\Omega, E) \\ \times \\ H^{s-1/2}(\Gamma, F) \end{matrix} \to \begin{matrix} H^{s-d}(\Omega, E') \\ \times \\ H^{s-d-1/2}(\Gamma, F') \end{matrix}, \tag{2.14}$$

where E and E' are C^∞ vector bundles over $\bar{\Omega}$, F and F' are C^∞ vector bundles over Γ; we have here taken all orders equal to d. The indicated Sobolev space continuity holds for $s > l - 1/2$, when G and T are of class $\leqq l$, and extends to $s > -1/2$ for P_Ω and to $s \in \mathsf{R}$ for K and S. We also have that $\mathscr{A}$ is continuous from $C^\infty(\bar{\Omega}, E) \times C^\infty(\Gamma, F)$ to $C^\infty(\bar{\Omega}, E') \times C^\infty(\Gamma, F')$.

When such a system $\mathscr{A}$ has a certain ellipticity (consisting of invertibility of the principal symbol of P for each x, each $|\xi| \geqq 1$, and invertibility of the principal boundary symbol operator for each x', each $|\xi'| \geqq 1$), then it has a parametrix (essentially an inverse) belonging to the calculus.

G and T do not apply to the full space $L_2(\Omega, E)$ unless their *class* is zero.

EXAMPLE 2.1. The solution operator $K_D : \varphi \mapsto u$ for the semi-homogeneous Dirichlet problem

$$-\Delta u = 0 \text{ in } \Omega, \quad \gamma_0 u = \varphi \text{ on } \Gamma, \tag{2.15}$$

is a Poisson operator of order 0, with principal symbol-kernel $e^{-|\xi'|x^n}$ (in suitable local coordinates where Ω is replaced by R^n_+). The solution operator $R_D : f \mapsto u$ for the other semi-honogeneous Dirichlet problem

$$-\Delta u = f \text{ in } \Omega, \quad \gamma_0 u = 0 \text{ on } \Gamma, \tag{2.16}$$

is of the form $R_D = Q_\Omega + G_D$, where Q is the ps.d.o. $OP(|\xi|^{-2})$ and G_D is a singular Green operator of order -2 and class 0 (G_D equals $-K_D \gamma_0 Q_\Omega$). Altogether, the system

$$(2.17)\quad \begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} : H^2(\Omega) \to \begin{matrix} L_2(\Omega) \\ \times \\ H^{3/2}(\Gamma) \end{matrix} \text{ has the inverse } (R_D \;\; K_D) : \begin{matrix} L_2(\Omega) \\ \times \\ H^{3/2}(\Gamma) \end{matrix} \to H^2(\Omega).$$

It follows e.g. from variational considerations that R_D extends to a continuous operator

$$(2.18)\qquad R_D : H^{-1}(\Omega) \to H_0^1(\bar{\Omega}),$$

still solving (2.16). (It extends to a mapping from $H^s(\Omega)$ to $H^{s+2}(\Omega)$ for all $s > -3/2$, cf. [G6, Ex. 3.15], but we shall not use more than (2.18) in the present paper.)

The Neumann problem

$$(2.19)\qquad -\Delta u = f \text{ in } \Omega, \quad \gamma_1 u = \varphi \text{ on } \Gamma,$$

has a solution u, uniquely determined up to a constant, provided that f and φ satisfy the one-dimensional condition

$$(2.20)\qquad (f, 1)_{L_2(\Omega)} - (\varphi, 1)_{L_2(\Gamma)} = 0.$$

It is customary to fix the solution u by requiring that either

$$(2.21)\qquad \text{(a) } (u, 1)_{L_2(\Omega)} = 0, \quad \text{or} \quad \text{(b) } (\gamma_0 u, 1)_{L_2(\Gamma)} = 0,$$

the choice (2.21 b) is convenient for some later calculations (in Theorem 2.6 and Section 5). Then we can define u by extension of

$$(2.22)\qquad u = R_N f + K_N \varphi,$$

where R_N and K_N solve the respective semi-homogeneous problems. Here $(R_N \;\; K_N)$ can be supplied with a mapping with a one-dimensional C^∞ range to give a parametrix of $\begin{pmatrix} -\Delta \\ \gamma_1 \end{pmatrix}$ on the full space $H^2(\Omega)$; and considered in this way, K_N defined a Poisson operator of order -1, and R_N equals $Q_\Omega + G_N$ with Q as above and G_N a singular Green operator of order -2 and class 0. Then we have that

$$(2.23)\qquad \begin{aligned} &\begin{pmatrix} -\Delta \\ \gamma_1 \end{pmatrix} (R_N \;\; K_N) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \text{on the set } \{\{f, \varphi\} \mid (2.20) \text{ holds}\}; \\ &(R_N \;\; K_N) \begin{pmatrix} -\Delta \\ \gamma_1 \end{pmatrix} = I \;\text{ on the set } \{u \mid (2.21\text{ a}) \text{ resp. } (2.21\text{ b}) \text{ holds}\}, \end{aligned}$$

with $u \in H^2(\Omega)$, $f \in L_2(\Omega)$ and $\varphi \in H^{1/2}(\Omega)$. Here K_N has the principal symbol-kernel $-|\xi'|^{-1} e^{-x_n|\xi'|}$ (in suitable local coordinates where Ω is replaced by $\mathbb{R}^n_+$), and $G_N = R_N - Q_\Omega \sim -K_N \gamma_1 Q_\Omega$ (modulo an operator of rank 1).

When the time variable t is included, the operators P_Ω, S, K, T and G defined above as mappings between C^∞ spaces in x extend to mappings between C^∞ spaces in (x, t), when considered as constant in t. Then they furthermore extend to continuous mappings between anisotropic Sobolev spaces $H^{r,s}(Q)$ and $H^{r,s}(S)$ as follows (cf. [G-S4] for the proofs):

PROPOSITION 2.2. *Let P_Ω, S, K, T and G be operators of order d, as defined above; T and G being of class l. Let $r \geqq \max\{d, 0\}$ and $s \geqq 0$. The operators have the continuity properties (where $(r-d)s/r$ is read as s when $r = 0$):*

(2.24) $$P_\Omega : H^{r,s}(Q) \to H^{r-d,s'}(Q) \text{ with } s' = \min\{(r-d)s/r, s\};$$

(2.25) $$S : H^{r,s}(S) \to H^{r-d,s'}(S) \text{ with } s' = \min\{(r-d)s/r, s\};$$

(2.26) $$K : H^{r-1/2,s}(S) \to H^{r-d,s}(Q) \text{ for } r \geqq 1/2,\ d \leqq 1/2;$$

(2.27) $$K : H^{r-1/2,(r-1/2)s/r}(S) \to H^{r-d,(r-d)s/r}(Q) \text{ for } r \geqq 1/2,\ d \geqq 1/2;$$

(2.28) $$T : H^{r,s}(Q) \to H^{r-d-1/2,s'}(S) \text{ for } r > l - 1/2, \text{ with}$$
$$s' = \min\{(r-d-1/2)s/r, s\} \text{ if } l = 0;$$
$$s' = \min\{(r-d-1/2)s/r, (r-l+1/2)s/r\} \text{ if } l \geqq 1;$$

(2.29) $$G : H^{r,s}(Q) \to H^{r-d,s'}(Q) \text{ for } r > l - 1/2, \text{ with}$$
$$s' = \min\{(r-d)s/r, s\} \text{ if } l = 0;$$
$$s' = \min\{(r-d)s/r, (r-l+1/2)s/r\} \text{ if } l \geqq 1.$$

One has in particular:

(2.30) $$P_\Omega : H^{r,r/2}(Q) \to H^{r-d,\min\{(r-d)/2,r/2\}}(Q);$$

(2.31) $$S : H^{r,r/2}(S) \to H^{r-d,\min\{(r-d)/2,r/2\}}(S);$$

(2.32) $$K : H^{r-1/2,(r-1/2)/2}(S) \to H^{r-d,r'}(Q) \text{ for } r \geqq 1/2 \text{ and}$$
$$r' = \min\{(r-1/2)/2, (r-d)/2\};$$

(2.33) $$T : H^{r,r/2}(Q) \to H^{r-d-1/2,r'}(S) \text{ for } r > l - 1/2 \text{ and}$$
$$r' = \min\{(r-d-1/2)/2, r/2\} \text{ if } l = 0;$$
$$r' = \min\{(r-d-1/2)/2, (r-l+1/2)/2\} \text{ if } l \geqq 1;$$

(2.34) $$G : H^{r,r/2}(Q) \to H^{r-d,r'}(Q) \text{ for } r > l - 1/2 \text{ and}$$
$$r' = \min\{(r-d)/2, r/2\} \text{ if } l = 0;$$
$$r' = \min\{(r-d)/2, (r-l+1/2)/2\} \text{ if } l \geqq 1.$$

All the rules extend without difficulty to operators between vector bundles.

There are also more refined estimates in spaces that are anisotropic in (x', s), where $x' \in \Gamma$ and s is the normal coordinate. To avoid the most complicated statements, let us just mention that when a Poisson operator of order d is considered from R^{n-1} to R^n_+, and we define the space $H^{(a,b)}(\mathsf{R}^n_+)$ (as in [H1]) by

$$H^{(a,b)}(\mathsf{R}^n_+) = \{u = r^+ v \mid \langle\xi\rangle^a \langle\xi'\rangle^b \hat{v}(\xi) \in L_2(\mathsf{R}^n)\} \tag{2.35}$$

(provided with an infimum norm as in (2.2)), then K has the continuity properties:

$$\begin{aligned} &\|Kv\|_{H^{(m,r'-m-d)}(\mathsf{R}^n_+)} \leq C\|v\|_{H^{r'-1/2}(\mathsf{R}^{n-1})}, \\ &\|Kv\|_{L_2(I;H^{(m,r'-m-d)}(\mathsf{R}^n_+))} \leq C\|v\|_{L_2(I;H^{(r'-1/2)}(\mathsf{R}^{n-1}))} \quad \text{for all } m \geqq 0,\ r' \in \mathsf{R}; \end{aligned} \tag{2.36}$$

note that here negative norms over the boundary are allowed. One interest of this is that it shows that normal derivatives γ_j of any order can be applied to Kv, giving that $\gamma_j K v \in L_2(I; H^{r'-1/2-d-j}(\mathsf{R}^{n-1}))$ when $v \in L_2(I; H^{r'-1/2}(\mathsf{R}^{n-1}))$; in fact, one can show that $\gamma_j K$ is a pseudo-differential operator on Γ of order $d+j$.

EXAMPLE 2.3. For the operators introduced in Example 2.1 we find the following continuity properties when the t variable is included:

$$\begin{aligned} &Q_\Omega, G_D, G_N, R_D, R_N : H^{r,s}(Q) \to H^{r+2,s}(Q) \ \text{ for } r \geqq 0,\ s \geqq 0, \\ &K_D : H^{r-1/2,s}(S) \to H^{r,s}(Q) \ \text{ for } r \geqq 1/2,\ s \geqq 0, \\ &K_N : H^{r-1/2,s}(S) \to H^{r+1,s}(Q) \ \text{ for } r \geqq 1/2,\ s \geqq 0. \end{aligned} \tag{2.37}$$

Since the operators are extensions by continuity of operators on C^∞ spaces, we have also in these t-dependent spaces that $(R_D \quad K_D)$ is the inverse of $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$; and that $(R_N \quad K_N)$ is a parametrix of $\begin{pmatrix} -\Delta \\ \gamma_1 \end{pmatrix}$; more precisely, (2.23) holds with (2.20), (2.21 a) and (2.21 b) replaced by (2.38), (2.39 a) and (2.39 b), respectively:

$$\int_I |(f, 1)_{L_2(\Omega)} - (\varphi, 1)_{L_2(\Gamma)}|^2\, dt = 0; \tag{2.38}$$

$$\text{(a)} \int_I |(u, 1)_{L_2(\Omega)}|^2\, dt = 0; \quad \text{(b)} \int_I |(\gamma_0 u, 1)_{L_2(\Gamma)}|^2\, dt = 0 \tag{2.39}$$

(i.e., (2.20), (2.21 a), resp. (2.21 b) hold for almost every $t \in I$). R_D has moreover the continuity property

$$R_D : H^s(I; H^{-1}(\Omega)) \to H^s(I; H^1_0(\bar{\Omega})) \quad \text{for } s \geqq 0, \tag{2.40}$$

where it acts as an inverse of $-\Delta$. By (2.36), K_D and K_N have the continuity properties (in local coordinates)

(2.41)
$$\begin{aligned} &K_D: H^{r-1/2}(\mathsf{R}^{n-1}) \to H^{\{m,r-m\}}(\mathsf{R}^n_+),\\ &K_D: L_2(I; H^{r-1/2}(\mathsf{R}^{n-1})) \to L_2(I; H^{\{m,r-m\}}(\mathsf{R}^n_+)),\\ &K_N: H^{r-1/2}(\mathsf{R}^{n-1}) \to H^{\{m,r+1-m\}}(\mathsf{R}^n_+),\\ &K_N: L_2(I; H^{r-1/2}(\mathsf{R}^{n-1})) \to L_2(I; H^{\{m,r+1-m\}}(\mathsf{R}^n_+)) \quad \text{for } r\in\mathsf{R}, m\in\mathsf{N}; \end{aligned}$$

in particular,

(2.42)
$$\begin{aligned} &K_D: H^{-1/2}(\Gamma) \to L_2(\Omega),\\ &K_D: L_2(I; H^{-1/2}(\Gamma)) \to L_2(Q),\\ &K_N: H^{-1/2}(\Gamma) \to H^1(\Omega),\\ &K_N: L_2(I; H^{-1/2}(\Gamma)) \to L_2(I; H^1(\Omega)). \end{aligned}$$

Also here, $\gamma_0 K_D = I$, and K_D solves (2.15). For K_N we have that $\gamma_1 K_N$ maps $H^{-1/2}(\Gamma)$ resp. $L_2(I; H^{-1/2}(\Gamma))$ into itself; and that for $\varphi \in H^{-1/2}(\Gamma)$ resp. $L_2(I; H^{-1/2}(\Gamma))$ satisfying $\langle \varphi, 1\rangle_\Gamma = 0$ (resp. satisfying it for almost every in $t\in I$), $K_N\varphi$ is an exact solution of the Neumann problem (2.19) with $f = 0$.

Finally, we list the following simple continuity properties for operations with respect to the t direction (recall (2.1))

(2.43)
$$\begin{aligned} &\partial_t: H^{r,s}(Q) \to H^{(s-1)r/s,s-1}(Q) \quad \text{for } s \geqq 1;\\ &r_{t_0}: H^{r,s}(Q) \to H^{(s-1/2)r/s}(\Omega_{(t_0)}) \quad \text{for } s > 1/2; \text{ in particular,}\\ &\partial_t: H^{r,r/2}(Q) \to H^{r-2,r/2-1}(Q) \quad \text{for } r \geqq 2;\\ &r_{t_0}: H^{r,r/2}(Q) \to H^{r-1}(\Omega_{(t_0)}) \quad \text{for } r > 1; \end{aligned}$$

the same formulas hold with Q and Ω replaced by S and Γ.

2.3. *The boundary value of the normal component.* The trace operator

(2.44)
$$\gamma_\nu : u \mapsto \gamma_0(\vec{n}\cdot u) (= \gamma_0 u_\nu)$$

is well-defined on $H^s(\Omega)^n$ for $s > 1/2$, cf. (A.7) ff. We shall need the observation that its defintion extends to the space

(2.45)
$$\begin{aligned} &H_{\mathrm{div}}(\Omega) = \{u \in L_2(\Omega)^n \mid \operatorname{div} u \in L_2(\Omega)\}, \text{ with norm}\\ &\|u\|_{H_{\mathrm{div}}} = (\|u\|^2_{L_2(\Omega)^n} + \|\operatorname{div} u\|^2_{L_2(\Omega)})^{1/2}. \end{aligned}$$

To see this, note that $C^\infty(\bar{\Omega})^n$ is dense in this space, and that $u \in L_2(\Omega)^n$, $\operatorname{div} u \in L_2(\Omega)$ imply, by (A.21),

$$u_\nu \in L_2(]0,\delta[;\ L_2(\Gamma))$$

$$\partial_\nu u_\nu = \operatorname{div} u - \operatorname{div}' u_\tau - (\operatorname{div} \vec{n}) u_\nu \in L_2(]0,\delta[;\ H^{-1}(\Gamma))$$

(where Σ is represented by $\Gamma \times]-\delta, \delta[$). Recall the basic consideration from Lions and Magenes [L-M, Chap. 1]:

LEMMA 2.4. *Let X and Y be Hilbert spaces such that $X \subset Y$ densely, with continuous injection. Let W be the Hilbert space of functions u: $s \in]0,\delta[\mapsto u(s) \in Y$, for which $u \in L_2(]0,\delta[; X)$ and $\partial_s u \in L_2(]0,\delta[; Y)$. Then $C^\infty([0,\delta]; X)$ is dense in W, and the mapping $u \mapsto u|_{s=0}$ from this space to X extends by continuity to a continuous and surjective mapping from W to the interpolated space $[X, Y]_{1/2}$. In fact, $W \subset C^0([0,\delta]; [X, Y]_{1/2})$.*

This applies to show that the mapping γ_ν extends by continuity to a continuous mapping:

$$\gamma_\nu : H_{\operatorname{div}}(\Omega) \to [L_2(\Gamma), H^{-1}(\Gamma)]_{1/2} = H^{-1/2}(\Gamma). \tag{2.46}$$

Note that $\gamma_\nu \operatorname{grad} f$ is well-defined as an element of $H^{-1/2}(\Gamma)$ when $f \in H^1(\Omega)$ with $\Delta f \in L_2(\Omega)$, since $\operatorname{grad} f \in H_{\operatorname{div}}(\Omega)$ then; and we can denote $\gamma_\nu \operatorname{grad} f = \gamma_1 f$ for such f, since they can be approximated by $C^\infty(\bar{\Omega})$ functions in the norm $(\|f\|_1^2 + \|\Delta f\|_0^2)^{1/2}$.

In particular, one can define the "solenoidal" spaces

$$\begin{aligned} J(\Omega) &= \{u \in L_2(\Omega)^n \mid \operatorname{div} u = 0\}, \\ J_0(\Omega) &= \{u \in L_2(\Omega)^n \mid \operatorname{div} u = 0, \gamma_\nu u = 0\}, \end{aligned} \tag{2.47}$$

that are closed subspaces of $L_2(\Omega)$. It is well-known that $C^\infty(\bar{\Omega})^n \cap J(\Omega)$ is dense in $J(\Omega)$, and that $C_0^\infty(\Omega)^n \cap J_0(\Omega)$ is dense in $J_0(\Omega)$, and that the orthogonal complements of $J(\Omega)$ resp. $J_0(\Omega)$ in $L_2(\Omega)^n$ are the spaces

$$\begin{aligned} J(\Omega)^\perp &= G_0(\Omega) \equiv \{w = \operatorname{grad} f \mid f \in H_0^1(\bar{\Omega})\}, \\ J_0(\Omega)^\perp &= G(\Omega) \equiv \{w = \operatorname{grad} f \mid f \in H^1(\Omega)\}, \end{aligned} \tag{2.48}$$

where $G_0(\Omega)$ equals the L_2 closure of $\{w = \operatorname{grad} f \mid f \in C_0^\infty(\Omega)\}$ and $G(\Omega)$ equals the L_2 closure of $\{w = \operatorname{grad} f \mid f \in C^\infty(\bar{\Omega})\}$. These facts can be shown for quite general sets Ω (cf. e.g. [L], [L-S], [T], ...). With the presently assumed smoothness of $\bar{\Omega}$, we can describe the orthogonal projections pr_J and pr_{J_0} of $L_2(\Omega)^n$ onto $J(\Omega)$ resp. $J_0(\Omega)$ in terms of the pseudo-differential boundary operator calculus, as follows:

THEOREM 2.5. *The orthogonal projections* pr_J *and* pr_{J_0} *of $L_2(\Omega)^n$ onto $J(\Omega)$ resp. $J_0(\Omega)$ are described by the formulas*

$$\text{(2.49)}\qquad \begin{aligned} \mathrm{pr}_J &= I + \operatorname{grad} R_D \operatorname{div}, \\ \mathrm{pr}_{J_0} &= (I - \operatorname{grad} K_N \gamma_\nu)\, \mathrm{pr}_J; \end{aligned}$$

where R_D and K_N are as defined in Example 2.3.

Both of the projectors pr_J *and* pr_{J_0} *have the form:* $I + (\operatorname{grad} \mathrm{OP}(|\xi|^{-2}) \operatorname{div})_\Omega$ *plus a singular Green operator of order* 0 *and class* 0; *and* $\operatorname{grad} K_N \gamma_\nu \mathrm{pr}_J$ *is a* s.g.o. *of order* 0 *and class* 0.

The operators are continuous for all $r, s \geqq 0$:

$$\text{(2.50)}\qquad \begin{aligned} &\mathrm{pr}_J, \mathrm{pr}_{J_0} \colon H^r(\Omega)^n \to H^r(\Omega)^n, \\ &\mathrm{pr}_J, \mathrm{pr}_{J_0} \colon H^{r,s}(Q)^n \to H^{r,s}(Q)^n. \end{aligned}$$

PROOF. The operator in the first line of (2.49) is well-defined on $L_2(\Omega)^n$ in view of (2.18). Clearly, $(I + \operatorname{grad} R_D \operatorname{div})u = u$ if $\operatorname{div} u = 0$, and on the other hand, $\operatorname{div}(I + \operatorname{grad} R_D \operatorname{div})u = 0$ since $-\operatorname{div} \operatorname{grad} R_D = I$ on $H^{-1}(\Omega)$ by (2.17), (2.18); so the expression does indeed define a projection of $L_2(\Omega)^n$ onto $J(\Omega)$. Since $\operatorname{grad} R_D \operatorname{div}$ maps $L_2(\Omega)^n$ into $G_0(\Omega) = J(\Omega)^\perp$, the projection is orthogonal, hence equals pr_J.

For the second line, the operator $\gamma_\nu \mathrm{pr}_J = \gamma_\nu(I + \operatorname{grad} R_D \operatorname{div})$ is a trace operator by the calculus; and it is *of class* 0, since γ_ν is continuous from $H_{\mathrm{div}}(\Omega)$ *to* $H^{-1/2}(\Gamma)$, so that the composed expression is continuous from $L_2(\Omega)^n$ to $H^{-1/2}(\Gamma)$. (One could also analyze the symbol directly, to see that the operator is as in (2.8). For K_N and R_N, *we use the convention* (2.21 b), noting that other choices give values of $K_N\varphi$ and $R_N f$ deviating from the chosen ones by constants, that disappear when div and grad are applied. One has for $u \in J(\Omega)$ that $\langle \gamma_\nu u, 1 \rangle_\Gamma = 0$, since this equation holds for smooth functions u in $J(\Omega)$, cf. (A.30), so it follows from the observations at the end of Example 2.3 that K_N acts on $\gamma_\nu u$ as the exact solution operator for the Neumann problem (2.19) with $f = 0$; in particular, $\gamma_1 K_N \gamma_\nu u = \gamma_\nu u$. Now $\gamma_\nu u = 0$ clearly implies $(I - \operatorname{grad} K_N \gamma_\nu)u = u$; and on the other hand, if $u \in J(\Omega)$, then $\gamma_\nu(I - \operatorname{grad} K_N \gamma_\nu)u = \gamma_\nu u - \gamma_1 K_N \gamma_\nu u = 0$. Thus $I - \operatorname{grad} K_N \gamma_\nu$ defines a projection of $J(\Omega)$ onto $J_0(\Omega)$; and since $\operatorname{grad} K_N \gamma_\nu$ maps into $G(\Omega) = J_0(\Omega)^\perp$, the projection is orthogonal. This shows that the operator defined by the second line of (2.49) is indeed pr_{J_0}.

By the pseudo-differential boundary operator calculus, both of the projectors pr_J and pr_{J_0} have the form: $I + (\operatorname{grad} \mathrm{OP}(|\xi|^{-2}) \operatorname{div})_\Omega$ plus a singular Green operator, and are of order 0; here the singular Green operator terms are of class 0 since the expressions are defined on $L_2(\Omega)^n$. In particular, $\operatorname{grad} K_N \gamma_\nu \mathrm{pr}_J$ is a s.g.o. of order 0 and class 0. The continuity properties now follow from the general rules; cf. (2.49) for the first line in (2.50), and cf. (2.24), (2.29) for the second line ($d = 0, l = 0$).

One can also consider the intersections of the solenoidal spaces with Sobolev spaces; here one has e.g. that $C^\infty(\bar\Omega)\cap J(\Omega)$ is dense in $H^s(\Omega)\cap J(\Omega)$, and $C_0^\infty(\Omega)\cap J_0(\Omega)$ is dense in $H_0^s(\bar\Omega)\cap J_0(\Omega)$, in the H^s resp. H_0^s norm, for $s\geqq 0$.

Let us moreover introduce the time-dependent solenoidal spaces

$$
\begin{aligned}
H_{\mathrm{div}}(Q) &= \{u\in L_2(Q)^n \mid \operatorname{div} u\in L_2(Q)\},\\
J(Q) &= \{u\in L_2(Q)^n \mid \operatorname{div} u = 0\},\\
J_0(Q) &= \{u\in L_2(Q)^n \mid \operatorname{div} u = 0, \gamma_\nu u = 0\},\\
G(Q) &= \{w = \operatorname{grad} f \mid f\in L_2(I; H^1(\Omega))\},\\
G_0(Q) &= \{w = \operatorname{grad} f \mid f\in L_2(I; H_0^1(\bar\Omega))\};
\end{aligned}
\tag{2.51}
$$

here one has that $C^\infty(\bar Q)^n\cap J(Q)$ is dense in $J(Q)$, that $C_0^\infty(Q)^n\cap J_0(Q)$ is dense in $J_0(Q)$, that $G_0(Q)$ is the L_2 closure of $\{w=\operatorname{grad} f\mid f\in C_0^\infty(Q)\}$ and that $G(Q)$ is the L_2 closure of $\{w=\operatorname{grad} f\mid f\in C^\infty(\bar Q)\cap L_2(Q)^n\}$; and $J(Q)^\perp = G_0(Q)$, $J_0(Q)^\perp = G(Q)$.

The orthogonal projections pr_J and pr_{J_0} of $L_2(Q)^n$ onto $J(Q)$ and $J_0(Q)$ satisfy again the formulas (2.49) (so the notation is consistent).

We find by Lemma 2.4 that γ_ν extends to a continuous mapping

$$
\gamma_\nu : H_{\mathrm{div}}(Q)\to L_2(I; H^{-1/2}(\Gamma)),
\tag{2.52}
$$

since $\partial_\nu u_\nu\in L_2(]0,\delta[; L_2(I; H^{-1}(\Gamma)))$ when $u\in H_{\mathrm{div}}(Q)$. Note that for $u\in J(Q)$,

$$
\int_I |\langle \gamma_\nu u, 1\rangle_\Gamma|^2\,dt = 0,
\tag{2.53}
$$

since this holds for the smooth functions in $J(Q)$ by (A.30).

Let us also deduce another useful pseudo-differential formula, describing the $G(\Omega)$ part of a function as a gradient, and linking this with a Neumann problem.

THEOREM 2.6. 1° *Choose R_N and K_N according to the convention* (2.21b). *The following operator (which is the sum of the* ps.d.o. $(\mathrm{OP}(|\xi|^{-2})\operatorname{div})_\Omega$ *and a singular Green operator)*

$$
\tilde G = R_D \operatorname{div} - K_N\gamma_\nu \mathrm{pr}_J,
\tag{2.54}
$$

is of order -1 *and class* 0; *it also equals*

$$
\tilde G = R_N \operatorname{div} - K_N\gamma_\nu,
\tag{2.55}
$$

so this operator is of class 0 *although the two terms are separately of class* 1.

2° *The operator $\tilde G$ satisfies*

$$
\mathrm{pr}_{J_0} = I + \operatorname{grad}\tilde G;
\tag{2.56}
$$

hence, in the decomposition of an element $g\in L_2(\Omega)^n$ as the sum of a function

$u \in J_0(\Omega)$ and a function $w \in G(\Omega)$, one has that $w = -\operatorname{grad} v$, where $v = \tilde{G}g$. For a given $w \in G(\Omega)$, the solution $v' \in H^1(\Omega)$ of the equation $-\operatorname{grad} v' = w$ is determined up to a constant; the present solution v is determined by the condition $(\gamma_0 v, 1)_\Gamma = 0$.

3° *When $g \in L_2(\Omega)^n$, the function $v = \tilde{G}g$ solves the generalized Neumann problem*

$$-\Delta v = \operatorname{div} g, \quad \gamma_1 v = -\gamma g, \tag{2.57}$$

in the sense that if $(g_m)_{m \in \mathbb{N}}$ is a sequence in $C^\infty(\bar{\Omega})^n$ with $g_m \to g$ in $L_2(\Omega)^n$, then $v_m = \tilde{G}g_m$ solves (2.57) with g replaced by g_m; here $v_m \to v$ in $H^1(\Omega)$ and $\gamma_1 v_m \to \gamma_1 v$ in $H^{-1/2}(\Gamma)$. (The solution is uniquely determined by the condition $(\gamma_0 v, 1)_\Gamma = 0$.)

4° *The statements extend to the situation where $L_2(\Omega)$, $G(\Omega)$ and $H^1(\Omega)$ are replaced by $L_2(Q)$, $G(Q)$ resp. $L_2(I; H^1(\Omega))$, with the condition $(\gamma_0 v, 1)_\Gamma = 0$ replaced by $\int_I |(\gamma_0 v, 1)|^2\, dt = 0$.*

PROOF. From (2.54) we see that $\tilde{G}$ is of order -1 and class 0, since both terms are well-defined on $L_2(\Omega)^n$, cf. (2.18). Now since $\gamma_0 u = 0$ for u in the range of R_D, $K_N\gamma_1 + R_N(-\Delta) = I$ holds on $R_D(L_2(\Omega))$, by (2.23) with (2.21 b). Moreover, $-\Delta R_D = I$ by (2.17). Then one has on $C^\infty(\bar{\Omega})$ (and even on $H_{\operatorname{div}}(\Omega)$):

$$\begin{aligned} R_N \operatorname{div} - K_N\gamma_\nu &= R_N(-\Delta) R_D \operatorname{div} - K_N\gamma_\nu = (I - K_N\gamma_1)R_D \operatorname{div} - K_N\gamma_\nu \\ &= R_D \operatorname{div} - K_N\gamma_\nu(\operatorname{grad} R_D \operatorname{div} + I) = R_D \operatorname{div} - K_N\gamma_\nu \operatorname{pr}_J = \tilde{G}, \end{aligned}$$

which shows (2.55). This shows 1°. (2.56) follows in view of (2.49); for we have on $C^\infty(\bar{\Omega})$:

$$\begin{aligned} \operatorname{pr}_{J_0} &= \operatorname{pr}_J - \operatorname{grad} K_N\gamma_\nu \operatorname{pr}_J \\ &= I + \operatorname{grad} R_D \operatorname{div} - \operatorname{grad} R_D \operatorname{div} + \operatorname{grad} R_N \operatorname{div} - \operatorname{grad} K_N\gamma_\nu \\ &= I + \operatorname{grad} R_N \operatorname{div} - \operatorname{grad} K_N\gamma_\nu = I + \operatorname{grad} \tilde{G}. \end{aligned}$$

For the second statement in 2° we use that the only solutions of $\operatorname{grad} v' = 0$ are the constants, since Ω is connected; and $(\gamma_0 \tilde{G}g, 1)_\Gamma = 0$ is seen from (2.54), using that $\langle \gamma_\nu \operatorname{pr}_J g, 1\rangle = 0$ as in the preceding proof. 3° follows from (2.55) and the various continuity properties of the operators. 4° is an easy consequence of the preceding statements.

3. Green's formulas and ellipticity properties.

3.1. *Sesquilinear forms.* We refer to Appendix A for notation used in the following.

The linearized Navier-Stokes operator L (the Stokes operator) can be written in the following form

$$L\begin{pmatrix}u\\p\end{pmatrix}=\begin{pmatrix}-\Delta u+\operatorname{grad}p\\-\operatorname{div}u\end{pmatrix}=\begin{pmatrix}-\Delta & \dots & 0 & \partial_1\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & -\Delta & \partial_n\\ -\partial_1 & \dots & -\partial_n & 0\end{pmatrix}\begin{pmatrix}u_1\\ \vdots\\ u_n\\ p\end{pmatrix} \tag{3.1}$$

(where we have given the divergence a minus sign, to obtain a symmetric operator). Recall that this is a Douglas-Nirenberg elliptic system with principal symbol

$$l^0(\xi)=\begin{pmatrix}|\xi|^2 & \dots & 0 & i\xi_1\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & |\xi|^2 & i\xi_n\\ -i\xi_1 & \dots & -i\xi_n & 0\end{pmatrix}, \tag{3.2}$$

where the (j,k)th entry in L is considered to be of order 2 if j and $k<n+1$, of order 1 if j or k but not both equal $n+1$, and of order 0 when $j=k=n+1$.

There are several sesquilinear forms of interest that can be associated with L:

$$\begin{aligned} s_1\left(\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\right)&=\sum_{j=1}^{n}(\operatorname{grad}u_j,\operatorname{grad}v_j)_{L_2(\Omega)^n}-(\operatorname{div}u,q)_{L_2(\Omega)}-(p,\operatorname{div}v)_{L_2(\Omega)},\\ s_2\left(\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\right)&=E(u,v)-(\operatorname{div}u,\operatorname{div}v)_{L_2(\Omega)}-(\operatorname{div}u,q)_{L_2(\Omega)}-(p,\operatorname{div}v)_{L_2(\Omega)}.\end{aligned} \tag{3.3}$$

Here s_1 is a generalization of the usual choice for the scalar Laplace operator, and it satisfies the Green's formula (cf. (A.30))

$$\left(L\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\right)_{L_2(\Omega)^{n+1}}=s_1\left(\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\right)+(\gamma_1 u-\gamma_0 p\vec{n},\gamma_0 v)_{L_2(\Gamma)^n}. \tag{3.4}$$

(Here and in the following, we assume that the functions are smooth enough for the formulas to have a sense, e.g. $u\in H^2(\Omega)^n$, $p\in H^1(\Omega)$, $v\in H^1(\Omega)^n$, $q\in L_2(\Omega)$.) In s_2, the term $E(u,v)$ is defined by

$$E(u,v)=\tfrac{1}{2}\textstyle\sum_{j,k=1}^{n}(\partial_j u_k+\partial_k u_j,\partial_j v_k+\partial_k v_j)_{L_2(\Omega)}=\tfrac{1}{2}(S(u),S(v))_{L_2(\Omega)^{n^2}}, \tag{3.5}$$

where the matrix $S(u)$ is the so-called strain tensor:

$$S(u)=(\partial_j u_k+\partial_k u_j)_{j,k=1,\dots,n}. \tag{3.6}$$

Here one has, using the inherent symmetries and (A.30),

$$\begin{aligned} E(u,v)&=\textstyle\sum_{j,k=1}^{n}(\partial_j u_k+\partial_k u_j,\partial_j v_k)_\Omega\\ &=\textstyle\sum_{j,k=1}^{n}[(-\partial_j^2 u_k-\partial_j\partial_k u_j,v_k)_\Omega-(n_j\gamma_0(\partial_j u_k+\partial_k u_j),\gamma_0 v_k)_\Gamma]\\ &=(-\Delta u-\operatorname{grad}\operatorname{div}u,v)_\Omega-(\gamma_0 S(u)\vec{n},\gamma_0 v)_\Gamma\\ &=(-\Delta u,v)_\Omega+(\operatorname{div}u,\operatorname{div}v)_\Omega+(\gamma_0(\operatorname{div}u)\vec{n},\gamma_0 v)_\Gamma-(\gamma_0 S(u)\vec{n},\gamma_0 v)_\Gamma,\end{aligned} \tag{3.8}$$

so, defining the boundary operator χ_1 by

$$\chi_1 u = \gamma_0 S(u)\vec{n}, \tag{3.9}$$

we find the Green's formula

$$\left(L\binom{u}{p}, \binom{v}{q}\right)_{L_2(\Omega)^{n+1}} = s_2\left(\binom{u}{p}, \binom{v}{q}\right) + (\chi_1 u - \gamma_0(\operatorname{div} u)\,\vec{n} - \gamma_0 p\vec{n}, \gamma_0 v)_{L_2(\Gamma)^n}. \tag{3.10}$$

For divergence free vectors it implies:

$$\left(L\binom{u}{p}, \binom{v}{q}\right)_{L_2(\Omega)^{n+1}} = E(u,v) + (\chi_1 u - \gamma_0 p\vec{n}, \gamma_0 v)_{L_2(\Gamma)^n}, \quad \text{when } \operatorname{div} u = \operatorname{div} v = 0. \tag{3.11}$$

Boundary conditions involving $\chi_1 u - \gamma_0 p\vec{n}$ are of particular interest in applications (e.g. to free boundary problems, cf. [S4,5]).

Concerning E, we recall that for functions vanishing on the boundary, one has (cf. (3.8))

$$E(u,v) = (\operatorname{grad} u, \operatorname{grad} v)_{L_2(\Omega)^{n^2}} + (\operatorname{div} u, \operatorname{div} v)_{L_2(\Omega)} \quad \text{for } u, v \in H_0^1(\Omega)^n, \tag{3.12}$$

so that E satisfies, when Ω is bounded,

$$E(u,u) = \|\operatorname{grad} u\|^2_{L_2(\Omega)^{n^2}} + \|\operatorname{div} u\|^2_{L_2(\Omega)} \geqq c\|u\|^2_{H^1(\Omega)^n}, \text{ for } u \in H_0^1(\Omega)^n, \tag{3.13}$$

with $c > 0$. By (3.13) and general ellipticity results (as in [A-D-N]), one has moreover the Korn inequality,

$$\|\operatorname{grad} u\|^2 \leqq c_1 E(u,u) + c_2\|u\|^2, \quad \text{for } u \in H^1(\Omega)^n; \tag{3.14}$$

here $c_1 > 0$ and $c_2 \geqq 0$, and one can take $c_2 = 0$ under extra conditions on u, see the discussion and additional results in Solonnikov-Ščadilov [S-Šč] and Solonnikov [S6].

Let us compare χ_1 with the standard normal derivative γ_1. In the "flat" case $\Omega = \mathbb{R}^n_+$, we simply have

$$S(u)\vec{n} = \begin{pmatrix} \partial_1 u_n + \partial_n u_1 \\ \vdots \\ \partial_{n-1} u_n + \partial_n u_{n-1} \\ 2\partial_n u_n \end{pmatrix},$$

so that (cf. Appendix A for the notation)

$$\begin{aligned} &\chi_1 u \equiv \gamma_0 S(u)\vec{n} = \gamma_1 u + \gamma_0 \operatorname{grad} u_n, \quad \text{with} \\ &(\chi_1 u)' = \gamma_1 u' + \gamma_0 \operatorname{grad}' u_n, \quad (\chi_1 u)_n = 2\gamma_1 u_n, \quad \text{when } \Omega = \mathbb{R}^n_+. \end{aligned} \tag{3.15}$$

The general picture is similar:

$$\begin{aligned}\chi_1 u &= \gamma_0(\textstyle\sum_{k=1}^n (n_k \partial_j u_k + n_k \partial_k u_j))_{j=1,\dots,n} \\ &= \gamma_1 u + \gamma_0(\partial_j(\textstyle\sum_{k=1}^n n_k u_k) - \sum_{k=1}^n (\partial_j n_k) u_k)_{j=1,\dots,n} \\ &= \gamma_1 u + \gamma_0 \operatorname{grad} u_\nu - s_0 \gamma_0 u,\end{aligned} \tag{3.16}$$

where s_0 is the symmetric matrix

$$s_0(x) = (\partial_j n_k(x))_{j,k=1,\dots,n} = (\partial_j \partial_k \varrho(x))_{j,k=1,\dots,n}. \tag{3.17}$$

Here we note that

$$s_0 \vec{n} = (\textstyle\sum_k (\partial_j n_k) n_k)_{j \leqq n} = \frac{1}{2}(\partial_j \sum_k n_k^2)_{j \leqq n} = 0, \quad \text{and likewise } {}^t\vec{n} s_0 = 0. \tag{3.18}$$

In particular we have, using (A.11) (or (3.18)),

$$\begin{aligned}(\chi_1 u)_\nu &= \gamma_0 \textstyle\sum_{j,k=1}^n n_j n_k (\partial_j u_k + \partial_k u_j) = 2\gamma_0 \sum_{j,k=1}^n n_k n_j \partial_j u_k \\ &= 2\gamma_0 \vec{n} \cdot \gamma_1 u = 2(\gamma_1 u)_\nu = 2\gamma_1 u_\nu;\end{aligned} \tag{3.19}$$

and by (A.12), (A.18) and (3.18),

$$\begin{aligned}(\chi_1 u)_\tau &= (\gamma_1 u)_\tau + (\gamma_0 \operatorname{grad} u_\nu)_\tau - (s_0 \gamma_0 u)_\tau \\ &= \gamma_1 u_\tau + \operatorname{grad}'_\Gamma \gamma_0 u_\nu - s_0 \gamma_0 u_\tau.\end{aligned} \tag{3.20}$$

Altogether,

$$\begin{aligned}\chi_1 u &= (\gamma_1 u)_\tau + 2(\gamma_1 u)_\nu \vec{n} + \operatorname{grad}'_\Gamma \gamma_0 u_\nu - s_0 \gamma_0 u \\ &= \gamma_1 u_\tau + 2\gamma_1 u_\nu \vec{n} + \operatorname{grad}'_\Gamma \gamma_0 u_\nu - s_0 \gamma_0 u_\tau,\end{aligned} \tag{3.21}$$

which in particular shows that χ_1 is *normal*, i.e., $\gamma_1 u$ enters with an invertible coefficient, namely, the morphism consisting of multiplication by 1 on $F_{\nu,\Gamma}$ and by 2 on $F_{\tau,\Gamma}$. We here refer to the decomposition of $\Gamma \times \mathbf{C}^n$ into the tangential and the normal bundles $F_{\tau,\Gamma}$ resp. $F_{\nu,\Gamma}$, cf. (A.10).

For a more systematic study of boundary values and sesquilinear forms associated with systems of differential operators of mixed order, we refer to Grubb [G1-3]. The set $\{\gamma_0 u, \gamma_1 u - \gamma_0 p\vec{n}\}$ is a set of so-called *reduced Cauchy data* for L (cf. [G1, p. 182], [G2], [G3, (1.17) with $p = -u_2$]); and $\{\gamma_0 u, \chi_1 u - \gamma_0(\operatorname{div} u)\vec{n} - \gamma_0 p\vec{n}\}$ is equivalent with it (in the sense that the two sets of data determine one another), since by (3.21) and (A.22),

$$\begin{aligned}\chi_1 u - \gamma_0(\operatorname{div} u)\vec{n} &= \gamma_1 u_\tau + 2\gamma_1 u_\nu \vec{n} + \operatorname{grad}'_\Gamma \gamma_0 u_\nu - s_0 \gamma_0 u_\tau \\ &\quad - (\operatorname{div}'_\Gamma \gamma_0 u_\tau + (\operatorname{div} \vec{n})\gamma_0 u_\nu + \gamma_1 u_\nu)\vec{n} \\ &= \gamma_1 u + \begin{pmatrix} -s_0 & \operatorname{grad}'_\Gamma \\ -\operatorname{div}'_\Gamma & -(\operatorname{div} \vec{n}) \end{pmatrix} \begin{pmatrix} \gamma_0 u_\tau \\ \gamma_0 u_\nu \end{pmatrix}.\end{aligned} \tag{3.22}$$

The block decomposition here refers to the decomposition in (A.10) (not to be confounded with the decomposition of $\{u, p\}$ in an n-vector u and a scalar p, that also enters below).

Let us introduce the notation for the trace operators (where we likewise use (A.10)):

$$T_0\begin{pmatrix}u\\p\end{pmatrix} = \gamma_0 u,\ T_1\begin{pmatrix}u\\p\end{pmatrix} = \chi_1 u - \gamma_0 p\vec{n},\ T_2\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}(\chi_1 u)_\tau\\ \gamma_0 u_\nu\end{pmatrix},$$
$$T_3\begin{pmatrix}u\\p\end{pmatrix} = \gamma_1 u - \gamma_0 p\vec{n},\quad T_4\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}(\gamma_1 u)_\tau\\ \gamma_0 u_\nu\end{pmatrix} = \begin{pmatrix}\gamma_1 u_\tau\\ \gamma_0 u_\nu\end{pmatrix}; \tag{3.23}$$

here T_0 defines the Dirichlet condition, T_1 and T_3 define Neumann type conditions, and T_2 and T_4 define intermediate conditions, for the Stokes and the Navier-Stokes operators. Note that T_0, T_2 and T_4 involve only u, and that, in view of (3.20),

$$T_2\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}1 & \operatorname{grad}'_\Gamma\\ 0 & 1\end{pmatrix} T_4\begin{pmatrix}u\\p\end{pmatrix} + \begin{pmatrix}-s_0\gamma_0 u_r\\ 0\end{pmatrix}, \tag{3.24}$$

so T_2 and T_4 have equivalent principal parts. The discussions of them will be very similar.

We have as a simple consequence of (3.4), (3.10) and (3.11):

LEMMA 3.1. *Let L_{T_k} be the realization of L in $L_2(\Omega)^{n+1}$ with domain*

$$D(L_{T_k}) = \left\{\begin{pmatrix}u\\p\end{pmatrix}\middle|\, u\in H^2(\Omega)^n, p\in H^1(\Omega), T_k\begin{pmatrix}u\\p\end{pmatrix} = 0\right\}.$$

1° *When $k = 0$, 1 or 2, one has for $\{u, p\}$ in $D(L_{T_k})$ that*

$$\left(L\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v\\q\end{pmatrix}\right)_{L_2(\Omega)^{n+1}} = s_2\left(\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v\\q\end{pmatrix}\right) - (\gamma_0 \operatorname{div} u, \gamma_0 v_\nu)_{L_2(\Gamma)}, \tag{3.25}$$

and in particular:

$$\left(L\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v\\q\end{pmatrix}\right)_{L_2(\Omega)^{n+1}} = E(u, v), \quad \textit{if } \operatorname{div} u = \operatorname{div} v = 0. \tag{3.26}$$

2° *When $k = 0, 3$ or 4, one has for $\{u, q\}$ and $\{v, q\}$ in $D(L_{T_k})$ that*

$$\left(L\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v\\q\end{pmatrix}\right)_{L_2(\Omega)^{n+1}} = s_1\left(\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v\\q\end{pmatrix}\right), \tag{3.27}$$

and in particular:

$$(3.28)\quad \left(L\begin{pmatrix}u\\p\end{pmatrix},\begin{pmatrix}v\\q\end{pmatrix}\right)_{L_2(\Omega)^{n+1}} = (\operatorname{grad} u, \operatorname{grad} v)_{L_2(\Omega)^{n^2}},\ \text{if } \operatorname{div} u = \operatorname{div} v = 0.$$

3.2. *The model problems.* We shall now investigate the ellipticity properties of the boundary value problem

$$(3.29)\qquad L\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}f\\g\end{pmatrix} \text{ in } \Omega,\quad T_k\begin{pmatrix}u\\p\end{pmatrix} = \varphi \text{ on } \Gamma,$$

for $k = 0,\dots,4$. Classically, this is a study of the solvability properties of the operator obtained at each boundary point $x_0 \in \Gamma$ by taking the principal part, freezing the coefficients at x_0, and introducing new coordinates y by a translation carrying x_0 into 0 and an orthogonal transformation carrying $\vec{n}(x_0)$ into $(0,\dots,0,1)$ (this is also applied to the vector fields); finally one performs a Fourier transformation in the y' variables. This leads precisely to the *model operators* or *principal boundary symbol operators* at x_0. For the present operators, whose form is preserved under orthogonal transformations, the model problem at each $x_0 \in \Gamma$ has the same form as when $\Omega = \mathsf{R}^n_+$, so we can assume that $\Omega = \mathsf{R}^n_+$.

Let $u = (u_1,\dots,u_n) \in \mathscr{S}(\bar{\mathsf{R}}_+)^n$ and $p \in \mathscr{S}(\bar{\mathsf{R}}_+)$; then the model trace operators $t_k^0(\xi', D_n)$ associated with the T_k act as follows:

$$t_0^0(\xi',D_n)\begin{pmatrix}u\\p\end{pmatrix} = u(0),\quad t_1^0(\xi',D_n)\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}\partial_n u'(0) + i\xi' u_n(0)\\ 2\partial_n u_n(0) - p(0)\end{pmatrix},$$

$$(3.30)\qquad t_2^0(\xi',D_n)\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}\partial_n u'(0) + i\xi' u_n(0)\\ u_n(0)\end{pmatrix},$$

$$t_3^0(\xi',D_n)\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}\partial_n u'(0)\\ \partial_n u_n(0) - p(0)\end{pmatrix},\quad t_4^0(\xi',D_n)\begin{pmatrix}u\\p\end{pmatrix} = \begin{pmatrix}\partial_n u'(0)\\ u_n(0)\end{pmatrix};$$

here, as we recall, u' is the column vector $(u_1,\dots,u_{n-1})$ (that is written ${}^t u'$ when used explicitly as a row vector). We denote by $I_{(n)}$ the $n \times n$ unit matrix.

LEMMA 3.2. *Let $z \in \mathsf{C}$, with either* $\operatorname{Re} z > 0$ *or* $z = 0$, *and let* $k = 0, 1, 2, 3$ *or* 4. *Let l^0 be the principal symbol of L (*cf. (3.2)*), and let $l^0\ (\xi', D_n)$ be the corresponding model operator on* R,

$$(3.31)\qquad l^0(\xi',D_n) = \begin{pmatrix}(|\xi'|^2 - \partial_n^2)I_{(n-1)} & 0 & i\xi'\\ 0 & |\xi'|^2 - \partial_n^2 & \partial_n\\ {}^t(-i\xi') & -\partial_n & 0\end{pmatrix}.$$

For each $\xi' \in \mathsf{R}^{n-1}\setminus\{0\}$, the boundary value problem

(3.32)
$$\begin{pmatrix} z^2 u \\ 0 \end{pmatrix} + l^0(\xi', D_n)\begin{pmatrix} u \\ p \end{pmatrix} = f \quad on\ \mathsf{R}_+,$$
$$t_k^0(\xi', D_n)\begin{pmatrix} u \\ p \end{pmatrix} = \varphi \quad at\ x_n = 0,$$

has a unique solution $\{u, p\} \in \mathscr{S}(\bar{\mathsf{R}}_+)^{n+1}$ *for any* $f \in \mathscr{S}(\bar{\mathsf{R}}_+)^{n+1}$ *and* $\varphi \in \mathsf{C}^n$.

PROOF. The symbol of the differential operator in the first line of (3.32) equals, when z is included,

(3.33)
$$\tilde{l}(\xi, z) = \begin{pmatrix} z^2 + |\xi|^2 & \cdots & 0 & i\xi_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & z^2 + |\xi|^2 & i\xi_n \\ -i\xi_1 & \cdots & -i\xi_n & 0 \end{pmatrix},$$

which has determinant $\det \tilde{l}(\xi, z) = -|\xi|^2(z^2 + |\xi|^2)^{n-1}$; it is nonzero for all $\xi \in \mathsf{R}^n \setminus \{0\}$ since $z^2 \in \mathsf{C} \setminus \mathsf{R}_-$. Then the problem in the first line of (3.32) has the solution $\{v, q\} = \mathscr{F}^{-1}_{\xi_n \to x_n}[\tilde{l}(\xi, z)^{-1}\hat{g}(\xi_n)]$, where $g \in \mathscr{S}(\mathsf{R})^{n+1}$ is chosen such that $r^+ g = f$. Since we can subtract this solution from $\{u, p\}$, we can now assume that $f = 0$ in (3.32).

For each $\xi' \in \mathsf{R}^{n-1} \setminus \{0\}$, the polynomial of degree $2n$ in $\tau \in \mathsf{C}$

(3.34)
$$\det \tilde{l}(\xi', \tau, z) = -(|\xi'|^2 + \tau^2)(z^2 + |\xi'|^2 + \tau^2)^{n-1}$$

has its $2n$ roots in $\mathsf{C} \setminus \mathsf{R}$, with n roots in each of the halfplanes $\mathsf{C}_\pm = \{\tau \in \mathsf{C} \mid \operatorname{Im}\tau > 0 \text{ resp. } \operatorname{Im}\tau < 0\}$ (since $\det \tilde{l}(\xi', \tau, z)$ is real for real τ and z, and the number of roots in C_+ depends continuously on z). It follows that for each $\xi' \in \mathsf{R}^{n-1} \setminus \{0\}$, the space of solutions of the differential equation

(3.35)
$$\tilde{l}(\xi', D_n, z)\begin{pmatrix} u \\ p \end{pmatrix} = 0 \quad \text{on } \mathsf{R}_+,$$

is the direct sum of two n-dimensional spaces Z_+ and Z_- of $(n+1)$-vectors of exponential polynomials, where $Z_+ \subset \mathscr{S}(\bar{\mathsf{R}}_+)^{n+1}$ and $Z_- \cap \mathscr{S}(\bar{\mathsf{R}}_+)^{n+1} = \{0\}$ (the vectors are, respectively, exponentially decreasing and exponentially increasing). So the space of solutions in $\mathscr{S}(\bar{\mathsf{R}}_+)^{n+1}$ of the first line of (3.32) with $f = 0$ is precisely the n-dimensional space Z_+. Since t_k^0 defines a linear mapping of Z_+ into C^n, the lemma will be proved if we show that this mapping is *injective*, i.e. if the elements $\{u, p\} \in Z_+$ with $t_k^0\{u, p\} = 0$ are zero.

This is achieved by use of the "model" versions of the formulas (3.26), (3.28), which give that if $\{u, p\}$ solves (3.32) with f and $\varphi = 0$, then for $k = 0$, 1 or 2,

$$(3.36)\qquad \begin{aligned} 0 &= \left(\left[\begin{pmatrix} z^2 & 0 \\ 0 & 0 \end{pmatrix} + l(\xi', D_n)\right]\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right)_{L_2(\mathbf{R}_+)^{n+1}} \\ &= z^2 \|u\|^2_{L_2(\mathbf{R}_+)^n} + \tfrac{1}{2}\textstyle\sum_{j,k<n} \|i\xi_j u_k + i\xi_k u_j\|^2_{L_2(\mathbf{R}_+)} \\ &\quad + \tfrac{1}{2}\textstyle\sum_{j<n} \|i\xi_j u_n + \partial_n u_j\|^2_{L_2(\mathbf{R}_+)} + \tfrac{1}{2}\|\partial_n u_n\|^2_{L_2(\mathbf{R}_+)}; \end{aligned}$$

and for $k = 3$ or 4,

$$(3.37)\qquad \begin{aligned} 0 &= \left(\left[\begin{pmatrix} z^2 & 0 \\ 0 & 0 \end{pmatrix} + l(\xi', D_n)\right]\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right)_{L_2(\mathbf{R}_+)^{n+1}} \\ &= z^2 \|u\|^2_{L_2(\mathbf{R}_+)^n} + \textstyle\sum_{j=1}^n \left(\sum_{k=1}^{n-1} \|i\xi_k u_j\|^2_{L_2(\mathbf{R}_+)} + \|\partial_n u_j\|^2_{L_2(\mathbf{R}_+)}\right). \end{aligned}$$

When (3.36) or (3.37) holds, $z^2 \in \mathbf{C} \setminus \mathbf{R}_-$ implies that all the terms are zero, so since $\partial_n u_n = 0$ implies $u_n = 0$ for $u_n \in \mathscr{S}(\bar{\mathbf{R}}_+)$, $u_1 = \ldots = u_n = 0$. As for p, we can now observe that

$$(i\xi_1 p, \ldots, i\xi_{n-1} p, \partial_n p) = 0$$

follows since $u = 0$ and the first line in (3.32) holds with $f = 0$. This implies $p = 0$.

The lemma shows in particular (from the case $z = 0$) that the trace operators T_k ($k = 0, \ldots, 4$) define elliptic boundary value problems together with L, in the sense of Agmon, Douglis and Nirenberg [A-D-N]. On the other hand, the z-dependent problems are not parameter-elliptic (in a sense similar to Agmon [A], Agranovič and Vishik [A-V], Grubb [G4]), because already the interior symbol $\tilde{l}(\xi', \xi_n, z)$ is not so; it degenerates too much for $\xi \to 0$. It is for this reason that we make the reductions in Sections 4 and 5.

REMARK 3.3. Also more general boundary conditions could be considered. For one thing, the above arguments work for many other "intermediate" trace operators than T_2 and T_4, namely all operators of the form

$$(3.38)\qquad \begin{aligned} T_5\begin{pmatrix} u \\ p \end{pmatrix} &= \mathrm{pr}_{F'}(\gamma_0 u) + \mathrm{pr}_{F''}(\chi_1 u - \gamma_0 p\vec{n}), \text{ or} \\ T_6\begin{pmatrix} u \\ p \end{pmatrix} &= \mathrm{pr}_{F'}(\gamma_0 u) + \mathrm{pr}_{F''}(\gamma_1 u - \gamma_0 p\vec{n}), \end{aligned}$$

where $\Gamma \times \mathbf{C}^n$ is decomposed into orthogonal bundles over Γ:

$$(3.39)\qquad \Gamma \times \mathbf{C}^n = F' \oplus F'',$$

and $\mathrm{pr}_{F'}$ and $\mathrm{pr}_{F''}$ are the corresponding orthogonal projections (morphisms). T_2 and T_4 correspond to the choice $F' = F_{\nu,\Gamma}$, $F'' = F_{\tau,\Gamma}$; one can for example instead take $F' = F_{\tau,\Gamma}$, $F'' = F_{\nu,\Gamma}$. Another generalization is to the cases where one adds to the sesquilinear forms s_1 and s_2 a term $(S_0\gamma_0 u, \gamma_0 v)$ with a suitable first order differential operator S_0 on Γ, leading to "oblique" Neumann problems. More generally, one can use the systematic study in Grubb [G3] of the boundary conditions for Douglis-Nirenberg elliptic systems satisfying the Gårding inequality. (The scalar function u_2 in [G3, Example 1] corresponds to $-p$ here, which is harmless for the general calculations but nonstandard in the applications.) One can also include systems that are not semibounded but do have the needed (degenerate) parabolicity property of the principal symbol.

4. The reduction of the Neumann problems.

4.1. *Reduction of a linear Neumann problem.* Consider the time-dependent Stokes problem (the linearized Navier-Stokes problem):

$$\text{(4.1)}\qquad \begin{aligned} &\text{(i)} && \partial_t u - \Delta u + \operatorname{grad} p = f && \text{in } Q,\\ &\text{(ii)} && \operatorname{div} u = 0 && \text{in } Q,\\ &\text{(iii)} && u|_{t=0} = u_0 && \text{on } \Omega, \end{aligned}$$

with one of the boundary conditions defined in Section 3 ($k = 0, 1, 2, 3$ or 4),

$$\text{(4.2}_k\text{)}\qquad T_k\begin{pmatrix} u \\ p \end{pmatrix} = \varphi_k \quad \text{on } S.$$

It is assumed that the data satisfy:

$$\text{(4.3}_k\text{)}\qquad \begin{aligned} &\operatorname{div} u_0 = 0, \quad \text{when } k = 1 \text{ or } 3;\\ &\operatorname{div} u_0 = 0,\ \gamma_0 u_{0,\nu} = 0,\ \varphi_{k,\nu} = 0, \quad \text{when } k = 0, 2 \text{ or } 4. \end{aligned}$$

The main idea is to reduce the study of (4.1), (4.2$_k$), (4.3$_k$), to the study of a parabolic pseudo-differential boundary problem

$$\text{(4.4}_k\text{)}\qquad \begin{aligned} \partial_t u + M_k u &= f_k && \text{in } Q,\\ T'_k u &= \psi_k && \text{on } S,\\ u|_{t=0} &= u_0 && \text{on } \Omega, \end{aligned}$$

to which the results of [G4] and [G-S4] can be applied.

Because of the novelty of the case $k = 1$, we begin with that, i.e. the case where (4.2$_k$) takes the form

$$\text{(4.5)}\qquad \chi_1 u - \gamma_0 p\vec{n} = \varphi_1 \quad \text{on } S.$$

The reduction will be done in two steps: first we eliminate p, and next we show how to remove the equation $\operatorname{div} u = 0$.

Let $\{u, p\}$ be a solution of (4.1), (4.5), with (4.3$_1$). We postpone for a moment the exact discussion of the spaces where the function are considered; for the time being, they should just be "sufficiently smooth." Application of $-\operatorname{div}$ to (4.1 i) gives, in view of (4.1 ii):

$$-\operatorname{div}\operatorname{grad} p \equiv -\Delta p = -\operatorname{div} f \quad \text{in } Q, \tag{4.6}$$

and a multiplication of (4.5) by $\vec{n}$ gives (cf. (3.19))

$$\gamma_0 p = \vec{n}\cdot\chi_1 u - \vec{n}\cdot\varphi_1 = 2\gamma_1 u_\nu - \varphi_{1,\nu} \quad \text{on } S \tag{4.7}$$

We see that p solves a Dirichlet problem for $-\Delta$, so it can be expressed by the other entries by use of the solution operators defined in Example 2.1:

$$p = -R_D \operatorname{div} f + K_D(2\gamma_1 u_\nu - \varphi_{1,\nu}). \tag{4.8}$$

Insertion of p in (4.1 i) then gives that u satisfies (cf. (2.49))

$$\begin{aligned}\partial_t u - \Delta u + 2\operatorname{grad} K_D\gamma_1 u_\nu &= f + \operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D\varphi_{1,\nu}\\ &= \operatorname{pr}_J f + \operatorname{grad} K_D\varphi_{1,\nu},\end{aligned} \tag{4.9}$$

where $2\operatorname{grad} K_D\gamma_1 \operatorname{pr}_{F_\nu}$ is a singular Green operator. In this way, the problem has been reduced to the form

$$\begin{aligned}&\text{(i)} & \partial_t u - \Delta u + G_1 u &= f_1 & &\text{in } Q,\\ &\text{(ii)} & \operatorname{div} u &= 0 & &\text{in } Q,\\ &\text{(iii)} & u|_{t=0} &= u_0 & &\text{on } \Omega,\\ &\text{(iv)} & (\chi_1 u)_\tau &= \psi_1 & &\text{on } S;\end{aligned} \tag{4.10}$$

where G_1, f_1 and ψ_1 are defined by

$$G_1 u = 2\operatorname{grad} K_D\gamma_1 u_\nu, \quad f_1 = \operatorname{pr}_J f + \operatorname{grad} K_D\varphi_{1,\nu}, \quad \psi_1 = \varphi_{1,\tau}. \tag{4.11}$$

Conversely, one has that if u solves (4.10) and p is defined from u and the data by (4.8), then in view of (4.11), (2.49),

$$\begin{aligned}\partial_t u - \Delta u + \operatorname{grad} p = f_1 - G_1 u + \operatorname{grad} p &= \operatorname{pr}_J f + \operatorname{grad} K_D\varphi_{1,\nu} - 2\operatorname{grad} K_D\gamma_1 u_\nu\\ &\quad - \operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D(2\gamma_1 u_\nu - \varphi_{1,\nu}) = f,\end{aligned}$$

which shows (4.1 i). Moreover, since p is defined such that $\gamma_0 p = 2\gamma_1 u_\nu - \varphi_{1,\nu}$,

$$(\chi_1 u)_\nu - \gamma_0 p = 2\gamma_1 u_\nu - 2\gamma_1 u_\nu + \varphi_{1,\nu} = \varphi_{1,\nu}$$

(cf. (3.19)), which together with (4.10 iv) shows that (4.5) is satisfied. The conditions (4.10 ii, iii) carry over unchanged, so $\{u, p\}$ solves (4.1), (4.5).

Note here that, by the definition of K_D,

$$\operatorname{div}\operatorname{grad} K_D = \Delta K_D = 0, \tag{4.12}$$

so in particular,

$$\operatorname{div} G_1 = 0. \tag{4.13}$$

The next step is to eliminate the equation $\operatorname{div} u = 0$. Here we note that if u solves the system of equations (4.10), then it solves a fortiori the system of equations

$$(4.14)\qquad \begin{aligned} &\text{(i)} & \partial_t u - \Delta u + G_1 u &= f_1 \quad \text{in } Q,\\ &\text{(ii)} & u|_{t=0} &= u_0 \quad \text{on } \Omega,\\ &\text{(iii)} & (\chi_1 u)_\tau + \gamma_0(\operatorname{div} u)\vec{n} &= \psi_1 \quad \text{on } S. \end{aligned}$$

The system (4.14) can be considered for arbitrary f_1 and u_0 (not necessarily divergence free) and arbitary ψ_1 (not necessarily tangential), and u will then also be quite general. However, the fundamental observation that we shall now make, is that when f_1 and u_0 in (4.14) are divergence free, and $\psi_{1,\nu} = 0$, then (4.14) *suffices to assure that* $\operatorname{div} u = 0$. For, if we then apply div to (4.14 i) and (4,14 ii), and $\vec{n}\cdot$ to (4.14 iii), we find in view of (4.13),

$$(4.15)\qquad \begin{aligned} \partial_t \operatorname{div} u - \Delta \operatorname{div} u &= 0 \quad \text{in } Q,\\ \operatorname{div} u|_{t=0} &= 0 \quad \text{on } \Omega,\\ \gamma_0 \operatorname{div} u &= 0 \quad \text{on } S; \end{aligned}$$

so $\operatorname{div} u$ solves the *ordinary heat equation with Dirichlet boundary condition.* It is well known that this is uniquely solvable (in suitable function spaces), so we can conclude that $\operatorname{div} u = 0$.

Before we formulate the result in a theorem, we shall look more closely at the spaces in which the reductions are performed.

The largest space in which we take f (in the present work) is $L_2(Q)^n = H^{0,0}(Q)^n$. In that case we look for u in $H^{2,1}(Q)^n$ (in order to have $\partial_t u$ and Δu in $L_2(Q)^n$). Since $u \in H^{2,1}(Q)^n$ implies $\gamma_0 u \in H^{3/2,3/4}(S)^n$, $\gamma_1 u \in H^{1/2,1/4}(S)^n$, and $r_0 u \in H^1(\Omega)^n$ (cf. Proposition 2.2 and (2.43)), we assume that $u_0 \in H^1(\Omega)^n$ and $\varphi_1 \in H^{1/2,1/4}(S)^n$. Finally, p should be such that grad $p \in L_2(Q)^n$ and $\gamma_0 p$ is well-defined, so we assume $p \in H^{1,0}(Q)$, whereby $\gamma_0 p \in H^{1/2,0}(S)$. With these hypotheses, (4.6) is considered in $L_2(I; H^{-1}(\Omega))$, cf. Example 2.3. (When $\operatorname{div} f = 0$, p moreover gets a little bit of t-regularity from u_ν and φ_1; see the systematic discussion later in Section 7.) In these spaces, the passage between (4.1)–(4.5) and (4.10), and the passage from (4.10) to (4.14), make good sense.

The passage from (4.14) to (4.15) and its consequence, under the assumption that $\operatorname{div} f_1 = 0$, $\operatorname{div} u_0 = 0$, $\psi_{1,\nu} = 0$, is straightforward when the functions are sufficiently smooth (e.g. when $u \in H^{3,3/2}(Q)^n$); but for $u \in H^{2,1}(Q)^n$ it requires a little more care. The difficulty is that $\operatorname{div} u$ lies in $H^{1,1/2}(Q)^n$ then, where r_0 is not generally defined (so that we cannot interchange r_0 and div), and where the heat problem (4.15) must be interpreted in some generalized sense. We shall apply the following lemma, that will be usefull also for a Neumann heat problem in Section 5:

LEMMA 4.1. *Let* $u \in H^{2,1}(Q)^n$ *with* $\operatorname{div} r_0 u = 0$, *and let* $v = \operatorname{div} u$. *Then it satisfies*

$v \in L_2(I; H^1(\Omega))$ and $\partial_t v \in L_2(I; H^{-1}(\Omega))$, and

$$v \in C^0(\bar{I}; L_2(\Omega)) \quad \text{with } r_0 v = 0. \tag{4.16}$$

In particular, if $\partial_t v - \Delta v = 0$ in Q, then the extension $\tilde{v}$ of v by 0 for $t < 0$ satisfies $\partial_t \tilde{v} - \Delta \tilde{v} = 0$ in $]-\infty, b[\times \Omega$.

PROOF. Since $u \in L_2(I; H^2(\Omega))$ with $\partial_t u \in L_2(I; L_2(\Omega))$, one has that $v \in L_2(I; H^1(\Omega))$ and $\partial_t v = \partial_t \operatorname{div} u = \operatorname{div} \partial_t u \in L_2(I; H^{-1}(\Omega))$. Then Lemma 2.4 shows that $v \in C^0(\bar{I}; L_2(\Omega))$, whereby $r_0 v$ is well-defined as an element of $L_2(\Omega)$. Let u_m be a sequence in $C^\infty(\bar{\Omega})^n$ with $u_m \to u$ in $H^{2,1}(Q)^n$, then $v_m = \operatorname{div} u_m$ converges to v in the above spaces. Moreover, $r_0 u_m \to r_0 u$ in $H^1(\Omega)$, so $\operatorname{div} r_0 u_m \to \operatorname{div} r_0 u$ in $L_2(\Omega)$. Then $r_0 v = \lim_{m \to \infty} r_0 v_m = \lim_{m \to \infty} r_0 \operatorname{div} u_m = \lim_{m \to \infty} \operatorname{div} r_0 u_m = 0$. This shows (4.16), and it follows that the extension $\tilde{v}$ belongs to $C^0(]-\infty, b]; L_2(\Omega))$. Similarly, $\Delta \tilde{v} \in C^0(]-\infty, b]; H^{-2}(\Omega))$. Now since $\Delta \tilde{v}|_{t>0}$ and $\partial_t \tilde{v}|_{t>0}$ are in $L_2(I; H^{-1}(\Omega))$, and $\Delta \tilde{v}|_{t<0}$ and $\partial_t \tilde{v}|_{t<0}$ are in $L_2(\mathbb{R}_-; H^{-1}(\Omega))$, in fact equal 0, we have for any $\varphi \in C_0^\infty(]-\infty, b[\times \Omega)$, using that $\tilde{v}(0) = 0$,

$$\begin{aligned} \langle \partial_t \tilde{v} - \Delta \tilde{v}, \bar{\varphi} \rangle_{]-\infty[\times\Omega} &= -\int_{-\infty}^{b} (\langle \tilde{v}, \partial_t \bar{\varphi}(\cdot, t) \rangle_\Omega + \langle \Delta \tilde{v}, \bar{\varphi}(\cdot, t) \rangle_\Omega)\, dt \\ &= \int_0^b \langle \partial_t v - \Delta v, \bar{\varphi}(\cdot, t) \rangle_\Omega \, dt; \end{aligned}$$

and this is zero when $\partial_t v - \Delta v = 0$ on Q.

This is used as follows: Since $\operatorname{div} G_1 = 0$ and $\operatorname{div} f_1 = 0$, (4.14 i) implies $\partial_t v - \Delta v = 0$ in Q. Since $\operatorname{div} u_0 = 0$, (4.14 ii) shows that $\operatorname{div} r_0 u = 0$; then the lemma gives that $\partial_t \tilde{v} - \Delta \tilde{v} = 0$ in $]-\infty, b[\times \Omega$. Since $\psi_{1,\nu} = 0$, (4.14 iii) shows that $\gamma_0 v = 0$, this clearly extends to $\tilde{v}$. Altogether we have that $\tilde{v}$ satisfies

$$\begin{aligned} \partial_t \tilde{v} - \Delta \tilde{v} &= 0 \quad \text{in }]-\infty, b[\times \Omega, \\ \gamma_0 \tilde{v} &= 0 \quad \text{on }]-\infty, b[\times \Gamma; \end{aligned} \tag{4.17}$$

as an element of $L_2(]-\infty, b[; H^{-1}(\Omega))$. Since the usual uniqueness result for the heat equations extends to this setting (cf. Piriou [Pi, Th. (32) ii]), we conclude that $\tilde{v} = 0$ in $]-\infty, b[\times \Omega$, and hence $\operatorname{div} u = 0$ in Q.

It is now straightforward to carry the analysis over to the case where the functions are taken in the spaces with $r \geqq 0$,

$$\begin{aligned} &u \in H^{r+2, r/2+1}(Q)^n, \\ &p \in H^{1,0}(Q) \text{ with } \operatorname{grad} p \in H^{r, r/2}(Q)^n, \\ &f \text{ and } f_1 \in H^{r, r/2}(Q)^n, \\ &\varphi_1 \text{ and } \psi_1 \in H^{r+1/2, r/2+1/4}(S)^n, \\ &u_0 \in H^{r+1}(\Omega)^n; \end{aligned} \tag{4.18}$$

by use of Proposition 2.2, (2.43) and (2.50).

Altogether, we have obtained the following result:

THEOREM 4.2. *Consider functions in the spaces* (4.18), for some $r \geqq 0$.

1° *Let* f, u_0 *and* φ_1 *be given, satisfying* (4.3_1). *Define* G_1, f_1 *and* ψ_1 *by* (4.11), G_1 *is a singular Green operator of order and class 2. If* $\{u, p\}$ *is a solution of* (4.1) *and* (4.5), *then* p *satisfies* (4.8), *and* u *is a solution of* (4.10). *Conversely, if* u *solves* (4.10) *and* p *is defined by* (4.8), *then* $\{u, p\}$ *solves* (4.1) *and* (4.5).

2° *Let* f_1, u_0 *and* ψ_1 *be given. When* u *solves* (4.10), *then it solves* (4.14). *Conversely, when* u *solves* (4.14), *and* $\operatorname{div} f_1 = 0$, $\operatorname{div} u_0 = 0$, $\psi_{1,\nu} = 0$, *then* u *solves* (4.10).

In this way, we have carried the study of (4.1), (4.2_1), (4.3_1), over to the study of (4.4_1) with $M_1 = -\Delta + G_1$, and

$$T_1' u = (\chi_1 u)_r + \gamma_0 (\operatorname{div} u)\vec{n}. \tag{4.19}$$

The parabolicity will be shown in Section 6.

Note that T_1' is a differential trace operator. It is of order 1 and *normal*; for we have, in the block notation (recalling (A.22) and (3.20), (3.17)):

$$\begin{aligned} T_1' u &= \begin{pmatrix} (\chi_1 u)_\tau \\ \gamma_0 \operatorname{div} u \end{pmatrix} = \begin{pmatrix} \gamma_1 u_\tau + \operatorname{grad}'_\Gamma \gamma_0 u_\nu - s_0 \gamma_0 u_\tau \\ \gamma_1 u_\nu + \operatorname{div}'_\Gamma \gamma_0 u_\tau + (\operatorname{div} \vec{n}) \gamma_0 u_\nu \end{pmatrix} \\ &= \gamma_1 u + \begin{pmatrix} 0 & \operatorname{grad}'_\Gamma \\ \operatorname{div}'_\Gamma & 0 \end{pmatrix} \begin{pmatrix} \gamma_0 u_\tau \\ \gamma_0 u_\nu \end{pmatrix} + \begin{pmatrix} -s_0 & 0 \\ 0 & (\operatorname{div} \vec{n}) \end{pmatrix} \begin{pmatrix} \gamma_0 u_\tau \\ \gamma_0 u_\nu \end{pmatrix}, \end{aligned} \tag{4.20}$$

where $\gamma_1 u$ has coefficient 1. The principal part is

$$\gamma_1 + \begin{pmatrix} 0 & \operatorname{grad}'_\Gamma \\ \operatorname{div}'_\Gamma & 0 \end{pmatrix} \gamma_0, \tag{4.21}$$

and in the case where Ω is given as $\mathbb{R}^n_+$, T_1' simply has the form

$$T_1' u = [\gamma_1 + \begin{pmatrix} 0 & \operatorname{grad}' \\ \operatorname{div}' & 0 \end{pmatrix} \gamma_0] \begin{pmatrix} u' \\ u_n \end{pmatrix} = \gamma_0 \begin{pmatrix} \partial_n u_1 + \partial_1 u_n \\ \vdots \\ \partial_n u_{n-1} + \partial_{n-1} u_n \\ \partial_1 u_1 + \cdots + \partial_n u_n \end{pmatrix}.$$

The model operator (the principal boundary symbol operator) associated with T_1' has the form (at each $x' \in \mathbb{R}^{n-1}$)

$$t_1'^{\,0} u = \partial_n u(0) + \begin{pmatrix} 0 & i\xi' \\ {}^t(i\xi') & 0 \end{pmatrix} u(0) = \begin{pmatrix} \partial_n u'(0) + i\xi' u_n(0) \\ \partial_n u_n(0) + i\xi' \cdot u'(0) \end{pmatrix};$$

we shall also write $u(0)$ resp. $\partial_n u(0)$ as $\gamma_0 u$ resp. $\gamma_1 u$.

The singular Green operator G_1 is of order 2 and contains normal derivatives up to order 1, hence is *of class 2* (cf. Section 2.2). It is of interest for the finer analysis of the solutions, that one can replace (4.4_1) with a variant where the s.g.o. is *of class 1*: By formula (A.21),

$$\gamma_1 u_\nu = -\operatorname{div}'_\Gamma \gamma_0 u, \quad \text{when } \gamma_0 \operatorname{div} u = 0. \tag{4.22}$$

Then p in (4.1) can also be described by

$$p = -2K_D \operatorname{div}'_\Gamma \gamma_0 u - K_D \varphi_{1,\nu}; \tag{4.23}$$

and when we insert this in (4.1), we get a formulation like (4.10) but with G_1 replaced by

$$G'_1 = -2 \operatorname{grad} K_D \operatorname{div}'_\Gamma \gamma_0, \tag{4.24}$$

It is essential to observe here that, by (4.22),

$$G_1 u = G'_1 u, \quad \text{when } \gamma_0 \operatorname{div} u = 0. \tag{4.25}$$

Since G'_1 only involves γ_0, it is well-defined on $H^s(\Omega)^n$ for $s > 1/2$, whereas G_1 requires $s > 3/2$; this can be advantageous for some purposes. One finds, exactly as before:

THEOREM 4.3. *Theorem 4.2 holds with G_1 replaced by G'_1, when the formula for p is replaced by* (4.23); *here G'_1 is a singular Green operator of order 2 and class 1.*

The model operators (principal boundary symbol operators) for G_1 and G'_1 have the form, for $|\xi'| \geqq 1$,:

$$\begin{aligned} g_1^0(\xi', D_n)u(x_n) &= 2\begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} e^{-|\xi'|x_n}\, \partial_n u_n(0) = 2e^{-|\xi'|x_n}\begin{pmatrix} i\xi' \\ -|\xi'| \end{pmatrix} \partial_n u_n(0), \\ g_1'^{\,0}(\xi', D_n)u(x_n) &= -2\begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} e^{-|\xi'|x_n} \sum_{j=1}^{n-1} i\xi_j u_j(0) = 2e^{-|\xi'|x_n}\begin{pmatrix} \xi'(\xi'\cdot u'(0)) \\ i|\xi'|(\xi'\cdot u'(0)) \end{pmatrix}; \end{aligned} \tag{4.26}$$

this follows easily by using that the model operator for K_D is the multiplication by $e^{-|\xi'|x_n}$. (The expressions are extended smoothly for $|\xi'| < 1$, or are used as they stand if we want the strictly homogeneous symbols.)

4.2. *Inclusion of first order linear and nonlinear terms.* Before we go on to treat the other boundary conditions, we shall show one can include lower order terms in the first line of (4.1), in the case $k = 1$. Let

$$Bu = \sum_{j=1}^{n} B_j(x)\partial_j u + B_0(x)u, \tag{4.27}$$

where the B_j for $j = 0, \ldots, n$ are C^∞ $n \times n$-matrix functions on $\bar{\Omega}$ (extendable, of course, to a neighborhood of $\bar{\Omega}$); this includes cases arizing from linearization of (1.1) around a fixed smooth vector field. If $B_j = b_j I$ with the b_j constant and $b_0 = 0$, this is Oseen's equation [O].

When B is included, we get the evolution problem (the generalized Stokes problem)

$$(4.28_k)\qquad \begin{aligned} &\text{(i)} & \partial_t u - \Delta u + Bu + \operatorname{grad} p &= f & &\text{in } Q,\\ &\text{(ii)} & \operatorname{div} u &= 0 & &\text{in } Q,\\ &\text{(iii)} & u|_{t=0} &= u_0 & &\text{on } \Omega,\\ &\text{(iv)} & T_k\begin{pmatrix} u \\ p \end{pmatrix} &= \varphi_k & &\text{on } S; \end{aligned}$$

where the data are, as usual, assumed to satisfy (4.3_k).

Including also the nonlinear term, we get the (generalized) Navier-Stokes problem:

$$(4.29_k)\qquad \begin{aligned} &\text{(i) } \partial_t u - \Delta u + Bu + \mathsf{K}u + \operatorname{grad} p = f \quad \text{in } Q,\\ &\qquad \text{(ii), (iii) and (iv) as in } (4.28_k), \end{aligned}$$

where we for the nonlinear term $\mathsf{K}u$ use the notation

$$(4.30)\quad \mathsf{K}(u, v) = \sum_{j=1}^{n} u_j \partial_{x_j} v, \quad \text{in particular } \mathsf{K}(u) = \mathsf{K}(u, u), \text{ also written } \mathsf{K}u.$$

The reductions with Bu and with $\mathsf{K}u$ are so similar that we treat them at the same time, considering

$$(4.31_k)\qquad \begin{aligned} &\text{(i) } \partial_t u - \Delta u + Bu + \delta \mathsf{K}u + \operatorname{grad} p = f & &\text{in } Q,\\ &\text{(ii)} \qquad\qquad\qquad\qquad \operatorname{div} u = 0 & &\text{in } Q,\\ &\text{(iii)} \qquad\qquad\qquad\qquad u|_{t=0} = u_0 & &\text{on } \Omega,\\ &\text{(iv)} \qquad\qquad\qquad\qquad T_k\begin{pmatrix} u \\ p \end{pmatrix} = \varphi_k & &\text{on } S; \end{aligned}$$

for $\delta = 0, 1$. We use that B and K are continuous:

$$(4.32)\qquad \begin{aligned} &B\colon H^{r+2, r/2+1}(Q) \to H^{r+1, r/2+1/2}(Q) \quad \text{for } r \geqq 0,\\ &\mathsf{K}\colon H^{r+2, r/2+1}(Q) \to H^{r, r/2}(Q) \quad \text{for } r \geqq 0, r + 2 \geqq n/2; \end{aligned}$$

the first statement is a simple instance of (2.30), and the second statement is proved in Appendix B, Theorem B.3.

Consider the case $k = 1$, with functions satisfying (4.18) for some $r \geqq 0$, where in addition $r + 2 \geqq n/2$ if $\delta = 1$. When $\{u, p\}$ is a solution of (4.31_1), an application of $-\operatorname{div}$ to (i) there gives, in view of (ii),

$$(4.33)\qquad -\Delta p = \operatorname{div}(Bu + \delta \mathsf{K}u - f),$$

Multiplication of (4.31_1 iv) by $\vec{n}$ gives (4.7), as before. Then p is determined by

$$p = R_D \operatorname{div}(Bu + \delta \mathsf{K} u - f) + K_D(2\gamma_1 u_\nu - \varphi_{1,\nu}). \tag{4.34}$$

Note that $\operatorname{div} Bu$ vanishes if B has constant, scalar coefficients.

Inserting this in (4.31_1), we find, using (2.49), that u solves the problem

$$\begin{aligned} &\text{(i) } \partial_t u - \Delta u + G_1 u + \mathrm{pr}_J(B + \delta\mathsf{K})u = f_1 && \text{in } Q,\\ &\text{(ii) } \operatorname{div} u = 0 && \text{in } Q,\\ &\text{(iii) } u|_{t=0} = u_0 && \text{on } \Omega,\\ &\text{(iv) } (\chi_1 u)_\tau = \psi_1 && \text{on } S; \end{aligned} \tag{4.35}$$

that implies

$$\begin{aligned} &\text{(i) } \partial_t u - \Delta u + G_1 u + \mathrm{pr}_J(B + \delta\mathsf{K})u = f_1 && \text{in } Q,\\ &\text{(ii) } u|_{t=0} = u_0 && \text{on } \Omega,\\ &\text{(iii) } T_1' u = \psi_1 && \text{on } S; \end{aligned} \tag{4.36}$$

here we have defined G_1, f_1, ψ_1 and T_1' as in (4.11), (4.19). One goes back from (4.35) to (4.31_1) just as in the proof of Theorem 4.1.

Now let u be given as a solution of (4.36), with $\operatorname{div} f_1 = 0$, $\operatorname{div} u_0 = 0$ and $\psi_{1,\nu} = 0$. We find by application of div to (4.36 i) that

$$\begin{aligned} 0 &= \partial_t \operatorname{div} u - \Delta \operatorname{div} u + \operatorname{div} G_1 u + \operatorname{div} \mathrm{pr}_J(B + \delta \operatorname{div} \mathsf{K})u \\ &= \partial_t \operatorname{div} u - \Delta \operatorname{div} u, \end{aligned} \tag{4.37}$$

since $\operatorname{div} G_1$ and $\operatorname{div} \mathrm{pr}_J$ are zero. By hypothesis, $\gamma_0 \operatorname{div} u = 0$. Since $\operatorname{div} r_0 u = 0$, we can now reason in the same way as the passage from (4.14) to (4.15), using the interpretation (4.16) if necessary, and conclude that $\operatorname{div} u = 0$, so we get back (4.31_1) for $\{u, p\}$, when we define p by (4.34).

The analogous arguments hold with G_1 replaced by G_1'.

Altogether, we have obtained:

THEOREM 4.4. *Consider the problem* (4.31_1) *for functions in the spaces* (4.18), *for some* $r \geqq 0$; *assume in addition* $r + 2 \geqq n/2$ *if* $\delta = 1$.

1° *Let* f, u_0 *and* φ_1 *be given, satisfying* (4.3_1). *Define* G_1, f_1 *and* ψ_1 *by* (4.11). *If* $\{u, p\}$ *is a solution of* (4.31_1), *then* p *satisfies* (4.34), *and* u *is a solution of* (4.35). *Conversely, if* u *solves* (4.35) *and* p *is defined by* (4.34), *then* $\{u, p\}$ *solves* (4.31_1).

2° *Let* f_1, u_0 *and* ψ_1 *be given. When* u *solves* (4.35), *then it solves* (4.36). *Conversely, when* u *solves* (4.36), *and* $\operatorname{div} f_1 = 0$, $\operatorname{div} u_0 = 0$, $\psi_{1,\nu} = 0$, *then* u *solves* (4.35).

3° *Analogous statements hold with* G_1 *replaced by* G_1' *defined in* (4.24), *when* $2K_D\gamma_1 u_\nu$ *in* (4.34) *is replaced by* $-2K_D \operatorname{div}'_\Gamma \gamma_0 u$.

In this way we have reduced the problem (4.31_1) to a problem of the form (4.4_1)

with M_1 replaced by $-\Delta + G_1 + \mathrm{pr}_J B + \delta\,\mathrm{pr}_J \mathrm{K}$, or the form with G_1' instead of G_1. It should be noted here that since B is of order 1, so is $\mathrm{pr}_J B$, so it *does not contribute to the principal symbol* of the linear problem (the problem with $\delta = 0$). More precisely, $\mathrm{pr}_J B$ is the sum of a pseudo-differential operator and a singular Green operator:

$$\text{(4.39)}\qquad \begin{aligned} \mathrm{pr}_J B &= P_{B,\Omega} + G_{B,k}, \quad k = 1, 3, \quad \text{where} \\ P_B &= B + \mathrm{grad}\,\mathrm{OP}(|\xi|^{-2})\,\mathrm{div}\,B, \end{aligned}$$

P_B of order 1 and $G_{B,k}$ of order and class 1.

4.3. *Another Neumann problem.* The treatment of the other Neumann trace opeator T_3 is very similar to the treatment of T_1 (whereas we need some different arguments for T_0, T_2 and T_4), so we can explain it rapidly now.

Consider the Navier-Stokes problem (4.3_3) with

$$\text{(4.40)}\qquad T_3\begin{pmatrix} u \\ p \end{pmatrix} \equiv \gamma_1 u - \gamma_0 p\vec{n} = \varphi_3 \quad \text{on } S;$$

the data satisfying (4.3_3). The replacement of $\chi_1 u$ by $\gamma_1 u$ in the calculations of the preceding sections gives the boundary condition for p (compare with (4.7))

$$\text{(4.41)}\qquad \gamma_0 p = \gamma_1 u_\nu - \varphi_{3,\nu} \quad \text{on } S,$$

whereas the interior equation for p (4.33) is unchanged; so we now get

$$\text{(4.42)}\qquad p = R_D\,\mathrm{div}(Bu + \delta \mathrm{K} u - f) + K_D(\gamma_1 u_\nu - \varphi_{3,\nu}).$$

Then we define the singular Green operator G_3 and the functions f_3 and ψ_3 by

$$\text{(4.43)}\qquad G_3 u = \mathrm{grad}\,K_D \gamma_1 u_\nu, \quad f_3 = \mathrm{pr}_J f + \mathrm{grad}\,K_D \varphi_{3,\nu}, \quad \psi_3 = \varphi_{3,\tau}$$

(compare with (4.11)); and (4.31_3) reduces to

$$\text{(4.44)}\qquad \begin{aligned} &\text{(i)}\ \partial_t u - \Delta u + G_3 u + \mathrm{pr}_J(B + \delta \mathrm{K})u = f_3 && \text{in } Q, \\ &\text{(ii)}\qquad\qquad\qquad\qquad\qquad u|_{t=0} = u_0 && \text{on } \Omega, \\ &\text{(iii)}\qquad\qquad\qquad\qquad\qquad\quad T_3' u = \psi_3 && \text{on } S; \end{aligned}$$

where (cf. also (A.13))

$$\text{(4.45)}\qquad T_3' u = \gamma_1 u_r + \gamma_0(\mathrm{div}\,u)\vec{n}.$$

It is seen just as above that, conversely, (4.44) gives back (4.31_3) when f_3, u_0 and ψ_3 are given with $\mathrm{div}\,f_3 = 0$, $\mathrm{div}\,u_0 = 0$, $\psi_{3,\nu} = 0$. Note that, in block notation using (A.22),

$$\text{(4.46)}\qquad T_3' u = \begin{pmatrix} \gamma_1 u_\tau \\ \gamma_0\,\mathrm{div}\,u \end{pmatrix} = \gamma_1 u + \begin{pmatrix} 0 & 0 \\ \mathrm{div}_\Gamma' & (\mathrm{div}\,\vec{n}) \end{pmatrix}\begin{pmatrix} \gamma_0 u_\tau \\ \gamma_0 u_\nu \end{pmatrix}.$$

Again we observe that G_3 is of order 2 and class 2, so it is of interest in some connections to know that it can be replaced by a singular Green operator G_3' of order 2 and class 1 (compare with (4.24)), with a corresponding formula for p:

$$(4.47)\qquad \begin{aligned} G_3' &= -\operatorname{grad} K_D \operatorname{div}_T' \gamma_0, \\ p &= R_D \operatorname{div}(Bu + \delta \mathrm{K} u - f) - K_D \operatorname{div}_T' \gamma_0 u - K_D \varphi_{3,\nu}. \end{aligned}$$

This is seen just as in the case $k = 1$ by use of the condition $\gamma_0 \operatorname{div} u = 0$.

5. The reduction of the other problems.

5.1. *Reduction of Dirichlet and intermediate problems.* The boundary conditions with T_0, T_2 and T_4 differ from those with T_1 and T_3 by acting on u only, and they require a different analysis of p. Let us go directly to the problems containing possibly nonzero first order terms and nonlinear terms, (4.31_k), which we as usual want to reduce to the form (4.4_k), now for $k = 0, 2$ and 4. Let us also keep account of the smoothness properties right away. We consider, for $r \geqq 0$, and in addition $r + 2 \geqq n/2$ if $\delta = 1$,

$$(5.1)\qquad \begin{aligned} &u \in H^{r+2, r/2+1}(Q)^n, \\ &p \in H^{1,0}(Q) \text{ with } \operatorname{grad} p \in H^{r, r/2}(Q)^n, \\ &f \in H^{r, r/2}(Q)^n, \\ &\varphi_0 \in H^{r+3/2, r/2+3/4}(S)^n, \\ &\varphi_k \in H^{r+3/2, r/2+3/4}(S, \underline{F}_{\tau,\Gamma}) \times H^{r+1/2, r/2+1/4}(S, \underline{F}_{\nu,\Gamma}) \text{ for } k = 2, 4, \\ &u_0 \in H^{r+1}(\Omega)^n. \end{aligned}$$

cf. Proposition 2.2, (2.43) and (4.32); cf. also (A.10).

Assume that u satisfies (4.31_k), with data satisfying (4.3_k), for $k = 0, 2$ or 4. The only information we now have on p comes from the first line:

$$(5.2)\qquad \begin{aligned} -\operatorname{grad} p = \partial_t u - \Delta u + Bu + \delta \mathrm{K} u - f = \partial_t u + g, \\ \text{where we set } g = -\Delta u + Bu + \delta \mathrm{K} u - f. \end{aligned}$$

We deduce both a differential equation and a boundary condition from this by applying div resp. γ_ν, using that $\operatorname{div} \Delta u = \operatorname{div} \partial_t u = 0$ and $\gamma_\nu \partial_t u = 0$:

$$(5.3)\qquad \begin{aligned} -\Delta p = \operatorname{div} g = \operatorname{div}(Bu + \delta \mathrm{K} u - f) \quad &\text{in } Q, \\ \gamma_1 p = \gamma_\nu \operatorname{grad} p = -\gamma_\nu g \quad &\text{on } S. \end{aligned}$$

Then p is the solution of a Neumann problem, cf. also (2.55),

$$(5.4)\qquad p = (R_N \operatorname{div} - K_N \gamma_\nu) g = \tilde{G} g,$$

uniquely determined when we pose the side condition

$$(5.5)\qquad \int_I |(\gamma_0 p, 1)_\Gamma|^2 \, dt = 0.$$

When $r \geqq 1$, $p \in H^{2,0}(Q)$ and $g \in H^{1,0}(Q)$, so the Neumann problem (5.3) and the solution (5.4) have a straightforward meaning.

When $r \in [0, 1[$, so that g is possibly just in $L_2(Q)$, the Neumann problem and (5.4) are understood in a more general sense, as in Theorem 2.6. Indeed, (5.2) expresses that g has the orthogonal decomposition

$$g = -\partial_t u - \operatorname{grad} p, \quad \partial_t u \in J_0(Q) \text{ and } \operatorname{grad} p \in G(Q);$$

and from this formula, p is determined (uniquely in view of (5.5)) as $p = \tilde{G}g$, cf. Theorem 2.6. Of course, the latter considerations are valid for all $r \geqq 0$.

Let us study the formula for p some more. Since $\operatorname{div} \Delta u = 0$, we have in view of (2.54),

$$(5.6) \quad p = \tilde{G}(-\Delta u + Bu + \delta \mathsf{K} u - f) = \tilde{G}(Bu + \delta \mathsf{K} u - f) + K_N \gamma_\nu \operatorname{pr}_J \Delta u.$$

By the calculus, $K_N \gamma_\nu \operatorname{pr}_J \Delta$ is a singular Green operator of order 1, and it is *of class 2*, since it is well-defined on $H^2(\Omega)^n$ (and on $H^{2,1}(Q)^n$), hence it involves only normal derivatives of order $\leqq 1$. This is not clear from its form, so we want to replace it by an expression showing this, before we go on. Using that $\operatorname{div} u = 0$ and $\gamma_\nu u = 0$, we write

$$(5.7) \quad \begin{aligned} K_N \gamma_\nu \operatorname{pr}_J \Delta u &= K_N \gamma_\nu \Delta u = K_N(\gamma_\nu \Delta u - \gamma_1 \operatorname{div} u) \\ &= -K_N \operatorname{div}'_\Gamma \gamma_1 u_\tau + K_N A'_\Gamma \gamma_0 u_\tau, \end{aligned}$$

where we used (A.27) in the passage to the last expression. Then we get the formula for p:

$$(5.8) \quad p = \tilde{G}(Bu + \delta \mathsf{K} u - f) - K_N \operatorname{div}'_\Gamma \gamma_1 u_\tau + K_N A'_\Gamma \gamma_0 u_\tau.$$

Insertion of (5.8) in (4.31_k i) gives, since $I + \operatorname{grad} \tilde{G} = \operatorname{pr}_{J_0}$,

$$(5.9) \quad \partial_t u - \Delta u + \operatorname{pr}_{J_0}(Bu + \delta \mathsf{K} u) + \operatorname{grad}(-K_N \operatorname{div}'_\Gamma \gamma_1 u_\tau + K_N A'_\Gamma \gamma_0 u_\tau) = \operatorname{pr}_{J_0} f;$$

and we therefore now define

$$(5.10) \quad G_0 u = G_2 u = G_4 u = -\operatorname{grad} K_N \operatorname{div}'_\Gamma \gamma_1 u_\tau + \operatorname{grad} K_N A'_\Gamma \gamma_0 u_\tau.$$

For convenience, we list some of the relevant continuity properties (derivable from Proposition 2.2 and the examples).

LEMMA 5.1. *The operators $K_N \operatorname{div}'_\Gamma$ and $K_N A'_\Gamma$ are Poisson operators of order 0 and define continuous mappings*

$$(5.11) \quad K_N \operatorname{div}'_\Gamma, K_N A'_\Gamma : H^{r+1/2, r/2+1/4}(S_T)^n \to H^{r+1, r/2+1/4}(Q_T) \quad \textit{for } r \geqq 0.$$

The operators $\operatorname{grad} K_N \operatorname{div}'_\Gamma$ *and* $\operatorname{grad} K_N A'_\Gamma$ *are Poisson operators of order 1 and define continuous mappings*

(5.12) $\operatorname{grad} K_N \operatorname{div}'_\Gamma, \operatorname{grad} K_N A'_\Gamma\colon H^{r+1/2,r/2+1/4}(S_T)^n \to H^{r,r/2}(Q_T)^n$ *for* $r \geqq 0$.

The operators $\operatorname{grad} K_N \operatorname{div}'_\Gamma \gamma_1 \operatorname{pr}_{F_\tau}$ *and* $\operatorname{grad} K_N A'_\Gamma \gamma_0 \operatorname{pr}_{F_\tau}$ *are singular Green operators of order and class* 2, *resp. order and class* 1, *and define continuous mappings*

$$\begin{aligned} &\operatorname{grad} K_N \operatorname{div}'_\Gamma \gamma_1 \operatorname{pr}_{F_\tau}\colon H^{r+2,r/2+1}(Q)^n \to H^{r,r/2}(Q)^n \quad \textit{for } r \geqq 0, \\ &\operatorname{grad} K_N A'_\Gamma \gamma_0 \operatorname{pr}_{F_\tau}\colon H^{r+1,r/2+1/2}(Q)^n \to H^{r,r/2}(Q)^n \quad \textit{for } r \geqq 0. \end{aligned} \tag{5.13}$$

In particular, $G_0 = G_2 = G_4 = \operatorname{grad} K_N(-\operatorname{div}'_\Gamma \gamma_1 + A'_\Gamma \gamma_0)\operatorname{pr}_{F_\tau}$ *is of order and class* 2.

We define the reduced trace operators simply as equal to the given ones (but now written with u only):

$$T'_0 u = \gamma_0 u, \quad T'_2 u = \begin{pmatrix} (\chi_1 u)_\tau \\ \gamma_\nu u \end{pmatrix}, \quad T'_4 u = \begin{pmatrix} \gamma_1 u_\tau \\ \gamma_\nu u \end{pmatrix}; \tag{5.14}$$

and set

$$\psi_k = \varphi_k, \quad f_k = \operatorname{pr}_{J_0} f, \quad \text{for } k = 0, 2, 4; \tag{5.15}$$

then we have altogether transformed (4.31_k) to the system

$$\begin{aligned} &\text{(i)} \ \partial_t u - \Delta u + G_k u + \operatorname{pr}_{J_0}(B + \delta \mathsf{K})u = f_k && \text{in } Q, \\ &\text{(ii)} \qquad \operatorname{div} u = 0 && \text{in } Q, \\ &\text{(iii)} \qquad u|_{t=0} = u_0 && \text{on } \Omega, \\ &\text{(iv)} \qquad T'_k u = \psi_k && \text{on } S; \end{aligned} \tag{5.16$_k$}$$

for $k = 0, 2, 4$. (In the references [G-S1] and [G-S3], G_k is kept on the equivalent form $G = \operatorname{grad} K_N(\gamma_\nu \Delta - \gamma_1 \operatorname{div})$, cf. (5.7) above.)

It is easy to check, conversely, that if data are given satisfying (4.3_k), and f_k and φ_k are defined from them by (5.15), then when u solves (5.16_k) and p is defined by (5.8), it follows that $\{u, p\}$ solves (4.31_k) ($k = 0, 2, 4$).

The fully reduced system is now simply (5.16_k) with the divergence equation removed:

$$\begin{aligned} &\text{(i)} \ \partial_t u - \Delta u + G_k u + \operatorname{pr}_{J_0}(B + \delta \mathsf{K})u = f_k && \text{in } Q, \\ &\text{(ii)} \qquad u|_{t=0} = u_0 && \text{in } \Omega, \\ &\text{(iii)} \qquad T'_k u = \psi_k && \text{in } S. \end{aligned} \tag{5.17$_k$}$$

This can be considered for general data:

$$\begin{aligned} &f_k \in H^{r,r/2}(Q)^n, \\ &\psi_0 \in H^{r+3/2,r/2+3/4}(S)^n, \\ &\psi_k \in H^{r+3/2,r/2+3/4}(S, \underline{F}_{\tau,\Gamma}) \times H^{r+1/2,r/2+1/4}(S, \underline{F}_{\nu,\Gamma}) \text{ for } k = 2, 4, \end{aligned} \tag{5.18}$$

and we shall show in Section 6 that the linear system (5.17_k) (with $\delta = 0$) is parabolic, hence uniquely solvable in suitable spaces, as accounted for in [G4], [G-S4].

We shall now show that when u solves (5.17_k) for a set of data with $f_k \in J_0(Q)$, $u_0 \in J_0(\Omega)$ and $\psi_{k,v} = 0$ (and the relevant parts of (5.1), (5.18)), then $\operatorname{div} u = 0$. Set

$$\eta = -\operatorname{div}'_\Gamma \gamma_1 u_\tau + A'_\Gamma \gamma_0 u_\tau, \tag{5.19}$$

and note that by (A.27) and (A.31), $\langle \eta, 1 \rangle_\Gamma = 0$ for $t \in I$, since $\gamma_v u = 0$, so that K_N applies to η as a precise solution operator of the Neumann problem, cf. Example 2.3. Thus

$$\begin{aligned} \operatorname{div} G_k u &= \operatorname{div} \operatorname{grad} K_N \eta = 0, \\ \gamma_v G_k u &= \gamma_v \operatorname{grad} K_N \eta = \gamma_1 K_N \eta = \eta. \end{aligned} \tag{5.20}$$

Application of div to (5.17_ki) then gives

$$\begin{aligned} 0 &= \partial_t \operatorname{div} u - \Delta \operatorname{div} u + \operatorname{div} G_k u + \operatorname{div} \operatorname{pr}_{J_0}(Bu + \delta \mathrm{K} u) - \operatorname{div} f_k \\ &= \partial_t \operatorname{div} u - \Delta \operatorname{div} u. \end{aligned} \tag{5.21}$$

If $r \geqq 1$, we moreover easily find, by application of div to (5.17_kii) and γ_v to (5.17_k i):

$$\begin{aligned} &\text{(i)}\ \ 0 = \operatorname{div} u|_{t=0}, \\ &\text{(ii)}\ 0 = \partial_t \gamma_v u - \gamma_v \Delta u + \gamma_v G_k u + \gamma_v \operatorname{pr}_{J_0}(Bu + \delta \mathrm{K} u) - \gamma_v f_k \\ &\qquad = (-\gamma_1 \operatorname{div} u + \operatorname{div}'_\Gamma \gamma_1 u_\tau - A'_\Gamma \gamma_0 u_\tau) + (-\operatorname{div}'_\Gamma \gamma_1 u_\tau + A'_\Gamma \gamma_0 u_\tau) \\ &\qquad = -\gamma_1 \operatorname{div} u, \end{aligned} \tag{5.22}$$

where we rewrote $\gamma_v \Delta u$ by use of (A.27). This shows that $\operatorname{div} u$ solves the heat equation with Neumann boundary condition and all data equal to zero,

$$\begin{aligned} \partial_t \operatorname{div} u - \Delta \operatorname{div} u &= 0 \quad \text{in } Q, \\ \operatorname{div} u|_{t=0} &= 0 \quad \text{on } \Omega, \\ \gamma_1 \operatorname{div} u &= 0 \quad \text{on } S; \end{aligned} \tag{5.23}$$

so it follows that $\operatorname{div} u = 0$.

When $r \in [0, 1[$, we define $\tilde{v}$ as the extension of $v = \operatorname{div} u$ by 0 for $t < 0$; then Lemma 4.1 shows that $\tilde{v} \in C^0(]-\infty, b];\ L_2(\Omega))$ with $(\partial_t - \Delta)\tilde{v} = 0$ in $]-\infty, b[\ \times \Omega$. Moreover, the calculation in (5.22ii) is still valid, in view of the smoothness of $\partial_v^2 \operatorname{div} u$ that follows from (5.21) ("partial hypoellipticity at the boundary"), and it implies that $\gamma_1 \tilde{v} = 0$. Thus $\tilde{v}$ is a solution of the generalized Neumann problem

$$\begin{aligned} \partial_t \tilde{v} - \Delta \tilde{v} &= 0 \quad \text{in }]-\infty, b[\ \times \Omega, \\ \gamma_1 \tilde{v} &= 0 \quad \text{on }]-\infty, b[\ \times \Gamma, \end{aligned} \tag{5.24}$$

that has uniqueness in view of [Pi, Th. (32) ii]; thus $\tilde{v} = 0$, so finally $\operatorname{div} u = 0$.
Altogether we have obtained:

THEOREM 5.2. *Consider the problem* (4.31_k), $k = 0, 2$ *or* 4, *for functions in the spaces* (5.1), (5.18) *for some* $r \geqq 0$; *assume in addition* $r + 2 \geqq n/2$ *if* $\delta = 1$.

1° *Let* f, u_0 *and* φ_k *be given, satisfying* (4.3_k). *Define* G_k, f_k, T_k' *and* ψ_k *by* (5.10), (5.14) *and* (5.15). *If* $\{u, p\}$ *is a solution of* (4.31_k), *then* p *satisfies* (5.8) *(under the condition* (5.5)*), and* u *is a solution of* (5.16_k). *Conversely, if* u *solves* (5.16_k) *and* p *is defined by* (5.8), *then* $\{u, p\}$ *solves* (4.31_k).

2° *Let* f_k, u_0 *and* ψ_k *be given. When* u *solves* (5.16_k), *then it solves* (5.17_k). *Conversely, when* u *solves* (5.17_k), *and* $f_k \in J_0(Q)$, $u_0 \in J_0(\Omega)$ *and* $\psi_{k,\nu} = 0$, *then* u *solves* (5.16_k).

5.2. *Additional observations and summary.* It is of interest for the application of the systematic theory to see whether one can find another formulation in this set-up with a singular Green operator of lower class, as in the cases $k = 1$ and 3. With this in mind, we observe that the boundary data enter in the formulas for p and G_k in all three cases, in different ways. In fact, when $\{u, p\}$ is a solution of (4.31_k) we can write (recalling that $\varphi_k = \varphi_{k,\tau}$ here),

$$\begin{aligned}
&\text{for } k = 0,\ p = -K_N \operatorname{div}_\Gamma' \gamma_1 u_\tau + K_N A_\Gamma' \varphi_0 + \tilde{G}(Bu + \delta \mathsf{K} u - f),\\
&\text{for } k = 2,\ p = -K_N \operatorname{div}_\Gamma' \gamma_1 u_\tau + K_N A_\Gamma' \gamma_0 u_\tau + \tilde{G}(Bu + \delta \mathsf{K} u - f),\\
&\qquad\qquad\ \ = -K_N \operatorname{div}_\Gamma'((\chi_1 u)_\tau + s_0 \gamma_0 u_\tau) + K_N A_\Gamma' \gamma_0 u_\tau + \tilde{G}(Bu + \delta \mathsf{K} u - f)\\
&\qquad\qquad\ \ = -K_N \operatorname{div}_\Gamma' \varphi_{2,\tau} + K_N A_\Gamma'' \gamma_0 u_\tau + \tilde{G}(Bu + \delta \mathsf{K} u - f),\\
&\text{for } k = 4,\ p = -K_N \operatorname{div}_\Gamma' \varphi_{4,\tau} + K_N A_\Gamma' \gamma_0 u_\tau + \tilde{G}(Bu + \delta \mathsf{K} u - f),
\end{aligned} \tag{5.25}$$

where we have used (3.20), and introduced the notation

$$A'' v = A' v - \operatorname{div}(s_0 v), \tag{5.26}$$

defining yet another tangential first order operator that vanishes when $\vec{n}$ is constant. Accordingly, the first line in (5.16_k) can, by use of the boundary condition, be replaced by

$$\partial_t u - \Delta u + G_k' u + \operatorname{pr}_{J_0}(B + \delta \mathsf{K}) u = f_k' \quad \text{for } k = 0, 2, 4, \tag{5.27}$$

where

$$\begin{aligned}
G_0' u &= -\operatorname{grad} K_N \operatorname{div}_\Gamma' \gamma_1 u_\tau, & f_0' &= f - \operatorname{grad} K_N A_\Gamma' \varphi_0;\\
G_2' u &= \operatorname{grad} K_N A_\Gamma'' \gamma_0 u_\tau, & f_2' &= f + \operatorname{grad} K_N \operatorname{div}_\Gamma' \varphi_2;\\
G_4' u &= \operatorname{grad} K_N A_\Gamma' \gamma_0 u_\tau, & f_4' &= f + \operatorname{grad} K_N \operatorname{div}_\Gamma' \varphi_4.
\end{aligned} \tag{5.28}$$

Here G_0' is again of order and class 2 (and we do not expect that the class can be reduced further in the present treatment of the Dirichlet problem), whereas both G_2' and G_4' are *of order 1 and class 1.* As we shall see later (cf. Theorems 7.6 and

7.7), p has in these last cases more smoothness than in general, when $\varphi_k = 0$ (and $\operatorname{div} f = 0$).

In particular, G_2' and G_4' vanish altogether, when $\Omega = \mathbf{R}^n_+$, since A' and A'' are zero then.

We remark that when the formulas (5.27), (5.28) are used, K_N may have to be applied to functions ψ that do not satisfy $\langle \psi, 1 \rangle_\Gamma = 0$, so one needs the slightly more general definition explained in Example 2.1; but this is of no importance in smoothness questions.

Concerning the contribution from B, we note that $\mathrm{pr}_{J_0} B$ is, like $\mathrm{pr}_J B$, the sum of a ps.d.o. and a s.g.o.:

$$\mathrm{pr}_{J_0} B = P_{B,\Omega} + G_{B,k}, \quad k = 0, 2, 4, \tag{5.30}$$

where P_B is as in (4.39) and $G_{B,k}$ is of order and class 1.

Let us collect the results of this and the preceding section in a theorem. For the systematic formulation, we define

$$\begin{aligned} J_k &= J, \quad \mathrm{pr}_{J_k} = \mathrm{pr}_J, \quad \text{for } k = 1, 3, \\ J_k &= J_0, \quad \mathrm{pr}_{J_k} = \mathrm{pr}_{J_0}, \quad \text{for } k = 0, 2, 4. \end{aligned} \tag{5.31}$$

Moreover, we formulate the boundary conditions in a unified way by introducing the bundle notation (using that bundles can have fiber dimension zero):

$$\begin{aligned} &\text{for } k = 0, \quad F_{00} = \Gamma \times \mathbf{C}^n, \quad F_{01} = \Gamma \times \{0\}, \\ &\text{for } k = 1, \quad F_{10} = \Gamma \times \{0\}, \quad F_{11} = \Gamma \times \mathbf{C}^n, \\ &\text{for } k = 2, \quad F_{20} = F_{\nu,\Gamma}, \quad F_{21} = F_{\tau,\Gamma}, \\ &\text{for } k = 3, \quad F_{30} = \Gamma \times \{0\}, \quad F_{31} = \Gamma \times \mathbf{C}^n, \\ &\text{for } k = 4, \quad F_{40} = F_{\nu,\Gamma}, \quad F_{41} = F_{\tau,\Gamma}; \end{aligned} \tag{5.32}$$

cf. (A.10) ff. In all cases, $F_{k0} \oplus F_{k1} = \Gamma \times \mathbf{C}^n = E|_\Gamma$. The liftings to bundles over $S = \Gamma \times I$ are as usual called $\underline{F}_{k0}$ and $\underline{F}_{k1}$. Of course, the bundle notation is only really necessary in the intermediate cases $k = 2$ and 4; but it is practical for the systematic formulation of the results involving trace operators of different orders. In accordance with (5.32), we write trace operators in two components

$$T_k \begin{pmatrix} u \\ p \end{pmatrix} = \left\{ T_{k0} \begin{pmatrix} u \\ p \end{pmatrix}, T_{k1} \begin{pmatrix} u \\ p \end{pmatrix} \right\}, \quad T_k' u = \{ T_{k0}' u, T_{k1}' u \} \tag{5.33}$$

(they should be regarded as column vectors), where T_{k0}' is of order 0 and maps into sections in $\underline{F}_{k0}$, and T_{k1}' is of order 1, mapping into sections in $\underline{F}_{k1}$. In details, (5.34)

$$T_{00} \begin{pmatrix} u \\ p \end{pmatrix} = \gamma_0 u, \quad T_{01} \begin{pmatrix} u \\ p \end{pmatrix} = 0, \qquad T_{00}' u = \gamma_0 u, \quad T_{01}' u = 0,$$

$$T_{10}\begin{pmatrix}u\\p\end{pmatrix}=0,\quad T_{11}\begin{pmatrix}u\\p\end{pmatrix}=\chi_1 u-p\vec{n},\quad T'_{10}u=0,\quad T'_{11}u=(\chi_1 u)_\tau+\gamma_0(\operatorname{div}u)\vec{n},$$

$$T_{20}\begin{pmatrix}u\\p\end{pmatrix}=\gamma_\nu u,\quad T_{21}\begin{pmatrix}u\\p\end{pmatrix}=(\chi_1 u)_\tau,\quad T'_{20}u=\gamma_0 u_\nu,\quad T'_{21}u=(\chi_1 u)_\tau,$$

$$T_{30}\begin{pmatrix}u\\p\end{pmatrix}=0,\quad T_{31}\begin{pmatrix}u\\p\end{pmatrix}=\gamma_1 u-p\vec{n},\quad T'_{30}u=0,\quad T'_{31}u=\gamma_1 u_\tau+\gamma_0(\operatorname{div}u)\vec{n},$$

$$T_{40}\begin{pmatrix}u\\p\end{pmatrix}=\gamma_\nu u,\quad T_{41}\begin{pmatrix}u\\p\end{pmatrix}=\gamma_1 u_\tau,\quad T'_{40}u=\gamma_\nu u,\quad T'_{41}u=\gamma_1 u_\tau,$$

The boundary data φ_k and ψ_k are decomposed accordingly:

(5.35) $$\varphi_k=\{\varphi_{k0},\varphi_{k1}\},\quad \psi_k=\{\psi_{k0},\psi_{k1}\},\quad \text{sections in } \underline{F}_{k0}\oplus\underline{F}_{k1}.$$

The singular Green operators G_k and G'_k are defined by

(5.36) $$\begin{aligned}
&G_0u=\operatorname{grad}K_N(-\operatorname{div}'_\Gamma\gamma_1u_\tau+A'_\Gamma\gamma_0u_\tau) && G'_0u=-\operatorname{grad}K_N\operatorname{div}'_\Gamma\gamma_1u_\tau,\\
&G_1u=2\operatorname{grad}K_D\gamma_1u_\nu, && G'_1u=-2\operatorname{grad}K_D\operatorname{div}'_\Gamma\gamma_0u,\\
&G_2u=G_0u, && G'_2u=\operatorname{grad}K_NA''_\Gamma\gamma_0u_\tau,\\
&G_3u=\operatorname{grad}K_D\gamma_1u_\nu, && G'_3u=-\operatorname{grad}K_D\operatorname{div}'_\Gamma\gamma_0u,\\
&G_4u=G_0u, && G'_4u=\operatorname{grad}K_NA'_\Gamma\gamma_0u_\tau;
\end{aligned}$$

here G_k is of order 2 and class 2 for $k=0,1,2,3,4$ and so is G'_0, whereas G'_1 and G'_3 are of order 2 and class 1, and G'_2 and G'_4 are of order 1 and class 1. The relations between the old data f, φ_k and the new data f_k, f'_k, ψ_k are

(5.37) $$\begin{aligned}
&f_0=\operatorname{pr}_{J_0}f,\ \ f'_0=\operatorname{pr}_{J_0}f-\operatorname{grad}K_NA'_\Gamma\varphi_0, && \psi_0=\varphi_0,\\
&f_1=f'_1=\operatorname{pr}_Jf+\operatorname{grad}K_D\varphi_{1,\nu}, && \psi_1=\varphi_{1,\tau},\\
&f_2=\operatorname{pr}_{J_0}f,\ \ f'_2=\operatorname{pr}_{J_0}f+\operatorname{grad}K_N\operatorname{div}'_\Gamma\varphi_{2,\tau}, && \psi_2=\varphi_2,\\
&f_3=f'_3=\operatorname{pr}_Jf+\operatorname{grad}K_D\varphi_{3,\nu}, && \psi_3=\varphi_{3,\tau},\\
&f_4=\operatorname{pr}_{J_0}f,\ \ f'_4=\operatorname{pr}_{J_0}f+\operatorname{grad}K_N\operatorname{div}'_\Gamma\varphi_{4,\tau}, && \psi_4=\varphi_4.
\end{aligned}$$

Note that $f_k\in J_k$ in all cases (but not necessarily $f'_k\in J_k$), when f_k is defined from f and φ_k in this way. The formulas for p are

(5.38) $$\begin{aligned}
p&=-K_N\operatorname{div}'_\Gamma\gamma_1u_\tau+K_NA'_\Gamma\gamma_0u_\tau+p_k(u,f)\\
&=-K_N\operatorname{div}'_\Gamma\gamma_1u_\tau+K_NA'_\Gamma\varphi_0+p_k(u,f), && \text{for } k=0,\\
p&=2K_D\gamma_1u_\nu-K_D\varphi_{1,\nu}+p_k(u,f)\\
&=-2K_D\operatorname{div}'_\Gamma\gamma_0u-K_D\varphi_{1,\nu}+p_k(u,f), && \text{for } k=1,\\
p&=-K_N\operatorname{div}'_\Gamma\gamma_1u_\tau+K_NA'_\Gamma\gamma_0u_\tau+p_k(u,f)\\
&=-K_N\operatorname{div}'_\Gamma\varphi_{2,\tau}+K_NA''_\Gamma\gamma_0u_\tau+p_k(u,f), && \text{for } k=2,\\
p&=K_D\gamma_1u_\nu-K_D\varphi_{3,\nu}+p_k(u,f)\\
&=-K_D\operatorname{div}'_\Gamma\gamma_0u-K_D\varphi_{3,\nu}+p_k(u,f), && \text{for } k=3,\\
p&=-K_N\operatorname{div}'_\Gamma\gamma_1u_\tau+K_NA'_\Gamma\gamma_0u_\tau+p_k(u,f),\\
&=-K_N\operatorname{div}'_\Gamma\varphi_{4,\tau}+K_NA'_\Gamma\gamma_0u_\tau+p_k(u,f), && \text{for } k=4;
\end{aligned}$$

where

$$(5.39)\qquad \begin{aligned} p_k(u,f) &= \tilde{G}(Bu + \delta \mathsf{K}u - f), \quad \text{for } k = 0,2,4\\ p_k(u,f) &= R_D \operatorname{div}(Bu + \delta \mathsf{K}u - f), \quad \text{for } k = 1,3; \end{aligned}$$

here div Bu vanishes if B has constant, scalar coefficients. We write moreover

$$(5.40_k)\qquad \begin{aligned} \operatorname{pr}_{J_k} B = P_{B,\Omega} + G_{B,k} \quad &\text{with } P_B = B + \operatorname{grad} OP(|\xi|^{-2})\operatorname{div};\\ &\mathsf{Q}_k = \operatorname{pr}_{J_k}\mathsf{K}. \end{aligned}$$

THEOREM 5.3. *Consider the general Stokes and Navier-Stokes problems (with $\delta = 0$ resp. $\delta = 1$):*

$$(5.41_k)\qquad \begin{aligned} &\text{(i)}\ \partial_t u - \Delta u + Bu + \delta \mathsf{K}u + \operatorname{grad} p = f &&\text{in } Q,\\ &\text{(ii)}\qquad \operatorname{div} u = 0 &&\text{in } Q,\\ &\text{(iii)}\qquad u|_{t=0} = u_0 &&\text{on } \Omega,\\ &\text{(iv)}\qquad T_k\begin{pmatrix} u\\ p\end{pmatrix} = \varphi_k &&\text{on } S; \end{aligned}$$

for $k = 0, 1, 2, 3, 4$. The functions are taken in the spaces

$$(5.42_k)\qquad \begin{aligned} &u \in H^{r+2,r/2+1}(Q)^n,\\ &p \in H^{1,0}(Q) \text{ with } \operatorname{grad} p \in H^{r,r/2}(Q)^n,\\ &f, f_k, f_k' \in H^{r,r/2}(Q)^n,\\ &\varphi_k, \psi_k \in H^{r+3/2,r/2+3/4}(S, \underline{E}_{k0}) \times H^{r+1/2,r/2+1/4}(S, \underline{E}_{k1}),\\ &u_0 \in H^{r+1}(\Omega)^n; \end{aligned}$$

for some $r \geqq 0$, with $r + 2 \geqq n/2$ in case $\delta = 1$.

1° *Let $\{u, p\}$ be a solution of* (5.41$_k$) *with data satisfying* (5.37$_k$) *and*

$$(5.43_k)\qquad u_0 \in J_k(\Omega) \text{ in all cases,} \quad \varphi_\nu = 0 \text{ if } k = 0, 2 \text{ or } 4.$$

Then p (assumed to satisfy $\int_I |(\gamma_0 p, 1)_{L_2(\Gamma)}|^2\, dt = 0$ if $k = 0,2$ or 4) is determined from the other entries by the formulas (5.38), (5.39).

2° *If $\{u, p\}$ solves* (5.41$_k$) *with* (5.43$_k$), *then u solves the problem*

$$(5.44_k)\qquad \begin{aligned} &\text{(i)}\ \partial_t u + M_k u + \delta \mathsf{Q}_k u = f_k &&\text{in } Q,\\ &\text{(ii)}\qquad T_k' u = \psi_k &&\text{on } S,\\ &\text{(iii)}\qquad u|_{t=0} = u_0 &&\text{on } \Omega, \end{aligned}$$

with f_k and ψ_k defined by (5.37) *and (cf.* (5.36) *and* (5.40$_k$)*)*

$$(5.45_k)\qquad M_k = -\Delta + G_k + \operatorname{pr}_{J_k} B;$$

and it also solves the corresponding problem with G_k and f_k replaced by G_k' and f_k'.

3° *Conversely, let u be given as a solution of* (5.44$_k$) *with M_k as indicated under* 2° *and the data satisfying*

$$\begin{aligned} &f_k \in J_k(Q), \quad u_0 \in J_k(\Omega), \textit{for all } k, \\ &\psi_{k,\nu} = 0 \textit{ if } k = 0, 2 \textit{ or } 4. \end{aligned} \tag{5.46}$$

Then if f and φ_k are chosen according to (5.37) and (5.43$_k$), and p is defined by (5.38), then $\{u, p\}$ solves (5.41$_k$).

Note that when the problems for $k = 0, 2, 4$, are considered with zero boundary data, then $f_k = f_k'$, and G_k and G_k' can be used interchangeably, as for $k = 1$ and 3.

For later reference we observe that the reduced trace operators are all of the form

$$T_k' = \gamma_0 P_k, \tag{5.47$_k$}$$

where the P_k are differential operators with respect to x defined in the neighborhood Σ of Γ, namely

$$\begin{aligned} P_0 u = u, \quad P_1 u = (S(u)\vec{n})_\tau + (\operatorname{div} u)\vec{n}, \quad P_2 u = (S(u)\vec{n})_\tau + u_\nu \vec{n}, \\ P_3 u = \partial_\nu u_\tau + (\operatorname{div} u)\vec{n}, \quad P_4 u_\tau = \partial_\nu u_\tau + u_\nu \vec{n}. \end{aligned} \tag{5.48}$$

We can also write this with P_k decomposed into its zero order and first order part, $P_k = \{P_{k0}, P_{k1}\}$, where

$$\begin{aligned} &P_{00} u = u, && P_{01} u = 0, \\ &P_{10} u = 0, && P_{11} u = (S(u)\vec{n})_\tau + (\operatorname{div} u)\vec{n}, \\ &P_{20} u = u_\nu, && P_{21} u = (S(u)\vec{n})_\tau, \\ &P_{30} u = 0, && P_{31} u = \partial_\nu u_\tau + (\operatorname{div} u)\vec{n}, \\ &P_{40} u = u_\nu, && P_{41} u = \partial_\nu u_\tau; \end{aligned} \tag{5.49}$$

mapping into the bundles over Σ whose restrictions to Γ equal F_{kl}; we call them F_{kl} again. (Each F_{kl} is one of the bundles $\Sigma \times \{0\}$, $\Sigma \times \mathbf{C}^n$, F_τ or F_ν, cf. (A.10).) Then $T_{kl}' = \gamma_0 P_{kl}'$.

6. Parabolicity.

We shall now show that all the linear problems of the form (5.44$_k$) (with $\delta = 0$) introduced in Sections 4 and 5 are *parabolic* (with positive regularity) in the sense of Grubb [G4], so that the solvability theory developed there and in [G-S4] is directly applicable. For this we consider the system

$$\mathscr{A}_{k,\mu} = \begin{pmatrix} M_k + \mu^2 e^{i\theta} \\ T_{k0}' \\ T_{k1}' \end{pmatrix} \tag{6.1}$$

and its principal interior symbol (the principal symbol of the pseudo-differential part) and principal boundary symbol operator, taking their dependence on μ into account. M_k is the sum of a ps.d.o and a s.g.o. of order 2. Since B, and hence

$\mathrm{pr}_{J_k} B$, are of order 1, they do not contribute to the principal symbols, so M_k has the same principal part as the operator $-\Delta + G_k$, resp. $-\Delta + G'_k$, for each k. (Note moreover that since G'_2 and G'_4 are of order 1, they do not contribute either.) So in all cases, the principal interior μ-dependent symbol is

$$p^0(x, \xi, \mu) = (|\xi|^2 + \mu^2 e^{i\theta}) I_{(n)}. \tag{6.2}$$

The principal boundary symbol operator is the "model operator" acting on $\mathbb{R}_+$:

(6.3)

$$a_k^0(x', \xi', \mu, D_n) = \begin{pmatrix} (|\xi'|^2 + \mu^2 e^{i\theta} + D_n^2) I_{(n)} + g_k^0(x', \xi', D_n) \\ t'_{k0}{}^0(x', \xi', D_n) \\ t'_{k1}{}^0(x', \xi', D_n) \end{pmatrix} : H^2(\mathbb{R}_+)^n \to \begin{matrix} L_2(\mathbb{R}_+)^n \\ \times \\ \mathbb{C}^{N_{k0}} \\ \times \\ \mathbb{C}^{N_{k1}} \end{matrix}$$

where N_{kl} is the fiber dimension of F_{kl} (cf. (5.32)); we denote by $a_k'^0$ the corresponding expression where g_k^0 is replaced by $g_k'^0$. Here we find, by calculations according to the formulas in (5.36) for the simple one-dimensional case where Ω is replaced by $\mathbb{R}_+$ (using that K_D resp. K_N have symbol-kernels $e^{-|\xi'|x_n}$ resp. $-|\xi'|^{-1}e^{-|\xi'|x_n}$):

$$g_0{}^0(\xi', D_n)u(x_n) = g_2{}^0(\xi', D_n)u(x_n) = g_4{}^0(\xi', D_n)u(x_n) = g_0'{}^0(\xi', D_n)u(x_n)$$

$$= \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} \frac{1}{|\xi'|} e^{-|\xi'|x_n} i\xi' \cdot \partial_n u'(0) = e^{-|\xi'|x_n} \begin{pmatrix} \dfrac{\xi'}{|\xi'|} \\ i \end{pmatrix} \xi' \cdot \partial_n u'(0),$$

$$g_2'{}^0(\xi', D_n)u(x_n) = g_4'{}^0(\xi', D_n)u(x_n) = 0,$$

(6.4)

$$g_1{}^0(\xi', D_n)u(x_n) = 2\begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} e^{-|\xi'|x_n} \partial_n u_n(0) = 2e^{-|\xi'|x_n} \begin{pmatrix} i\xi' \\ -|\xi'| \end{pmatrix} \partial_n u_n(0),$$

$$g_1'{}^0(\xi', D_n)u(x_n) = -2\begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} e^{-|\xi'|x_n} \sum_{j=1}^{n-1} i\xi_j u_j(0) = 2e^{-|\xi'|x_n} \begin{pmatrix} \xi' \\ i|\xi'| \end{pmatrix} \xi' \cdot u'(0),$$

$$g_3{}^0(\xi', D_n)u(x_n) = \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} e^{-|\xi'|x_n} \partial_n u_n(0) = e^{-|\xi'|x_n} \begin{pmatrix} i\xi' \\ -|\xi'| \end{pmatrix} \partial_n u_n(0),$$

$$g_3'{}^0(\xi', D_n)u(x_n) = -\begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} e^{-|\xi'|x_n} \sum_{j=1}^{n-1} i\xi_j u_j(0) = e^{-|\xi'|x_n} \begin{pmatrix} \xi' \\ i|\xi'| \end{pmatrix} \xi' \cdot u'(0).$$

We here recall that $u' = \{u_1, \ldots, u_{n-1}\}$. These formulas describe the symbols for $|\xi'| \geqq 1$, and for $|\xi'| \leqq 1$ we use them with $|\xi'|$ replaced by a smooth, positive extension. The formulas with $|\xi'|$ used unchanged for *all* $\xi' \neq 0$ define the so-called *strictly homogeneous boundary symbol operators* g_k^h resp. $g_k'^h$. All the symbols are independent of x'.

The "model operators" $t_{kl}'^0$ for the trace operators T_{kl}' are as follows (cf. (5.34)):

$$\begin{aligned}
&t_{00}'^0(\xi', D_n)u(x_n) = u(0), \quad t_{01}'^0(\xi', D_n)u(x_n) = 0,\\
&t_{10}'^0(\xi', D_n)u(x_n) = 0, \qquad t_{11}'^0(\xi', D_n)u(x_n) = \begin{pmatrix} \partial_n u'(0) + i\xi' u_n(0) \\ \partial_n u_n(0) + i\xi' \cdot u'(0) \end{pmatrix},\\
&t_{20}'^0(\xi', D_n)u(x_n) = u_n(0), \quad t_{21}'^0(\xi', D_n)u(x_n) = \partial_n u'(0) + i\xi' u_n(0),\\
&t_{30}'^0(\xi', D_n)u(x_n) = 0, \qquad t_{31}'^0(\xi', D_n)u(x_n) = \begin{pmatrix} \partial_n u'(0) \\ \partial_n u_n(0) + i\xi' \cdot u'(0) \end{pmatrix}\\
&t_{40}'^0(\xi', D_n)u(x_n) = u_n(0), \quad t_{41}'^0(\xi', D_n)u(x_n) = \partial_n u'(0).
\end{aligned} \tag{6.5}$$

All the entries in (6.5) are strictly homogeneous and smooth (so $t_{kl}'^0 = t_{kl}'^h$).

With the concepts of parameter-ellipticity and parabolicity defined in [G4] (Definitions 1.5.5 and 3.1.3 etc.), we now show:

THEOREM 6.1. *Let $k = 0, 1, 2, 3,$ or 4. Consider one of the systems $\mathscr{A}_{k,\mu}$ (6.1) with M_k and T_k' defined in Section 5.2. For each $\theta \in]-\pi, \pi[$ it is parameter-elliptic,* i.e. *satisfies:*

(I) *The principal interior symbol $p^0(x, \xi, \mu)$ is bijective for all x, all $(\xi, \mu) \in \bar{\mathsf{R}}_+^{n+1}$ with $|\xi|^2 + |\mu|^2 \geqq 1$.*

(II) *The principal boundary symbol operator $a_k^0(x', \xi', \mu, D_n)$ is a bijection for all x', all $(\xi', \mu) \in \bar{\mathsf{R}}_+^n$, with $|\xi'| \geqq 1$, $\mu \geqq 0$.*

(III) *For each $\mu > 0$, each x', the strictly homogeneous principal boundary symbol operator $a_k^h(x', \xi', \mu, D_n)$ (coinciding with a_k^0 for $|\xi'| \geqq 1$) converges in the operator norm for $\xi' \to 0$ to a limit operator*

$$a_k^h(x', 0, \mu, D_n): H^2(\mathsf{R}_+)^n \to L_2(\mathsf{R}_+)^n \times \mathsf{C}^{N_{k0}} \times \mathsf{C}^{N_{k1}}, \tag{6.6}$$

which is bijective.

Similar statements hold with a_k replaced by a_k'.

In particular, the systems $\{\partial_t + M_k, T_k'\}$ are parabolic.

PROOF. (I) is seen immediately from (6.2). Concerning (II) and (III), we can say very briefly, that the invertibility required in Condition (II) follows essentially from the solvability property shown in Lemma 3.2, when we *apply the reductions* explained in Sections 4–5 *to the model operators*; that Condition (III) is satisfied follows then from the *normality* of the boundary condition, in the same way as in [G4, Proposition 1.5.9].

The parabolicity means that (I)–(III) hold for $\theta \in [-\pi/2, \pi/2]$.

Let us also give a more self-contained and detailed explanation of the proof of (II) and (III). Consider a_k^0 defined in (6.3), and the analogous expressions for ${a'_k}^0$, a_k^h and ${a'_k}^h$.

If we let $\xi' \to 0$ in the strictly homogeneous expressions, we find that the operator $|\xi'|^2 + D_n^2$ converges to D_n^2 for $\xi' \to 0$, and moreover, for each k, that the singular Green terms go to zero,

$$(6.7)\quad \|g_k{}^h(x', \xi', D_n)\|_{H^2(\mathbf{R}_+)^n, L_2(\mathbf{R}_+)^n} \to 0, \quad \|{g'_k}^h(x', \xi', D_n)\|_{H^2(\mathbf{R}_+)^n, L_2(\mathbf{R}_+)^n} \to 0,$$

since they are $\mathcal{O}(|\xi'|^{1/2})$ (or better). For example:

$$(6.8)\quad \begin{aligned} \|g_1{}^h(x', \xi', D_n)u\|_{L_2(\mathbf{R}_+)^n} &= 2\|e^{-|\xi'|x_n}\|_{L_2(\mathbf{R}_+)^n}\sqrt{2}\,|\xi'|\,|\partial_n u_n(0)| \\ &= 2|\xi'|^{1/2}\,|\partial_n u_n(0)| \leqq c|\xi'|^{1/2}\,\|u_n\|_{H^2(\mathbf{R}_+)^n}; \\ \|{g'_1}^h(x', \xi', D_n)u\|_{L_2(\mathbf{R}_+)^n} &\leqq 2\|e^{-|\xi'|x_n}\|_{L_2(\mathbf{R}_+)^n}\sqrt{2}\,|\xi'|^2\,|u'(0)| \\ &= 2|\xi'|^{3/2}\,|u'(0)| \leqq c|\xi'|^{3/2}\,\|u'\|_{H^1(\mathbf{R}_+)^n}. \end{aligned}$$

(It is here that we use that the s.g.o.s are of class $\leqq 2$.) For the trace operators we have, when $\xi' \to 0$,

$$(6.9)\quad \begin{aligned} &{t'_0}^0 - \gamma_0 = 0, \\ &\|{t'_1}^0(\xi', D_n) - \gamma_1\|_{H^2(\mathbf{R}_+)^n, \mathbf{C}^n} \to 0, \\ &\|{t'_{20}}^0(\xi', D_n) - \gamma_0\|_{H^2(\mathbf{R}_+), \mathbf{C}^n} \to 0, \quad \|{t'_{21}}^0(\xi', D_n) - \gamma_1\|_{H^2(\mathbf{R}_+)^{n-1}, \mathbf{C}^{n-1}} \to 0, \\ &\|{t'_3}^0(\xi', D_n) - \gamma_1\|_{H^2(\mathbf{R}_+)^n, \mathbf{C}^n} \to 0, \\ &\|{t'_{40}}^0(\xi', D_n) - \gamma_0\|_{H^2(\mathbf{R}_+), \mathbf{C}^n} \to 0, \quad \|{t'_{41}}^0(\xi', D_n) - \gamma_1\|_{H^2(\mathbf{R}_+)^{n-1}, \mathbf{C}^{n-1}} \to 0, \end{aligned}$$

since the differences are $\mathcal{O}(|\xi'|)$; for example,

$$(6.10)\quad |{t'_1}^0(\xi', D_n)u - \gamma_1 u| = \left|\begin{pmatrix} i\xi' u_n(0) \\ i\xi' \cdot u'(0) \end{pmatrix}\right| \leqq c_1|\xi'|\,|u(0)| \leqq c_2|\xi'|\,\|u(x_n)\|_{H^1(\mathbf{R}_+)^n}.$$

Hence the limit operator $a_k{}^h(x', 0, \mu, D_n)$ exists in each case; it is described by the formulas (we omit $I_{(n)}$ from now on):

$$(6.11)\quad \begin{aligned} a_0{}^h(x', 0, \mu, D_n) &= {a'_0}^h = \begin{pmatrix} D_n^2 + \mu^2 e^{i\theta} \\ \gamma_0 \end{pmatrix}, \\ a_1{}^h(x', 0, \mu, D_n) &= {a'_1}^h = a_3{}^h = {a'_3}^h = \begin{pmatrix} D_n^2 + \mu^2 e^{i\theta} \\ \gamma_1 \end{pmatrix}, \\ a_2{}^h(x', 0, \mu, D_n)u &= {a'_2}^h u = a_4{}^h u = {a'_4}^h u = \begin{pmatrix} D_n^2 u + \mu^2 e^{i\theta} u \\ \gamma_0 u_n \\ \gamma_1 u' \end{pmatrix}. \end{aligned}$$

It is well known that each of these operators is bijective from $H^2(\mathbf{R}_+)^n$ to $L_2(\mathbf{R}_+)^n \times \mathbf{C}^n$ for all $\theta \in]-\pi, \pi[$, so this proves that Condition (III) holds for all $\theta \in]-\pi, \pi[$.

Finally, consider Condition (II). Let us fix a ξ' (with $|\xi'| \geqq 1$) and a $\mu^2 e^{i\theta} \in \mathbf{C} \setminus \mathbf{R}_-$, and let us write

$$\beta = |\xi'|^2 + \mu^2 e^{i\theta}, \quad \text{noting that} \quad \beta \in \mathbf{C} \setminus]-\infty, 1[.$$

Then $a_k{}^0$ has the form

$$(6.12_k) \quad a_k{}^0(x', \xi', \mu, D_n) = \begin{pmatrix} D_n^2 + \beta + g_k{}^0(\xi', D_n) \\ t_k'{}^0(\xi', D_n) \end{pmatrix} : H^2(\mathbf{R}_+)^n \to \begin{matrix} L_2(\mathbf{R}_+)^n \\ \times \\ \mathbf{C}^n \end{matrix},$$

with analogous expressions for $a_k'^0$. We must show that (6.12_k) is *bijective.* The main point in the proof is to carry this back to the unique solvability statement in Lemma 3.2.

Actually, we shall simplify the procedure by making the observation that it suffices to show *injectiveness*, which is particularly easy to carry over. This observation hinges on a reduction to a finite dimensional problem. Let us show the details in the case of $a_1{}^0$. Note that the first line in (6.12_1), the operator $D_n^2 + \beta + g_1{}^0$, is *surjective*, for the problem

$$(6.13) \qquad (D_n^2 + \beta + g_1{}^0)u = f \quad \text{on } \mathbf{R}_+, \qquad \gamma_1 u = 0 \quad \text{at } x_n = 0,$$

is equivalent with the problem

$$(6.14) \qquad (D_n^2 + \beta)u = f \quad \text{on } \mathbf{R}_+, \qquad \gamma_1 u = 0 \quad \text{at } x_n = 0,$$

since $\gamma_1 u = 0$ implies $g_1{}^0 u = 0$ (cf. (6.4)), and it is well known that the (Neumann) problem (6.14) has a (unique) solution $u \in H^2(\mathbf{R}_+)^n$ for any $f \in L_2(\mathbf{R}_+)^n$. Then for the bijectiveness of (6.12_1) it suffices to show that the mapping

$$(6.15) \qquad t_1'{}^0 : Z \to \mathbf{C}^n$$

is bijective, where Z is the kernel of $D_n^2 + \beta + g_1{}^0$, i.e.,

$$Z = \{u \in H^2(\mathbf{R}_+)^n \mid (D_n^2 + \beta + g_1{}^0)u = 0\}.$$

Now since $g_1{}^0 : H^2(\mathbf{R}_+)^n \to L_2(\mathbf{R}_+)^n$ has finite rank, the *index* of the operator $D_n^2 + \beta + g_1{}^0$ from $H^2(\mathbf{R}_+)^n$ to $L_2(\mathbf{R}_+)^n$ is the same as the index of $D_n^2 + \beta : H^2(\mathbf{R}_+)^n \to L_2(\mathbf{R}_+)^n$. The latter index is known to equal n (since $D_n^2 + \beta$

is surjective and has n-dimensional kernel). Then in view of the surjectiveness of $D_n^2 + \beta + g_1{}^0$, we find that

$$\dim Z = n.$$

But then the bijectiveness of $t_1'^0$ in (6.15) is assured if we merely show its *injectiveness*, and this is the same as showing the injectiveness of (6.12_1). Note also that if $u \in H^2(\mathsf{R}_+)^n$ satisfies $a_1{}^0 u = 0$, then $(D_n^2 + \beta)u$ is a matrix times $e^{-|\xi'|x_n}$, so u must lie in $\mathscr{S}(\bar{\mathsf{R}}_+)^n$.

It remains to show that (6.12_1) has kernel zero in $\mathscr{S}(\bar{\mathsf{R}}_+)^n$, and here we use Lemma 3.2. It is shown there (with $z = \mu e^{i\theta/2}$) that the problem

$$\text{(6.16)}\quad \begin{aligned} &\text{(i)} && (\mu^2 e^{i\theta} + |\xi'|^2 + D_n^2)u(x_n) + \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} p(x_n) = 0 \quad \text{on } \mathsf{R}_+, \\ &\text{(ii)} && ({}^t(i\xi') \quad \partial_n)u(x_n) = 0 \quad \text{on } \mathsf{R}_+, \\ &\text{(iii)} && t_1{}^0 \begin{pmatrix} u \\ p \end{pmatrix} = 0 \quad \text{at } 0, \end{aligned}$$

has only the zero solution $\mathscr{S}(\bar{\mathsf{R}}_+)^{n+1}$. Let u be a function in $\mathscr{S}(\bar{\mathsf{R}}_+)^n$ satisfying $a_1{}^0 u = 0$. Define

$$p(x_n) = 2e^{-|\xi'|x_n}\partial_n u_n(0),$$

modeled after (4.8), then clearly (cf. (6.4))

$$g_1{}^0(\xi', D_n)u = \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} p,$$

so $\{u, p\}$ satisfies (6.16 i). Moreover we have (cf. (3.30)) that

$$\text{(6.17)}\qquad t_1{}^0 \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \partial_n u'(0) + i\xi' u_n(0) \\ 2\partial_n u_n(0) - p(0) \end{pmatrix} = \begin{pmatrix} \partial_n u'(0) + i\xi' u_n(0) \\ 0 \end{pmatrix} = 0,$$

since the upper entry is the same as the tangential component of $t_1'^0 u$; this shows (6.16 iii). To show (6.16 ii), we consider $v(x_n) = i\xi' \cdot u'(x_n) + \partial_n u_n(x_n)$. Here the normal component of the equation $t_1' u = 0$ shows that

$$\text{(6.18)}\qquad v(0) = 0.$$

Moreover, it is seen by application of $({}^t(i\xi') \;\; \partial_n)$ to the equation $(\mu^2 e^{i\theta} + |\xi'|^2 + D_n^2)u + g_1^0 u = 0$ that

$$\text{(6.19)}\qquad (\mu^2 e^{i\theta} + |\xi'|^2 + D_n^2)v(x_n) = 0 \quad \text{on } \mathsf{R}_+;$$

it is used here that $-(|\xi'|^2 + D_n^2)e^{-|\xi'|x_n} = 0$. By (6.18)–(6.19), v is in the kernel of an elliptic Dirichlet problem, so v is zero. Altogether, $\{u, p\}$ is a solution of (6.16), and hence u and p are zero by Lemma 3.2. This shows Condition (II) for $a_1{}^0$.

For the other boundary operators, one proceeds in a similar way: The problem with $g_3{}^0$ is quite analogous. In the problems with $g_1'{}^0$ and $g_3'{}^0$, one gets the surjectivity of $D_n^2 + g_k'{}^0 + \beta$ by a comparison with the Dirichlet problem. In the problems with $g_0{}^0, g_0'{}^0, g_2{}^0$ and $g_4{}^0$, one gets the surjectivity by use of (6.14). In the remaining cases, the singular Green term is zero, so the property is classically known.

REMARK 6.2. Let us include some remarks about the so-called regularity number ν for these systems. This concept is important in the systematic study of parameter-dependent ps.d. boundary problems; it measures essentially how many of the "classical" estimates the symbols satisfy, when μ is included as a cotangent variable (like $\xi_1, \ldots, \xi_n$). Expressed in more detail, a μ-dependent pseudo-differential boundary operator is of regularity ν when the strictly homogeneous principal boundary symbol operator is $\mathcal{O}(|\xi'|^\nu + 1)$ near $\xi' = 0$ in symbol norm, and likewise the $|\alpha|$th derivative of the j-th term is $\mathcal{O}(|\xi'|^{\nu - |\alpha| - j} + 1)$, for all α, j. (When $\nu > 0$, this means for the principal part that it is Hölder continuous of degree ν.) Since we shall only describe the regularity number in a few special cases, we shall make do with these indications, referring the interested reader to [G4, Sect. 1.5 and Ch. 2] for more information.

In the present systems, the pseudo-differential part, $-\Delta + P_B$ contributes with regularity $\geqq 2$ ($-\Delta$ has a strictly homogeneous C^∞ symbol and hence regularity $+\infty$, and P_B is a ps.d.o. of order 1, hence the jth term in the full symbol is $\mathcal{O}(|\xi|^{2-j})$, etc.). For each k, T_k' is a differential trace operator, so that the strictly homogeneous terms in the symbol are C^∞ and contribute with regularity $+\infty$. It is the singular Green terms G_k, G_k', $G_{B,k}$, that give a lower regularity.

The s.g.o.s G_k ($k = 0, \ldots, 4$) and G_0' are all of order 2 and class 2, containing a nontrivial term of the form $K_1\gamma_1$ with K_1 a Poisson operator of order 1. In the strictly homogeneous principal symbol, this gives a term whose operator norm is $\mathcal{O}(|\xi'|^{1/2})$, see (6.4) and the details for $g_1{}^h$ in (6.8); and the regularity is 1/2. The operators G_1' and G_3' are of order 2 and class 1, they are of the form $K_0\gamma_0$ with K_0 a Poisson operator of order 2. The norm of the strictly homogeneous principal boundary symbol operator is $\mathcal{O}(|\xi'|^{3/2})$, see the details for $g_1'{}^h$ in (6.8); and the regularity is 3/2. (G_0 also contains a term of the form $K_0\gamma_0$, but here the term of the form $K_1\gamma_1$ has the lowest regularity.) The operators G_2', G_4' and $G_{B,k}$ are of order 1 and class 1, namely of the form $K_0\gamma_0$ with K_0 a Poisson operator of order 1. Considered as operators of order 2, they have vanishing principal symbols. Since they act together with the second order operator $-\Delta$, we must count them as operators of order 2 and class 1, hence of regularity 3/2, like G_1'.

To sum up, we have found that for $k = 0$, $\mathscr{A}_{k,\mu}$ has regularity 1/2; and for $k = 1$, 2, 3, 4, $\mathscr{A}_{k,\mu}$ has regularity 1/2, when G_k enters and regularity 3/2 when G_k' enters.

Observe also that in the very special case where $\Omega = \mathsf{R}^n_+$, G'_2 and G'_4 vanish, then when $B = 0$, one gets *differential operator* problems with regularity $+\infty$.

As shown in [G4, Sect. 4.2], a better regularity gives better estimate for the kernels of the solution operators for the parabolic equations; it is also related to some improved estimates of p in the present problems, cf. Theorems 7.6 and 7.7 below.

From the parameter-ellipticity we conclude, by application of [G4, Th. 3.3.1, Cor. 3.3.2]:

THEOREM 6.3. *Consider M_k and T'_k as defined in Section 5.2. Let A_k be the associated L_2-realization*

$$A_k u = M_k u, \quad D(A_k) = \{u \in H^2(\Omega)^n \mid T'_k u = 0\}, \tag{6.20}$$

and let $R_{k,\lambda} = (A_k - \lambda)^{-1}$ be the resolvent, i.e. *the solution operator for the problem*

$$\begin{aligned} (M_k - \lambda)u &= f \quad \text{in } \Omega, \\ T'_k u &= 0 \quad \text{on } \Gamma; \end{aligned} \tag{6.21}$$

defined for the $\lambda \in \mathsf{C}$ for which it exists as a bounded operator in $L_2(\Omega)^n$.

For each $\theta \in]0, 2\pi[$ there is a constant $r_\theta \geqq 0$ such that the resolvent exists on the ray $\lambda = re^{i\theta}$, $r \geqq r_0$, satisfying an estimate for each $s \in \bar{\mathsf{R}}_+$:

$$\begin{aligned} &\|R_{k,\lambda} f\|_{s+2} + \langle\lambda\rangle^{s/2+1}\|R_{k,\lambda} f\|_0 \\ &\leqq C_s(\|f\|_s + \langle\lambda\rangle^{s/2}\|f\|_0) \quad \text{for } f \in H^s(\Omega)^n. \end{aligned} \tag{6.22}$$

In particular, the spectrum of A_k is discrete and has a lower bound a; and for any $\varepsilon > 0$ there is a bounded set K_ε such that the resolvent satisfies (6.22) *in the region*

$$U_\varepsilon = \{\lambda \mid \operatorname{Re}\lambda \leqq a - \varepsilon\} \cup \{\lambda \mid \lambda \notin K_\varepsilon, \varepsilon \leqq \arg\lambda \leqq 2\pi - \varepsilon\}. \tag{6.23}$$

The discreteness of the spectrum follows as usual from the compactness of $R_{k,c}$ for a constant c in the resolvent set.

7. Solution of the initial-boundary value problems.

7.1. *Compatibility conditions.* We are now in a position to apply the general parabolic solvability theory of [G4], [G-S4] to the reduced pseudo-differential problems (5.44_k), and to draw the conclusions for the original problems (5.41_k). In the linear case (the case $\delta = 0$), we can apply [G-S4, Th. 6.3] directly to (5.44_k). Here we recall that for the existence of a solution $u \in H^{r,r/2}(Q)^n$ it is necessary and sufficient that the given data $\{f_k, \psi_k, u_0\}$ satisfy a certain *compatibility condition*, assuring that the initial value u_0, given on the "bottom" of the cylinder $Q = \Omega \times I$, fits together with the "vertical" boundary values ψ_k, at the "corner" $\Gamma_{(0)}$ (for larger r also t-derivatives of ψ_k and expressions derived from f_k are involved). In the nonlinear case, one formulates in a similar way a compatibility condition, that is clearly necessary for the solvability; and it is found that it is also sufficient, when the data or I satisfy certain smallness hypotheses.

The compatibility conditions for the reduced pseudo-differential problems carry over to compatibility conditions for the original problems in a natural way. We make this explicit in the following.

Consider the problem (5.44$_k$). When $u \in C^\infty(\bar{Q})^n$ solves (5.44$_k$), then one has for all j:

$$T'_k \partial_t^j u = \partial_t^j T'_k u = \partial_t^j \psi_k, \quad \text{on } S,$$

$$\begin{aligned} r_0 \partial_t^{j+1} u &= -r_0 \partial_t^j (M_k + \delta \mathrm{Q}_k) u + r_0 \partial_t^j f_k \\ &= -M_k r_0 \partial_t^j u - \delta r_0 (\mathrm{Q}_k u)^{(j)} + r_0 \partial_t^j f_k \quad \text{on } \Omega_{(0)}, \end{aligned} \tag{7.1}$$

where

$$(\mathrm{Q}_k u)^{(j)} = \mathrm{pr}_{J_k} \textstyle\sum_{m=0}^{j} \binom{j}{m} \mathrm{K}(\partial_t^m u, \partial_t^{j-m} u). \tag{7.2}$$

At the corner $\Gamma_{(0)}$, one must have

$$r_0 T'_k \partial_t^j u = T'_k r_0 \partial_t^j u, \tag{7.3}$$

for all j. From the data u_0 and f_k we now define successively the functions on $\Omega_{(0)}$:

$$u^{(0)} = u_0,$$

$$u^{(j+1)} = -M_k u^{(j)} - \delta \mathrm{pr}_{J_k} \textstyle\sum_{m=0}^{j} \binom{j}{m} \mathrm{K}(u^{(m)}, u^{(j-m)}) + r_0 \partial_t^j f_k, \tag{7.4$_k$}$$

for $j = 0, 1, \ldots$; then in view of (7.1), we can express (7.3) in terms of these functions as the system of conditions on the data

$$r_0 \partial_t^j \psi_k = T'_k u^{(j)}, \tag{7.5}$$

for $j = 0, 1, \ldots$

When the problem (5.44$_k$) is considered for $u \in H^{r+2, r/2+1}(Q)^n$ (with $r \geqq 0$, and $r + 2 \geqq n/2$ in case $\delta = 1$), the expressions $u^{(j)}$ are defined only for j up to a certain value depending on r. More precisely, when (5.42$_k$) holds, one has by Proposition 2.2, (2.43) and (4.32),

$$u^{(0)} \in H^{r+1}(\Omega)^n, \ldots, u^{(j)} \in H^{r-2j+1}(\Omega), \ldots, \quad \text{for } j < (r+1)/2. \tag{7.6}$$

Moreover, when $r + 1/2$ is integer, the equation (7.5) need not make sense as it stands (e.g. when $T'_k u^{(j)}$ is not well-defined), but it may still be given a sense as a coincidence relation. This is explained in detail in [G-S4, Sect. 5], in terms of the integral

$$\mathscr{I}[\varphi, v] = \int_{t\in I}\int_{x'\in\Gamma_{(t)}}\int_{y\in\Omega_{(0)}} \frac{|\varphi(x',t) - v(y)|^2}{(|x'-y|^2 + t)^{1+n/2}}\,dy\,d\sigma_{x'}\,dt, \tag{7.7}$$

that is defined for functions $\varphi(x', t)$ on S and $v(y)$ on $\Omega_{(0)}$ with $\varphi \in H^{1,1/2}(S)$ and $v \in H^{1/2}(\Omega_{(0)})$ ($\sigma_{x'}$ is the surface measure on Γ). When this integral is finite, we say that φ and v *coincide at* $\Gamma_{(0)}$. If $u \in H^{3/2,3/4}(Q)$, then the coincidence holds for $\gamma_0 u$ and $r_0 u$, with

$$\mathscr{I}[\gamma_0 u, r_0 u] \leqq C_1 \|u\|_{H^{3/2,3/4}(Q)}; \tag{7.8}$$

and conversely, for given $\varphi \in H^{1,1/2}(S)$ and $v \in H^{1/2}(\Omega_{(0)})$ with $\mathscr{I}[\varphi, v] < \infty$, there exists $u \in H^{3/2,3/4}(Q)$ such that $\gamma_0 u = \varphi$ and $r_0 u = v$. In particular, $\varphi \in H^{1,1/2}(S)$ coincides with 0 at $\Gamma_{(0)}$ precisely when $\varphi \in H^{1,1/2}_{(0)}(S)$.

We recall from (5.47$_k$) ff. that $T'_k = \{T'_{k0}, T'_{k1}\} = \{\gamma_0 P_{k0}, \gamma_0 P_{k1}\}$, with differential operators P_{kl} of order l, mapping into the extended bundles F_{kl} over Σ.

Definition 7.2. Consider the problem (5.44$_k$) for $k = 0, 1, 2, 3$ or 4. Let $r \geqq 0$ (with $r + 2 \geqq n/2$ in case $\delta = 1$), and let $\{f_k, \psi_k, u_0\}$ be given in the spaces indicated in (5.42$_k$). Define $u^{(j)}$ by (7.4$_k$) for j $< (r + 1)/2$. The data $\{f_k, \psi_k, u_0\}$ are said to satisfy the compatibility condition of order r for (5.44$_k$), when

$$T'_{kl} u^{(j)} = r_0 \partial_t^j \psi_{kl} \quad \text{for all } j \in \mathrm{N} \text{ and } l = 0, 1 \quad \text{with } 2j + l \leqq r + 1/2,\ \dim F_{kl} \neq 0; \tag{7.10}$$

here, when $r + 1/2$ is integer, the equation for $2j + l = r + 1/2$ is understood in the sense of coincidence:

$$\mathscr{I}[\partial_t^j \psi_{kl}, P_{kl} u^{(j)}] < \infty, \quad \text{for } 2j + l = r + 1/2, \dim F_{kl} \neq 0 \tag{7.11}$$

(when $r + 1/2$ is even, $\{j, l\} = \{(r + 1/2)/2, 0\}$, and when $r + 1/2$ is odd, $\{j, l\} = \{(r - 1/2)/2, 1\}$).

There is a similar formulation where f_k and G_k are replaced by f'_k and G'_k.

In the case $\delta = 0$, this is simply the specialization of the general compatibility condition [G-S4, Def. 6.1] to our particular problem (5.44$_k$) (where we use the special form of our trace operators), and the solvability theorem proved there applies directly. But before we go on to that, we first shall translate the compatibility condition back to the original problems (5.41$_k$) by Theorem 5.3.

Let the set of data $\{f, \varphi_k, u_0\}$ be given as in Theorem 5.3, and let f_k be derived from $\{f, \varphi_k\}$ as described there. Then we again define the functions $u^{(j)}$ by (7.4$_k$). Since $f_k \in J_k$, we have that $\operatorname{div} u^{(j)} = 0$ in all cases; moreover, one shows that $\gamma_\nu u^{(j)} = 0$ in the cases $k = 0, 2, 4$, by calculations as in (5.22). Therefore one has automatically that $\gamma_0 \operatorname{div} u^{(j)}$ and (in case $k = 0, 2, 4$) $\gamma_\nu u^{(j)}$ are 0 (or $\operatorname{div} u^{(j)}$ resp. $\vec{n} \cdot u^{(j)}$ coincide with 0 at $\Gamma_{(0)}$), so this need not be listed as explicit conditions. Altogether, the compatibility condition from Definition 7.2 carries over to the form:

DEFINITION 7.3. Consider the problem (5.41_k) for $k = 0, 1, 2, 3$ or 4. Let $r \geqq 0$ (with $r + 2 \geqq n/2$ in case $\delta = 1$), and let $\{f, \varphi_k, u_0\}$ be given in the spaces indicated in (5.42_k), satisfying (5.43_k). Define f_k from $\{f, \varphi_k\}$ by (5.37), and define $u^{(j)}$ by (7.4_k) for $j < (r+1)/2$. The data $\{f, \varphi_k, u_0\}$ are said to satisfy the compatibility condition of order r for (5.41_k), when

$$(7.12)\qquad \begin{aligned} r_0\partial_t^j\varphi_k &= \gamma_0 u^{(j)} && \text{for } k = 0, j \leqq (r+1/2)/2,\\ r_0\partial_t^j\varphi_{k,\tau} &= (\chi_1 u^{(j)})_\tau && \text{for } k = 1 \text{ and } 2, j \leqq (r-1/2)/2,\\ r_0\partial_t^j\varphi_{k,\tau} &= \gamma_1 u_\tau^{(j)} && \text{for } k = 3 \text{ and } 4, j \leqq (r-1/2)/2. \end{aligned}$$

When $(r \pm 1/2)/2$ is integer, the equations are interpreted as coincidences, i.e. (cf. (7.7)),

$$(7.13)\qquad \begin{aligned} \mathscr{I}[\partial_t^j\varphi_k, u^{(j)}] &< \infty && \text{for } k = 0, j = (r+1/2)/2,\\ \mathscr{I}[\partial_t^j\varphi_{k,\tau}, (S(u^{(j)})\vec{n})_\tau] &< \infty && \text{for } k = 1 \text{ and } 2, j = (r-1/2)/2,\\ \mathscr{I}[\partial_t^j\varphi_{k,\tau}, \partial_\nu u_\tau^{(j)}] &< \infty && \text{for } k = 3 \text{ and } 4, j = (r-1/2)/2. \end{aligned}$$

We have here inserted the explicit expressions for the relevant differential operators P_{kl}.

7.2. *Solution of the linear problems.* In the linear case, [G-S4, Th. 6.3 1°] implies immediately:

THEOREM 7.4. *Let $k = 0, 1, 2, 3$ or 4, let $\delta = 0$, let $r \geqq 0$, and let $I =]0, b[$ with $b < \infty$. For any system of functions $\{f_k, \psi_k, u_0\}$, given as in (5.42_k) and satisfying the compatibility condition of order r for (5.44_k) (Definition 7.2), the problem (5.44_k) has a unique solution $u \in H^{r+1,r/2+1}(Q)^n$, satisfying estimates as follows:*

$$(7.14_k)\qquad \|u\|^2_{H^{r+2,r/2+1}(Q)^n} \leqq C_b(\|f_k\|^2_{H^{r,r/2}(Q)^n} + \|\psi_{k0}\|^2_{H^{r+3/2,r/2+3/4}(S,\underline{E}_{k0})} + \|\psi_{k1}\|^2_{H^{r+1/2,r/2+1/4}(S,\underline{E}_{k1})} + \|u_0\|^2_{H^{r+1}(\Omega)^n} + \mathscr{I}_{k,r}).$$

*Here $\mathscr{I}_{k,r} = 0$ if $r + 1/2$ is not integer, or if $r + 1/2$ is even and $k = 1$ or 3, or if $r + 1/2$ is odd and $k = 0$. In the remaining cases, $\mathscr{I}_{k,r}$ is defined by (*cf. (5.49)*)*

$$(7.15)\qquad \mathscr{I}_{k,r} = \mathscr{I}[\partial_t^j\psi_{kl}, P_{kl}u^{(j)}],\ \text{for the value } \{j, l\} \text{ such that } 2j + l = r + 1/2,$$

that is, $\{j, l\} = \{(r+1/2)/2, 0\}$ for $r + 1/2$ even and $k = 0$, 2 or 4, and $\{j, l\} = \{(r-1/2)/2, 1\}$ for $r + 1/2$ odd and $k = 1, 2, 3$ or 4.

In particular, for any system of data

$$(7.16_k)\qquad \begin{aligned} &f_k \in H_{(0)}^{r,r/2}(Q)^n,\\ &\psi_k \in H_{(0)}^{r+3/2,r/2+3/4}(S, \underline{E}_{k0}) \times H_{(0)}^{r+1/2,r/2+1/4}(S, \underline{E}_{k1}),\\ &u_0 = 0, \end{aligned}$$

the compatibility condition of order r holds, and the solution satisfies

$$(7.17_k) \qquad \|u\|_{H^{r+2,r/2+1}_{(0)}(Q)^n} \leqq C'_b(\|f_k\|_{H^{r,r/2}_{(0)}(Q)^n} + \|\psi_{k0}\|_{H^{r+3/2,r/2+3/4}_{(0)}(S,E_{k0})} + \|\psi_{k1}\|_{H^{r+1/2,r/2+1/4}_{(0)}(S,E_{k1})}$$

Similar estimates hold when f_k and G_k are replaced by f'_k and G'_k.
The constants C_b can be taken to be nondecreasing in $b \in \mathbb{R}_+$.

This will now be translated back to a statement on the generalized Stokes problems (5.41_k) with $\delta = 0$, by Theorem 5.3. The transition from the original data to the new data, by (5.37), is straightforward and takes place in the spaces indicated in (5.42_k), so the theorem above gives us $u \in H^{r+2,r/2+1}(Q)^n$. Then p is defined from all the other entries by (5.38) ff. Here it remains to analyze p more closely.

The contributions from the given boundary values φ_k are described by use of (5.11), where we can include K_D and $K_N A''_\Gamma$, that have similar mapping properties; these contributions belong to $H^{r+1,r/2+1/4}(Q)$ if $k = 1, 2, 3$ or 4, and belong to $H^{r+2,r/2+3/4}(Q)$ if $k = 0$. By (5.12) (extended to include grad K_D and grad $K_N A''_\Gamma$), their gradients belong to $H^{r,r/2}(Q)^n$ for $k = 1, 2, 3$ or 4, resp. to $H^{r+1,r/2+1/2}(Q)^n$ for $k = 0$.

The contributions from the boundary values of u (in the final formulas for p in (5.38)) are similarly described by use of (5.11) and (5.12); here we need moreover the information:

$$(7.18) \qquad \begin{aligned} K_D \operatorname{div}'_\Gamma : H^{r+3/2,r/2+3/4}(S)^n &\to H^{r+1,r/2+1/2}(Q) \quad \text{for } r \geqq -1, \\ \operatorname{grad} K_D \operatorname{div}'_\Gamma : H^{r+3/2,r/2+3/4}(S)^n &\to H^{r,r/2}(Q) \quad \text{for } r \geqq 0, \end{aligned}$$

that follows from Proposition 2.2 since $K_D \operatorname{div}'_\Gamma$ is a Poisson operator of order 1. Altogether, the contributions from the boundary values of u are in $H^{r+1,r/2+1/4}(Q)$ if $k = 0$, in $H^{r+1,r/2+1/2}(Q)$ if $k = 1$ or 3, and in $H^{r+2,r/2+3/4}(Q)$ if $k = 2$ or 4; and their gradients are in $H^{r,r/2}(Q)^n$ if $k = 0$, 1 or 3, and in $H^{r+1,r/2+1/2}(Q)^n$ if $k = 2$ or 4.

The contributions from Bu and f consist of $R_D \operatorname{div} Bu$ and $R_D \operatorname{div} f$, or $\tilde{G}Bu$ and $\tilde{G}f$ (recall (2.54) ff.). Here we have by Proposition 2.2,

$$(7.19) \qquad \begin{aligned} R_D \operatorname{div}, \tilde{G} : H^{r,r/2}(Q)^n &\to H^{r+1,r/2}(Q), \quad \text{for } r \geqq 0, \\ R_D \operatorname{div} B, \tilde{G}B : H^{r+2,r/2+1}(Q)^n &\to H^{r+2,r/2+3/4}(Q), \quad \text{for } r > -3/2, \\ \operatorname{grad} R_D \operatorname{div} B, \operatorname{grad} \tilde{G}B : H^{r+2,r/2+1}(Q)^n &\to H^{r+1,r/2+1/2}(Q), \quad \text{for } r \geqq -1, \end{aligned}$$

since the operators in the first line are of order -1 and class 0, in the second line of order 0 and class 1, and in the third line of order 1 and class 1, cf. (2.18) and Theorem 2.6.

Thus for general data we get $p \in H^{r+1,r/2}(Q)$ with grad $p \in H^{r,r/2}(Q)^n$.

We can then formulate the main result for the linear problems:

THEOREM 7.5. *Let $k = 0, 1, 2, 3$ or 4, let $\delta = 0$, let $r \geqq 0$, and let $I =]0, b[$ with $b < \infty$. For any system of functions $\{f, \varphi_k, u_0\}$, given as in (5.42_k) and satisfying (5.43_k) and the compatibility condition of order r for (5.41_k) (Definition 7.3), the problem (5.41_k) has a solution $\{u, p\}$ in $H^{r+2,r/2+1}(Q)^n \times H^{r+1,r/2}(Q)$, unique if $k = 1$ or 3, and unique under one of the side conditions: $(p, 1)_{L_2(\Omega)} = 0$ for almost all $t \in I$, or: $(\gamma_0 p, 1)_{L_2(\Gamma)} = 0$ for almost all $t \in I$, if $k = 0, 2$ or 4. It satisfies estimates, with C_b nondecreasing in $b \in \mathbb{R}_+$:*

$$(7.20_k) \qquad \begin{aligned} &\|u\|^2_{H^{r+2,r/2+1}(Q)^n} + \|p\|^2_{H^{r+1,r/2}(Q)} + \|\operatorname{grad} p\|^2_{H^{r,r/2}(Q)^n} \\ &\leqq C_b(\|f\|^2_{H^{r,r/2}(Q)^n} + \|\varphi_{k0}\|^2_{H^{r+3/2,r/2+3/4}(S,\underline{E}_{k0})} \\ &\quad + \|\varphi_{k1}\|^2_{H^{r+1/2,r/2+1/4}(S,\underline{E}_{k1})} + \|u_0\|^2_{H^{r+1}(\Omega)^n} + \mathscr{I}_{k,r}). \end{aligned}$$

Here $\mathscr{I}_{k,r} = 0$ if $r - 1/2$ is not integer, or if $r - 1/2$ is odd and $k = 1, 2, 3$ or 4, or if $r - 1/2$ is even and $k = 0$ (recall also that $\varphi_{k0} = 0$ if $k = 1$ or 3, and $\varphi_{k1} = 0$ if $k = 0, 2$ or 4). In the remaining cases, $\mathscr{I}_{k,r}$ is defined by

$$(7.21) \qquad \begin{aligned} &\mathscr{I}_{k,r} = \mathscr{I}[\partial_t^j \varphi_k, u^{(j)}] && \text{for } k = 0, j = (r + 1/2)/2; \\ &\mathscr{I}_{k,r} = \mathscr{I}[\partial_t^j \varphi_{k,\tau}, (S(u^{(j)})\vec{n}_\tau] && \text{for } k = 1 \text{ or } 2, j = (r - 1/2)/2; \\ &\mathscr{I}_{k,r} = \mathscr{I}[\partial_t^j \varphi_{k,\tau}, \partial_\nu u_\tau^{(j)}] && \text{for } k = 3 \text{ or } 4, j = (r - 1/2)/2. \end{aligned}$$

In the cases $k = 0, 2$ and 4, Theorem 5.3 at first gives p uniquely under the side condition $(\gamma_0 p, 1)_\Gamma = 0$ a.e., but if we subtract the function $c(t) = \operatorname{vol}(\Omega)^{-1}(p, 1)_{\Omega_{(t)}}$, which lies in $H^{r/2}(I)$ (and hence in $H^{s,r/2}(Q)$ for any s), we get $p_1(x,t) = p(x,t) - c(t)$ satisfying $(p_1, 1)_\Omega$ a.e., and determined uniquely as such.

Theorems 7.4 and 7.5 were presented in [G-S1] under slightly more restrictive circumstances: the cases $k = 3$ and 4 were not mentioned, B was 0, and there were some extra hypotheses on the data (e.g. that $\operatorname{div} f = 0$).

It is seen from the discussion before Theorem 7.5 that the individual terms in the formulas (5.38) for p contribute in different ways, so the estimates can be improved in many special cases. If $\operatorname{div} f = 0$ and (for $k = 0, 2, 4$) $\gamma_\nu f = 0$, the explicit contribution from f is 0, so since Bu is more smooth, p gets more smoothness in the t-direction (the weakest contribution then comes from $\varphi_{k,\nu} \in H^{r+1/2,r/2+1/4}(S, \underline{E}_{k1})$ for $k = 1$ and 3, so the t-order is improved by 1/4).

If in addition $B = 0$ (or, in case $k = 1$ or 3, if B has constant scalar coefficients, as in the Oseen equation), then $p_k(u, f) = 0$, so p is expressed purely by Poisson operators. Here it is sometimes the contribution from the boundary value of u, sometimes the contribution from φ_k, that is more smooth, and we can use the fine representations in (5.38) to advantage. In these cases, we can moreover apply the estimates (2.36) to the present operators: When $\varphi \in L_2(I; H^{s+1/2}(\Gamma))$ (possibly a vector, we leave out the indication), then, in local coordinates at the boundary,

$$\begin{gathered} K_N\varphi \in \textstyle\bigcap_{m\ge 0} L_2(I; H^{\{m,s+2-m\}}(\mathsf{R}^n_+)), \\ K_D\varphi, K_N \operatorname{div}'_\Gamma\varphi, K_N A'_\Gamma\varphi, \text{ and } K_N A''_\Gamma\varphi \\ \in \textstyle\bigcap_{m\ge 0} L_2(I; H^{\{m,s+1-m\}}(\mathsf{R}^n_+)), \\ \operatorname{grad} K_D\varphi, \operatorname{grad} K_N \operatorname{div}'_\Gamma\varphi, \operatorname{grad} K_N A'_\Gamma\varphi, \operatorname{grad} K_N A''_\Gamma\varphi \text{ and } K_D \operatorname{div}'_\Gamma\varphi \\ \in \textstyle\bigcap_{m\ge 0} L_2(I; H^{\{m,s-m\}}(\mathsf{R}^n_+)). \end{gathered} \tag{7.22}$$

This can be combined with the mapping properties, valid for $r > -1/2$ (as special consequences of Proposition 2.2),

$$\begin{aligned} \gamma_0 &: H^{r+2,r/2+1}(Q) \to L_2(I; H^{r+3/2}(\Gamma)), \\ \gamma_1, \gamma_\nu \operatorname{pr}_J B &: H^{r+2,r/2+1}(Q) \to L_2(I; H^{r+1/2}(\Gamma)), \end{aligned} \tag{7.23}$$

Recall also that A'_Γ and A''_Γ vanish when $\Omega = \mathsf{R}^n_+$.

We collect some of the resulting improvements of p in the following theorem:

THEOREM 7.6. *Consider the situation described in Theorem 7.5.*

1° *If* $\operatorname{div} f = 0$ *and, when* $k = 0, 2, 4$, $\gamma_\nu f = 0$, *then:*

$$p \in H^{r+1,r/2+1/4}(Q) \quad \textit{for all } k, \tag{7.24}$$

and, moreover,

$$\begin{aligned} p + K_D\varphi_{k,\nu} &\in H^{r+1,r/2+1/2}(Q) \quad \textit{for } k = 1, 3, \\ p + K_N \operatorname{div}'_\Gamma\varphi_{k,\tau} &\in H^{r+2,r/2+3/4}(Q) \quad \textit{for } k = 2, 4, \\ \operatorname{grad} p + \operatorname{grad} K_N \operatorname{div}'_\Gamma\varphi_{k,\tau} &\in H^{r+1,r/2+1/2}(Q) \quad \textit{for } k = 2, 4, \end{aligned} \tag{7.25}$$

giving improved smoothness when $\varphi_{k,\nu}$ *resp.* $\varphi_{k,\tau}$ *is zero.*

2° *If in addition* $B = 0$ *(or, in the cases* $k = 1$ *and* 3, B *just has constant scalar coefficients), then, in local coordinates at the boundary,*

$$\begin{aligned} p &\in \textstyle\bigcap_{m\ge 0} L_2(I; H^{\{m,r+1-m\}}(\mathsf{R}^n_+)) \quad \textit{for } k = 0, 1, 2, 3, 4 \\ p + K_N \operatorname{div}'_\Gamma\varphi_k &\in \textstyle\bigcap_{m\ge 0} L_2(I; H^{\{m,r+2-m\}}(\mathsf{R}^n_+)) \quad \textit{for } k = 2, 4; \end{aligned} \tag{7.26}$$

in particular, if $\Omega = \mathsf{R}^n_+$,

$$p + K_N \operatorname{div}'_\Gamma\varphi_k = 0 \quad \text{for } k = 2, 4. \tag{7.27}$$

7.3. *Consequences for the nonlinear problems.* The method for solving the nonlinear problems (4.31_k) with $\delta = 1$ is described in detail in [G-S3], so we shall just outline the strategy here, for completeness.

Let $k = 0, 1, 2, 3$ or 4. (The cases 3 and 4 are not mentioned in [G-S3], but their treatment is, as we see, analogous to that of 1 and 2.) Assume that the data are as in (5.42_k) for some $r \geqq 0$ with $r + 2 \geqq n/2$, and satisfy (5.43_k) and the compatibility condition of order r according to Definition 7.3. The first step is to reduce the problem to the form (5.44_k) concerning u alone, where the data satisfy the compatibility condition of order r according to Definition 7.2; this is briefly indicated in [G-S3], and is accounted for in detail in the present terminology above. The next step is to reduce to a problem for $v = u - w$, where w is chosen such that u_0 is replaced by 0, and the other data are replaced by data in the $H_{(0)}$ variant of the original spaces. For this there is shown, in the course of the proof of Theorem 5 in [G-S3], a nonlinear variant of the linear reduction [G-S4, Prop. 6.2]. This leads to a problem of the form

$$\begin{aligned} \partial_t v + M_k v + \mathsf{Q}_k(v, w) + \mathsf{Q}_k(w, v) + \mathsf{Q}_k v &= g \quad \text{in } Q, \\ T'_k v &= \psi \quad \text{in } S, \\ v|_{t=0} &= 0 \quad \text{in } \Omega. \end{aligned} \tag{7.28}$$

Now one applies Theorem 7.4 (in particular (7.17_k) successively (as in [S1]) to construct solutions of the problems for $m = 0, 1, \ldots$, beginning with $v_0 = 0$:

$$\begin{aligned} \partial_t v_{m+1} + M_k v_{m+1} &= g - \mathsf{Q}_k v_m - \mathsf{Q}_k(v_m, w) - \mathsf{Q}_k(w, v_m) && \text{in } Q, \\ T'_k v_{m+1} &= \psi && \text{on } S, \\ v_{m+1}|_{t=0} &= 0 && \text{on } \Omega; \end{aligned} \tag{7.29}$$

and it is found using Theorems B.3 and B.4 (from Appendix B below) that under suitable smallness hypotheses, where either the data are small, $r + 2 \geqq n/2$, or the interval I is replaced by a small interval I', $r + 2 > n/2$, v_m converges for $m \to \infty$ to a (unique) solution of (7.28). Finally this is carried back to a solution of the original problem by Theorem 5.3.

The solution operator satisfies estimates as in Theorem 7.5 (on a small ball in the space of data or on a small interval).

The resulting estimates for p were not in [G-S3] analyzed beyond noting (as in the treatment of the linear problems in [G-S1]) that $\|\operatorname{grad} p\|_{r,r/2}$ is bounded in terms of the data. In view of the formulas (5.38) this gives immediately a bound on $\|p\|_{r+1,r/2}$, when we simply use that K is continuous from $H^{r+2,r/2+1}(Q_{I'})$ to $H^{r,r/2}(Q_{I'})$. Now we observe that Theorem B.3 contains a better mapping property of K, which allows us to extend the results of Theorem 7.6 to the nonlinear case also, to some extent, when $r + 2 > n/2$.

In fact, (B.8) shows that K is continuous from $H^{r+2,r/2+1}(Q_{I'})$ to $H^{r+\sigma,(r+\sigma)/2}(Q_{I'})$ with $\sigma = 1$ if $r + 2 - n/2 > 1$, and with $\sigma < 1$, $\sigma \leqq r + 2 - n/2$, when $r + 2 - n/2 \leqq 1$. In view of the mapping property in the first line in (7.19), the contribution from $\mathsf{K}(u)$ then satisfies:

$$R_D \operatorname{div} \mathsf{K}(u) \text{ and } \tilde{G}\mathsf{K}(u) \in H^{r+\sigma+1,(r+\sigma)/2}(Q_{I'}).$$

Then the considerations leading to Theorem 7.6 1° give in the nonlinear case:

THEOREM 7.7. *Let $r \geqq 0$ with $r + 2 > n/2$, and let $\{u, p\}$ be a solution of the nonlinear problem (5.41_k) with $\delta = 1$, constructed as in* [G-S3, Th. 5], *with data satisfying (5.42_k), (5.43_k) and the compatibility condition of order r (Definition 7.3). Then $p \in H^{r+1,r/2}(Q_{I'})$ (where I' is determined by the solution method), and one has furthermore:*

Let $\sigma = 1$ if $r + 2 - n/2 > 1$, and let $\sigma < 1$ with $\sigma \leqq r + 2 - n/2$ if $r + 2 - n/2 \leqq 1$.

If $\operatorname{div} f = 0$ and, when $k = 0, 2, 4$, $\gamma_\nu f = 0$, then:

$$p \in H^{r+1,r/2+\min\{\sigma/2,1/4\}}(Q_{I'}) \quad \text{for all } k. \tag{7.30}$$

Moreover,

$$\begin{aligned} p + K_D \varphi_{k,\nu} &\in H^{r+1,r/2+\min\{\sigma/2,1/2\}}(Q_{I'}) \quad \text{for } k = 1, 3, \\ p + K_N \operatorname{div}'_\Gamma \varphi_{k,\tau} &\in H^{r+2,r/2+\min\{\sigma/2,3/4\}}(Q_{I'}) \quad \text{for } k = 2, 4, \\ \operatorname{grad} p + \operatorname{grad} K_N \operatorname{div}'_\Gamma \varphi_{k,\tau} &\in H^{r+1,r/2+\min\{\sigma/2,1/2\}}(Q_{I'}) \quad \text{for } k = 2, 4, \end{aligned} \tag{7.31}$$

giving improved smoothness when $\varphi_{k,\nu}$ resp. $\varphi_{k,\tau}$ is zero.

In the best cases, the t-smoothness of $p(x, t)$ is lifted to $r/2 + 1/2$.

8. Further consequences.

There are some further considerations in [G4] and [G-S4] that have interesting consequences for the present problems. We begin with an application of the results of [G-S4, Section 7] in the case $\delta = 0$. Here [G-S4, Th. 7.1] implies immediately that when the given functions f_k and ψ_k in (5.44_k) are C^∞ for $t > 0$, then so is u; and this in turn implies that when f and φ_k in (5.41_k) are C^∞ for $t > 0$, then so are u and p. Moreover, if f and φ_k are such that f_k and ψ_k are 0, then u depends analytically on t. In fact, (5.44_k) is then solved by a holomorphic semigroup $U_k(t): u_0 \mapsto u(t)$, generated by $-A_k$, where A_k is the $L_2(\Omega)$ realization of the stationary problem (this follows from the resolvent estimates in Theorem 6.3 above, where A_k and its resolvent $R_{k,\lambda}$ are defined). Moreover, $U_k(t)$ allows initial values u_0 in $H^s(\Omega)^n$ also for $0 \leqq s < 1$ (cf. [G-S4, Th. 7.3]). More precisely, the initial values can be taken in the following spaces (where the compatibility conditions are taken into account), defined for $m \in \mathbb{N}$, $0 \leqq s < 2$:

$$H^{2m+s}_{T'_k,M_k} = \{u \in H^{2m+s}(\Omega)^n \mid T'_k M^j_k u = 0 \text{ for } 0 \leqq j < m, \tag{8.1}$$
$$T'_{k0} M^m_k u = 0 \text{ if } s \geqq 1/2,\ T'_{k1} M^m_k u = 0 \text{ if } s \geqq 3/2\}.$$

(The boundary condition is understood in the sense of coincidence, when $s = 1/2$ or $s = 3/2$.). The semigroup $U_k(t)$ restricts to $J_k(\Omega)$ as a holomorphic semigroup $\tilde{U}_k(t)$ there (since $U_k(t)$ maps $J_k(\Omega)$ into itself – this is obvious for the smooth elements, and follows then by continuity in general). In particular, the inequalities

$$\|\tilde{U}_k(t)u_0\|_{\mathcal{M}^r_k} \leqq C_{r,s,a_1} t^{-(r-s)/2} e^{-a_1 t} \|u_0\|_{\mathcal{M}^s_k} \quad \text{for all } t > 0,\ r \geqq s \geqq 0, \tag{8.2}$$

follow from the corresponding inequalities for $U_k(t)$ [G-S4, (7.13)] by restriction to

$$\mathcal{M}^s_k = H^s_{T'_k,M_k} \cap J_k. \tag{8.3}$$

In (8.2), a_1 is any number $< a$, where a is the lower bound of the spectrum of the realization A_k. The inequalities (8.2) were presented first in [G-S1], and they are used in [G-S3] (to which we refer the reader for further details on this application) to show solvability of the Navier-Stokes problems with not very smooth initial data (see also Remark B.2 below).

Another question, we shall take up, is whether the infinite interval $I_\infty =]0, \infty[$ can be included. The general result [G-S4, Th 6.3] gives existence of solutions to (5.44$_k$), when the data satisfy suitable exponential estimates depending on the lower bound a of the spectrum of the realization A_k. (Global existence can also be obtained when $\varphi_k = 0$ and u_0 and f take values in a suitable subspace of $L_2(\Omega)^n$ of finite codimension, by [G-S4, Cor. 7.5].) This carries over to (5.41$_k$), but is not so convenient for the treatment of the nonlinear problems.

Actually, one can get much more precise results here, applicable to the basic Laplace transform argument used in [G-S4, Th. 6.3] when $f \in J_k$, and φ_k and u_0 are 0, by observing that the resolvent $R_{k,\lambda} = (A_k - \lambda)^{-1}$ is then only used on the space $J_k(\Omega)$.

LEMMA 8.1. *Denote by $\sigma((-\Delta)_D)$ resp. $\sigma((-\Delta)_N)$ the spectra of the Dirichlet resp. the Neumann realizations of $-\Delta$ in $L_2(\Omega)$, and denote the spectrum of A_k by $\sigma(A_k)$.*

For $k = 1$ or 3 and $\lambda \notin \sigma(A_k) \cup \sigma((-\Delta)_D)$, $R_{k,\lambda}$ maps $J_k(\Omega) = J(\Omega)$ into itself.

For $k = 0, 2$ or 4 and $\lambda \notin \sigma(A_k) \cup (\sigma((-\Delta)_N) \setminus \{0\})$, $R_{k,\lambda}$ maps $J_k(\Omega) = J_0(\Omega)$ into itself.

In these cases, the restriction $R'_{k,\lambda}$ of the resolvent to the space $J_k(\Omega)$, identifies with the solution operator for the problem

$$(M_k - \lambda)u = f \quad \text{in } \Omega, \tag{8.4}$$
$$T'_k u = 0 \quad \text{on } \Gamma;$$

with f and u in $J_k(\Omega)$, and the estimates (6.22) *for λ satisfying* (6.23) *carry over to $R'_{k,\lambda}$.*

PROOF. The proof follows an argumentation very similar to the one we used to show $\operatorname{div} u = 0$ in Sections 4 and 5 for time-dependent problems:

Let $k = 1$ or 3, let $\lambda \notin \sigma(A_k)$, and let $f \in J_k(\Omega) = J(\Omega)$. Then an application of div to the first line in (6.21) shows that any solution $u \in H^2(\Omega)^n$ of (6.21) must satisfy

$$(8.5)\qquad (-\Delta - \lambda)\operatorname{div} u = 0$$

(for $\operatorname{div} \Delta u = \Delta \operatorname{div} u$; $G_k u = G'_k u$ since the boundary value is zero; $\operatorname{div} G'_k u = \operatorname{div} G_k u = 0$; $\operatorname{div} \operatorname{pr}_{J_k} Bu = 0$). Combining (8.5) with the information that $\gamma_0 \operatorname{div} u = 0$, we conclude that if $\lambda \notin \sigma((-\Delta)_D)$, then $\operatorname{div} u = 0$. Thus for these λ, $J(\Omega)$ is invariant under $R_{k,\lambda}$.

If $k = 0, 2$ or 4, we take $f \in J_k(\Omega) = J_0(\Omega)$. Again (8.5) holds, and we here find by a calculation similar to that of (5.22) (noting that $\gamma_\nu u = 0$ by the boundary condition), that $\gamma_1 \operatorname{div} u = 0$. This allows us to conclude that $\operatorname{div} u = 0$, when λ is outside $\sigma((-\Delta)_N)$. For $\lambda = 0$, we can at first only conclude that $\operatorname{div} u$ equals a constant c, but since $\gamma_\nu u = 0$, $c \operatorname{vol}(\Omega) = (\operatorname{div} u, 1)_\Omega = -(\gamma_\nu u, 1)_\Gamma = 0$. Thus for $\lambda \notin \sigma((-\Delta)_N) \setminus \{0\}$, u must lie in $J_0(\Omega)$; and therefore $J_0(\Omega)$ is invariant under $R_{k,\lambda}$, for such λ.

It is now obvious that the restriction of $R_{k,\lambda}$ to $J_k(\Omega)$, $R'_{k,\lambda}$, is the solution operator for (8.4) on $J_k(\Omega)$, in each case. The estimates (6.22) imply the same estimates for $R'_{k,\lambda}$ by restriction.

Observe that $R'_{k,\lambda}$ is the resolvent of the operator A'_k defined in the Hilbert space $J_k(\Omega)$ by:

$$(8.6)\qquad A'_k u = \operatorname{pr}_{J_k} M_k u, \quad D(A'_k) = \{u \in J_k(\Omega) \cap H^2(\Omega)^n \mid T'_k u = 0\}.$$

In particular, A'_0 is the usual operator studied in connection with the Dirichlet problem, e.g. in the works mentioned in the introduction. The preceding analysis shows that the spectrum of A'_k outside $\sigma((-\Delta)_D)$ resp. $\sigma((-\Delta)_N) \setminus \{0\}$ *is contained in the spectrum of A_k*. Let us pursue this in the case $B = 0$, by some further observations:

Let $u \in D(A'_k)$, and define p such that $G_k u = \operatorname{grad} p$, then we find from Lemma 3.1, in view of the reductions described in Sections 4 and 5,

$$(8.7)\qquad \begin{aligned} ((M_k - \lambda)u, u)_\Omega &= (-\Delta u + \operatorname{grad} p - \lambda u, u)_\Omega \\ &= \left(L\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right) - \lambda \|u\|_0^2 \\ &= \begin{cases} \|\operatorname{grad} u\|_0^2 - \lambda \|u\|_0^2 & \text{for } k = 0, 2 \text{ or } 4, \\ E(u,u) - \lambda \|u\|_0^2 & \text{for } k = 1 \text{ or } 3. \end{cases} \end{aligned}$$

Denote the lowest eigenvalue of $(-\Delta)_D$ by a_D, it equals $\inf\{\|\operatorname{grad} u\|_0^2 \mid u \in H_0^1(\Omega)^n, \|u\|_0 = 1\}$ and is positive. Then for $k = 0$ and $u \in D(A_0')$,

$$\begin{aligned}\|(A_0' - \lambda)u\|_0 \|u\|_0 &= \|\mathrm{pr}_{J_0}(M_0 - \lambda)u\|_0 \|u\|_0 \\ &\geqq |((M_0 - \lambda)u, u)| = |\|\operatorname{grad} u\|_0^2 - \operatorname{Re}\lambda \|u\|_0^2 - i\operatorname{Im}\lambda \|u\|_0^2| \\ &\geqq c_\lambda \|u\|_0^2,\end{aligned}$$

with $c_\lambda > 0$ if $\operatorname{Re}\lambda < a_D$ or $\operatorname{Im}\lambda \neq 0$. Dividing by $\|u\|_0$, and using similar considerations for $k = 1, 2, 3, 4$, we conclude that there are positive constants c_λ such that

$$\|(A_k' - \lambda)u\|_0 = \|\mathrm{pr}_{J_k}(M_k - \lambda)u\|_0 \geqq c_\lambda \|u\|_0,$$

when $\lambda \notin [a_D, \infty]$ in the case $k = 0$, and when $\lambda \notin [0, \infty[$ in the other cases, so $A_k' - \lambda$ is injective then. Moreover, these values of λ are in the resolvent set of A_k', since the index of $A_k' - \lambda$ is 0 at each $\lambda \in \mathbf{C}$ (by continuation from the resolvent set), so we conclude that the spectrum of A_k' satisfies

$$(8.8) \qquad \sigma(A_k') \subset [a_D, \infty[\quad \text{for } k = 0, \quad \sigma(A_k') \subset [0, \infty[\quad \text{for } k = 1, 2, 3, 4$$

We also find that A_k' is symmetric as an operator in $J_k(\Omega)$, since $(A_k' u, u)$ is real; hence A_k' is selfadjoint since the deficiency indices are 0.

This means in particular that the resolvents $R_{k,\lambda}'$ satisfy estimates (6.22) on sets (6.23) *with* $a > 0$ if $k = 0$, *and with* $a = 0$ if $k = 1, 2, 3, 4$. In the latter cases, one can restrict $R_{k,\lambda}'$ further to the orthogonal complement J_k' of the zero eigenspace of A_k', to get estimates *with* $a > 0$ there. Finally, the Laplace transform method used in [G-S4, Sect. 6] can be used to carry this over to *global estimates over* Q_∞ *for the solution* $\{u, p\}$ *of* (5.41_k):

$$(8.9) \qquad \begin{aligned}&\|u\|_{L_2(\mathbf{R}_+; H^{r+2}(\Omega)^n \cap J_k) \cap H^{r/2+1}(\mathbf{R}_+; J_k)} + \|p\|_{H^{r+1, r/2}} + \|\operatorname{grad} p\|_{H^{r, r/2}} \\ &\qquad \leqq C \|f\|_{L_2(\mathbf{R}_+; H^r(\Omega)^n \cap J_k) \cap H^{r/2}(\mathbf{R}_+; J_k)},\end{aligned}$$

when φ_k *and* u_0 *are 0, the compatibility conditions are satisfied, and, if* $k = 1, 2, 3, 4$, $f \in L_2(\mathbf{R}_+; J_k')$.

This of course also has consequences for problems with nonzero boundary- and initial data; and it implies a uniformity for the constants in the estimates of solutions of the linear problems on Q_b for $b \to \infty$. It also allows the inclusion of the case $I =]0, \infty[$ in the treatment of the nonlinear Dirichlet problem, when $r + 2 > n/2$.

Precisions on the zero eigenspace can be found in [S-Šs] and [S6].

The ps.d.o. calculus allows us to derive very easily a spectral estimate for each of the operators A_k', $k = 1, 2, 3, 4$, from a fine estimate of Kozevnikov [K] for A_0', in the case $B = 0$. He showed that the counting function for the Dirichlet operator, $N(A_0'; t)$ = the number of eigenvalues of A_0' in $[0, t]$, satisfies the asymptotic estimate

$$N(A_0'; t) = C_0 t^{n/2} + \mathcal{O}(t^{(n-1)/2}) \quad \text{for } t \to \infty, \tag{8.10}$$

where $C_0 = (2\pi)^{-n}(n-1)\operatorname{vol}(\{|x| < 1\})\operatorname{vol}(\Omega)$. Let c be a positive constant for which $-c$ is in the resolvent set of all of the operators A_k. Then we have in view of the preceding observations:

$$(A_k' + c)^{-N} = \operatorname{pr}_{J_k}(A_k + c)^{-N} \quad \text{on } J_k,$$

for any integer $N \geqq 1$. In particular, for $k = 1, 2, 3, 4$,

$$\begin{aligned} (A_k' + c)^{-N} &= \operatorname{pr}_{J_k}((A_k + c)^{-N} - (A_0 + c)^{-N}) + \operatorname{pr}_{J_k}(A_0 + c)^{-N} \\ &= \operatorname{pr}_{J_0}(A_0 + c)^{-N}\operatorname{pr}_{J_0} + G_N \quad \text{on } J_k, \end{aligned} \tag{8.11}$$

where

$$\begin{aligned} G_N = {} & \operatorname{pr}_{J_k}((A_k + c)^{-N} - (A_0 + c)^{-N}) \\ & + (\operatorname{pr}_{J_k} - \operatorname{pr}_{J_0})(A_0 + c)^{-N} + \operatorname{pr}_{J_0}(A_0 + c)^{-N}(\operatorname{pr}_{J_k} - \operatorname{pr}_{J_0}) \end{aligned}$$

is a singular Green operator of order $-2N$ and class 0 (since $(A_k + c)^{-N}$ and $(A_0 + c)^{-N}$ have the same ps.d.o. part, and $\operatorname{pr}_{J_k} - \operatorname{pr}_{J_0}$ is a s.g.o.). The s-numbers of G_N in $L_2(\Omega)^n$ (the eigenvalues s_j of the compact operator $(G_N^* G_N)^{1/2}$ satisfy

$$s_j(G_N) \leqq C_1 j^{-2N/(n-1)} \tag{8.12}$$

(as shown in [G5]), and this implies a similar estimate for the operator on J_k. Since (8.10) is equivalent with the validity of

$$s_j((A_0' + c)^N) = C_0^{2N/n} j^{-2N/n} + \mathcal{O}(j^{-(2N+1)/n}) \quad \text{for } j \to \infty$$

([G4, Lemma A.5]), we can apply a perturbation argument ([G4, Lemma A.6]), which gives that

$$s_j((A_k' + c)^N) = C_0^{2N/n} j^{-2N/n} + \mathcal{O}(j^{-(2N+\theta)/n}), \quad \text{for } j \to \infty,$$

where $\theta = (1 + (n-1)/2N)^{-1}$. By carrying this over to an estimate on $N(A_k'; t)$ (again using [G4, Lemma A.5]), we get, since N can be taken arbitrarily large:

COROLLARY 8.2. *For each of the operators A_k', $k = 1, 2, 3, 4$ (with $B = 0$), one has the spectral estimate*

$$N(A_k'; t) = C_0 t^{n/2} + \mathcal{O}(t^{(n-\theta)/2}) \quad \text{for } t \to \infty; \tag{8.13}$$

for any $\theta < 1$.

Appendix A: Normal and tangential components.

Near Γ there is defined a normal vector field $\vec{n}(x) = (n_1(x), \dots, n_n(x))$, as follows: For $x_0 \in \Gamma$, $\vec{n}(x_0)$ is the unit vector normal to Γ, pointing towards the interior of Ω, and we set

(A.S) $$\vec{n}(x) = \vec{n}(x_0) \text{ for } x \text{ of the form } x = x_0 + s\vec{n}(x_0) \equiv \lambda(x_0, s), \quad \text{where } x_0 \in \Gamma, s \in]-\delta, \delta[.$$

Here $\delta > 0$ is taken so small that the representation of x in terms of $x_0 \in \Gamma$ and $s \in]-\delta, \delta[$ is unique and smooth, i.e., λ is bijective and is C^∞ with C^∞ inverse, from $\Gamma \times]-\delta, \delta[$ to the set

$$\Sigma = \lambda(\Gamma \times]-\delta, \delta[) \subset \mathrm{R}^n.$$

When $\kappa_j : U_j \to V_j, j = 1, \dots, j_0$, is a system of local coordinates for Γ (here $U_j \subset \Gamma$ and $V_j \subset \mathrm{R}^{n-1}$, with κ_j bijective and C^∞ from U_j to V_j, and $\Gamma = \bigcup_1^{j_0} U_j$), then the following composed mappings μ_j, going from $V_j \times]-\delta, \delta[$ to $\Sigma_j = \lambda(U_j \times]-\delta, \delta[)$,

$$\mu_j : V_j \times]-\delta, \delta[\xrightarrow{\kappa_j^{-1} \times \mathrm{id}} U_j \times]-\delta, \delta[\xrightarrow{\lambda} \Sigma_j,$$

are diffeomorphisms (in R^n); they define a particularly convenient system of local coordinates $\mu_j^{-1} : \Sigma_j \to V_j \times]-\delta, \delta[$ for the neighborhood $\Sigma (= \bigcup_{j=1}^{j_0} \Sigma_j)$ of Γ in R^n.

(We note in passing that the mapping $\lambda^{-1} : \Sigma \to \Gamma \times]-\delta, \delta[$ defines a function ϱ from Σ to $]-\delta, \delta[$ for which the level surfaces $\{x \mid \varrho(x) = s\}$ are precisely the surfaces $\Gamma_s = \lambda(\Gamma \times \{s\})$ "parallel" to Γ (in particular, $\Gamma = \Gamma_0 = \{x \mid \varrho(x) = 0\}$); and $\vec{n} = (\partial_1 \varrho, \dots, \partial_n \varrho)$.)

The derivative along $\vec{n}$ is denoted ∂_ν (the *normal derivative*):

(A.2) $$\partial_\nu f = \textstyle\sum_{j=1}^n n_j(x) \partial_j f(x),$$

defined for $x \in \Sigma$. For the function $f \circ \mu_j$ of $(y', s) \in V_j \times]-\delta, \delta[$, we have by the chain rule

$$\partial_s(f \circ \mu_j) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial s} = \sum_{i=1}^n n_i(x) \partial_{x_i} f,$$

so the normal derivative is simply the s-derivative in the local coordinate system defined by μ_j. Note in particular that the normal derivative of $\vec{n}$ itself is zero,

(A.3) $$\partial_\nu n_i = 0 \quad \text{for } i = 1, \dots, n,$$

since $\vec{n}$ is constant along the rays $x_0 + s\vec{n}(x_0)$.

We call differential operators *tangential* when they are of the form

$$\text{(A.4)}\quad Af = \textstyle\sum_{i=1}^{n} a_i(x)\partial_{x_i} f + a_0(x)f \text{ with } \sum_{i=1}^{n} a_i(x)n_i(x) = 0 \text{ for } x \in \Sigma,$$

or are compositions of such operators. They "act along" the surfaces Γ_s parallel to Γ, since they correspond, in the local coordinates $(y', s) \in V_j \times\,]-\delta, \delta[$ defined by the μ_j, to (compositions of) operators of the form

$$\text{(A.5)}\qquad \textstyle\sum_{k=1}^{n-1} b_k(y', s)\partial_{y_k}(f \circ \mu_j)(y', s) + b_0(y', s)(f \circ \mu_j)(y', s),$$

differentiating with respect to $y' \in V_j$ only. Note that when A is a first order tangential differential operator (as in (A.4)), then

$$\text{(A.6)}\qquad \partial_\nu A f = A\partial_\nu f + A_1 f,$$

where A_1 is another first order tangential operator (as is easily seen using the local coordinates).

We set

$$\text{(A.7)}\qquad \gamma_k f = \gamma_0(\partial_\nu)^k f \equiv (\partial_\nu^k f)|_\Gamma, \quad \text{for } k = 0, 1, 2, \ldots,$$

the k-th normal derivative of f at Γ. We recall that the mapping γ_k is well-defined as a continuous mapping $\gamma_k : H^r(\Omega) \to H^{r-k-1/2}(\Gamma)$ for $r > 1/2$, also some generalizations are possible. We assume in the following that the functions are smooth enough for the formulas to make sense (more precision is given when they are applied in Sections 2–8).

The fact that $\gamma_0 D_{y_i} u = D_{y_i}\gamma_0 u$, $i = 1, \ldots, n-1$, when u is considered in the local coordinates $(y', s) \in V_j \times\,]0, \delta[$, carries over to Ω as the observation that when A is a *tangential* differential operator, then

$$\text{(A.8)}\qquad \gamma_0(Au) = A_\Gamma \gamma_0 u,$$

where A_Γ is a differential operator acting in Γ (obtained from A by restricting the coefficients to Γ).

We shall also consider complex vector fields $v(x) = (v_1(x), \ldots, v_n(x))$ defined for $x \in \mathbb{R}^n$ or a subset (they can be considered as sections of $\mathbb{R}^n \times \mathbb{C}^n$, the complexified tangent bundle, i.e. as 1-forms). When $v(x)$ is defined on Σ, it is decomposed into a normal component $v_\nu \vec{n}$ (where v_ν is a scalar function) and a tangential component v_τ, defined at each x by

$$\text{(A.9)}\qquad \begin{aligned} v_\nu &= \vec{n}\cdot v = \textstyle\sum_{j=1}^{n} n_j v_j, \\ v_\tau &= v - (\vec{n}\cdot v)\vec{n} = (\textstyle\sum_{j=1}^{n} a_{ij} v_j)_{i=1,\ldots,n}, \\ &\quad \text{with } a_{ij} = \delta_{ij} - n_i n_j = a_{ji}, \text{ for } i, j = 1, \ldots, n. \end{aligned}$$

(Here δ_{ij} is the Kronecker delta.) Correspondingly, the vector bundle $\Sigma \times \mathbb{C}^n$ splits into what we can call the normal bundle F_ν and the tangential bundle F_τ:

$$\Sigma \times \mathbf{C}^n = F_\tau \oplus F_\nu; \text{ where}$$
$$\text{(A.10)} \qquad F_\tau = \{(x,z) \in \Sigma \times \mathbf{C}^n \mid \vec{n}(x) \cdot z = 0\},$$
$$F_\nu = \{(x,z) \in \Sigma \times \mathbf{C}^n \mid z \in \mathbf{C}\,\vec{n}(x)\}, \text{ restricted to } \Gamma \text{ as}$$
$$\Gamma \times \mathbf{C}^n = F_{\tau,\Gamma} \oplus F_{\tau,\Gamma}.$$

(For precision, we sometimes denote $v_\nu = \mathrm{pr}_{F_\nu} v$, $v_\tau = \mathrm{pr}_{F_\tau} v$.) Of course, $(\vec{n})_\nu = 1$ and $(\vec{n})_\tau = 0$. Observe also that the vector $(a_{ij})_{j=1,\dots,n}$ is orthogonal to $\vec{n}$ for all i, in fact

$$\text{(A.11)} \qquad \left(\textstyle\sum_{j=1}^n a_{ij} n_j\right)_{i=1,\dots,n} = \vec{n} - (\vec{n}\cdot\vec{n})\vec{n} = 0 = \left(\textstyle\sum_{j=1}^n a_{ji} n_j\right)_{i=1,\dots,n}.$$

When $\Omega = \mathbf{R}^n_+$, one simply has that $v_\nu = v_n$ and $v_\tau = (v_1, \dots, v_{n-1}, 0)$, where the latter is usually identified with $v' = (v_1, \dots, v_{n-1})$.

Clearly,

$$\gamma_0 v_\nu = (\gamma_0 v)_\nu \quad \text{and} \quad \gamma_0 v_\tau = (\gamma_0 v)_\tau.$$

Because of (A.3), one has

$$\text{(A.12)} \qquad \begin{aligned} \partial_\nu v_\nu &= \partial_\nu(\vec{n}\cdot v) = (\partial_\nu \vec{n})\cdot v + \vec{n}\cdot\partial_\nu v = \vec{n}\cdot\partial_\nu v = (\partial_\nu v)_\nu, \\ \partial_\nu^k v_\nu &= \partial_\nu^{k-1}(\partial_\nu v)_\nu = \cdots = (\partial_\nu^k v)_\nu, \\ \partial_\nu v_\tau &= \partial_\nu(v - v_\nu \vec{n}) = \partial_\nu v - (\partial_\nu v)_\nu \vec{n} = (\partial_\nu v)_\tau, \\ \partial_\nu^k v_\tau &= \partial_\nu^{k-1}(\partial_\nu v)_\tau = \cdots = (\partial_\nu^k v)_\tau, \end{aligned}$$

so that moreover,

$$\text{(A.13)} \qquad \begin{aligned} \gamma_k v_\nu &= \gamma_0(\partial_\nu^k v_\nu) = \gamma_0(\partial_\nu^k v)_\nu = (\gamma_k v)_\nu, \\ \gamma_k v_\tau &= \gamma_0(\partial_\nu^k v_\tau) = \gamma_0(\partial_\nu^k v)_\tau = (\gamma_k v)_\tau \quad \text{for } k = 0, 1, 2, \dots \end{aligned}$$

On Σ, each differentiation ∂_i may be split into a normal and a tangential differentiation (as defined in (A.2) and (A.4)), cf. (A.9), (A.11),

$$\text{(A.14)} \qquad \begin{aligned} \partial_i f &= n_i \partial_\nu f + \textstyle\sum_{j=1}^n (\delta_{ij} - n_i n_j)\partial_j f = n_i \partial_\nu f + A_i f, \quad \text{for each } i, \text{ where} \\ A_i f &= \textstyle\sum_{j=1}^n a_{ij}\partial_j f = \sum_{j=1}^n (\delta_{ij} - n_i n_j)\partial_j f \text{ is a tangential derivative.} \end{aligned}$$

In particular, one has in view of (A.7), (A.8),

$$\text{(A.15)} \qquad \gamma_0 \partial_i f = n_i \gamma_1 f + A_{i,\Gamma}\gamma_0 f.$$

The divergence of a vector field v, the gradient of a function f, and the Laplace operator applied to f, are defined as usual by:

$$\begin{aligned} \operatorname{div} v &= \nabla \cdot v = \partial_1 v_1 + \cdots + \partial_n v_n, \\ \operatorname{grad} f &= \nabla f = (\partial_i f)_{i=1,\ldots,n}, \qquad \text{(A.16)} \\ \Delta f &= \partial_1^2 f + \cdots + \partial_n^2 f = \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f. \end{aligned}$$

Under coordinate changes, these differential operators of course change form; observe however that a linear, orthogonal transformation of the x variable, applied also to the vector fields v and $\operatorname{grad} f$, leads to the same expressions in the new coordinates.

It will be convenient for our calculations to decompose div and grad on Σ into their normal and tangential differential operator parts, on the basis of (A.14):

$$\begin{aligned} \operatorname{div} v &= \textstyle\sum_{i=1}^n \partial_i v_i = \sum_{i=1}^n n_i \partial_\nu v_i + \sum_{i=1}^n A_i v_i \\ &= \vec{n} \cdot \partial_\nu v + \operatorname{div}' v = (\partial_\nu v)_\nu + \operatorname{div}' v = \partial_\nu v_\nu + \operatorname{div}' v; \qquad \text{(A.17)} \\ \operatorname{grad} f &= (\partial_i f)_{i=1,\ldots,n} = (n_i \partial_\nu f + A_i f)_{i=1,\ldots,n} = \vec{n} \partial_\nu f + \operatorname{grad}' f; \end{aligned}$$

here we have introduced the notation

$$\operatorname{div}' v = \textstyle\sum_{i=1}^n A_i v_i, \quad \operatorname{grad}' f = (A_i f)_{i=1,\ldots,n}, \qquad \text{(A.18)}$$

and used (A.12). Note that in view of (A.2) and (A.9), $\operatorname{grad} f$ satisfies

$$(\operatorname{grad} f)_\nu = \partial_\nu f, \quad (\operatorname{grad} f)_\tau = \operatorname{grad}' f. \qquad \text{(A.19)}$$

When f is replaced by a vector $u = (u_1, \ldots, u_n)$, the operators are applied to each component u_i.

Concerning the relation of $\operatorname{div} v$ to v_ν and v_τ, one finds furthermore

$$\begin{aligned} \operatorname{div}' v &= \operatorname{div} v - \partial_\nu v_\nu = \operatorname{div}(v_\tau + v_\nu \vec{n}) - \partial_\nu v_\nu \qquad \text{(A.20)} \\ &= \operatorname{div} v_\tau + \textstyle\sum_{i=1}^n (\partial_i n_i) v_\nu + \sum_{i=1}^n n_i \partial_i v_\nu - \partial_\nu v_\nu = \operatorname{div} v_\tau + (\operatorname{div} \vec{n}) v_\nu; \end{aligned}$$

here the coefficient $\operatorname{div} \vec{n}(x)$ (which, by the way, equals the mean curvature $\sum_{j=1}^n \partial_j^2 \varrho(x)$ of the surface Γ_s going through x) is generally nonvanishing, so $\operatorname{div} v_\tau$ and $\operatorname{div}' v$ differ by a zero order term. A further application shows

$$\begin{aligned} &\operatorname{div}' v_\tau = \operatorname{div}(v_\tau)_\tau + (\operatorname{div} \vec{n})(v_\tau)_\nu = \operatorname{div} v_\tau, \quad \text{and hence} \qquad \text{(A.21)} \\ &\operatorname{div} v = \operatorname{div}' v_\tau + (\operatorname{div} \vec{n}) v_\nu + \partial_\nu v_\nu. \end{aligned}$$

All this gives for the boundary values (cf. (A.8)):

$$\begin{aligned} \gamma_0 \operatorname{div} v &= \operatorname{div}'_\Gamma \gamma_0 v + \vec{n} \cdot \gamma_1 v = \operatorname{div}'_\Gamma \gamma_0 v + \gamma_1 v_\nu \\ &= \operatorname{div}'_\Gamma \gamma_0 v_\tau + (\operatorname{div} \vec{n}) \gamma_0 v_\nu + \gamma_1 v_\nu, \qquad \text{(A.22)} \\ \gamma_0 \operatorname{div}' v &= \operatorname{div}'_\Gamma \gamma_0 v = \operatorname{div}'_\Gamma \gamma_0 v_\tau + (\operatorname{div} \vec{n}) \gamma_0 v_\nu, \quad \text{and} \\ \gamma_0 \operatorname{grad} f &= \operatorname{grad}'_\Gamma \gamma_0 f + \vec{n} \gamma_1 f; \end{aligned}$$

where the restriction of $\vec{n}$ and $\operatorname{div}\vec{n}$ to Γ is tacitly understood. When $\Omega = \mathbb{R}^n_+$, then $\operatorname{div}\vec{n} = 0$, and one simply identifies

$$\begin{aligned} &\operatorname{div}' v = \partial_1 v_1 + \cdots + \partial_{n-1} v_{n-1} = \operatorname{div}' v', \ \operatorname{div} v = \operatorname{div}' v' + \partial_n v_n, \\ &\operatorname{grad}' f = (\partial_1 f, \ldots, \partial_{n-1} f) = (\operatorname{grad} f)', \quad (\operatorname{grad} f)_\nu = \partial_n f = (\operatorname{grad} f)_n. \end{aligned} \tag{A.23}$$

More generally, when $\Omega = \{x \in \mathbb{R}^n \mid x \cdot \vec{n}_0 > 0\}$ for a fixed unit vector $\vec{n}_0$, $\operatorname{div} v$ and $\operatorname{grad} f$ decompose as in (A.23), when the original coordinates x are replaced by new coordinates y obtained by an orthogonal transformation carrying $\vec{n}_0$ into $(0, \ldots, 0, 1)$, and the vector fields are transformed in the same way.

We use the formulas in an analysis of the special boundary operator $\gamma_0(\Delta u)_\nu$, that is used in Section 5.

Lemma A.1. *Let $u = (u_1, \ldots, u_n)$. The operator A' defined on a neighborhood of Γ by*

$$\begin{aligned} A'u &= \vec{n} \cdot \operatorname{div}' \operatorname{grad}' u - \operatorname{div}' \operatorname{grad}'(\vec{n} \cdot u) + \operatorname{div}' \partial_\nu u_\tau - \partial_\nu \operatorname{div}' u_\tau \\ &= \textstyle\sum_{i,j=1}^n [n_i A_j^2 u_i - A_j^2(n_i u_i)] + \sum_{i,j=1}^n [A_i(n_j \partial_j u_{\tau,i}) - n_j \partial_j(A_i u_{\tau,i})], \end{aligned} \tag{A.24}$$

cf. (A.18), (A.14), *is a tangential first order differential operator, that vanishes when $\vec{n}$ is constant. It enters in the formula, valid for general u,*

$$\begin{aligned} \vec{n} \cdot \Delta u &= \partial_\nu \operatorname{div} u - \partial_\nu \operatorname{div}' u_\tau + \vec{n} \cdot \operatorname{div}' \operatorname{grad}' u - (\partial_\nu \operatorname{div} \vec{n}) u_\nu \\ &= \partial_\nu \operatorname{div} u - \operatorname{div}' \partial_\nu u_\tau + \operatorname{div}' \operatorname{grad}' u_\nu + A'u - (\partial_\nu \operatorname{div} \vec{n}) u_\nu. \end{aligned} \tag{A.25}$$

In particular,

$$\begin{aligned} \gamma_0(\Delta u)_\nu = \gamma_1 \operatorname{div} u - \operatorname{div}'_\Gamma \gamma_1 u_\tau + \operatorname{div}'_\Gamma \operatorname{grad}'_\Gamma \gamma_0 u_\nu \\ + A'_\Gamma \gamma_0 u - (\partial_\nu \operatorname{div} \vec{n}) \gamma_0 u_\nu; \end{aligned} \tag{A.26}$$

and if $\gamma_0 u_\nu = 0$, then

$$\gamma_0(\Delta u)_\nu - \gamma_1 \operatorname{div} u = -\operatorname{div}'_\Gamma \gamma_1 u_\tau + A'_\Gamma \gamma_0 u_\tau. \tag{A.27}$$

Proof. In the first two terms in the definition of A', $\operatorname{div}' \operatorname{grad}'$ is a tangential scalar second order operator, so the commutator between this and the multiplication by n_i gives a tangential first order operator for each i. That the commutator defined by the last two terms is likewise tangential, follows from (A.6). Clearly, the commutators are zero, when $\vec{n}$ is constant.

Now consider (A.25). For each j one has by (A.19) and (A.21),

$$\begin{aligned} \operatorname{div} \operatorname{grad} u_j &= \operatorname{div}' (\operatorname{grad} u_j)_\tau + (\operatorname{div} \vec{n})(\operatorname{grad} u_j)_\nu + \partial_\nu (\operatorname{grad} u_j)_\nu \\ &= \operatorname{div}' \operatorname{grad}' u_j + (\operatorname{div} \vec{n}) \partial_\nu u_j + \partial_\nu^2 u_j. \end{aligned} \tag{A.28}$$

Then we find for the vector u, in view of (A.12) and (A.21),

$$\begin{aligned}\vec{n}\cdot\operatorname{div}\operatorname{grad} u &= \vec{n}\cdot\operatorname{div}'\operatorname{grad}' u + (\operatorname{div}\vec{n})\,\partial_\nu u_\nu + \partial_\nu^2 u_\nu\\ &= \vec{n}\cdot\operatorname{div}'\operatorname{grad}' u + (\operatorname{div}\vec{n})\,\partial_\nu u_\nu + \partial_\nu(\operatorname{div} u - \operatorname{div}' u_\tau - (\operatorname{div}\vec{n})u_\nu)\\ &= \vec{n}\cdot\operatorname{div}'\operatorname{grad}' u + \partial_\nu \operatorname{div} u - \partial_\nu \operatorname{div}' u_\tau - (\partial_\nu \operatorname{div}\vec{n})u_\nu.\end{aligned}\tag{A.29}$$

This shows the first equation in (A.25), and the second equation follows by insertion of (A.24). Restriction to the boundary gives (A.26), and (A.27) is an obvious special case.

Recall finally the Gauss formula

$$\begin{aligned}&-\int_\Omega \partial_i f\,\bar{g}\,dx = \int_\Omega f\,\overline{\partial_i g}\,dx + \int_\Gamma n_i\gamma_0 f\,\overline{\gamma_0 g}\,d\sigma, \quad \text{implying e.g.}\\ &(-\Delta f, g)_{L_2(\Omega)} = (\operatorname{grad} f, \operatorname{grad} g)_{L_2(\Omega)^n} + (\gamma_1 f, \gamma_0 g)_{L_2(\Gamma)},\\ &(\operatorname{div} w, 1)_{L_2(\Omega)} = (\vec{n}\cdot\gamma_0 w, 1)_{L_2(\Gamma)}.\end{aligned}\tag{A.30}$$

A special calculation, we need in Section 5, is that

$$\begin{aligned}&\text{for } \eta = \gamma_0(\vec{n}\cdot\Delta u) - \gamma_1 \operatorname{div} u, \quad \text{one has}\\ &(\eta, 1)_\Gamma = \int_\Gamma \gamma_0\big(\textstyle\sum_{i,j=1}^n n_j\partial_i\partial_i v_j - \sum_{i,j=1}^n n_j\partial_j\partial_i v_i\big)\,d\sigma\\ &\qquad = \int_\Omega \textstyle\sum_{i,j=1}^n \partial_j\partial_i(\partial_i v_j - \partial_j v_i)\,dx = 0,\end{aligned}\tag{A.31}$$

by symmetry. The formulas are valid for smooth functions, and extend by continuity to Sobolev spaces and domains of differential operators, as accounted for e.g. in [L-M].

Appendix B: Nonlinear estimates.

We here show the anisotropic estimates of the nonlinear term $\mathsf{K}(u, v)$ used above. We also show the estimates that are used in Grubb-Solonnikov [G-S3] to deduce the solvability properties of the Navier-Stokes problems from those of the Stokes problems.

PROPOSITION B.1. *Let $\lambda, \mu, \omega, \nu \in \mathsf{R}$. One has for $f \in H^{\lambda+\mu}(\Omega)$ and $g \in H^{\lambda+\omega}(\Omega)$ (where Ω is a smooth open subset of R^n):*

$$\|fg\|_\lambda \leqq C\|f\|_{\lambda+\mu}\|g\|_{\lambda+\omega},$$

(B.1)
(i) *when $\mu + \omega + \lambda \geqq n/2$,*
(ii) *with $\mu \geqq 0, \omega \geqq 0, 2\lambda \geqq -\mu - \omega$;*
(iii) *except that $\mu + \omega + \lambda > n/2$ if equality holds somewhere in* (ii).

Thus the bilinear form $\mathsf{K}(u,v)=\sum_{j=1}^{n}u_j\partial_j v$ *satisfies:*

$$\|\mathsf{K}(u,v)\|_\lambda \leqq C\|u\|_{\lambda+\mu}\|v\|_{\lambda+1+\omega}, \text{ when (B.1 i,ii,iii) hold.} \tag{B.2}$$

In particular,

$$\begin{aligned}&\|\mathsf{K}(u,v)\|_\lambda \leqq C\|u\|_\nu\,\|v\|_{1+\nu};\\ &\text{when } 2\nu \geqq n/2+\lambda, \nu\geqq\lambda, \nu\geqq 0; \text{ with } 2\nu > n/2+\lambda \text{ if } \nu=\lambda \text{ or } \nu=0.\end{aligned} \tag{B.3}$$

This type of result has been known for a long time (the estimate (B.1) with $\lambda > n/2$ and $\mu=\omega=0$ goes back to Schauder). For $\lambda \geqq 0$ it can be shown by use of Sobolev and Hölder inequalities; and we had originally planned to present a proof of that, and use the result to derive the anisotropic estimates in the (x,t)-variables that we needed, namely (B.8) with σ close to 0, and Theorem B.4. However, there is another proof based on the Fourier transform (cf. Rauch [R]) that allows a sharper analysis of the possible exponents, as carried out in Hörmander [H2, Th. 8.3.1]. We generalize that proof to anisotropic spaces below, which gives a more precise information on σ in (B.8) (permitting $\sigma=1$ for large r), that we use to derive some useful supplementary information on $p(x,t)$ in Section 7.3.

It should also be mentioned that these estimates have been shown (except for an extreme case) by Yamazaki [Y, Th. 6.1] in the more complicated framework of L_p-related spaces (the scales $B^s_{p,q}$ and $F^s_{p,q}$, $p,q\in\,]0,\infty[$), as a byproduct of a thorough analysis of quasi-homogeneous paradifferential operators. However, we think that the short L_2 proof below may be of interest anyway. The extreme case where $\lambda+\mu=-\lambda-\omega$ is not covered by [Y].

REMARK B.2. Using that λ can take negative values of r and s, one can extend the proof of Theorem 6 of [G-S3] such that the condition $r>0$ is relaxed to $r\geqq 0$ there.

Before showing the theorem, we shall include anisotropic spaces with negative exponents $s, s/d$:

Denote $(x,t)=\tilde{x}$, $(\xi,\tau)=\tilde{\xi}$, etc. The space $H^{s,s/d}(\mathsf{R}^n\times\mathsf{R})$ can be defined for general $s\in\mathsf{R}$ as the space of distributions $f\in\mathscr{S}'(\mathsf{R}^{n+1})$ with finite norm

$$\begin{aligned}&\|f\|_{s,s/d}=\|\theta(\xi,\tau)^s\hat{f}(\xi,\tau)\|_{L_2(\mathsf{R}^{n+1})}, \text{ where}\\ &\theta(\xi,\tau)=(|\xi|^{2d}+\tau^2+1)^{1/2d}, \text{ also denoted } \theta(\tilde{\xi}).\end{aligned} \tag{B.4}$$

When $Q_I=\Omega\times I$, where Ω is a smooth open subset of R^n and I is an interval of R, the space $H^{s,s/d}(Q_I)$ consists of the restrictions to Q_I of elements of $H^{s,s/d}(\mathsf{R}^n\times\mathsf{R})$, provided with the infimum norm

$$\|f\|_{H^{s,s/d}(Q_I)}=\inf\{\|v\|_{H^{s,s/d}(\mathsf{R}^n\times\mathsf{R})} \mid v\in H^{s,s/d}(\mathsf{R}^n\times\mathsf{R}), v|_{Q_I}=f\}. \tag{B.5}$$

When $I =]0, b[$, the space $H^{s,s/d}_{(0)}(Q_I)$ consists of the elements of $H^{s,s/d}(\Omega \times]-\infty, b[)$ (with the norm from that space) vanishing for $t < 0$; it is a subset of $H^{s,s/d}(Q_I)$ when $s \geqq 0$.

THEOREM B.3. *Let $d \geqq 1$, and let $Q_I = \Omega \times I$, where Ω is a smooth open subset of R^n and I is an interval of R. One has for $f \in H^{\lambda+\mu,(\lambda+\mu)/d}(Q_I)$ and $g \in H^{\lambda+\omega,(\lambda+\omega)/d}(Q_I)$, with constants independent of Q_I:*

$$\|fg\|_{\lambda,\lambda/d} \leqq C\,\|f\|_{\lambda+\mu,(\lambda+\mu)/d}\,\|g\|_{\lambda+\omega,(\lambda+\omega)/d},$$

(B.6)
(i) *when $\mu + \omega + \lambda \geqq (n+d)/2$,*
(ii) *with $\mu \geqq 0$, $\omega \geqq 0$, $2\lambda \geqq -\mu - \omega$;*
(iii) *except that $\mu + \omega + \lambda > (n+d)/2$ if equality holds somewhere in* (ii).

Thus the bilinear form $\mathrm{K}(u, v) = \sum_{j=1}^n u_j \partial_j v$ satisfies:

(B.7)
$$\|\mathrm{K}(u,v)\|_{\lambda,\lambda/2} \leqq C\,\|u\|_{\lambda+\mu,(\lambda+\mu)/2}\,\|v\|_{\lambda+1+\omega,(\lambda+1+\omega)/2},$$
when (B.6 i,ii,iii) *hold with $d = 2$.*

In particular,

(B.8)
$$\|\mathrm{K}(u,v)\|_{r+\sigma,(r+\sigma)/2} \leqq C\,\|u\|_{r+2,r/2+1}\,\|v\|_{r+2,r/2+1};$$
when $r + 2 - \sigma \geqq n/2$, $r \geqq -3/2$, $\sigma \leqq 1$,

except that $r + 2 - \sigma > n/2$ if $r = -3/2$ or $\sigma = 1$.

When $I =]0, b[$ and $r + \sigma \geqq 0$, the estimate (B.8) *holds for $u \in H^{r+2,r/2+1}_{(0)}(Q_I)^n$, $v \in H^{r+2,r/2+1}(Q_I)^n$, when the norms on u resp. $\mathrm{K}(u,v)$ are replaced by norms in $H^{r+2,r/2+1}_{(0)}(Q_I)^n$ resp. $H^{r+\sigma,(r+\sigma)/2}_{(0)}(Q_I)^n$. This estimate also holds with $\mathrm{K}(u,v)$ replaced by $\mathrm{K}(v,u)$.*

PROOF. In view of (B.5) it suffices to consider the case $Q_I = \mathrm{R}^n \times \mathrm{R}$. Then the proof also covers Proposition B.1 when we take $d = 1$ (with an easy modification for $n = 1$ there).

Denote $\lambda + \mu = \lambda_1$ and $\lambda + \omega = \lambda_2$, and observe that the conditions in (B.6) can be written

(B.9)
(i) $\lambda_1 + \lambda_2 - \lambda \geqq (n+d)/2$,
(ii) with $\lambda_1 \geqq \lambda$, $\lambda_2 \geqq \lambda$, $\lambda_1 + \lambda_2 \geqq 0$;
(iii) except that $\lambda_1 + \lambda_2 - \lambda > (n+d)/2$ if λ_1, λ_2 or $-\lambda$ equals $(n+d)/2$.

We consider a set $\lambda, \lambda_1, \lambda_2$ such that (B.9) holds.
Recall first the elementary fact that if $T_F(f,g)$ is defined by

$$T_F(f,g)(\tilde{\xi}) = \int_{\mathrm{R}^{n+1}} F(\tilde{\xi}, \tilde{\eta}) f(\tilde{\xi} - \tilde{\eta}) g(\tilde{\eta})\, d\tilde{\eta}$$

for f and $g \in C_0^\infty(\mathsf{R}^{n+1})$, then

(B.10) $$\|T_F(f,g)\|_{L_2(\mathsf{R}^{n+1})} \leqq B\|f\|_{L_2(\mathsf{R}^{n+1})}\,\|g\|_{L_2(\mathsf{R}^{n+1})},$$

when either (a), (b) or (c) holds:

(B.11) $$\text{(a)}\ \int_{\mathsf{R}^{n+1}} |F(\tilde\xi,\tilde\eta)|^2\,d\tilde\eta \leqq B^2 \text{ for all } \tilde\xi,$$
$$\text{(b)}\ \int_{\mathsf{R}^{n+1}} |F(\tilde\xi,\tilde\eta)|^2\,d\tilde\xi \leqq B^2 \text{ for all } \tilde\eta,$$
$$\text{(c)}\ \int_{\mathsf{R}^{n+1}} |F(\tilde\xi,\tilde\xi-\tilde\eta)|^2\,d\tilde\xi \leqq B^2 \text{ for all } \tilde\eta.$$

It suffices to show (B.6) for $f, g \in \mathscr{F}^{-1}C_0^\infty(\mathsf{R}^{n+1})$, which is dense in all spaces. We have by (B.4) that $\|fg\|_{\lambda,\lambda/d} = \|\theta(\xi,\tau)^\lambda\,\widehat{fg}\|_{L_2(\mathsf{R}^{n+1})}$; here we can write

$$\theta(\tilde\xi)^\lambda \widehat{fg}(\tilde\xi) = (2\pi)^{-n-1}\int \theta(\tilde\xi)^\lambda \hat f(\tilde\xi-\tilde\eta)\hat g(\tilde\eta)\,d\tilde\eta$$
$$= (2\pi)^{-n-1}\int F(\tilde\xi,\tilde\eta)\hat f_1(\tilde\xi-\tilde\eta)\hat g_1(\tilde\eta)\,d\tilde\eta,$$

where

$$\hat f_1(\tilde\xi) = \theta(\tilde\xi)^{\lambda_1}\hat f(\tilde\xi),\ \hat g_1(\tilde\xi) = \theta(\tilde\xi)^{\lambda_2}\hat g(\tilde\xi),\ F(\tilde\xi,\tilde\eta) = \theta(\tilde\xi)^\lambda\theta(\tilde\xi-\tilde\eta)^{-\lambda_1}\theta(\tilde\eta)^{-\lambda_2},$$

so we have to show (B.10) for F, f_1 and g_1. To do this, we write F as a sum of four terms:

$$F = \chi_1 F + \chi_2 F + \chi_3 F + \chi_4 F, \quad \chi_i = 1_{M_i},$$

where

$$M_1 = \{(\tilde\xi,\tilde\eta) \mid \theta(\tilde\eta) \leqq \theta(\tilde\xi)/2\}, \quad M_2 = \{(\tilde\xi,\tilde\eta) \mid \theta(\tilde\xi-\tilde\eta) \leqq \theta(\tilde\xi)/2\},$$
$$M_3 = \{(\tilde\xi,\tilde\eta) \mid \theta(\tilde\xi) < \min\{2\theta(\tilde\eta), 2\theta(\tilde\xi-\tilde\eta)\}, \theta(\tilde\xi) \leqq 4\},$$
$$M_4 = \{(\tilde\xi,\tilde\eta) \mid \theta(\tilde\xi) < \min\{2\theta(\tilde\eta), 2\theta(\tilde\xi-\tilde\eta)\}, \theta(\tilde\xi) > 4\}.$$

Observe that with the change of variables $\pm\tau = \xi_{n+1}^d$, with (ξ, ξ_{n+1}) denoted ξ'',

(B.12) $$\int_{\mathsf{R}^{n+1}} \theta(\tilde\xi)^{2s}\,d\tilde\xi = 2d\int_{\mathsf{R}_+^{n+1}} (|\xi|^{2d} + \xi_{n+1}^{2d} + 1)^{s/d}\,\xi_{n+1}^{d-1}\,d\xi\,d\xi_{n+1}$$
$$\leqq C\int_{\mathsf{R}^{n+1}} \langle\xi''\rangle^{2s+d-1}\,d\xi'' \text{ is } < \infty$$

if and only if $s < -(n+d)/2$; and, on the other hand, (B.13) $$\int_{\theta(\tilde{\xi})\leqq a} \theta(\tilde{\xi})^{2s}\, d\tilde{\xi} \leqq C_1 \int_{\langle \xi''\rangle \leqq C_2 a} \langle \xi''\rangle^{2s+d-1}\, d\xi''$$

$$\leqq C_3 a^{2s+d+n} \text{ resp. } C_3 \log|a|, \text{ when } s > -(n+d)/2 \text{ resp. } s = -(n+d)/2.$$

Consider $\chi_1 F$; for this we show (a). Note that $\theta(\tilde{\xi}-\tilde{\eta}) \simeq \theta(\tilde{\xi})$ when $\theta(\tilde{\eta}) \leqq \theta(\tilde{\xi})/2$. Then if $\lambda_2 < (n+d)/2$,

$$\int_{\mathrm{R}^{n+1}} |\chi_1 F(\tilde{\xi},\tilde{\eta})|^2\, d\tilde{\eta} \simeq \int_{\theta(\tilde{\eta})\leqq\theta(\tilde{\xi})/2} \theta(\tilde{\xi})^{2\lambda-2\lambda_1}\theta(\tilde{\eta})^{-2\lambda_2}\, d\tilde{\eta} \leqq C\theta(\tilde{\xi})^{2(\lambda-\lambda_1-\lambda_2)+d+n} \leqq C',$$

by (B.13) and (B.9i); and if $\lambda_2 = (n+d)/2$, the expression is likewise bounded by (B.13) and (B.9i) since $\lambda_1 > \lambda$ by (B.9iii). If $\lambda_2 > (n+d)/2$, the integral is bounded because of (B.12) and (B.9ii).

For the term $\chi_2 F$ one shows (c) in a very similar way.

Now consider $\chi_3 F$ and $\chi_4 F$; here we shall estimate (b). For $\chi_3 F$ we note that since $\theta(\tilde{\xi}-\tilde{\eta})/\theta(\tilde{\eta})$ and its inverse are uniformly bounded in $\tilde{\eta}$, $\lambda_1+\lambda_2 \geqq 0$ implies that the integral over $\theta(\tilde{\xi}) \leqq 4$ is uniformly bounded. Finally consider $\chi_4 F$. Since $\theta(\tilde{\xi}) \geqq 4$ on M_4, $\theta(\tilde{\eta}) \geqq 2$ and $\theta(\tilde{\xi}-\tilde{\eta}) \geqq 2$ there. Expressed in relation to the coordinates ξ'', η'', we have that $\theta(\tilde{\xi}) \simeq |\xi''|$, $\theta(\tilde{\eta}) \simeq |\eta''|$, $\theta(\tilde{\xi}-\tilde{\eta}) \simeq |\xi''-\eta''|$ on M_4; and more precisely:

$$|\xi''|^{2d}+1 \geqq 4^{2d}, \quad |\eta''|^{2d}+1 \geqq 2^{2d}, \quad |\xi''-\eta''|^{2d}+1 \geqq 2^{2d},$$

$$|\xi''|^{2d}+1 \leqq 2^{2d}(|\eta''|^{2d}+1), \quad |\xi''|^{2d}+1 \leqq 2^{2d}(|\xi''-\eta''|^{2d}+1) \text{ on } M_4.$$

This implies that there exist positive constants C_1, C_2, C_3 and C_4 such that

$$C_1 \geqq \frac{|\xi''|}{|\eta''|} \geqq C_2 \frac{1}{|\eta''|} \text{ and } C_3 \geqq \left|\frac{\xi''}{|\eta''|} - \frac{\eta''}{|\eta''|}\right| \geqq C_4 \frac{|\xi''|}{|\eta''|} \text{ on } M_4.$$

Denote $\xi''/|\eta''| = \zeta''$ and $\eta''/|\eta''| = \varrho$, note that $|\varrho| = 1$. When $|\zeta''-\varrho| = r$ holds with $r \leqq 1$, then

$$r = |\zeta''-\varrho| \geqq C_4|\zeta''| = C_4|\varrho+\zeta''-\varrho| \geqq C_4\big||\varrho| - |\zeta''-\varrho|\big| = C_4(1-r),$$

hence $r \geqq C_4/(1+C_4)$. Thus

(B.14) $$C_1 \geqq |\zeta''| \geqq C_2|\eta''|^{-1} \text{ and } C_3 \geqq |\zeta''-\eta''/|\eta''|| \geqq C_4/(1+C_4) \text{ on } M_4.$$

The integral of $\chi_4 F$ gives in these coordinates:

$$\int |\chi_4 F(\tilde{\xi},\tilde{\eta})|^2\, d\tilde{\xi} \leqq C \int_{(\tilde{\xi},\tilde{\eta})\in M_4\cap \mathrm{R}^{n+1}_+} |\xi''|^{2\lambda}|\xi''-\eta''|^{-2\lambda_1}|\eta''|^{-2\lambda_2}\xi_{n+1}^{d-1}\, d\xi''$$

$$= C|\eta''|^{2(\lambda-\lambda_1-\lambda_2)+d+n} \int_{(\tilde{\xi},\tilde{\eta})\in M_4\cap \mathrm{R}^{n+1}_+} |\zeta''|^{2\lambda}|\zeta''-\eta''/|\eta''||^{-2\lambda_1}\zeta_{n+1}^{d-1}\, d\zeta''.$$

From (B.14) we see that the factor with $|\zeta'' - \eta''/\eta''|$ is harmless. The integral of $|\zeta''|^{2\lambda}\zeta_{n+1}^{d-1}$ is bounded when $\lambda > -(n+d)/2$, so (b) holds in view of (B.9i); it is $O(|\eta''|^{-d-n-2\lambda})$ when $\lambda < -(n+d)/2$ (cf. (B.14)), so (b) holds since $\lambda_1 + \lambda_2 \geqq 0$; and when $\lambda = -(n+d)/2$ it gives $O(\log|\eta''|)$, so (b) holds since $\lambda_1 + \lambda_2 > 0$, cf. (B.9iii).

Altogether, we have obtained the estimate in (B.6), and (B.7) and (B.8) are straightforward consequences.

The inequalities hold in particular for $I =]-\infty, b[$; here if u vanishes for $t < 0$, so does $\mathsf{K}(u,v)$. We get the last statement in the theorem by applying (B.8) to $u \in H_{(0)}^{r+2,r/2+1}(Q_I)^n$ and $v' \in H^{r+2,r/2+1}(\Omega \times]-\infty, b[)^n$, where $v'|_{t>0} = v$, with the norm of v' arbitrarily close to that of v; here $\mathsf{K}(u,v') = \mathsf{K}(u,v)$.

The following simple consequence is also used in [G-S3].

THEOREM B.4. *Let $I =]0,b[$, $b \leqq \infty$, and let $r \geqq 0$ and $r + 2 > n/2$. When $u \in H^{r+2,r/2+1}(Q_I)^n$ and $v \in H^{r+2,r/2+1}(Q_I)^n$, one has for any $\varepsilon > 0$, with constants independent of b,*

$$\begin{aligned} &\|\mathsf{K}(u,v)\|_{H_{(0)}^{r,r/2}(Q_I)^n} + \|\mathsf{K}(v,u)\|_{H_{(0)}^{r,r/2}(Q_I)} \\ \text{(B.15)}\quad &\leqq (\varepsilon\|u\|_{H_{(0)}^{r+2,r/2+1}(Q_I)^n} + C_\varepsilon\|u\|_{L_2(Q_I)^n})\|v\|_{H_{(0)}^{r+2,r/2+1}(Q_I)^n} \\ &\leqq \Big(2\varepsilon\|u\|_{H_{(0)}^{r+2,r/2+1}(Q_I)^n} + C'_\varepsilon\int_0^b \|u\|_{H_{(0)}^{r+2,r/2+1}(\Omega\times]0,t[)^n}\,dt\Big)\|v\|_{H^{r+2,r/2+1}(Q_I)^n}. \end{aligned}$$

PROOF. It is easily seen from (B.4) that when $r \geqq 0$ and $0 < \sigma < 2$,

$$\|u\|_{r+\sigma,(r+\sigma)/2} \leqq \varepsilon\|u\|_{r+2,r/2+1} + C(\varepsilon)\|u\|_{0,0};$$

then the first inequality follows from the $H_{(0)}$ variant of (B.8). For the second inequality, we observe that (recalling (2.1))

$$\begin{aligned} &\|u\|_{L_2(Q_I)^n} = \left(\int_0^b \|r_t u\|^2_{L(\Omega_I)^n}\,dt\right)^{1/2} \leqq \sup_{t\in I}\|r_t u\|^{1/2}_{L_2(\Omega)^n}\left(\int_0^b \|r_t u\|_{L_2(\Omega)^n}\,dt\right)^{1/2} \\ \text{(B.16)}\quad &\leqq \delta\sup_{t\in I}\|r_t u\|_{L_2(\Omega)^n} + \frac{1}{4\delta}\int_0^b \|r_t u\|_{L_2(\Omega)^n}\,dt; \end{aligned}$$

for any $\delta > 0$. One has in general that $\|r_t f\|_{L_2(\Omega)} \leqq C_0\|f\|_{H^{r+2,r/2+1}(\Omega\times]-\infty,t[)}$ for any $t \in \mathsf{R}$, with C_0 independent of t (recall that $r \geqq 0$). We apply this fact to u and insert (B.16) with $\delta = \varepsilon/C_0C_\varepsilon$ in the first inequality in (B.15), then we get the second inequality.

REFERENCES

[A] S. Agmom, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. 15 (1962), 119–147.

[A-D-N] S. Agmon, A. Douglas and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. II, Comm. Pure Appl. Math. 17 (1964), 35–92.

[A-V] M. S. Agranovič and M. I. Vishik, *Elliptic problems with a parameter and parabolic problems of a general type*, Uspekhi. Mat. Nauk 19 (1963), 53–161; = Russian Math. Surveys 19 (1963), 53–159.

[BM] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. 126 (1971), 11–51.

[F-L-R] E. B. Fabes, J. E. Lewis and N. M. Riviere, *Boundary value problems for the Navier-Stokes equations*, Amer. J. Math. 99 (1977), 626–668.

[F-K] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. 16 (1964), 269–315.

[Gi1] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in L_r-spaces*, Math. Z. 178 (1981), 297–329.

[Gi2] Y. Giga, *The nonstationary Navier-Stokes system with some first order boundary conditions*, Proc. Jap. Acad. 58 (1982), 101–104.

[G1] G. Grubb, *Weakly semibounded boundary problems and sesquilinear forms*, Ann. Inst. Fourier Grenoble 23 (1973), 145–194.

[G2] G. Grubb, *Boundary problems for systems of partial differential operators of mixed order*, J. Funct. Anal. 26 (1977), 131–165.

[G3] G. Grubb, *On coerciveness of Douglis-Nirenberg elliptic boundary value problems*, Boll. Un. Mat. Ital. (5) 16-B (1979), 1049–1080.

[G4] G. Grubb, *Functional Calculus of Pseudo-Differential Boundary Problems*, Progr. Math., Vol. 65, Birkhäuser, Boston, 1986.

[G5] G. Grubb, *Singular Green operators and their spectral asymptotics*, Duke Math. J. 51 (1984), 477–528.

[G6] G. Grubb, *Pseudo-differential boundary problems in L_p spaces*, Comm. Partial Differential Equations 15 (1990), 289–340.

[G7] G. Grubb, *Solution dans les espaces de Sobolev L_p anisotropes des problèmes aux limites pseudo-différentiels paraboliques et des problèmes de Stokes*, C. R. Acad. Sci. Paris 312, Série I (1991), 89–92.

[G-H] G. Grubb and L. Hörmander, *The transmission property*, Math. Scand. 67 (199), 273–289.

[G-K] G. Grubb and N. J. Kokholm, *Calcul symbolique et estimations L_p pour les problèmes aux limites pseudo-différentiels paramétrisés*, C. R. Acad. Sci. Paris 312, Série I (1991), 223–226.

[G-S1] G. Grubb and V. A. Solonnikov, *Reduction of fundamental initial-boundary value problems for the Stokes equation to parabolic initial-boundary value problems for systems of pseudodifferential equations*, Zap. Nauchn. Sem. L.O.M.I. 163 (1987), 37–48; = J. Soviet Math. 49 (1990), 1140–1147.

[G-S2] G. Grubb and V. A. Solonnikov, *A pseudo-differential treatment of general inhomogeneous initial-boundary value problems for the Navier-Stokes equation, Exposé no. 3*, Proc. of Journées Equations Dér. Part., St. Jean de Monts 1988, Ecole Polytechnique, Palaiseau.

[G-S3] G. Grubb and V. A. Solonnikov, *Reduction of basic initial-boundary value problems for the Navier-Stokes equations to nonlinear parabolic systems of pseudodifferential equations*, Zap. Nauchn. Zap. Nauchn. Sem. L.O.M.I. 171 (1989), 36–52; English transl. available as Copenh. Univ. Math. Dept. Report Ser. 1989 no. 5.

[G-S4] G. Grubb and V. A. Solonnikov, *Solution of parabolic pseudo-differential initial-boundary value problems*, J. Differential Equations 87 (1990), 256–304.

[H1] L. Hörmander, *Linear Partial Differential Operators*, Springer Verlag, Berlin, New York, 1963.

[H2] L. Hörmander, *Non-Linear Hyperbolic Differential Equations*, Lectures 1986–1987, Report 1988-2, Department of Mathematics, Lund.

[K] A. N. Kozhevnikov, *On the operator of the linearized steady-state Navier-Stokes problem*, Mat. Sbornik 125 (167) (1984), 3–18; = Math. USSR Sbornik 53 (1984), 1–16.

[L] O. A. Ladyzhenskaya, *Mathematical Problems of Viscous Incompressible Flow*, "Nauka", Moscow, 1970.

[L-S] O. A. Ladyzhenskaya and V. A. Solonnikov, *Some problems of vector analysis, and generalized formulation of boundary value problems for the Navier-Stokes equation*, Zap. Nauchn. Sem. LOMI 59 (1976), 81–116; = J. Soviet Math. 10 (1978), 254–285.

[L-M] J.-L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications. Vol. 1 et 2*, Editions Dunod, Paris, 1968.

[Mi] T. Miyakawa, *The L^p approach to the Navier-Stokes equations with the Neumann boundary condition*, Hiroshima Math. J. 10 (1980), 517–537.

[Mo] I. Sh. Mogilevskiĭ, *Estimates for solutions of a general initial-boundary value problem for the linear nonstationary system of Navier-Stokes equations in a bounded domain*, Proc. Steklov Inst. Math. 159 (1982), 61–94.

[O] C. W. Oseen, *Hydrodynamik*, Akademische Verlagsgesellschaft M. B. H., Leipzig, 1927.

[Pi] A. Piriou, *Problèmes aux limites généraux pour des opérateurs différentiels paraboliques dans un domaine borné*, Ann. Inst. Fourier 21 (1971), 59–78.

[R] J. Rauch, *Singularities of solutions to semilinear wave equations*, J. Math. Pures Appl. 58 (1979), 299–308.

[S1] V. A. Solonnikov, *Estimates for solutions of nonstationary Navier-Stokes systems*, Zap. Nauchn. Sem. LOMI 38 (1973), 153–231; = J. Soviet Math. 8 (1977), 467–529.

[S2] V. A. Solonnikov, *Estimates of the solutions of an initial-boundary value problem for a nonstationary system of Navier-Stokes equations*, Zap. Nauchn. Sem. LOMI 59 (1976), 178–254; = J. Soviet Math. 10 (1978), 336–393.

[S3] V. A. Solonnikov, *On the solvability of the second initial-boundary value problem for a linear non-stationary system of Navier-Stokes equations*, Zap. Nauchn. Sem. LOMI 69 (1977), 200–218; = J. Soviet Math. 10 (1978), 141–193.

[S4] V. A. Solonnikov, *Solvability of a problem on the plane motion of a heavy viscous incompressible capillary liquid partially filling a container*, Izv. Akad. Nauk SSSR 43 (1979); = Math. USSR Izvestiya 14 (1980), 193–221.

[S5] V. A. Solonnikov, *Solvability of the problem of evolution of an isolated of viscous, incompressible capillary fluid*, Zap. Nauchn. Sem. L.O.M.I. 140 (1984), 179–186; = J. Soviet Math. 32 (1986), 223–228.

[S6] V. A. Solonnikov, *On the transient motion of an isolated volume of viscous incompressible fluid*, Izv. Akad. Nauk SSSR 51 (1987), 1065–1087; = Math. USSR Izvestiya 31 (1988), 381–405.

[S-Šč] V. A. Solonnikov and S. Ščadilov, *On a boundary problem for a stationary system of Navier-Stokes equations*, Proc. Steklov Inst. Math. 125 (1973), 186–199.

[T1] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam, 1977.

[T2] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM Regional Conference Series in Applied Mathematics, Philadelphia, 1983.

[Y] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators*, I. *Boundedness on spaces of Besov type*, J. Fac. Sci. Tokyo 33 (1986), 131–174.

MATHEMATICS DEPARTMENT
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARKEN 5
DK-2100 COPENHAGEN
DENMARK

STEKLOV INSTITUTE OF MATEMATICS
ST. PETERBURG DEPARTMENT
FONTANKA 27
ST. PETERSBURG
191011 RUSSIA