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Research Article

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Boundary value problems of Hilfer-type fractional integro-differential equations and inclusions with nonlocal integro-multipoint boundary conditions

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Abstract: In this paper, we study boundary value problems of fractional integro-differential equations and inclusions involving Hilfer fractional derivative. Existence and uniqueness results are obtained by using the classical fixed point theorems of Banach, Krasnosel'skiĭ, and Leray-Schauder in the single-valued case, while Martelli's fixed point theorem, nonlinear alternative for multi-valued maps, and Covitz-Nadler fixed point theorem are used in the inclusion case. Examples illustrating the obtained results are also presented.

Keywords: fractional differential equations and inclusions, Hilfer fractional derivative, Riemann-Liouville fractional derivative, Caputo fractional derivative, boundary value problems, existence and uniqueness, fixed point theorems

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1 Introduction

In the last few decades, fractional differential equations with initial/boundary conditions have been studied by many researchers. This is because, fractional differential equations describe many real-world processes related to memory and hereditary properties of various materials more accurately as compared to classical order differential equations. Therefore, the fractional-order models become more practical and realistic as compared to the integer-order models. Fractional differential equations arise in lots of engineering and clinical disciplines which include biology, physics, chemistry, economics, signal and image processing, control theory, and so on; see the monographs in [1–8].

In the literature, there exist several different definitions of fractional integrals and derivatives, from the most popular of them Riemann-Liouville and Caputo fractional derivatives to other less-known definitions such as Hadamard fractional derivative, Erdelyi-Kober fractional derivative, and so on. A generalization of derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [9], known as the *Hilfer fractional derivative* of order α and a type $\beta \in [0, 1]$, which can be reduced to the Riemann-Liouville and

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Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively. Such a derivative interpolates between the Riemann-Liouville and Caputo derivative. Some properties and applications of the Hilfer derivative are given in [10,11] and references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors, see for example [12–14] and references therein. However, in the literature there are few papers on boundary value problems of Hilfer fractional derivatives. Nonlocal boundary value problems of Hilfer fractional derivatives were initiated by the authors in [15]. For some more recent work on boundary value problems with Hilfer fractional derivatives we refer to the papers in [16–18].

Motivated by the research going on in this direction, in this paper, we study the existence and the uniqueness of solutions for a new class of boundary value problems of Hilfer-type fractional differential equations with nonlocal integro-multipoint boundary conditions of the form:

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = f(t, x(t), I^{\delta}x(t)), \quad t \in [a, b], \\ x(a) = 0, \quad \displaystyle\int_a^b x(s)\mathrm{d}s + \mu = \sum_{i=1}^{m-2} \zeta_i x(\theta_i), \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer fractional derivative of order α, $1 < \alpha < 2$ and parameter β, $0 \le \beta \le 1$, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, I^{δ} is the Riemann-Liouville fractional integral of order $\delta > 0$, the points $a < \theta_1 < \theta_2 < \cdots < \theta_{m-2} < b$, $a \ge 0$, and $\mu, \zeta_i \in \mathbb{R}$, $i = 1, 2, \ldots, m-2$ are given constants.

We pay attention to the topic of nonlocal problems, because in many cases a nonlocal condition in this kind of problem reflects physical phenomena more precisely than classical boundary conditions.

Existence and uniqueness results are proved by using classical fixed point theorems. We make use of Banach's fixed point theorem to obtain the uniqueness result, while nonlinear alternatives of Leray-Schauder type [19] and Krasnosel'skiĭ's fixed point theorem [20] are applied to obtain the existence results for the problem (1).

Then we look at the corresponding multi-valued problem by studying the existence of solutions for a new class of boundary value problems of Hilfer-type fractional differential inclusions with nonlocal integro-multipoint boundary conditions of the form:

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) \in F(t, x(t), I^{\delta}x(t)), \quad t \in [a, b], \\ x(a) = 0, \quad \displaystyle\int_a^b x(s)\mathrm{d}s + \mu = \sum_{i=1}^{m-2} \zeta_i x(\theta_i), \end{cases} \tag{2}$$

where $F : [a, b] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

Existence results for the problem (2) with convex and nonconvex valued maps are, respectively, derived by applying Martelli's fixed point theorem, the nonlinear alternative for Kakutani maps, and Covitz and Nadler fixed point theorem for contractive maps.

The paper is organized as follows: Section 2 contains some preliminary concepts related to our problem. We present our main work for the problem (1) in Section 3, while the main results for the problem (2) are presented in Section 4. Our method of proof is standard, but its application in the framework of the present problem is new. Examples are constructed to illustrate the main results. The work accomplished in this paper is new and enrich the literature on boundary value problems of Hilfer-type fractional derivatives.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and multi-valued analysis. We present first preliminary results from fractional calculus needed in our proofs later [3,6].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $u : [a, \infty) \to \mathbb{R}$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}u(s)\,\mathrm{d}s,$$

provided the right-hand side exists on (a, ∞).

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function u is defined by

$${}^{RL}D^{\alpha}u(t) := D^nI^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-s)^{n-\alpha-1}u(s)\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on (a, ∞).

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function u is defined as

$${}^{C}D^{\alpha}u(t) := I^{n-\alpha}D^nu(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n u(s)\,\mathrm{d}s, \quad n-1 < \alpha < n,$$

provided the right-hand side is point-wise defined on (a, ∞).

In [9] (see also [11]), another new definition of the fractional derivative was suggested. The generalized Riemann-Liouville fractional derivative is defined as follows.

Definition 2.4. The generalized Riemann-Liouville fractional derivative or the Hilfer fractional derivative of order α and parameter β of a function is defined by

$${}^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^nI^{(1-\beta)(n-\alpha)}u(t),$$

where $n-1 < \alpha < n$, $0 \le \beta \le 1$, $t > a$, $D^n = \frac{\mathrm{d}^n}{\mathrm{d}t^n}$.

Remark 2.1. In Definition 2.4, type β allows $D^{\alpha,\beta}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative. When $\beta = 0$ the Hilfer fractional derivative corresponds to the Riemann-Liouville fractional derivative

$${}^{H}D^{\alpha,0}u(t) = D^nI^{n-\alpha}u(t),$$

while when $\beta = 1$ the Hilfer fractional derivative corresponds to the Caputo fractional derivative

$${}^{H}D^{\alpha,1}u(t) = I^{n-\alpha}D^nu(t).$$

In the following lemma, we present the compositional property of the Riemann-Liouville fractional integral operator with the Hilfer fractional derivative operator.

Lemma 2.1. [11] *Let $f \in L(a, b)$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$, $0 \le \beta \le 1$, $I^{(n-\alpha)(1-\beta)}f \in AC^k[a, b]$. Then*

$$(I^{\alpha}\, {}^{H}D^{\alpha,\beta}f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t\to a^+}(I^{(1-\beta)(n-\alpha)}f)(t).$$

Let $C([a, b], \mathbb{R})$ denote the Banach space of continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|f\| = \sup\{|f(t)| : t \in [a, b]\}$. $L^1([a, b], \mathbb{R})$ denotes the Banach space of functions $y : [a, b] \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_a^b |y(t)|\mathrm{d}t.$$

For each $y \in C([a,b],\mathbb{R})$, we define the set of selections of the multi-valued map F as

$$S_{F,y} = \{f \in L^1([a,b],\mathbb{R}) : f(t) \in F(t,y) \text{ for a.e. } t \in [a,b]\}.$$

In the following by $\mathcal{P}_p$ we denote the set of all nonempty subsets of X which have the property "p", where "p" will be bounded (b), closed (cl), convex (c), compact (cp), etc. Thus, $\mathcal{P}_{\rm cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \mathcal{P}_{\rm b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}, \mathcal{P}_{\rm cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \mathcal{P}_{\rm cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$, and $\mathcal{P}_{\rm b,cl,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded, closed and convex}\}$.

For more details on multi-valued maps and the proof of the known results cited in this section, we refer interested reader to the books by Castaing and Valadier [21], Deimling [22], Gorniewicz [23], and Hu and Papageorgiou [24].

3 Main results

The following lemma deals with a linear variant of the boundary value problem (1).

Lemma 3.1. *Let $a \geq 0$, $1 < \alpha < 2$, $\gamma = \alpha + 2\beta - \alpha\beta$, $h \in C([a,b],\mathbb{R})$, and*

$$\Lambda := \frac{(b-a)^\gamma}{\gamma} - \sum_{i=1}^{m-2} \zeta_i(\theta_i - a)^{\gamma-1} \neq 0. \tag{3}$$

Then the function $x \in C([a,b],\mathbb{R})$ is a solution of the boundary value problem

$$^{H}D^{\alpha,\beta}x(t) = h(t), \quad t \in [a,b], \quad 1 < \alpha < 2, \quad 0 \leq \beta \leq 1, \tag{4}$$

$$x(a) = 0, \quad \int_a^b x(s)\mathrm{d}s + \mu = \sum_{i=1}^{m-2} \zeta_i x(\theta_i), \tag{5}$$

if and only if

$$x(t) = I^\alpha h(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^\alpha h(\theta_i) - \int_a^b I^\alpha h(s)\mathrm{d}s - \mu\right]. \tag{6}$$

Proof. Assume that x is a solution of the nonlocal boundary value problem (4) and (5). Operating fractional integral I^α on both sides of equation (4) and using Lemma 2.1, we obtain

$$x(t) = c_0\frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_1\frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} + I^\alpha h(t) = c_0\frac{(t-a)^{\gamma-2}}{\Gamma(\gamma-1)} + c_1\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + I^\alpha h(t),$$

since $(1-\beta)(2-\alpha) = 2-\gamma$, where c_0 and c_1 are some real constants.

From the first boundary condition $x(a) = 0$ we can obtain $c_0 = 0$, since $\lim_{t\to a}(t-a)^{\gamma-2} = \infty$. Then we get

$$x(t) = c_1\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + I^\alpha h(t). \tag{7}$$

From $\int_a^b x(s)\mathrm{d}s + \mu = \sum_{i=1}^{m-2} \zeta_i x(\theta_i)$, we found

$$c_1 = \frac{\Gamma(\gamma)}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^\alpha h(\theta_i) - \int_a^b I^\alpha h(s)\mathrm{d}s - \mu\right].$$

Substituting the values of c_1 in (7), we obtain the solution (6). The converse follows by direct computation. This completes the proof. □

In view of Lemma 2.4, we define an operator $\mathcal{A} : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ by

$$(\mathcal{A}x)(t) = \frac{(t-a)^{\gamma-1}}{\Lambda}\left(\sum_{i=1}^{m-2}\zeta_i I^{\alpha} f(s, x(s), I^{\delta}x(s))(\theta_i) - I^{\alpha+1} f(s, x(s), I^{\delta}x(s))(b) - \mu\right) \\ + I^{\alpha} f(s, x(s), I^{\delta}x(s))(t), \quad t \in [a, b]. \tag{8}$$

It should be noted that problem (1) has solution if and only if the operator $\mathcal{A}$ has fixed points.

In the following, for the sake of convenience, we set a constant

$$\Omega = \frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i - a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}. \tag{9}$$

In the next subsections, we prove existence, as well as existence and uniqueness results, for the boundary value problem (1) by using classical fixed point theorems.

3.1 Existence and uniqueness result

Our first result is an existence and uniqueness result, based on Banach's fixed point theorem [22].

Theorem 3.1. *Assume that:*

(*H1*) *there exists a constant $L > 0$ such that*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le L(|x_1 - y_1| + |x_2 - y_2|)$$

for each $t \in [a, b]$ and $x_i, y_i \in \mathbb{R}$, $i = 1, 2$.

If

$$LL_1\Omega < 1, \tag{10}$$

where Ω is defined by (9) *and $L_1 = 1 + \frac{(b-a)^{\delta}}{\Gamma(\delta+1)}$, then the boundary value problem* (1) *has a unique solution on $[a, b]$.*

Proof. We transform the boundary value problem (1) into a fixed point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined as in (8). Observe that the fixed points of the operator $\mathcal{A}$ are solutions of problem (1). Applying the Banach contraction mapping principle, we shall show that $\mathcal{A}$ has a unique fixed point.

We let $\sup_{t\in[a,b]}|f(t, 0, 0)| = M < \infty$ and choose

$$r \ge \frac{M\Omega + ((b-a)^{\gamma-1}|\mu|)/|\Lambda|}{1 - LL_1\Omega}. \tag{11}$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in C([a, b], \mathbb{R}) : \|x\| \le r\}$. By using Assumption ($H_1$), we have

$$\begin{aligned} |f(t, x(t), I^{\delta}x(t))| &\le |f(t, x(t), I^{\delta}x(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \le L(|x(t)| + |I^{\delta}x(t)|) + M \\ &\le L\left(\|x\| + \frac{(b-a)^{\delta}}{\Gamma(\delta+1)}\|x\|\right) + M = L\|x\|\left(1 + \frac{(b-a)^{\delta}}{\Gamma(\delta+1)}\right) + M = LL_1\|x\| + M. \end{aligned}$$

For any $x \in B_r$, we have

$$\begin{aligned} &|(\mathcal{A}x)(t)| \\ &\le \sup_{t\in[a,b]}\left\{\frac{(t-a)^{\gamma-1}}{|\Lambda|}\left(\sum_{i=1}^{m-2}|\zeta_i| I^{\alpha}|f(s, x(s), I^{\delta}x(s))|(\theta_i) + I^{\alpha+1}|f(s, x(s), I^{\delta}x(s))|(b) + |\mu|\right) + I^{\alpha}|f(s, x(s), I^{\delta}x(s))|(t)\right\} \\ &\le \left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left(\sum_{i=1}^{m-2}|\zeta_i| I^{\alpha}(|f(s, x(s), I^{\delta}x(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\theta_i)\right.\right. \end{aligned}$$

$$+ I^{\alpha+1}(|f(s,x(s),I^{\delta}x(s)) - f(s,0,0)| + |f(s,0,0)|)(b) + |\mu|\Bigg)$$

$$+ I^{\alpha}(|f(s,x(s),I^{\delta}x(s)) - f(s,0,0)| + |f(s,0,0)|)(b)\Bigg\}$$

$$\le \left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right\}\left\{L\|x\|\left(1+\frac{(b-a)^{\delta}}{\Gamma(\delta+1)}\right) + M\right\} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|$$

$$\le (LL_1 r + M)\Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu| \le r,$$

which implies that $\mathcal{A}B_r \subset B_r$.

Next, we let $x, y \in C$. Then for $t \in [a,b]$, we have

$$|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| \le \Bigg\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg(\sum_{i=1}^{m-2}|\zeta_i|I^{\alpha}|f(s,x(s),I^{\delta}x(s)) - f(s,y(s),I^{\delta}y(s))|(\theta_i)$$

$$+ \int_a^b I^{\alpha}|f(s,x(s),I^{\delta}x(s)) - f(s,y(s),I^{\delta}y(s))|\mathrm{d}s\Bigg)$$

$$+ I^{\alpha}|f(s,x(s),I^{\delta}x(s)) - f(s,y(s),I^{\delta}y(s))|(b)\Bigg\}$$

$$\le LL_1\left\{\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right\}\|x-y\|$$

$$= LL_1\Omega\|x-y\|,$$

which implies that $\|\mathcal{A}x - \mathcal{A}y\| \le LL_1\Omega\|x-y\|$. As $LL_1\Omega < 1$, $\mathcal{A}$ is a contraction. Therefore, we deduce by the Banach contraction mapping principle that $\mathcal{A}$ has a fixed point which is the unique solution of the boundary value problem (1). The proof is complete. □

Example 3.1. Consider the boundary value problem of Hilfer fractional integro-differential equation with nonlocal integro-multipoint boundary condition of the form:

$$\begin{cases} {}^{H}D^{\frac{3}{2},\frac{1}{3}}x(t) = \dfrac{8}{3(87+8t)}\left(\dfrac{x^2(t)+2|x(t)|}{1+|x(t)|} + \sin\left(I^{\frac{1}{4}}x(t)\right)\right) + \dfrac{2}{3}, \quad t \in \left[\dfrac{1}{8},\dfrac{9}{8}\right], \\ x\left(\dfrac{1}{8}\right) = 0, \quad \displaystyle\int_{\frac{1}{8}}^{\frac{9}{8}} x(s)\mathrm{d}s + \dfrac{3}{2} = \dfrac{2}{3}x\left(\dfrac{3}{8}\right) + \dfrac{3}{4}x\left(\dfrac{5}{8}\right). \end{cases} \tag{12}$$

Here $\alpha = 3/2$, $\beta = 1/3$, $\delta = 1/4$, $a = 1/8$, $b = 9/8$, $\mu = 3/2$, $m = 4$, $\zeta_1 = 2/3$, $\zeta_2 = 3/4$, $\theta_1 = 3/8$, and $\theta_2 = 5/8$. From these settings, we compute constants as $\gamma \approx 1.66667$, $\Lambda \approx -0.13704$, $\Omega \approx 4.86106$, and $L_1 \approx 2.10326$. Let

$$f\left(t,x,I^{\frac{1}{4}}x\right) = \frac{8}{3(87+8t)}\left(\frac{x^2+2|x|}{1+|x|} + \sin\left(I^{\frac{1}{4}}x\right)\right) + \frac{2}{3}.$$

Then we have

$$\left|f(t,x,I^{\frac{1}{4}}x) - f(t,y,I^{\frac{1}{4}}y)\right| \le \frac{1}{11}\left(|x-y| + |I^{\frac{1}{4}}x - I^{\frac{1}{4}}y|\right).$$

Condition (10) is fulfilled by setting $L = 1/11$, since $LL_1\Omega \approx 0.92946 < 1$. Therefore, by the benefit of Theorem 3.1, the problem (12) has a unique solution $x(t)$ on $[1/8, 9/8]$.

3.2 Existence results

In this subsection, we present two existence results. The first existence result is based on the well-known Krasnosel'skiĭ's fixed point theorem [20].

Theorem 3.2. *Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition, we assume that:*
$(H2)$ $|f(t, x, y)| \le \varphi(t), \quad \forall (t, x, y) \in [a, b] \times \mathbb{R} \times \mathbb{R}$, *and* $\varphi \in C([a, b], \mathbb{R}^+)$.

Then the boundary value problem (1) *has at least one solution on* $[a, b]$ *provided*

$$LL_1 \frac{(b-a)^{\gamma-1}}{|\Lambda|} \left[\sum_{i=1}^{m-2} |\zeta_i| \frac{(\theta_i - a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \right] < 1. \tag{13}$$

Proof. Setting $\sup_{t\in[a,b]} \varphi(t) = \|\varphi\|$ and choosing

$$\rho \ge \|\varphi\| \Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|} |\mu|, \tag{14}$$

(where Ω is defined by (9)), we consider $B_\rho = \{x \in C([a, b], \mathbb{R}) : \|x\| \le \rho\}$. We define the operators $\mathcal{A}_1, \mathcal{A}_2$ on B_ρ by

$$\mathcal{A}_1 x(t) = I^\alpha f(t, x(t), I^\delta x(t)), \quad t \in [a, b]$$

and

$$\mathcal{A}_2 x(t) = \frac{(t-a)^{\gamma-1}}{\Lambda} \left(\sum_{i=1}^{m-2} \zeta_i I^\alpha f(\theta_i, x(\theta_i), I^\delta x(\theta_i)) - \int_a^b I^\alpha f(s, x(s), I^\delta x(s)) ds - \mu \right), \quad t \in [a, b].$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}
&|(\mathcal{A}_1 x)(t) + (\mathcal{A}_2 y)(t)| \\
&\le \sup_{t\in[a,b]} \left\{ \frac{(t-a)^{\gamma-1}}{|\Lambda|} \left(\sum_{i=1}^{m-2} |\zeta_i| I^\alpha |f(\theta_i, y(\theta_i), I^\delta y(\theta_i))| + \int_a^b I^\alpha |f(s, y(s), I^\delta y(s))| ds + |\mu| \right) + I^\alpha |f(t, x(t), I^\delta x(t))| \right\} \\
&\le \|\varphi\| \left(\frac{(b-a)^{\gamma-1}}{|\Lambda|} \left[\sum_{i=1}^{m-2} |\zeta_i| \frac{(\theta_i - a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \right] + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \right) + \frac{(b-a)^{\gamma-1}}{|\Lambda|} |\mu| \\
&= \|\varphi\| \Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|} |\mu| \le \rho.
\end{aligned}$$

This shows that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_\rho$. It is easy to see, using (13), that $\mathcal{A}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{A}_1$ is continuous. Also, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{A}_1 x\| \le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{A}_1$.

We define $\sup_{(t,x)\in[a,b]\times B_\rho \times B_\rho} |f(t, x, y)| = \bar{f} < \infty$, and consequently we have

$$\begin{aligned}
&|(\mathcal{A}_1 x)(t_2) - (\mathcal{A}_1 x)(t_1)| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s, x(s), I^\delta x(s)) ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s), I^\delta x(t)) ds \right| \\
&\le \frac{\bar{f}}{\Gamma(\alpha+1)} [2(t_2 - t_1)^\alpha + |(t_2 - a)^\alpha - (t_1 - a)^\alpha|],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{A}_1$ is equicontinuous. So $\mathcal{A}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on B_ρ. Thus, all the assumptions of Krasnosel'skiĭ's fixed point theorem are satisfied. So its conclusion implies that the boundary value problem (1) has at least one solution on $[a, b]$. □

The Leray-Schauder nonlinear alternative [19] is used for proving our second existence result.

Theorem 3.3. *Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H3) *there exist a continuous, nondecreasing, subhomogeneous (i.e., $\psi(kx) \le k\psi(x)$ for all $k \ge 1$ and $x \in \mathbb{R}^+$) function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C([a, b], \mathbb{R}^+)$ such that*

$$|f(t, u, v)| \le p(t)\psi(|u| + |v|) \quad \text{for each} \quad (t, u, v) \in [a, b] \times \mathbb{R} \times \mathbb{R};$$

(H4) *there exists a constant $K > 0$ such that*

$$\frac{K}{L_1\psi(K)\|p\|\Omega + ((b-a)^{\gamma-1}|\mu|)/|\Lambda|} > 1,$$

where Ω is defined by (9).

Then the boundary value problem (1) has at least one solution on $[a, b]$.

Proof. Let the operator $\mathcal{A}$ be defined by (8). First, we shall show that $\mathcal{A}$ maps bounded sets (balls) into bounded set in C. For a number $r > 0$, let $B_r = \{x \in C([a, b], \mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([a, b], \mathbb{R})$. Then for $t \in [a, b]$ we have

$$\begin{aligned}
&|(\mathcal{A}x)(t)| \\
&\le \sup_{t\in[a,b]}\left\{\frac{(t-a)^{\gamma-1}}{|\Lambda|}\left(\sum_{i=1}^{m-2}|\zeta_i| I^{\alpha}|f(s, x(s), I^{\delta}x(s))|(\theta_i) + \int_a^b I^{\alpha}|f(s, x(s), I^{\delta}x(s))|ds + |\mu|\right) + I^{\alpha}|f(s, x(s), I^{\delta}x(s))|(t)\right\} \\
&\le \psi\left(\|x\| + \frac{(b-a)^{\delta}}{\Gamma(\delta+1)}\|x\|\right)\|p\|\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} +\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu| \\
&\le \psi(L_1\|x\|)\|p\|\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} +\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu| \\
&\le L_1\psi(\|x\|)\|p\|\left(\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + |\mu|\right] + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|,
\end{aligned}$$

and consequently,

$$\|\mathcal{A}x\| \le L_1\psi(r)\|p\|\Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|.$$

Next, we will show that $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C([a, b], \mathbb{R})$. Let $\tau_1, \tau_2 \in [a, b]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
&|(\mathcal{A}x)(\tau_2) - (\mathcal{A}x)(\tau_1)| \\
&\le \frac{1}{\Gamma(\alpha)}\left|\int_a^{\tau_1}[(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]f(s, x(s), I^{\delta}x(s))ds + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}f(s, x(s), I^{\delta}x(s))ds\right| \\
&\quad + \frac{(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}}{|\Lambda|}\left(\sum_{i=1}^{m-2}|\zeta_i| I^{\alpha}|f(s, x(s), I^{\delta}x(s))|(\theta_i) + \int_a^b I^{\alpha}|f(s, x(s), I^{\delta}x(s)|ds\right) \\
&\le \frac{\|p\|L_1\psi(r)}{\Gamma(\alpha+1)}[2(t_2-t_1)^{\alpha} + |(t_2-a)^{\alpha} - (t_1-a)^{\alpha}|] \\
&\quad + \frac{(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}}{|\Lambda|}L_1\|p\|\psi(r)\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right].
\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the aforementioned inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{A} : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative [19] once we have proved the boundedness of the set of all solutions to equations $x = \lambda\mathcal{A}x$ for $\lambda \in (0, 1)$.

Let x be a solution. Then, for $t \in [a, b]$, and following the similar computations to that in the first step, we have

$$|x(t)| \le L_1\psi(\|x\|)\|p\|\Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|,$$

which leads to

$$\frac{\|x\|}{\psi(L_1\|x\|)\|p\|\Omega + ((b-a)^{\gamma-1}|\mu|)/|\Lambda|} \le 1.$$

In view of (H_4), there exists K such that $\|x\| \neq K$. Let us set

$$U = \{x \in C([a, b], \mathbb{R}) : \|x\| < K\}.$$

We see that the operator $\mathcal{A} : \bar{U} \to C([a, b], \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda\mathcal{A}x$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{A}$ has a fixed point $x \in \bar{U}$, which is a solution of the boundary value problem (1). This completes the proof. ☐

Example 3.2. Consider the boundary value problem of Hilfer fractional integro-differential equation with nonlocal integro-multipoint boundary condition of the form:

$$\begin{cases} {}^{H}D^{\frac{4}{3},\frac{1}{4}}x(t) = \dfrac{3M}{2(3t+2)}\left(\dfrac{|x(t)|}{|x(t)|+1} + \tan^{-1}(I^{\frac{1}{2}}x(t))\right) + \dfrac{1}{2}, \quad t \in \left[\dfrac{1}{3}, \dfrac{7}{3}\right], \\ x\left(\dfrac{1}{3}\right) = 0, \quad \displaystyle\int_{\frac{1}{3}}^{\frac{7}{3}} x(s)\,ds + \dfrac{2}{5} = \dfrac{1}{4}x\left(\dfrac{2}{3}\right) + \dfrac{1}{2}x\left(\dfrac{4}{3}\right) + \dfrac{3}{4}x\left(\dfrac{5}{3}\right). \end{cases} \tag{15}$$

Here $\alpha = 4/3$, $\beta = 1/4$, $\delta = 1/2$, $a = 1/3$, $b = 7/3$, $\mu = 2/5$, $m = 5$, $\zeta_1 = 1/4$, $\zeta_2 = 1/2$, $\zeta_3 = 3/4$, $\theta_1 = 2/3$, $\theta_2 = 4/3$, $\theta_3 = 5/3$, and M is a given constant. Next, we can find that $\gamma = 1.50000$, $\Lambda \approx 0.62891$, $\Omega \approx 9.32772$, $L_1 \approx 2.59577$, and

$$\Omega^* := \frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i - a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right] \approx 7.21134.$$

By setting

$$f(t, x, y) = \frac{3M}{2(3t+2)}\left(\frac{|x|}{|x|+1} + \tan^{-1}(y)\right) + \frac{1}{2},$$

where $y = I^{\frac{1}{2}}x$, we obtain

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le M(|x_1 - x_2| + |y_1 - y_2|),$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}$, and

$$|f(t, x, y)| \le \frac{3M(2+\pi)}{4(3t+2)} + \frac{1}{2},$$

which satisfy the conditions (H_1), (H_2), respectively. Then we can conclude that if $M \in (0, (1/(L_1\Omega))) \approx (0, 0.04130)$, then the problem (15) has a unique solution by Theorem 3.1. If $M \in [(1/(L_1\Omega)), (1/(L_1\Omega^*))) \approx [0.04130, 0.05342)$, then the problem (15) has at least one solution on $[1/3, 7/3]$ by applying Theorem 3.2.

Example 3.3. Consider the boundary value problem of Hilfer fractional integro-differential equation with nonlocal integro-multipoint boundary condition of the form:

$$\begin{cases} {}^{H}D^{\frac{5}{4},\frac{1}{5}}x(t) = \dfrac{4}{4t+479}\left(\dfrac{\left(|x(t)| + \left|I^{\frac{3}{2}}x(t)\right|\right)^6}{\left(|x(t)| + \left|I^{\frac{3}{2}}x(t)\right|\right)^4 + 3} + 1\right), \quad t \in \left[\dfrac{1}{4}, \dfrac{11}{4}\right], \\ x\left(\dfrac{1}{4}\right) = 0, \quad \displaystyle\int_{\frac{1}{4}}^{\frac{11}{4}} x(s)\mathrm{d}s + \dfrac{3}{7} = \dfrac{1}{5}x\left(\dfrac{3}{4}\right) + \dfrac{2}{5}x\left(\dfrac{7}{4}\right) + \dfrac{3}{5}x\left(\dfrac{9}{4}\right). \end{cases} \tag{16}$$

Put $\alpha = 5/4$, $\beta = 1/5$, $\delta = 3/2$, $a = 1/4$, $b = 11/4$, $\mu = 3/7$, $m = 5$, $\zeta_1 = 1/5$, $\zeta_2 = 2/5$, $\zeta_3 = 3/5$, $\theta_1 = 3/4$, $\theta_2 = 7/4$, and $\theta_3 = 9/4$. Then we obtain $\gamma = 1.40000$, $\Lambda \approx 1.16254$, $\Omega \approx 8.98278$, and $L_1 \approx 3.97354$. By setting

$$f(t, x, y) = \frac{4}{4t + 479}\left(\frac{(|x| + |y|)^6}{(|x| + |y|)^4 + 3} + 1\right),$$

we obtain

$$|f(t, x, y)| \le \frac{4}{4t + 479}((|x| + |y|)^2 + 1),$$

which satisfies (H_3) with $p(t) = 4/(4t + 479)$ and $\psi(u) = u^2 + 1$. Furthermore, we can find that there exists a constant $K \in (1.48699, 1.87496)$ satisfying condition (H_4). Therefore, applying the conclusion of Theorem 3.3, the nonlocal boundary value problem (16) has at least one solution on $[1/4, 11/4]$.

4 Existence results for the problem (2)

Before stating and proving our main existence results for problem (2), we will give the definition of its solution.

Definition 4.1. A function $x \in AC([a, b], \mathbb{R})$ is said to be a solution of the problem (2) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v \in F(t, x)$ a.e. for $t \in [a, b]$ such that x satisfies the differential equation $D^\alpha x(t) = v(t)$ for $t \in [a, b]$ and the boundary conditions $x(a) = 0$, $\int_a^b x(s)\mathrm{d}s + \mu = \sum_{i=1}^{m-2} \zeta_i x(\theta_i)$.

4.1 The upper semicontinuous case

Consider first the case when F has convex values and we give an existence result based on Martelli's fixed point theorem, which is applicable to completely continuous operators. For convenience of the reader we include this lemma.

Lemma 4.1. (Martelli fixed point theorem) [25] *Let X be a Banach space, and $T : X \to \mathcal{P}_{b,cl,c}(X)$ be a completely continuous multi-valued map. If the set $\varepsilon = \{x \in X : \lambda x \in T(x), \lambda > 1\}$ is bounded, then T has a fixed point.*

Theorem 4.1. *Assume that the following hypotheses hold:*
(A_1) *$F : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory, i.e.,*
(*i*) *$t \mapsto F(t, x, y)$ is measurable for each $(x, y) \in \mathbb{R} \times \mathbb{R}$;*
(*ii*) *$(x, y) \mapsto F(t, x, y)$ is u.s.c. for almost all $t \in [a, b]$;*

(*iii*) *for each* $r > 0$, *there exists* $\phi_r \in L^1([a,b],\mathbb{R}^+)$ *such that*

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \le \phi_r(t)$$

for all $x, y \in \mathbb{R}$ *with* $\|x\|, \|y\| \le r$ *and for a.e.* $t \in [a,b]$;

(A_2) *there exists a function* $q \in C([a,b],\mathbb{R})$ *such that*

$$\|F(t,x,y)\| \le q(t), \text{ for a.e. } t \in [a,b] \text{ and each } x, y \in \mathbb{R}.$$

Then the problem (2) *has at least one solution on* $[a,b]$.

Proof. In order to transform the problem (2) into a fixed point problem, we consider the multi-valued map: $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ defined by

$$N(x) = \left\{ h \in C([a,b],\mathbb{R}) : h(t) = I^\alpha v(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^\alpha v(\theta_i) - \int_a^b I^\alpha v(s)\,\mathrm{d}s - \mu\right], \quad v \in S_{F,x} \right\}.$$

It is clear that fixed points of N are solutions of problem (2). In turn, we need to show that the operator N satisfies all conditions of Lemma 4.1. The proof is constructed in several steps.

Step 1. $N(x)$ is convex for each $x \in C([a,b],\mathbb{R})$.

Indeed, if h_1, h_2 belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in [a,b]$, we have

$$h_i(t) = I^\alpha v_i(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^\alpha v_i(\theta_i) - \int_a^b I^\alpha v_i(s)\,\mathrm{d}s - \mu\right], \quad i = 1, 2.$$

Let $0 \le \theta \le 1$. Then for each $t \in [a,b]$, we have

$$\begin{aligned}[\theta h_1 + (1-\theta)h_2](t) &= I^\alpha[\theta v_1(s) + (1-\theta)v_2(s)](t) \\ &+ \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^\alpha[\theta v_1(s) + (1-\theta)v_2(s)](\theta_i) - \int_a^b I^\alpha[\theta v_1(s) + (1-\theta)v_2(s)](s)\,\mathrm{d}s - \mu\right].\end{aligned}$$

Since F has convex values, that is, $S_{F,x}$ is convex, we have

$$\theta h_1 + (1-\theta)h_2 \in N(x).$$

Step 2. $N(x)$ maps bounded sets (balls) into bounded sets in $C([a,b],\mathbb{R})$.

For a positive number r, let $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([a,b],\mathbb{R})$. Then for each $h \in N(x)$, $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$h(t) = I^\alpha v(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2} \zeta_i I^\alpha v(\theta_i) - \int_a^b I^\alpha v(s)\,\mathrm{d}s - \mu\right], \quad t \in [a,b].$$

Then, for $t \in [a,b]$, we have

$$\begin{aligned}|h(t)| &\le I^\alpha|v(t)| + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\left(\sum_{i=1}^{m-2} |\zeta_i| I^\alpha|v(\theta_i)| + \int_a^b I^\alpha|v(s)|\,\mathrm{d}s + |\mu|\right) \\ &\le \|q\|\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2} |\zeta_i| \frac{(\theta_i-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|,\end{aligned}$$

and consequently,

$$\|N(x)\| \le \|q\|\Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|.$$

Step 3. $N(x)$ maps bounded sets into equicontinuous sets of $C([a, b], \mathbb{R})$.

Let x be any element in B_r and $h \in N(x)$, then there exists a function $v \in S_{F,x}$ such that, for each $t \in [a, b]$, we have

$$h(t) = I^{\alpha}v(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v(\theta_i) - \int_a^b I^{\alpha}v(s)\mathrm{d}s - \mu\right], \quad t \in [a, b].$$

Let $t_1, t_2 \in [a, b]$, $t_1 < t_2$. Thus,

$$\begin{aligned}|h(t_2) - h(t_1)| &\le \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]v(s)\mathrm{d}s + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}v(s)\mathrm{d}s\right| \\ &\quad + \frac{|(t_2-a)^{\gamma-1} - (t_1-a)^{\gamma-1}|}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i||I^{\alpha}|v(\theta_i)| + \int_a^b I^{\alpha}|v(s)|\mathrm{d}s\right] \\ &\le \frac{\|q\|}{\Gamma(\alpha+1)}[2(t_2-t_1)^{\alpha} + |(t_2-a)^{\alpha} - (t_1-a)^{\alpha}|] \\ &\quad + \frac{(\tau_2-a)^{\gamma-1} - (\tau_1-a)^{\gamma-1}}{|\Lambda|}\|q\|\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right].\end{aligned}$$

The right-hand side of the aforementioned inequality clearly tends to zero independently of $x \in B_r$ as $t_1 \to t_2$. As a consequence of Steps 1–3 together with the Arzelá-Ascoli theorem, we conclude that $N : C([a, b], \mathbb{R}) \to \mathcal{P}(C([a, b], \mathbb{R}))$ is completely continuous.

Next, we show that the operator N is upper semi-continuous. In order to do so, it is enough to establish that N has a closed graph, because from [22, Proposition 1.2] we know that if an operator is completely continuous and has a closed graph, then it is upper semi-continuous.

Step 4. N has a closed graph.

Let $x_n \to x_\star$, $h_n \in N(x_n)$, and $h_n \to h_\star$. We need to show that $h_\star \in N(x_\star)$. Now $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that for each $t \in [a, b]$,

$$h_n(t) = I^{\alpha}v_n(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v_n(\theta_i) - \int_a^b I^{\alpha}v_n(s)\mathrm{d}s - \mu\right].$$

We must show that there exists $v_\star \in S_{F,x_\star}$ such that for each $t \in [a, b]$,

$$h_\star(t) = I^{\alpha}v_\star(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v_\star(\theta_i) - \int_a^b I^{\alpha}v_\star(s)\mathrm{d}s - \mu\right].$$

Consider the continuous linear operator $\Theta : L^1([a, b], \mathbb{R}) \to C([a, b])$ by

$$v \to \Theta(v)(t) = I^{\alpha}v(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v(\theta_i) - \int_a^b I^{\alpha}v(s)\mathrm{d}s - \mu\right], \quad t \in [a, b].$$

Observe that $\|h_n - h_\star\| \to 0$ as $n \to \infty$, and thus, it follows from a closed graph Lemma [26] that $\Theta \circ S_{F,x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(S_{F,x_n}).$$

Since $x_n \to x_\star$, the closed graph Lemma [26] implies that

$$h_*(t) = I^\alpha v_*(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^\alpha v_*(\theta_i) - \int_a^b I^\alpha v_*(s)\mathrm{d}s - \mu\right],$$

for some $v_* \in S_{F,x_*}$.

Hence, we conclude that N is a compact multivalued map, u.s.c. with convex closed values.

Step 5. We show that the set $\mathcal{E} = \{x \in C([a,b],\mathbb{R}) : \lambda x \in N(x),\ \lambda > 1\}$ is bounded.
Let $x \in \mathcal{E}$, then $\lambda x \in N(x)$ for some $\lambda > 1$ and there exists a function $v \in S_{F,x}$ such that

$$x(t) = \frac{1}{\lambda} I^\alpha v(t) + \frac{1}{\lambda}\frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^\alpha v(\theta_i) - \int_a^b I^\alpha v(s)\mathrm{d}s - \mu\right], \quad t \in [a,b].$$

For each $t \in [a,b]$, we have from Step 2 that

$$\|x\| \le \|q\|\Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|.$$

Hence, the set $\mathcal{E}$ is bounded. As a consequence of Lemma 4.1, we deduce that N has at least one fixed point which implies that the problem (2) has a solution on $[a,b]$. □

Our second existence result in this subsection is based on the Leray-Schauder nonlinear alternative for multivalued maps.

Theorem 4.2. *Assume that* (A_1) *holds. In addition, we assume that:*
(A_3) *there exist a continuous, nondecreasing, subhomogeneous function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([a,b],\mathbb{R}^+)$ *such that*

$$\|F(t,x,y)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(|x|+|y|) \text{ for each } (t,x,y) \in [a,b]\times\mathbb{R}\times\mathbb{R};$$

(A_4) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{L_1\psi(M)\|p\|\Omega + ((b-a)^{\gamma-1}|\mu|)/|\Lambda|} > 1.$$

Then the boundary value problem (2) *has at least one solution on* $[a,b]$.

Proof. Consider the operator N defined in the proof of Theorem 4.1. Let $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. We show that there exists an open set $U \subseteq C([a,b],\mathbb{R})$ with $x \notin N(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda N(x)$. Then there exists $v \in L^1([a,b],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t) = I^\alpha v(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^\alpha v(\theta_i) - \int_a^b I^\alpha v(s)\mathrm{d}s - \mu\right], \quad t \in [a,b].$$

In view of (A_3), we have for each $t \in [a,b]$, as in Theorem 3.3 that

$$|x(t)| \le L_1\psi(\|x\|)\|p\|\Omega + \frac{(b-a)^{\gamma-1}}{|\Lambda|}|\mu|,$$

which leads to

$$\frac{\|x\|}{L_1\psi(\|x\|)\|p\|\Omega + ((b-a)^{\gamma-1}|\mu|)/|\Lambda|} \le 1.$$

In view of (A_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Proceeding as in the proof of Theorem 4.1, we claim that the operator $N : \bar{U} \to \mathcal{P}(C([a,b],\mathbb{R}))$ is a compact, upper semi-continuous multi-valued map with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [19], we deduce that N has a fixed point $x \in \bar{U}$, which is a solution of the boundary value problem (2). This completes the proof. □

4.2 The Lipschitz case

In this subsection, we prove the existence of solutions for the boundary value problem (2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [27].

Theorem 4.3. *Assume that the following conditions hold:*

(A_4) $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x, y) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x, y \in \mathbb{R}$;

(A_5) $H_d(F(t,x,y), F(t,\bar{x},\bar{y})) \le m(t)(|x-\bar{x}| + |y-\bar{y}|)$ *for almost all* $t \in [a,b]$ *and* $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ *with* $m \in C([a,b],\mathbb{R}^+)$ *and* $d(0, F(t,0,0)) \le m(t)$ *for almost all* $t \in [a,b]$.

Then the boundary value problem (2) *has at least one solution on* $[a,b]$ *if*

$$L_1\Omega\|m\| < 1.$$

Proof. We transform the boundary value problem (2) into a fixed point problem by considering the operator $N : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ defined at the beginning of the proof of Theorem 4.1. We show that the operator N satisfies the assumptions of Lemma of Covitz and Nadler [27] in two steps.

Step I. N is nonempty and closed for every $v \in S_{F,x}$.

Note that since the set-valued map $F(\cdot, x(\cdot))$ is measurable by the measurable selection theorem (e.g., [21, Theorem III.6]) and it admits a measurable selection $v : [a,b] \to \mathbb{R}$. Moreover, by Assumption (A_5), we have

$$|v(t)| \le m(t) + m(t)(|x(t)| + |I^{\delta}x(t)|) \le m(t) + L_1 m(t)|x(t)|,$$

i.e., $v \in L^1([a,b],\mathbb{R})$ and hence F is integrably bounded. Therefore, $S_{F,x} \neq \varnothing$. Moreover, $N(x) \in \mathcal{P}_{cl}(C([a,b],\mathbb{R}))$ for each $x \in C([a,b],\mathbb{R})$. Let $\{u_n\}_{n\ge 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([a,b],\mathbb{R})$. Then $u \in C([a,b],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [a,b]$,

$$u_n(t) = I^{\alpha}v_n(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v_n(\theta_i) - \int_a^b I^{\alpha}v_n(s)\,\mathrm{d}s - \mu\right].$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([a,b],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [a,b]$, we have

$$u_n(t) \to v(t) = I^{\alpha}v(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v(\theta_i) - \int_a^b I^{\alpha}v(s)\,\mathrm{d}s - \mu\right].$$

Hence, $u \in N(x)$.

Step II. Next, we show that there exists $0 < \theta < 1$ $(\theta = L_1\Omega\|m\|)$ such that

$$H_d(N(x), N(\bar{x})) \le \theta\|x - \bar{x}\| \text{ for each } x, \bar{x}, \in AC(J,\mathbb{R}).$$

Let $x, \bar{x} \in AC([a,b],\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t, x(t), y(t))$ such that, for each $t \in [a,b]$,

$$h_1(t) = I^{\alpha}v_1(t) + \frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^{\alpha}v_1(\theta_i) - \int_a^b I^{\alpha}v_1(s)\,\mathrm{d}s - \mu\right].$$

By (A_5), we have

$$H_d(F(t,x,y),F(t,\bar{x},\bar{y})) \le m(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|).$$

So, there exists $w(t)\in F(t,\bar{x}(t),\bar{y}(t))$ such that

$$|v_1(t)-w| \le m(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|),\quad t\in[a,b].$$

Define $U: J\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R} : |v_1(t)-w|\le m(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|)\}.$$

Since the multivalued operator $U(t)\cap F(t,\bar{x}(t),\bar{y}(t))$ is measurable [21, Proposition III.4], there exists a function $v_2(t)$, which is a measurable selection for U. So $v_2(t)\in F(t,\bar{x}(t),\bar{y}(t))$ and for each $t\in[a,b]$, we have $|v_1(t)-v_2(t)|\le m(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|)$.

For each $t\in[a,b]$, let us define

$$h_2(t)=I^\alpha v_2(t)+\frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^\alpha v_2(\theta_i)-\int_a^b I^\alpha v_2(s)\mathrm{d}s-\mu\right].$$

Thus,

$$\begin{aligned}|h_1(t)-h_2(t)|&=I^\alpha|v_2(t)-v_1(t)|+\frac{(t-a)^{\gamma-1}}{\Lambda}\left[\sum_{i=1}^{m-2}\zeta_i I^\alpha|v_2(\theta_i)-v_1(\theta)|+\int_a^b I^\alpha|v_2(s)-v_1(s)|\mathrm{d}s\right]\\&\le\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}+\frac{(b-a)^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m-2}|\zeta_i|\frac{(\theta_i-a)^\alpha}{\Gamma(\alpha+1)}+\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right)\|m\|L_1\|x-\bar{x}\|.\end{aligned}$$

Hence,

$$\|h_1-h_2\|\le\Omega\|m\|L_1\|x-\bar{x}\|.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(N(x),N(\bar{x}))\le\Omega\|m\|L_1\|x-\bar{x}\|.$$

Since N is a contraction, it follows by Lemma of Covitz and Nadler [27] that N has a fixed point x, which is a solution of (2). This completes the proof. □

Example 4.1. Consider the boundary value problem of Hilfer fractional integro-differential inclusion with nonlocal integro-multipoint boundary condition of the form:

$$\begin{cases}{}^H D^{\frac{6}{5},\frac{1}{6}}x(t)\in F(t,x(t),I^{\frac{5}{2}}x(t)),\quad t\in\left[\frac{1}{5},\frac{16}{5}\right],\\ x\left(\frac{1}{5}\right)=0,\quad \int_{\frac{1}{5}}^{\frac{16}{5}}x(s)\mathrm{d}s+\frac{4}{5}=\frac{2}{7}x\left(\frac{8}{5}\right)+\frac{3}{7}x\left(\frac{11}{5}\right)+\frac{4}{7}x\left(\frac{13}{5}\right)+\frac{5}{7}x\left(\frac{14}{5}\right),\end{cases}\tag{17}$$

where

$$F(t,x(t),I^{\frac{5}{2}}x(t))=\left[\frac{5\left(1+\tan^{-1}|x(t)|+I^{\frac{5}{2}}x(t)\right)}{8(t+100)},\frac{5}{5t+749}\left(1+\sin|x(t)|+\frac{\left|I^{\frac{5}{2}}x(t)\right|}{1+\left|I^{\frac{5}{2}}x(t)\right|}\right)\right].$$

Take $\alpha=6/5$, $\beta=1/6$, $\delta=5/2$, $a=1/5$, $b=16/5$, $\mu=4/5$, $m=6$, $\zeta_1=2/7$, $\zeta_2=3/7$, $\zeta_3=4/7$, $\zeta_4=5/7$, $\theta_1=8/5$, $\theta_2=11/5$, $\theta_3=13/5$, and $\theta_4=14/5$. Then we have $\gamma\approx1.33333$, $\Lambda\approx0.63821$, $\Omega\approx24.70353$, and $L_1\approx5.69058$. Let $y=I^{\frac{5}{2}}x$. It is obvious that $F(t,x,y)$ is measurable for each $x,y\in\mathbb{R}$. Next we can find that

$$H_d(F(t,x,y),F(t,\bar{x},\bar{y})) \le \left(\frac{5}{5t+749}\right)(|x-\bar{x}|+|y-\bar{y}|), \quad x,\bar{x},y,\bar{y}\in\mathbb{R}, t\in\left[\frac{1}{5},\frac{16}{5}\right].$$

By setting the function $m(t) = (5/(5t+749))$, we get $\|m\| = 1/150$ and we also obtain $d(0, F(t,0,0)) \le m(t)$ for all $t \in [1/5, 16/5]$. Hence, we can compute that $L_1\Omega\|m\| \approx 0.93718 < 1$. Thus, the problem (17) has at least one solution on $[1/5, 16/5]$ by applying Theorem 4.3.

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