

Boundary Value Problems on the Half Line in the Theory of Colloids

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We present existence results for some boundary value problems defined on infinite intervals. In particular our discussion includes a problem which arises in the theory of colloids.

Key words: Boundary value problem, half line, colloids, existence

1 INTRODUCTION

In the theory of colloids [4, 7] it is possible to relate particle stability with the charge on the colloidal particle. We model the particle and its attendant electrical double layer using Poisson's equation for a flat plate. If Ψ is the potential, ρ the charge density, D the dielectric constant and γ the displacement, then we have

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} = -\frac{4\pi\rho}{D}.$$

We assume the ions are point charged and their concentrations in the double layer satisfies the Boltzmann distribution

$$c_i = c_i^* \exp\left(\frac{-z_i e \Psi}{\kappa T}\right)$$

where c_i is the concentration of ions of type i, $c_i^* = \lim_{\Psi \to 0} c_i$, κ the Boltzmann constant, T the absolute temperature, e the electrical charge, and z the valency of the ion. In the neutral case, we have

$$\rho = c_{+}z_{+}e + c_{-}z_{-}e$$
 or $\rho = ze(c_{+} - c_{-})$

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where $z = z_{+} - z_{-}$. Then we have using

$$c_{+} = c \exp\left(\frac{-ze\Psi}{\kappa T}\right)$$
 and $c_{-} = c \exp\left(\frac{ze\Psi}{\kappa T}\right)$,

that

$$\frac{\mathrm{d}^2 \Psi}{\mathrm{d} y^2} = \frac{8\pi cze}{D} \sinh\left(\frac{ze\Psi}{\kappa T}\right)$$

where the potential initially takes some positive value $\Psi(0) = \Psi_0$ and tends to zero as the distance from the plate increases *i.e.* $\Psi(\infty) = 0$. Using the transformation

$$\phi(y) = \frac{ze\Psi(y)}{\kappa T}$$
 and $x = \sqrt{\frac{4\pi cz^2 e^2}{\kappa TD}}y$,

the problem becomes

$$\begin{cases} \frac{d^2 \phi}{dx^2} = 2 \sinh \phi, & 0 < x < \infty \\ \phi(0) = c_1 & \lim_{x \to \infty} \phi(x) = 0, \end{cases}$$
 (1.1)

where $c_1 = ze\Psi_0/\kappa T > 0$. From a physical point of view we wish the solution ϕ in (1.1) to also satisfy $\lim_{x\to\infty} \phi'(x) = 0$.

In this paper using the notion of upper and lower solutions (see [1, 2, 6]) we establish general existence results which guarantee the existence of $BC[0, \infty)$ solutions to

$$\begin{cases}
\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t)), & 0 < t < \infty \\
-a_0y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) = c_0, & a_0 > 0, b_0 \ge 0 \\
\lim_{t \to \infty} y(t) = 0;
\end{cases}$$
(1.2)

here $BC[0, \infty)$ denotes the space of continuous, bounded functions from $[0, \infty)$ to **R**. Our theory not only complements some of the known results, e.g., [5, 8], but also automatically produces the existence of a solution to (1.1). To establish these results we recall, for the convenience of the reader, the existence principle [3] we will use in Section 2. Consider the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = qf(t,y), & 0 < t < \infty \\ -a_0y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) = c_0, & a_0 > 0, b_0 \ge 0 \\ y(t) & \text{bounded on } [0, \infty). \end{cases}$$
(1.3)

By an upper solution β to (1.3) we mean a function $\beta \in BC[0, \infty) \cap C^2(0, \infty)$, $p\beta' \in C[0, \infty)$ with

$$\begin{cases} \frac{1}{p}(p\beta')' \le qf(t,\beta), & 0 < t < \infty \\ -a_0\beta(0) + b_0 \lim_{t \to 0^+} p(t)\beta'(t) \le c_0, \\ \beta(t) & \text{bounded on } [0,\infty) \end{cases}$$
(1.4)

and by a lower solution α to (1.3) we mean a function $\alpha \in BC[0, \infty) \cap C^2(0, \infty)$, $p\alpha' \in C[0, \infty)$ with

$$\begin{cases} \frac{1}{p}(p\alpha')' \ge qf(t,\alpha), & 0 < t < \infty \\ -a_0\alpha(0) + b_0 \lim_{t \to 0^+} p(t)\alpha'(t) \ge c_0, \\ \alpha(t) & \text{bounded on } [0,\infty). \end{cases}$$
(1.5)

THEOREM 1.1 [3] Let $f:[0,\infty)\times \mathbf{R}\to \mathbf{R}$ be continuous. Suppose the following conditions are satisfied:

$$q \in C(0, \infty)$$
 with $q > 0$ on $(0, \infty)$ (1.6)

$$p \in C[0, \infty) \cap C^1(0, \infty) \text{ with } p > 0 \text{ on } (0, \infty)$$

$$\tag{1.7}$$

$$\int_0^\mu \frac{\mathrm{d}s}{p(s)} < \infty \text{ and } \int_0^\mu p(s)q(s) \,\mathrm{d}s < \infty \text{ for any } \mu > 0$$
 (1.8)

$$\begin{cases} \text{there exists } \alpha, \beta \text{ respectively lower and upper} \\ \text{solutions of } (1.3) \text{ with } \alpha(t) \leq \beta(t) \text{ for } t \in [0, \infty) \end{cases}$$

$$\tag{1.9}$$

and

$$\begin{cases} \text{there exists a constant } M > 0 \text{ with } |f(t, u)| \leq M \\ \text{for } t \in [0, \infty) \text{ and } u \in [\alpha(t), \beta(t)]. \end{cases}$$
 (1.10)

Then (1.3) has a solution $y \in BC[0, \infty) \cap C^2(0, \infty)$, $py' \in C[0, \infty)$ with $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [0, \infty)$. Also there exist constants A_0 and A_1 with $|p(t)y'(t)| \leq A_0 + A_1 \int_0^t p(s)q(s) ds$ for $t \in (0, \infty)$.

2 THE BOUNDARY CONDITION AT INFINITY

Motivated by the colloid example [4, 7] we discuss the boundary value problem

$$\begin{cases}
\frac{1}{p}(py')' = q(t)f(t,y), & 0 < t < \infty \\
-a_0y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) = c_0, & a_0 > 0, \quad b_0 \ge 0, \quad c_0 \le 0 \\
\lim_{t \to \infty} y(t) = 0.
\end{cases}$$
(2.1)

THEOREM 2.1 Let $f:[0,\infty)\times \mathbf{R}\to \mathbf{R}$ be continuous and suppose the following conditions hold:

$$q \in C(0, \infty)$$
 with $q > 0$ on $(0, \infty)$ (2.2)

$$p \in C[0,\infty) \cap C^1(0,\infty)$$
 with $p > 0$ on $(0,\infty)$ and
$$\int_0^\infty \frac{\mathrm{d}s}{p(s)} = \infty$$
 (2.3)

$$\int_0^\mu \frac{\mathrm{d}s}{p(s)} < \infty \quad and \quad \int_0^\mu p(s)q(s) \, \mathrm{d}s < \infty \quad for \quad any \quad \mu > 0$$
 (2.4)

$$f(t,0) \le 0 \text{ for } t \in (0,\infty)$$
 (2.5)

$$\exists \ r_0 \ge \frac{-c_0}{a_0} \ \ with \ f(t, r_0) \ge 0 \ \ for \ t \in (0, \infty)$$
 (2.6)

$$\exists M > 0 \text{ with } |f(t, u)| \le M \text{ for } t \in [0, \infty) \text{ and } u \in [0, r_0]$$
 (2.7)

$$\begin{cases} \exists \ a \ constant \ m > 0 \ with \ q(t)p^2(t)[f(t,u) - f(t,0)] \ge m^2 u \\ for \ t \in (0,\infty) \ and \ u \in [0,r_0] \end{cases} \tag{2.8}$$

$$\int_0^\infty p(x) \exp\left(-m \int_0^x \frac{\mathrm{d}s}{p(s)}\right) q(x) |f(x,0)| \, \mathrm{d}x < \infty \tag{2.9}$$

$$\lim_{t \to \infty} p^2(t)q(t) f(t,0) = 0 \tag{2.10}$$

and

$$\begin{cases} \lim_{t\to\infty} \left(B_0 \int_{\mu}^{t} \frac{1}{p(s)} \int_{\mu}^{s} \frac{1}{p(x)} \, \mathrm{d}x \, \mathrm{d}s + C_0 \int_{\mu}^{t} \frac{\mathrm{d}s}{p(s)} \right) = \infty \\ \text{for any constants } B_0 > 0, C_0 \in \mathbf{R} \text{ and } \mu > 0. \end{cases}$$
 (2.11)

Then (2.1) has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$ with $py' \in C[0, \infty)$ and $0 \le y(t) \le r_0$ for $t \in [0, \infty)$.

Proof Now Theorem 1.1 (with $\alpha = 0$ and $\beta = r_0$) guarantees that

$$\begin{cases} \frac{1}{p} (py')' = q(t)f(t, y), & 0 < t < \infty \\ -a_0 y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) = c_0 \\ y(t) & \text{bounded on } [0, \infty) \end{cases}$$
 (2.12)

has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$, $py' \in C[0, \infty)$ and $0 \le y(t) \le r_0$ for $t \in [0, \infty)$. Let g(x) = q(x)f(x, 0) and notice that

$$w(t) = \exp\left(-m\int_0^t \frac{\mathrm{d}s}{p(s)}\right) \left[\frac{(-co)}{a_0 + b_0 m} + \frac{(a_0 - b_0 m)}{2m(a_0 + b_0 m)} \int_0^\infty p(x) \exp\left(-m\int_0^x \frac{\mathrm{d}s}{p(s)}\right) g(x) \, \mathrm{d}x\right]$$

$$-\frac{1}{2m} \exp\left(m\int_0^t \frac{\mathrm{d}s}{p(s)}\right) \int_t^\infty p(x) \exp\left(-m\int_0^x \frac{\mathrm{d}s}{p(s)}\right) g(x) \, \mathrm{d}x$$

$$-\frac{1}{2m} \exp\left(-m\int_0^t \frac{\mathrm{d}s}{p(s)}\right) \int_0^t p(x) \exp\left(m\int_0^x \frac{\mathrm{d}s}{p(s)}\right) g(x) \, \mathrm{d}x$$

$$= \exp\left(-m\int_0^t \frac{\mathrm{d}s}{p(s)}\right) \left[\frac{(-c_0)}{a_0 + b_0 m} - \frac{b_0}{a_0 + b_0 m} \int_0^\infty p(x) \exp\left(-m\int_0^x \frac{\mathrm{d}s}{p(s)}\right) g(x) \, \mathrm{d}x\right]$$

$$-\int_0^t \frac{1}{p(\zeta)} \exp\left(-m\int_\zeta^t \frac{\mathrm{d}s}{p(s)}\right) \left(\int_\zeta^\infty p(x) \exp\left(-m\int_\zeta^x \frac{\mathrm{d}s}{p(s)}\right) g(x) \, \mathrm{d}x\right) \, \mathrm{d}\zeta$$

is a nonnegative solution of

$$\begin{cases} \frac{1}{p}(pw')' - \frac{m^2}{p^2(t)}w = g(t), & 0 < t < \infty \\ -a_0w(0) + b_0\lim_{t \to 0^+} p(t)w'(t) = c_0 \\ \lim_{t \to \infty} w(t) = 0. \end{cases}$$
(2.13)

Notice (2.10) and l'Hopital's rule guarantees that $w(\infty) = 0$. Now let

$$r(t) = y(t) - w(t).$$

We first show r cannot have a local positive maximum on $[0, \infty)$. Suppose r has a local positive maximum at $t_0 \in [0, \infty)$.

Case (i) $t_0 \in [0, \infty)$.

For t > 0 notice from assumption (2.8) that

$$\frac{1}{p}(pr')'(t) = q(t)[f(t, y(t)) - f(t, 0)] - \frac{m^2}{p^2(t)}w(t) \ge \frac{m^2}{p^2(t)}[y(t) - w(t)]. \tag{2.14}$$

We also have $r'(t_0) = 0$ and $r''(t_0) \le 0$. However (2.14) yields

$$r''(t_0) = \frac{1}{p(t_0)}(pr')'(t_0) \ge \frac{m^2}{p^2(t_0)}[y(t_0) - w(t_0)] > 0,$$

a contradiction.

Case (ii) $t_0 = 0$.

Of course if $b_0 = 0$ we have a contradiction immediately. So suppose $b_0 \neq 0$. Then

$$\lim_{t \to 0^+} p(t)r'(t) = \frac{a_0}{b_0} [y(0) - w(0)]. \tag{2.15}$$

Now since y(0) - w(0) > 0 there exists $\delta > 0$ with y(t) - w(t) > 0 for $t \in (0, \delta)$. Then (2.14) implies (pr')' > 0 on $(0, \delta)$ and this together with (2.15) (i.e. $\lim_{t\to 0^+} p(t)y'(t) > 0$) implies pr' > 0 on $(0, \delta)$, a contradiction.

Thus r(t) cannot have a local positive maximum on $[0, \delta)$. We now claim that $r(t) \leq 0$ on $[0, \infty)$. If $r(t) \not\leq 0$ on $[0, \infty)$ then there exists a $c_1 > 0$ with $r(c_1) > 0$. Now since r(t) cannot have a positive local maximum on $[0, \infty)$ it follows that $r(t_2) > r(t_1)$ for all $t_2 > t_1 \geq c_1$; otherwise r(t) would have a local positive maximum on $[0, t_2]$. Thus r(t) is strictly increasing for $t \geq c_1$. Since both y(t) and w(t) are bounded on $[0, \infty)$ and $\lim_{t \to \infty} w(t) = 0$ then

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} [y(t) - w(t)] = \kappa \in (0, r_0].$$
 (2.16)

Now there exists $c_2 \ge c_1$ with $y(t) \ge \kappa/2$ for $t \ge c_2$. The differential equation and (2.8) imply that for t > 0 that we have

$$(p(t)y'(t))' = p(t)q(t)f(t, y(t)) = p(t)q(t)[f(t, y(t)) - f(t, 0)] + p(t)q(t)f(t, 0)$$

$$\geq \frac{m^2}{p(t)}y(t) + p(t)q(t)f(t, 0).$$

Consequently for $t \ge c_2$ we have

$$(py')'(t) \ge \frac{m^2 \kappa}{2p(t)} + p(t)q(t)f(t,0) = \frac{1}{p(t)} \left[\frac{m^2 \kappa}{2} + p^2(t)q(t)f(t,0) \right].$$

Assumption (2.10) implies that there is a constant $c_3 \ge c_2$ with

$$(py')'(t) \ge \frac{m^2 \kappa}{4p(t)}$$
 for $t \ge c_3$.

Two integrations together with the fact that $y \ge 0$ on $[0, \infty)$ yields

$$y(t) \ge p(c_3)y'(c_3) \int_{c_3}^t \frac{ds}{p(s)} + \frac{m^2 \kappa}{4} \int_{c_3}^t \frac{1}{p(s)} \int_{c_3}^s \frac{1}{p(x)} dx ds$$

(not also from Theorem 1.1 that there exist constants A_0 and A_1 with $|p(t)y'(t)| \le A_0 + A_1 \int_0^t p(s)q(s) \, ds$ for $t \in (0, \infty)$). Now assumption (2.11) implies that y is unbounded on $[0, \infty)$, a contradiction. Thus $r(t) \le 0$ on $[0, \infty)$ and the result follows.

Notice in Theorem 3.1 that the solution y of (2.1) satisfies $r(t) \le 0$ for $t \in [0, \infty)$, and so $y(t) \le w(t)$ for $t \in [0, \infty)$.

COROLLARY 2.2 Let $f:[0,\infty)\times \mathbf{R}\to \mathbf{R}$ be continuous and suppose (2.2)–(2.11) hold. Then (2.1) has a solution $y\in C[0,\infty)\cap C^2(0,\infty)$ with $py'\in C[0,\infty)$ and $0\leq y(t)\leq w(t)$ for $t\in[0,\infty)$, with w given in Theorem 2.1.

The colloid [4, 7] example motivates our next result.

THEOREM 2.3 Let $f:[0,\infty)\times \mathbb{R}\to \mathbb{R}$ be continuous and suppose (2.2)–(2.11) hold. In addition assume the following conditions hold:

$$f(t, u) \ge 0$$
 for $t \in [0, \infty)$ and $u \in [0, w(t)]$; here w is as in Theorem 2.1 (2.17)

and

$$\lim_{t \to \infty} p(t) \in (0, \infty]. \tag{2.18}$$

Then (2.1) has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$ with $py' \in C[0, \infty)$, $0 \le y(t) \le w(t)$ for $t \in [0, \infty)$ and $\lim_{t \to \infty} y'(t) = 0$.

Proof From Corollary 2.2 we know that there exists a solution $y \in C[0, \infty) \cap C^2(0, \infty)$, $py' \in C[0, \infty)$ and $0 \le y(t) \le w(t)$ for $t \in [0, \infty)$, to (2.1). Also (2.17) and the differential equation yields

$$(py')'(t) = p(t)q(t)f(t, y(t)) \ge 0 \quad \text{for } t > 0,$$
(2.19)

so py' is nondecreasing on $(0, \infty)$, and $\lim_{t\to\infty} p(t)y'(t) \in [-\infty, \infty]$. Suppose there exists $t_1 \in [0, \infty)$ with $p(t_1)y'(t_1) > 0$. Then

$$p(t)y'(t) \ge a_0 \equiv p(t_1)y'(t_1)$$
 for $t \ge t_1$,

and so

$$y(t) \ge y(t_1) + a_0 \int_{t_1}^{t} \frac{ds}{p(s)}$$
 for $t \ge t_1$. (2.20)

That is

$$y(t) \ge a_0 \int_{t_1}^t \frac{\mathrm{d}s}{p(s)} \quad \text{for } t \ge t_1$$
 (2.21)

(notice (2.3) implies that the right hand side of (2.21) goes to ∞ as $t \to \infty$). This contradicts $0 \le y(t) \le r_0$ for $t \in [0, \infty)$. Thus $p(t)y'(t) \le 0$ for $t \in (0, \infty)$, and so

$$\lim_{t \to \infty} p(t)y'(t) = \kappa \in [-\infty, 0] \quad \text{and} \quad \lim_{t \to \infty} y'(t) \in [-\infty, 0].$$
 (2.22)

In fact $\kappa \in (-\infty, 0]$ from (2.19). Finally if $\kappa < 0$ then there exists $t_2 > 0$ with $p(t)y'(t) \le \kappa/2$ for $t \ge t_2$. Integrate from t_2 to $t \ge t_2$ to get

$$y(t) \le y(t_2) + \frac{\kappa}{2} \int_{t_2}^{t} \frac{ds}{p(s)} \le r_0 + \frac{\kappa}{2} \int_{t_2}^{t} \frac{ds}{p(s)}.$$
 (2.23)

Now (2.23) together with (2.3) contradicts $y \ge 0$ on $[0, \infty)$. Consequently $\lim_{t \to \infty} p(t)$ y'(t) = 0, and this together with (2.18) gives $\lim_{t \to \infty} y'(t) = \lim_{t \to \infty} p(t)y'(t)/p(t) = 0$.

Example 2.1 (Colloid problem [4, 7]).

The boundary value problem

$$\begin{cases} y'' = 2\sinh y, & 0 < t < \infty \\ y(0) = c > 0 & \\ \lim_{t \to \infty} y(t) = 0 \end{cases}$$
 (2.24)

has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$ with

$$0 \le y(t) \le ce^{-t}$$
 for $t \in [0, \infty)$. (2.25)

To see this we will apply Corollary 2.2 with

$$p = 1$$
, $q = 1$, $a_0 = 1$, $c_0 = -c$, $b_0 = 0$ and $c_0 = c$.

Clearly (2.1)–(2.7), (2.8) since $f(t, u) - f(t, 0) = \sinh u \ge u$ for $u \ge 0$, (2.9)–(2.11) hold. Corollary 2.2 guarantees that (2.24) has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$ with $0 \le y(t) \le w(t)$ for $t \in [0, \infty)$. It is immediate from (2.13) (since g = 0) that

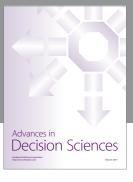
$$w(t) = ce^{-t}$$
 for $t \in [0, \infty)$.

Finally we remark that the solution y satisfies $\lim_{t\to\infty} y'(t) = 0$. To see this we need only check that (2.17)–(2.18) hold, but these are immediate.

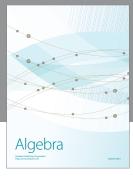
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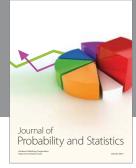
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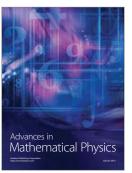


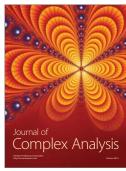




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