Mathematical Problems in Engineering, 2002, Vol. 8(2), pp. 143–150 Taylor & Francis
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Boundary Value Problems on the Half Line in the Theory of Colloids

RAVI P. AGARWAL[a,*] and DONAL O'REGAN[b]

[a]*Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, Fl 32901-6975, U.S.A;* [b]*Department of Mathematics, National University of Ireland, Galway, Ireland*

(Received 8 February 2002)

We present existence results for some boundary value problems defined on infinite intervals. In particular our discussion includes a problem which arises in the theory of colloids.

Key words: Boundary value problem, half line, colloids, existence

1 INTRODUCTION

In the theory of colloids [4, 7] it is possible to relate particle stability with the charge on the colloidal particle. We model the particle and its attendant electrical double layer using Poisson's equation for a flat plate. If Ψ is the potential, ρ the charge density, D the dielectric constant and y the displacement, then we have

$$\frac{d^2\Psi}{dy^2} = -\frac{4\pi\rho}{D}.$$

We assume the ions are point charged and their concentrations in the double layer satisfies the Boltzmann distribution

$$c_i = c_i^* \exp\left(\frac{-z_i e\Psi}{\kappa T}\right)$$

where c_i is the concentration of ions of type i, $c_i^* = \lim_{\Psi\to 0} c_i$, κ the Boltzmann constant, T the absolute temperature, e the electrical charge, and z the valency of the ion. In the neutral case, we have

$$\rho = c_+ z_+ e + c_- z_- e \quad \text{or} \quad \rho = ze(c_+ - c_-)$$

* Corresponding author. Tel.: 321-674-8091; Fax: 321-674-7412; E-mail: agarwal@fit.edu

ISSN 1024-123X print; ISSN 1563-5147 online

where $z = z_+ - z_-$. Then we have using

$$c_+ = c\ \exp\left(\frac{-ze\Psi}{\kappa T}\right) \quad \text{and} \quad c_- = c\ \exp\left(\frac{ze\Psi}{\kappa T}\right),$$

that

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} = \frac{8\pi cze}{D}\sinh\left(\frac{ze\Psi}{\kappa T}\right)$$

where the potential initially takes some positive value $\Psi(0) = \Psi_0$ and tends to zero as the distance from the plate increases *i.e.* $\Psi(\infty) = 0$. Using the transformation

$$\phi(y) = \frac{ze\Psi(y)}{\kappa T} \quad \text{and} \quad x = \sqrt{\frac{4\pi cz^2e^2}{\kappa TD}}y,$$

the problem becomes

$$\begin{cases} \dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = 2\sinh\phi, \quad 0 < x < \infty \\ \phi(0) = c_1 \\ \lim_{x\to\infty}\phi(x) = 0, \end{cases} \tag{1.1}$$

where $c_1 = ze\Psi_0/\kappa T > 0$. From a physical point of view we wish the solution ϕ in (1.1) to also satisfy $\lim_{x\to\infty}\phi'(x) = 0$.

In this paper using the notion of upper and lower solutions (see [1, 2, 6]) we establish general existence results which guarantee the existence of $BC[0, \infty)$ solutions to

$$\begin{cases} \dfrac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t)), \quad 0 < t < \infty \\ -a_0y(0) + b_0\lim_{t\to 0^+}p(t)y'(t) = c_0, \quad a_0 > 0, \ b_0 \geq 0 \\ \lim_{t\to\infty}y(t) = 0; \end{cases} \tag{1.2}$$

here $BC[0, \infty)$ denotes the space of continuous, bounded functions from $[0, \infty)$ to $\mathbf{R}$. Our theory not only complements some of the known results, *e.g.*, [5, 8], but also automatically produces the existence of a solution to (1.1). To establish these results we recall, for the convenience of the reader, the existence principle [3] we will use in Section 2. Consider the boundary value problem

$$\begin{cases} \dfrac{1}{p}(py')' = qf(t, y), \quad 0 < t < \infty \\ -a_0y(0) + b_0\lim_{t\to 0^+}p(t)y'(t) = c_0, \quad a_0 > 0, \ b_0 \geq 0 \\ y(t) \quad \text{bounded on } [0, \infty). \end{cases} \tag{1.3}$$

By an upper solution β to (1.3) we mean a function $\beta \in BC[0,\infty) \cap C^2(0,\infty)$, $p\beta' \in C[0,\infty)$ with

$$\begin{cases} \frac{1}{p}(p\beta')' \le qf(t,\beta), \quad 0 < t < \infty \\ -a_0\beta(0) + b_0 \lim_{t\to 0^+} p(t)\beta'(t) \le c_0, \\ \beta(t) \quad \text{bounded on } [0,\infty) \end{cases} \tag{1.4}$$

and by a lower solution α to (1.3) we mean a function $\alpha \in BC[0,\infty) \cap C^2(0,\infty)$, $p\alpha' \in C[0,\infty)$ with

$$\begin{cases} \frac{1}{p}(p\alpha')' \ge qf(t,\alpha), \quad 0 < t < \infty \\ -a_0\alpha(0) + b_0 \lim_{t\to 0^+} p(t)\alpha'(t) \ge c_0, \\ \alpha(t) \quad \text{bounded on } [0,\infty). \end{cases} \tag{1.5}$$

THEOREM 1.1 [3] *Let $f: [0,\infty) \times \mathbf{R} \to \mathbf{R}$ be continuous. Suppose the following conditions are satisfied:*

$$q \in C(0,\infty) \text{ with } q > 0 \text{ on } (0,\infty) \tag{1.6}$$

$$p \in C[0,\infty) \cap C^1(0,\infty) \text{ with } p > 0 \text{ on } (0,\infty) \tag{1.7}$$

$$\int_0^{\mu} \frac{ds}{p(s)} < \infty \text{ and } \int_0^{\mu} p(s)q(s)\,ds < \infty \text{ for any } \mu > 0 \tag{1.8}$$

$$\begin{cases} \text{there exists } \alpha, \beta \text{ respectively lower and upper} \\ \text{solutions of (1.3) with } \alpha(t) \le \beta(t) \text{ for } t \in [0,\infty) \end{cases} \tag{1.9}$$

and

$$\begin{cases} \text{there exists a constant } M > 0 \text{ with } |f(t,u)| \le M \\ \text{for } t \in [0,\infty) \text{ and } u \in [\alpha(t),\beta(t)]. \end{cases} \tag{1.10}$$

Then (1.3) *has a solution $y \in BC[0,\infty) \cap C^2(0,\infty)$, $py' \in C[0,\infty)$ with $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [0,\infty)$. Also there exist constants A_0 and A_1 with $|p(t)y'(t)| \le A_0 + A_1 \int_0^t p(s)q(s)\,ds$ for $t \in (0,\infty)$.*

2 THE BOUNDARY CONDITION AT INFINITY

Motivated by the colloid example [4, 7] we discuss the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = q(t)f(t,y), \quad 0 < t < \infty \\ -a_0y(0) + b_0 \lim_{t\to 0^+} p(t)y'(t) = c_0, \quad a_0 > 0, \quad b_0 \ge 0, \ c_0 \le 0 \\ \lim_{t\to\infty} y(t) = 0. \end{cases} \tag{2.1}$$

THEOREM 2.1 *Let $f: [0,\infty) \times \mathbf{R} \to \mathbf{R}$ be continuous and suppose the following conditions hold*:

$$q \in C(0,\infty) \text{ with } q > 0 \text{ on } (0,\infty) \tag{2.2}$$

$$p \in C[0,\infty) \cap C^1(0,\infty) \text{ with } p > 0 \text{ on } (0,\infty) \text{ and } \int_0^\infty \frac{\mathrm{d}s}{p(s)} = \infty \tag{2.3}$$

$$\int_0^\mu \frac{\mathrm{d}s}{p(s)} < \infty \text{ and } \int_0^\mu p(s)q(s)\,\mathrm{d}s < \infty \text{ for any } \mu > 0 \tag{2.4}$$

$$f(t,0) \le 0 \text{ for } t \in (0,\infty) \tag{2.5}$$

$$\exists\, r_0 \ge \frac{-c_0}{a_0} \text{ with } f(t,r_0) \ge 0 \text{ for } t \in (0,\infty) \tag{2.6}$$

$$\exists\, M > 0 \text{ with } |f(t,u)| \le M \text{ for } t \in [0,\infty) \text{ and } u \in [0,r_0] \tag{2.7}$$

$$\begin{cases} \exists \text{ a constant } m > 0 \text{ with } q(t)p^2(t)[f(t,u) - f(t,0)] \ge m^2 u \\ \text{for } t \in (0,\infty) \text{ and } u \in [0,r_0] \end{cases} \tag{2.8}$$

$$\int_0^\infty p(x)\exp\left(-m\int_0^x \frac{\mathrm{d}s}{p(s)}\right) q(x)|f(x,0)|\,\mathrm{d}x < \infty \tag{2.9}$$

$$\lim_{t\to\infty} p^2(t)q(t)f(t,0) = 0 \tag{2.10}$$

and

$$\begin{cases} \lim_{t\to\infty}\left(B_0 \int_\mu^t \frac{1}{p(s)}\int_\mu^s \frac{1}{p(x)}\,\mathrm{d}x\,\mathrm{d}s + C_0\int_\mu^t \frac{\mathrm{d}s}{p(s)}\right) = \infty \\ \text{for any constants } B_0 > 0,\ C_0 \in \mathbf{R} \text{ and } \mu > 0. \end{cases} \tag{2.11}$$

Then (2.1) *has a solution* $y \in C[0,\infty) \cap C^2(0,\infty)$ *with* $py' \in C[0,\infty)$ *and* $0 \le y(t) \le r_0$ *for* $t \in [0,\infty)$.

Proof Now Theorem 1.1 (with $\alpha = 0$ and $\beta = r_0$) guarantees that

$$\begin{cases} \frac{1}{p}(py')' = q(t)f(t,y), \quad 0 < t < \infty \\ -a_0 y(0) + b_0 \lim_{t\to 0^+} p(t)y'(t) = c_0 \\ y(t) \quad \text{bounded on } [0,\infty) \end{cases} \tag{2.12}$$

has a solution $y \in C[0,\infty) \cap C^2(0,\infty)$, $py' \in C[0,\infty)$ and $0 \le y(t) \le r_0$ for $t \in [0,\infty)$. Let $g(x) = q(x)f(x,0)$ and notice that

$$
\begin{aligned}
w(t) &= \exp\left(-m\int_0^t \frac{ds}{p(s)}\right)\left[\frac{(-co)}{a_0+b_0m}\right.\\
&\quad\left.+\frac{(a_0-b_0m)}{2m(a_0+b_0m)}\int_0^\infty p(x)\exp\left(-m\int_0^x\frac{ds}{p(s)}\right)g(x)\,dx\right]\\
&\quad-\frac{1}{2m}\exp\left(m\int_0^t\frac{ds}{p(s)}\right)\int_t^\infty p(x)\exp\left(-m\int_0^x\frac{ds}{p(s)}\right)g(x)\,dx\\
&\quad-\frac{1}{2m}\exp\left(-m\int_0^t\frac{ds}{p(s)}\right)\int_0^t p(x)\exp\left(m\int_0^x\frac{ds}{p(s)}\right)g(x)\,dx\\
&= \exp\left(-m\int_0^t\frac{ds}{p(s)}\right)\left[\frac{(-c_0)}{a_0+b_0m}-\frac{b_0}{a_0+b_0m}\int_0^\infty p(x)\exp\left(-m\int_0^x\frac{ds}{p(s)}\right)g(x)\,dx\right]\\
&\quad-\int_0^t\frac{1}{p(\zeta)}\exp\left(-m\int_\zeta^t\frac{ds}{p(s)}\right)\left(\int_\zeta^\infty p(x)\exp\left(-m\int_\zeta^x\frac{ds}{p(s)}\right)g(x)\,dx\right)d\zeta
\end{aligned}
$$

is a nonnegative solution of

$$
\begin{cases}
\frac{1}{p}(pw')' - \frac{m^2}{p^2(t)}w = g(t), \quad 0<t<\infty\\
-a_0w(0) + b_0\lim_{t\to0^+}p(t)w'(t) = c_0\\
\lim_{t\to\infty}w(t) = 0.
\end{cases}
\tag{2.13}
$$

Notice (2.10) and l'Hopital's rule guarantees that $w(\infty) = 0$.

Now let

$$r(t) = y(t) - w(t).$$

We first show r cannot have a local positive maximum on $[0,\infty)$. Suppose r has a local positive maximum at $t_0 \in [0,\infty)$.

Case (i) $t_0 \in [0,\infty)$.

For $t > 0$ notice from assumption (2.8) that

$$\frac{1}{p}(pr')'(t) = q(t)[f(t,y(t)) - f(t,0)] - \frac{m^2}{p^2(t)}w(t) \ge \frac{m^2}{p^2(t)}[y(t)-w(t)]. \tag{2.14}$$

We also have $r'(t_0) = 0$ and $r''(t_0) \le 0$. However (2.14) yields

$$r''(t_0) = \frac{1}{p(t_0)}(pr')'(t_0) \ge \frac{m^2}{p^2(t_0)}[y(t_0)-w(t_0)] > 0,$$

a contradiction.

Case (ii) $t_0 = 0$.

Of course if $b_0 = 0$ we have a contradiction immediately. So suppose $b_0 \ne 0$. Then

$$\lim_{t\to0^+} p(t)r'(t) = \frac{a_0}{b_0}[y(0)-w(0)]. \tag{2.15}$$

Now since $y(0) - w(0) > 0$ there exists $\delta > 0$ with $y(t) - w(t) > 0$ for $t \in (0, \delta)$. Then (2.14) implies $(pr')' > 0$ on $(0, \delta)$ and this together with (2.15) (*i.e.* $\lim_{t\to 0^+} p(t)y'(t) > 0$) implies $pr' > 0$ on $(0, \delta)$, a contradiction.

Thus $r(t)$ cannot have a local positive maximum on $[0, \delta)$. We now claim that $r(t) \le 0$ on $[0, \infty)$. If $r(t) \not\le 0$ on $[0, \infty)$ then there exists a $c_1 > 0$ with $r(c_1) > 0$. Now since $r(t)$ cannot have a positive local maximum on $[0, \infty)$ it follows that $r(t_2) > r(t_1)$ for all $t_2 > t_1 \ge c_1$; otherwise $r(t)$ would have a local positive maximum on $[0, t_2]$. Thus $r(t)$ is strictly increasing for $t \ge c_1$. Since both $y(t)$ and $w(t)$ are bounded on $[0, \infty)$ and $\lim_{t\to\infty} w(t) = 0$ then

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} [y(t) - w(t)] = \kappa \in (0, r_0]. \tag{2.16}$$

Now there exists $c_2 \ge c_1$ with $y(t) \ge \kappa/2$ for $t \ge c_2$. The differential equation and (2.8) imply that for $t > 0$ that we have

$$\begin{aligned}(p(t)y'(t))' &= p(t)q(t)f(t, y(t)) = p(t)q(t)[f(t, y(t)) - f(t, 0)] + p(t)q(t)f(t, 0)\\ &\ge \frac{m^2}{p(t)} y(t) + p(t)q(t)f(t, 0).\end{aligned}$$

Consequently for $t \ge c_2$ we have

$$(py')'(t) \ge \frac{m^2\kappa}{2p(t)} + p(t)q(t)f(t, 0) = \frac{1}{p(t)}\left[\frac{m^2\kappa}{2} + p^2(t)q(t)f(t, 0)\right].$$

Assumption (2.10) implies that there is a constant $c_3 \ge c_2$ with

$$(py')'(t) \ge \frac{m^2\kappa}{4p(t)} \quad \text{for } t \ge c_3.$$

Two integrations together with the fact that $y \ge 0$ on $[0, \infty)$ yields

$$y(t) \ge p(c_3)y'(c_3)\int_{c_3}^{t} \frac{ds}{p(s)} + \frac{m^2\kappa}{4}\int_{c_3}^{t} \frac{1}{p(s)}\int_{c_3}^{s} \frac{1}{p(x)}\, dx\, ds$$

(not also from Theorem 1.1 that there exist constants A_0 and A_1 with $|p(t)y'(t)| \le A_0 + A_1 \int_0^t p(s)q(s)\, ds$ for $t \in (0, \infty)$). Now assumption (2.11) implies that y is unbounded on $[0, \infty)$, a contradiction. Thus $r(t) \le 0$ on $[0, \infty)$ and the result follows. ∎

Notice in Theorem 3.1 that the solution y of (2.1) satisfies $r(t) \le 0$ for $t \in [0, \infty)$, and so $y(t) \le w(t)$ for $t \in [0, \infty)$.

COROLLARY 2.2 *Let* $f: [0, \infty) \times \mathbf{R} \to \mathbf{R}$ *be continuous and suppose* (2.2)–(2.11) *hold. Then* (2.1) *has a solution* $y \in C[0, \infty) \cap C^2(0, \infty)$ *with* $py' \in C[0, \infty)$ *and* $0 \le y(t) \le w(t)$ *for* $t \in [0, \infty)$, *with* w *given in Theorem* 2.1.

The colloid [4, 7] example motivates our next result.

THEOREM 2.3 *Let* $f: [0, \infty) \times \mathbf{R} \to \mathbf{R}$ *be continuous and suppose* (2.2)–(2.11) *hold. In addition assume the following conditions hold*:

$$f(t, u) \ge 0 \textit{ for } t \in [0, \infty) \quad \textit{and} \quad u \in [0, w(t)];\ \textit{here } w \textit{ is as in Theorem } 2.1 \tag{2.17}$$

and

$$\lim_{t\to\infty} p(t) \in (0, \infty]. \tag{2.18}$$

Then (2.1) *has a solution* $y \in C[0, \infty) \cap C^2(0, \infty)$ with $py' \in C[0, \infty)$, $0 \le y(t) \le w(t)$ *for* $t \in [0, \infty)$ *and* $\lim_{t\to\infty} y'(t) = 0$.

Proof From Corollary 2.2 we know that there exists a solution $y \in C[0, \infty) \cap C^2(0, \infty)$, $py' \in C[0, \infty)$ and $0 \le y(t) \le w(t)$ for $t \in [0, \infty)$, to (2.1). Also (2.17) and the differential equation yields

$$(py')'(t) = p(t)q(t)f(t, y(t)) \ge 0 \quad \text{for } t > 0, \tag{2.19}$$

so py' is nondecreasing on $(0, \infty)$, and $\lim_{t\to\infty} p(t)y'(t) \in [-\infty, \infty]$.

Suppose there exists $t_1 \in [0, \infty)$ with $p(t_1)y'(t_1) > 0$. Then

$$p(t)y'(t) \ge a_0 \equiv p(t_1)y'(t_1) \quad \text{for } t \ge t_1,$$

and so

$$y(t) \ge y(t_1) + a_0 \int_{t_1}^{t} \frac{ds}{p(s)} \quad \text{for } t \ge t_1. \tag{2.20}$$

That is

$$y(t) \ge a_0 \int_{t_1}^{t} \frac{ds}{p(s)} \quad \text{for } t \ge t_1 \tag{2.21}$$

(notice (2.3) implies that the right hand side of (2.21) goes to ∞ as $t \to \infty$). This contradicts $0 \le y(t) \le r_0$ for $t \in [0, \infty)$. Thus $p(t)y'(t) \le 0$ for $t \in (0, \infty)$, and so

$$\lim_{t\to\infty} p(t)y'(t) = \kappa \in [-\infty, 0] \quad \text{and} \quad \lim_{t\to\infty} y'(t) \in [-\infty, 0]. \tag{2.22}$$

In fact $\kappa \in (-\infty, 0]$ from (2.19). Finally if $\kappa < 0$ then there exists $t_2 > 0$ with $p(t)y'(t) \le \kappa/2$ for $t \ge t_2$. Integrate from t_2 to t $(t \ge t_2)$ to get

$$y(t) \le y(t_2) + \frac{\kappa}{2}\int_{t_2}^{t} \frac{ds}{p(s)} \le r_0 + \frac{\kappa}{2}\int_{t_2}^{t} \frac{ds}{p(s)}. \tag{2.23}$$

Now (2.23) together with (2.3) contradicts $y \ge 0$ on $[0, \infty)$. Consequently $\lim_{t\to\infty} p(t) y'(t) = 0$, and this together with (2.18) gives $\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} p(t)y'(t)/p(t) = 0$. ∎

Example 2.1 (Colloid problem [4, 7]).

The boundary value problem

$$\begin{cases} y'' = 2\sinh y, & 0 < t < \infty \\ y(0) = c > 0 \\ \lim_{t\to\infty} y(t) = 0 \end{cases} \tag{2.24}$$

has a solution $y \in C[0,\infty) \cap C^2(0,\infty)$ with

$$0 \le y(t) \le ce^{-t} \quad \text{for } t \in [0,\infty). \tag{2.25}$$

To see this we will apply Corollary 2.2 with

$$p = 1, \;\; q = 1, \;\; a_0 = 1, \;\; c_0 = -c, \;\; b_0 = 0 \text{ and } r_0 = c.$$

Clearly (2.1)–(2.7), (2.8) since $f(t,u) - f(t,0) = \sinh u \ge u$ for $u \ge 0$, (2.9)–(2.11) hold. Corollary 2.2 guarantees that (2.24) has a solution $y \in C[0,\infty) \cap C^2(0,\infty)$ with $0 \le y(t) \le w(t)$ for $t \in [0,\infty)$. It is immediate from (2.13) (since $g = 0$) that

$$w(t) = ce^{-t} \quad \text{for } t \in [0,\infty).$$

Finally we remark that the solution y satisfies $\lim_{t\to\infty} y'(t) = 0$. To see this we need only check that (2.17)–(2.18) hold, but these are immediate.

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