

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

FULVIO RICCI

MITCHELL TAIBLESON

**Boundary values of harmonic functions in mixed norm
spaces and their atomic structure**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 10,
n° 1 (1983), p. 1-54*

http://www.numdam.org/item?id=ASNSP_1983_4_10_1_1_0

© Scuola Normale Superiore, Pisa, 1983, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Boundary Values of Harmonic Functions in Mixed Norm Spaces and Their Atomic Structure.

FULVIO RICCI - MITCHELL TAIBLESON (*)

1. - Introduction.

In a recent paper R. R. Coifman and R. Rochberg [1] have obtained representation theorems for spaces of holomorphic and harmonic functions that are in L^p with respect to a weight induced by a Bergman kernel. In this paper we will extend their results to a more general class of « mixed norm » Lebesgue spaces of holomorphic and harmonic functions defined on the upper half-plane

$$\mathbb{R}_+^2 = \{z = x + iy, x \in \mathbb{R}, y > 0\}.$$

We will also characterize the distributions that arise as boundary values of functions in these spaces.

It will be convenient to introduce a « norm » for functions defined on \mathbb{R}_+^2 .

Suppose $0 < s, r < \infty$ and $\beta > 0$, and that $f(x, y)$ is a measurable function on \mathbb{R}_+^2 . Then, with the usual conventions if s or $r = \infty$, let

$$(1.1) \quad M_s(f; y) = \left\{ \int_{\mathbb{R}} |f(x, y)|^s dx \right\}^{1/s} \quad \text{and} \quad N_{sr}^\beta(f) = \left\{ \int_0^\infty (y^\beta M_s(f; y))^r \frac{dy}{y} \right\}^{1/r}.$$

DEFINITIONS (1.2). A_{sr}^β is the space of holomorphic functions on \mathbb{R}_+^2 such

(*) The work of this author was supported in part by a National Science Foundation Grant MCS75-02411A03.

Pervenuto alla Redazione il 22 Ottobre 1980 ed in forma definitiva il 24 Novembre 1982.

that $N_{sr}^\beta(f) < \infty$. \mathcal{A}_{sr}^β is the space of harmonic functions on \mathbb{R}_+^2 such that $N_{sr}^\beta(f) < \infty$.

Note that the natural homogeneity of $N_{sr}^\beta(f)$ is

$$(1.3) \quad h = \min \{1, s, r\}$$

in the sense that $(f, g) \rightarrow (N_{sr}^\beta(f - g))^h$ is a metric.

It is of interest to observe that these spaces of holomorphic and harmonic functions are not new. For example in [6] Flett extended a result of Hardy and Littlewood to the half plane and showed that the Hardy space H^p (consisting of holomorphic functions f such that $\sup_{y>0} M_p(f; y) < \infty$) is continuously contained in the space we have denoted A_{sr}^β , provided $s > p$, $r \geq p$ and $\beta + 1/s = 1/p$.

The main theorems of this part of the paper (the proofs are given in § 6) are representation theorems for functions in A_{sr}^β and \mathcal{A}_{sr}^β . These results extend results of Coifman and Rochberg.

Suppose $0 < s, r < \infty$ and that $\lambda = \{\lambda_{ij}^k\}$, $l, j \in \mathbb{Z}$, $1 \leq k \leq M^2$ (M a positive integer) is a sequence of complex numbers. Then, with the usual conventions if s or $r = \infty$, let

$$(1.4) \quad \|\lambda\|_{sr} = \left(\sum_j \left(\sum_{ik} |\lambda_{ij}^k|^s \right)^{r/s} \right)^{1/r}.$$

(1.5) **THEOREM.** *Suppose $0 < s, r < \infty$, $\beta > 0$ and $\eta > \max \{\beta + 1/s, \beta + 1\}$. Then there is a collection of points $\{\zeta_{ij}^k\}$ in \mathbb{R}_+^2 such that:*

i) *If $\lambda = \{\lambda_{ij}^k\}$ is a sequence of complex numbers such that $\|\lambda\|_{sr} < \infty$ then the series*

$$(1.6) \quad \sum_{ijk} \lambda_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta - (\beta + 1/s)}}{(z - \bar{\zeta}_{ij}^k)^\eta}$$

converges absolutely and uniformly on compact subsets of \mathbb{R}_+^2 to a (holomorphic) function f in A_{sr}^β and there is a constant $C > 0$ that depends only on s, r, β, η and M such that

$$(1.7) \quad N_{sr}^\beta(f) \leq C \|\lambda\|_{sr}.$$

ii) *If $f \in A_{sr}^\beta$ then there is a sequence $\lambda = \{\lambda_{ij}^k\}$, $\|\lambda\|_{sr} < \infty$ such that*

$$(1.8) \quad f(x + iy) = \sum_{ijk} \lambda_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta - (\beta + 1/s)}}{(z - \bar{\zeta}_{ij}^k)^\eta},$$

and there is a constant $C > 0$ that depends only on s, r, β, η and M such that

$$(1.9) \quad \|\lambda\|_{sr} \leq CN_{sr}^\beta(f).$$

Recall the definition of the Poisson kernel,

$$P(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \quad x \in \mathbf{R}, y > 0.$$

(1.10) **THEOREM.** *Suppose $0 < s, r < \infty, \beta > 0$ and κ is a positive integer, $\kappa + 1 > \max\{\beta + 1/s, \beta + 1\}$. Then there is a sequence of points $\{(\xi_{ij}^k, \eta_{ij}^k)\}$ in \mathbf{R}_+^2 such that:*

i) *If $\lambda = \{\lambda_{ij}^k\}$ satisfies $\|\lambda\|_{sr} < \infty$ then the series*

$$(1.11) \quad \sum_{ijk} \lambda_{ij}^k (\eta_{ij}^k)^{(\kappa+1) - (\beta+1/s)} \frac{\partial^\kappa}{\partial y^\kappa} P(x - \xi_{ij}^k, y + \eta_{ij}^k)$$

converges absolutely and uniformly on compact subsets of \mathbf{R}_+^2 to a (harmonic) function f in \mathcal{A}_{sr}^β and there is a constant $C > 0$ that depends only on s, r, β, κ and M such that

$$(1.12) \quad N_{sr}^\beta(f) \leq C \|\lambda\|_{sr}.$$

ii) *If $f \in \mathcal{A}_{sr}^\beta$ then there is a sequence $\lambda = \{\lambda_{ij}^k\}, \|\lambda\|_{sr} < \infty$ such that*

$$(1.13) \quad f(x, y) = \sum_{ijk} \lambda_{ij}^k (\eta_{ij}^k)^{(\kappa+1) - (\beta+1/s)} \frac{\partial^\kappa}{\partial y^\kappa} P(x - \xi_{ij}^k, y + \eta_{ij}^k),$$

and there is a constant $C > 0$ that depends only on s, r, β, κ and M such that

$$(1.14) \quad \|\lambda\|_{sr} \leq CN_{sr}^\beta(f).$$

In § 7 the spaces are extended to a range of $\beta > -1/s$, and in § 8 we consider representation theorems for the duals of \mathcal{A}_{sr}^β and \mathcal{A}_{sr}^β , $0 < s, r < \infty$.

Then we prove that it makes sense to talk of « boundary values » of functions in \mathcal{A}_{sr}^β , as linear functionals on certain Banach spaces (or as distributions), and that there is an atomic description for these objects which is closely related to the atomic description of the Hardy spaces H^p ,

$0 < p < 1$, as in [2, 4, 10]: roughly speaking, a « boundary functional » of a function in \mathcal{A}_{sr}^β is a sum of atoms for H^p , $1/p = \beta + 1/s$, with coefficients satisfying a mixed norm condition involving the exponents s and r (the precise statement is in Theorems (10.9) and (10.23)). We call the spaces obtained in this way H_{sr}^p .

We point out that, in contrast with the situation for the Hardy spaces, we can also deal with the case $p > 1$, but on the other hand we require a very rigid control on the size and the location of the supports of the single atoms, in order to control the norm in H_{sr}^p in terms of the mixed norm of the coefficients. Thus, it is almost impossible to develop an independent mixed-norm-atomic theory without making use of the theory of the Poisson integral and harmonic functions.

The dual space of H_{sr}^p is (naturally) characterized by conditions on the mean oscillations over intervals (that is why we denote it by the letters MO , properly decorated with indices, in analogy with BMO), but we use again Poisson integral together with the results of B. H. Qui [11] to prove that the MO -spaces are the same as the homogeneous Besov-Lipschitz spaces (of positive order) introduced by C. Herz [7], and studied and extended by R. Johnson [9]. We obtain therefore a Campanato-Morrey-type description of these Besov-Lipschitz spaces (see Cor. 12.25). There are a few things we want to point out:

1) Our theory is independent of the theory of Hardy spaces, but does not include it.

2) We only define H_{sr}^p for $p < s$. It would be very interesting to know if it makes sense to define H_{sr}^p for $p \geq s$, and to know what kind of spaces one obtains. Some of these spaces should be in duality with the Morrey spaces $\mathcal{L}^{p,\lambda}$ [1]. It would be also interesting to know whether or not ordinary L^p -spaces, $p > 1$, have an atomic structure.

By an abuse of language, we will refer to (1.1) as « norms », when what we mean is that this quantity induces a metric topology when one uses the proper homogeneity (see comment ii) following (10.8)).

(1.15) CONVENTION. The fact that there are values of s and r in the ranges $0 < s < 1$ and $0 < r < 1$ creates some special technical problems that can be dealt with efficiently by a notational convention. If $0 < s \leq \infty$ we denote by s' the number which is conjugate to s if $1 \leq s \leq \infty$ ($1/s + 1/s' = 1$) and ∞ if $0 < s < 1$. We use the same device for values of r . This convention will be used throughout the paper, and occasional reference to it will be made for emphasis.

2. – Inclusions.

In this section we establish the basic inclusion relations for the A_{sr}^β and \mathcal{A}_{sr}^β spaces. Our basic tool is the following lemma:

(2.1) LEMMA. *Suppose B is a ball in \mathbb{R}^n with center x_0 , and u is harmonic in B and continuous on the closure of B . Then for every $p > 0$ there is a constant $C > 0$ that depends only on p and n such that*

$$|u(x_0)|^p \leq C \frac{1}{|B|} \int_B |u(x)|^p dx.$$

A proof can be found in the paper of Fefferman and Stein [5], p. 172.

(2.2) PROPOSITION. *Suppose $0 < s \leq s_1 < \infty$, $0 < r \leq r_1 < \infty$, $\beta + 1/s = \beta_1 + 1/s_1$. Then $A_{sr}^\beta \subset A_{s_1 r_1}^{\beta_1}$, $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{s_1 r_1}^{\beta_1}$, and the inclusions are continuous.*

PROOF. It will suffice to prove the result under the assumption that f is harmonic. It will also suffice to show the two inclusions, $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{s\infty}^\beta$ and $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{\infty r}^{\beta+1/s}$ with the corresponding estimates on the norms.

Assume first that $0 < s < r$. From Lemma (2.1) we have that if $u \in \mathcal{A}_{sr}^\beta$,

$$(2.3) \quad |u(x + iy)|^s \leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} \int_{|v-t| < y/2} |u(\xi, \eta)|^s d\xi d\eta.$$

Thus, since the inner integral can be viewed as the convolution of the L^1 function $|u(\cdot, \eta)|^s$ with the characteristic function of $(-y/2, y/2)$, we have

$$(M_s(u; y))^s \leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} y (M_s(u; \eta))^s d\eta \leq C \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^s \frac{d\eta}{\eta}.$$

Then use Hölder with index r/s and we have,

$$(2.4) \quad M_s(u; y) \leq C \left\{ \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^r \frac{d\eta}{\eta} \right\}^{1/r}.$$

If $r \leq s < \infty$, then we have,

$$(2.5) \quad |u(x + iy)|^r \leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} \int_{|x-t| < y/2} |u(\xi, \eta)|^r d\xi d\eta,$$

so that if we first use Minkowski's integral inequality and then Young's theorem (with index s/r) we obtain,

$$\begin{aligned} (M_s(u; y))^r &\leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} \| (u(\cdot, \eta))^r * \chi_{(-y/2, y/2)} \|_{s/r} d\eta \\ &\leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} y (M_s(u; \eta))^r d\eta \leq C \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^r \frac{d\eta}{\eta} \end{aligned}$$

which is another version of (2.4). Thus,

$$y^\beta M_s(u; y) \leq C \left\{ \int_{y/2}^{\frac{3}{2}y} (\eta^\beta M_s(u; \eta))^r \frac{d\eta}{\eta} \right\}^{1/r} \leq CN_{sr}^\beta(u),$$

and consequently $N_{s\infty}^\beta(u) \leq CN_{sr}^\beta(u)$, $u \in \mathcal{A}_{s\infty}^\beta$.

To show that $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{\infty r}^\beta$ (where we assume that $s < \infty$, of course), we again consider two cases: $0 < s < r$, and $r \leq s < \infty$. When $0 < s < r$ we have by (2.3) that

$$(M_\infty(u; y))^s \leq \frac{C}{y} \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^s \frac{d\eta}{\eta}$$

so that,

$$\begin{aligned} N_{\infty, r}^{\beta+1/s}(u) &= \left(\int_0^\infty (y^{\beta+1/s} M_\infty(u; y))^r \frac{dy}{y} \right)^{1/r} \\ &\leq C \left\{ \int_0^\infty \left[\left(y^\beta \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^s \frac{d\eta}{\eta} \right)^{1/s} \right]^r \frac{dy}{y} \right\}^{1/r} \\ &\leq C \left\{ \int_0^\infty y^{\beta r} \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^r \frac{d\eta}{\eta} \frac{dy}{y} \right\}^{1/r} \\ &= C \left\{ \int_0^\infty (M(u; \eta))^r \int_{\frac{2}{3}\eta}^{2\eta} y^{\beta r} \frac{dy}{y} \frac{d\eta}{\eta} \right\}^{1/r} \\ &\leq CN_{s,r}^\beta(u). \end{aligned}$$

If $r \leq s < \infty$, we start from (2.5) and obtain, using Hölder with index s/r

on the inner integral,

$$\begin{aligned} |u(x + iy)|^r &\leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} \int_{|x-t| < y/2} |u(\xi, \eta)|^r d\xi d\eta \\ &\leq \frac{C}{y^2} \int_{y/2}^{\frac{3}{2}y} \left[\int_{|x-t| < y/2} |u(\xi, \eta)|^s d\xi \right]^{r/s} y^{1-r/s} d\eta \\ &\leq Cy^{-r/s} \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^r \frac{d\eta}{\eta}, \end{aligned}$$

and so it follows that

$$(M_\infty(u; y))^r \leq Cy^{-r/s} \int_{y/2}^{\frac{3}{2}y} (M_s(u; \eta))^r \frac{d\eta}{\eta},$$

and the rest of the proof goes just as for the case, $0 < s < r$.

REMARK. It can be proved that if $f \in A_{sr}^\beta$ or \mathcal{A}_{sr}^β , $s < \infty$, then for every $y > 0$ $\lim_{|x| \rightarrow \infty} f(x + iy) = 0$ and therefore A_{sr}^β and \mathcal{A}_{sr}^β are contained in closed subspaces of $A_{\infty r}^\beta$ and $\mathcal{A}_{\infty r}^\beta$ respectively. The argument is essentially the same as in [12], p. 80. In the same way, if $r < \infty$ and $f \in \mathcal{A}_{sr}^\beta$ or A_{sr}^β then $M_s(f; y) = o(y^{-\beta})$ as $y \rightarrow \infty$ or $y \rightarrow 0$.

3. - Reproducing kernels for spaces of harmonic functions.

In this section we demonstrate the existence of reproducing kernels for the \mathcal{A}_{sr}^β spaces. Recall that $P(x, y)$ is the Poisson kernel.

(3.1) THEOREM. If $u \in \mathcal{A}_{sr}^\beta$ and κ is an integer, $\kappa + 1 > \max\{\beta + 1/s, \beta + 1\}$ then

$$u(x + iy) = \frac{(-2)^\kappa}{(\kappa - 1)!} \int_0^\infty \int_{-\infty}^{+\infty} f(\xi + i\eta) \eta^\kappa \frac{\partial^\kappa}{\partial y^\kappa} P(x - \xi, y + \eta) d\xi \frac{d\eta}{\eta},$$

and the integral converges absolutely.

PROOF. Let us first observe that the integral converges absolutely. This depends on the observation that $M_p((\partial^\kappa / \partial y^\kappa)P; y) \leq C_\kappa y^{-(\kappa+1/p)}$ and on the inclusions $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{s\infty}^\beta$, $1 \leq s < \infty$ and $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{1\infty}^{\beta+1/s-1}$, $0 < s \leq 1$. Thus if $f \in \mathcal{A}_{sr}^\beta$ and $1 \leq s < \infty$ we have $M_s(f; \eta) \leq Cy^{-\beta}$ and if $0 < s \leq 1$ we have $M_1(f; \eta)$

$\leq Cy^{1-(\beta+1/s)}$. In the two cases the absolute convergence reduces to consideration of the integrals

$$\int_0^{\infty} \frac{\eta^{\kappa-\beta}}{(y+\eta)^{\kappa+1/s}} \frac{d\eta}{\eta} \quad \text{and} \quad \int_0^{\infty} \frac{\eta^{(\kappa+1)-(\beta+1/s)}}{(y+\eta)^{\kappa+1}} \frac{d\eta}{\eta}$$

respectively and the convergence follows.

A straightforward integration by parts argument shows that if $y > 0$, $N > 0$ then

$$\begin{aligned} u(x+iy) &= \int_{-\infty}^{+\infty} u\left(\xi + i\frac{y}{2}\right) P\left(x - \xi, \frac{y}{2}\right) d\xi \\ &= \frac{(-1)^\kappa}{(\kappa-1)!} \int_{-\infty}^{\infty} u\left(\xi + i\frac{y}{2}\right) \int_0^N \eta^{\kappa-1} \frac{\partial^\kappa}{\partial y^\kappa} P\left(x - \xi, \eta + \frac{y}{2}\right) d\eta d\xi \\ &\quad + u(x + i(N+y)) + \dots + \frac{N^{\kappa-1}}{(\kappa-1)!} \frac{\partial^{\kappa-1}}{\partial y^{\kappa-1}} u(x + i(N+y)). \end{aligned}$$

The main ingredient here is that $\mathcal{A}_{sr}^\beta \subset \mathcal{A}_{\infty\infty}^{\beta+1/s}$ so that $|u(x+iy)| \leq Cy^{-(\beta+1/s)}$ and so $u(x, y)$ is bounded and continuous on each proper subhalfplane of \mathbb{R}_+^2 and the semigroup formula for the Poisson integral applies. Namely, $(\partial^\kappa/\partial y^\kappa)u(\cdot, y+\eta) = u(\cdot, y) * (\partial^\kappa/\partial y^\kappa)P(\cdot, \eta)$ whenever $y, \eta > 0$ and κ is a non-negative integer. From this we find that $(\partial^\kappa/\partial y^\kappa)u(x+iy) = O(y^{-(\kappa+\beta+1/s)})$ for all non-negative integers κ . Consequently, as $N \rightarrow \infty$ each of the boundary terms tends to zero and we have

$$\begin{aligned} u(x+iy) &= \frac{(-1)^\kappa}{(\kappa-1)!} \int_{-\infty}^{+\infty} \int_0^{\infty} u\left(\xi + i\frac{y}{2}\right) \eta^\kappa \frac{\partial^\kappa}{\partial y^\kappa} P\left(x - \xi, \eta + \frac{y}{2}\right) \frac{d\eta}{\eta} d\xi \\ &= \frac{(-1)^\kappa}{(\kappa-1)!} \int_0^{\infty} \int_{-\infty}^{\infty} u\left(\xi + i\frac{\eta}{2}\right) \eta^\kappa \frac{\partial^\kappa}{\partial y^\kappa} P\left(x - \xi, \frac{\eta}{2} + y\right) d\xi \frac{d\eta}{\eta} \\ &= \frac{(-2)^\kappa}{(\kappa-1)!} \int_0^{\infty} \int_{-\infty}^{+\infty} u(\xi + i\eta) \eta^\kappa \frac{\partial^\kappa}{\partial y^\kappa} P(x - \xi, \eta + y) d\xi \frac{d\eta}{\eta}. \end{aligned}$$

In summary, we have been able to find a reproducing kernel for each integer κ that is large enough.

4. – Reproducing kernels for spaces of holomorphic functions.

In this section we demonstrate the existence of reproducing kernels for each η , real, that is large enough.

The first thing we do is to construct a subharmonic function that is non-increasing in $y > 0$ and whose norm is equivalent to that of $f \in A_{sr}^\beta$.

(4.1) LEMMA. *If $f \in A_{sr}^\beta$ and $f^*(x + iy) = \sup_{\eta \geq y} |f(x + i\eta)|$ then $f^*(x + iy)$ is defined for all $x + iy \in \mathbb{R}_+^2$ and $N_{sr}^\beta(f^*) \leq CN_{sr}^\beta(f)$ where $C > 0$ is a constant that depends only on s .*

PROOF. Suppose $f \in A_{sr}^\beta$. Then from (2.2) we have $f \in A_{\infty\infty}^{\beta+1/s}$ so $f(x + iy) = O(y^{-(\beta+1/s)})$ as $y \rightarrow \infty$ and so $f^*(x + iy)$ is well defined. But this also shows that $h_{y_0}(z) = |f(z + iy_0)|^s$, a subharmonic function that is continuous on the closure of \mathbb{R}_+^2 , is bounded on \mathbb{R}_+^2 . It follows from (2.2) that $f \in A_{s\infty}^\beta$ so there is a dense (in $(0, \infty)$) set of values $\{y_0\}$ such that $h_{y_0}(x) \in L^1$. It now follows by standard arguments that

$$h_{y_0}(x + iy) \leq \int_{-\infty}^{+\infty} h_{y_0}(\xi) P(x - \xi, y) d\xi, \quad y > 0$$

and consequently, $\int h_{y_0}(x + iy) dx \leq \int h_{y_0}(x) dx$, which implies that $M_s(f; y + y_0) \leq M_s(f; y_0)$ for all $y > 0$. Then by continuity if $s = \infty$ or Fatou's Lemma if $s < \infty$ we get that $M_s(f; y)$ is non-increasing in $y > 0$. From this it follows that $g_{y_0}(z) = f(z + iy_0)$ is in the Hardy space H^s for all $y_0 > 0$ and by standard H^s arguments it follows that $f^*(x + iy) = \sup_{\eta > 0} |g_\eta(x + iy)|$ is in L^s and $M_s(f^*; y) \leq C_s \|g_y\|_{L^s} = C_s M_s(f; y)$. Now it follows that $N_{sr}^\beta(f^*) \leq C_s N_{sr}^\beta(f)$.

The fact that $f^*(x + iy)$ is subharmonic is an easy exercise and since we do not use that fact the proof is left to the reader.

The spaces A_{22}^β are Hilbert spaces of functions in $L^2(\mathbb{R}_+^2, y^{2\beta-1} dy dx)$, and it follows from the estimate, $f \in A_{22}^\beta \Rightarrow |f(x + iy)| \leq CN_{22}^\beta(f) y^{-(\beta+1/2)}$ that pointwise evaluation is a continuous linear functional on A_{22}^β . Thus, there is a unique function $K_\beta(z, \zeta)$ which is holomorphic in z , antiholomorphic in ζ and such that for every $f \in A_{22}^\beta$ and $z \in \mathbb{R}_+^2$

$$(4.2) \quad f(z) = \int f(\zeta) K_\beta(z, \zeta) (\text{Im } \zeta)^{2\beta-1} d\zeta,$$

and it is known that

$$(4.3) \quad K_\beta(z, \zeta) = c_\beta(z - \bar{\zeta})^{-(2\beta+1)}.$$

See [13] for (4.3).

(4.4) LEMMA. *If $f \in A_{sr}^\beta$ and $\eta > \max\{\beta + 1/s, \beta + 1\}$ then*

$$f(z) = C_\eta \int_{\mathbf{R}_+} f(\zeta) \frac{(\operatorname{Im} \zeta)^{\eta-1}}{(z - \bar{\zeta})^\eta} d\eta$$

for each $z \in \mathbf{R}_+^2$, and the integral converges absolutely.

PROOF. Suppose $f \in A_{sr}^\beta$. Then for every $\varepsilon > 0$ let

$$f_\varepsilon(z) = \frac{e^{i\varepsilon z}}{1 - i\varepsilon z} f(z + i\varepsilon).$$

It is an easy exercise to show that $f_\varepsilon \in A_{22}^\alpha$ for every $\alpha > 0$. (If $0 < s < 1$, use the inclusion $A_{sr}^\beta \subset A_{1\infty}^{\beta+1/s-1}$). It follows from (4.2) and (4.3) that for any $\eta > 1$,

$$f_\varepsilon(z) = C_\eta \int_{\mathbf{R}_+} f_\varepsilon(\zeta) \frac{(\operatorname{Im} \zeta)^{\eta-2}}{(z - \bar{\zeta})^\eta} d\zeta.$$

It is obvious that $f_\varepsilon(z) \rightarrow f(z)$ as $\varepsilon \rightarrow 0$, all $z \in \mathbf{R}_+^2$, so the result will follow if the integral is dominated by an integrable function. The obvious dominating function is $f^*(\zeta)(\operatorname{Im} \zeta)^{\eta-2}|z - \bar{\zeta}|^{-\eta}$. Let us first assume that $s \geq 1$. We write $z = x + iy$, $\zeta = \xi + it$. Then

$$\begin{aligned} \int f^*(\zeta)(\operatorname{Im} \zeta)^{\eta-2}|z - \bar{\zeta}|^{-\eta} d\zeta &= \int_0^\infty t^{\eta-2} \int_{-\infty}^{+\infty} \frac{f^*(\xi + it)}{((x - \xi)^2 + (t + y)^2)^{\eta/2}} d\xi dt \\ &\leq \int_0^\infty t^{\eta-1} M_s(f^*; t) \left[\int_{-\infty}^{+\infty} \frac{1}{(\xi^2 + (y + t)^2)^{\eta s'/2}} d\xi \right]^{1/s'} \frac{dt}{t} \leq C \int_0^\infty t^\beta M_s(f^*; t) \frac{t^{\eta-(\beta+1)}}{(y + t)^{\eta-1/s'}} \frac{dt}{t} \\ &\leq CN_{s\infty}^\beta(f^*) \int_0^\infty \frac{t^{\eta-(\beta+1)}}{(y + t)^{\eta-1/s'}} \frac{dt}{t} \leq CN_{sr}^\beta(f), \end{aligned}$$

provided only $\eta > \beta + 1$. This proves the result for $1 \leq s \leq \infty$. If $0 < s < 1$ we use the inclusion $A_{sr}^\beta \subset A_{1r}^{\beta+1/s-1}$, so the result follows if $\eta > (\beta + 1/s - 1) + 1 = \beta + 1/s$. This completes the proof.

5. – A technical lemma.

The contents of this section is the long proof of a size estimate needed for the proofs of the main theorems.

(5.1) LEMMA. *Suppose we are given $\lambda = \{\lambda_{lj}\}$, $l, j \in \mathbb{Z}$, $\lambda_{lj} \geq 0$ and $0 < s$, $r \leq \infty$, $0 < \beta < \infty$ and $\eta > \max\{\beta + 1/s, \beta + 1\}$ with $\|\lambda\|_{sr} < \infty$. Then let*

$$(5.2) \quad \varphi(x, y) = \sum_{lj} \lambda_{lj} \frac{2^{j(\eta - (\beta + 1/s))}}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta/2}}$$

for $(x, y) \in \mathbb{R}_+^2$. Then the series defining φ converges uniformly on compact subsets of \mathbb{R}_+^2 and

$$(5.3) \quad N_{sr}^\beta(\varphi) \leq C \|\lambda\|_{sr},$$

where the constant C depends only on η, β, s and r .

PROOF. We first consider the question of uniform convergence. If $0 < s < 1$ the proof is particularly simple and we may write $l2^j = \xi_{lj}$ since the value of $\xi_{lj} \in \mathbb{R}$ plays no role in this case. We will now show that the series converges uniformly on each proper subhalfplane: $\{(x, y) \in \mathbb{R}_+^2 : y > y_0 > 0\}$. Fix $\varepsilon > 0$ and let F_ε be the finite set of indices,

$$F_\varepsilon = \{(l, j) : J_1 < j \leq J_2, |l| \leq L_j\}$$

where $J_1 < J_2$ and $\{L_j\}$ are integers which we select below.

$$\begin{aligned} & \sum_{(l,j) \notin F_\varepsilon} \lambda_{lj} \frac{2^{j(\eta - (\beta + 1/s))}}{((x - \xi_{lj})^2 + (y + 2^j)^2)^{\eta/2}} \\ & \leq \sum_{\substack{j < J_1 \\ \text{or } j > J_2}} \left(\sum_l \lambda_{lj} \right) \frac{2^{j(\eta - (\beta + 1/s))}}{(y_0 + 2^j)^\eta} + \sum_{j=J_1}^{J_2} \left(\sum_{|l| > L_j} \lambda_{lj} \right) \frac{2^{j(\eta - (\beta + 1/s))}}{(y_0 + 2^j)^\eta} \\ & \leq \sup_j \left(\sum_l \lambda_{lj} \right) \left\{ \frac{1}{y_0^\eta} \sum_{j > J_1} 2^{j(\eta - (\beta + 1/s))} + \sum_{j > J_2} 2^{-j(\beta + 1/s)} \right\} \\ & + \sup_{J_1 \leq j \leq J_2} \left(\sum_{|l| > L_j} \lambda_{lj} \right) (J_2 - J_1 + 1) \frac{2^{J_2(\eta - (\beta + 1/s))}}{y_0^\eta} \\ & \leq C_1 \|\lambda\|_{sr} \{2^{J_1(\eta - (\beta + 1/s))} + 2^{-J_2(\beta + 1/s)}\} + C_2 \sup_{J_1 \leq j \leq J_2} \left[\sum_{|l| > L_j} (\lambda_{lj})^s \right]^{1/s}, \end{aligned}$$

where C_1 and C_2 depend on η, β, s, r and y_0 , and (in addition) C_2 depends on J_1 and J_2 . One then chooses J_1 « small enough », J_2 « large enough ».

Then choose the $\{L_j\}$ and the sum will be less than $\varepsilon > 0$. The only conditions we needed were: $\eta > \beta + 1/s > 0$, so the proof of uniform convergence for $s \leq 1$ is complete.

Consider now the case $1 < s \leq \infty$. Let $\|\lambda(j)\|_s = \left(\sum_l (\lambda_{lj})^s\right)^{1/s}$, if $1 < s < \infty$, $\|\lambda(j)\|_\infty = \sup_l |\lambda_{lj}|$. We now show that the series converges uniformly on « proper strips » of \mathbb{R}_+^2 ; that is, on regions of the form: $\{(x, y) \in \mathbb{R}_+^2 : y \geq y_0 > 0, |x| \leq x_0 < \infty\}$. Fix $\varepsilon > 0$ and let F_ε be the finite set of integers

$$F_\varepsilon = \{(l, j) : J_1 \leq j \leq J_2, |l| \leq [2^{-j}x_0] + K\}$$

where $J_1 < J_2$ and K are integers which we choose below. Let s' be the index conjugate to s ($1/s + 1/s' = 1$).

$$\begin{aligned} & \sum_{(l,j) \notin F_\varepsilon} \lambda_{lj} \frac{2^{j(\eta - (\beta + 1/s))}}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta/2}} \\ & \leq \sum_{\substack{j > J_2 \\ \text{or } j < J_1}} 2^{j(\eta - (\beta + 1/s))} \|\lambda(j)\|_s \left\{ \sum_l \frac{1}{((x - l2^j)^2 + (y + 2^j)^2)^{(\eta/2)s'}} \right\}^{1/s'} \\ & \quad + \sum_{j=J_1}^{J_2} 2^{j(\eta - (\beta + 1/s))} \|\lambda(j)\|_s \left\{ \sum_{|l| > [2^{-j}x_0] + K} \frac{1}{|x - l2^j|^{\eta s'}} \right\}^{1/s'} \\ & \leq C \|\lambda\|_{s\infty} \left\{ \sum_{\substack{j < J_1 \\ \text{or } j > J_2}} \frac{2^{j(\eta - (\beta + 1))}}{(y + 2^j)^{\eta - 1 + 1/s}} + \sum_{j=J_1}^{J_2} \frac{2^{j(\eta - (\beta + 1/s))}}{2^{j\eta}} \frac{1}{K^{\eta - 1/s'}} \right\} \\ & \leq C \|\lambda\|_{sr} \{2^{J_1(\eta - (\beta + 1))} + 2^{-J_2(\beta + 1/s)} + (J_2 - J_1 + 1)2^{-J_1(\beta + 1/s)} K^{-(\eta - 1/s')}\}. \end{aligned}$$

This requires $\eta > \beta + 1$, $\beta + 1/s > 0$ and $\eta > 1/s'$, all of which are satisfied. One then chooses J_1 « small enough », J_2 « large enough » and then K « large enough » and the sum is made less than any prescribed $\varepsilon > 0$. The proof of uniform convergence is complete.

We now turn to the estimates for $N_{sr}^\beta(\varphi)$.

In this part of the proof one first considers the case $0 < s \leq 1$ with subcases: $r = \infty$, $s \leq r < \infty$, and $0 < r \leq s$; then the case $s = \infty$ with subcases: $r = \infty$, $0 < r \leq 1$, and $1 < r < \infty$; and finally the case $1 < s < \infty$, $0 < r \leq \infty$. Here the main subcases are: $\eta > (\beta + 1/s)(\beta + 1/s')/\beta$ and $\beta + 1 < \eta < (\beta + 1/s)(\beta + 1/s')/\beta$. The ultimate subcase is the most difficult and we provide details for this one subcase.

Thus we have $1 < s < \infty$, $\beta > 0$, and $\eta = \beta + 1 + \varepsilon$ where $0 < \varepsilon \leq 1/\beta s s'$. We will select two parameters $0 < \theta, \tau < 1$ that meet certain specifications. We go through the proof formally and then show that an appropriate choice

of θ and τ is possible. We have:

$$\varphi(x, y)^s \leq \left\{ \sum_{i'} (\lambda_{i'})^s \frac{2^{j(\eta - (\beta + 1/s)\tau)s}}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta\theta s/2}} \right\} \cdot \left\{ \sum_{i'} \frac{2^{j(\eta - (\beta + 1/s)(1-\tau)s'}}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta(1-\theta)s'/2}} \right\}^{s/s'}$$

The last factor is bounded by $Cy^{-[\eta(1-\theta) - (\eta - (\beta + 1/s)(1-\tau)]s}$ provided the following conditions are satisfied:

$$(5.4) \quad \begin{aligned} & \text{i) } \eta(1 - \theta)s' > 1; \\ & \text{ii) } \left(\eta - \left(\beta + \frac{1}{s} \right) \right) (1 - \tau)s' > 1; \\ & \text{iii) } \eta(1 - \theta) > \left(\eta - \left(\beta + \frac{1}{s} \right) \right) (1 - \tau). \end{aligned}$$

Thus,

$$\begin{aligned} (y^\beta M_s(\varphi; y))^s & \leq C \sum_j \frac{2^{j(\eta - (\beta + 1/s)\tau)s} y^{\beta s}}{y^{[\eta(1-\theta) - (\eta - (\beta + 1/s)(1-\tau)]s}} \sum_{i'} (\lambda_{i'})^s \int_{\mathbf{R}} \frac{dx}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta\theta s/2}} \\ & \leq C \sum_j \frac{2^{j(\eta - (\beta + 1/s)\tau)s} y^{\eta(\theta - \tau)s + \tau(\beta s + 1) - 1}}{(y + 2^j)^{\eta\theta s - 1}} \sum_{i'} (\lambda_{i'})^s \end{aligned}$$

where we need the additional restriction

$$(5.5) \quad \text{iv) } \eta\theta s > 1.$$

It is now clear that if θ, τ satisfy the additional conditions:

$$(5.6) \quad \begin{aligned} & \text{v) } \left(\eta - \left(\beta + \frac{1}{s} \right) \right) \tau s > 0 \\ & \text{vi) } \eta\theta s - 1 > \left(\eta - \left(\beta + \frac{1}{s} \right) \right) \tau s, \end{aligned}$$

then the rest of the proof will go through as we will show.

We may eliminate v) immediately and then it is clear that vi) implies iv). Thus, four conditions need to be satisfied: i), ii), iii) and vi) of (5.4) and (5.6).

Using $\eta = \beta + 1 + \varepsilon$ (so that $\eta - (\beta + 1/s) = \varepsilon + 1/s'$) we may rewrite

these conditions as follows:

$$\begin{aligned}
& \text{i) } \eta(1-\theta)s' > 1 \quad \text{iff} \quad \theta < \frac{\beta + 1/s + \varepsilon}{\beta + 1 + \varepsilon} \\
& \text{ii) } \left(\eta - \left(\beta + \frac{1}{s} \right) \right) (1-\tau)s' > 1 \quad \text{iff} \quad \tau < \frac{\varepsilon}{\varepsilon + 1/s'} \\
(5.7) \quad & \text{iii) } \eta(1-\theta) > \left(\eta - \left(\beta + \frac{1}{s} \right) \right) (1-\tau) \quad \text{iff} \quad \theta < \frac{(\beta + 1/s) + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} \\
& \text{vi) } \eta\theta s - 1 > \left(\eta - \left(\beta + \frac{1}{s} \right) \right) \tau s \quad \text{iff} \quad \theta > \frac{1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon}.
\end{aligned}$$

Suppose $\tau \geq \theta$. If we rewrite vi) in the form $\eta(\theta - \tau) > 1/s - \tau(\beta + 1/s)$ it would follow that $\tau > \frac{1/s}{\beta + 1/s}$: But from ii) we have that $\tau < \frac{\varepsilon}{\varepsilon + 1/s'}$, so that $\frac{1/s}{\beta + 1/s} < \frac{\varepsilon}{\varepsilon + 1/s'}$ which implies that $\varepsilon > 1/\beta ss'$. But $0 < \varepsilon \leq 1/\beta ss'$ so that we are looking for $0 < \tau < \theta < 1$ which satisfy the conditions:

$$\begin{aligned}
(5.8) \quad \theta < \frac{\beta + 1/s + \varepsilon}{\beta + 1 + \varepsilon}; \quad \tau < \frac{\varepsilon}{\varepsilon + 1/s'}; \quad \frac{1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} \\
& < \theta < \frac{(\beta + 1/s) + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon}.
\end{aligned}$$

Some simple observations.

$$\begin{aligned}
& \text{i) } \frac{\beta + 1/s + \varepsilon}{\beta + 1 + \varepsilon} > \frac{\varepsilon}{\varepsilon + 1/s'} \quad (\text{just multiply out}); \\
& \text{ii) } \frac{(\beta + 1/s) + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} > \frac{1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} \quad (\text{obvious}); \\
& \text{iii) } \tau < \frac{1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} \quad \text{iff} \quad \tau < \frac{1/s}{\beta + 1/s} \quad (\text{solve for } \tau); \\
& \text{iv) } \text{If } 0 < \varepsilon \leq \frac{1}{\beta ss'} \quad \text{then} \quad \min\left(\frac{\varepsilon}{\varepsilon + 1/s'}, \frac{1/s}{\beta + 1/s}\right) = \frac{\varepsilon}{\varepsilon + 1/s'}; \\
& \text{v) } \text{If } \tau < \frac{\varepsilon}{\varepsilon + 1/s'} \quad \text{then} \quad \min\left(\frac{\beta + 1/s + \varepsilon}{\beta + 1 + \varepsilon}, \frac{\beta + 1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon}\right) \\
& \quad \quad \quad = \frac{\beta + 1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon}.
\end{aligned}$$

The recipes for the choices of τ or θ are then:

First. Choose τ so that $0 < \tau < \frac{\varepsilon}{\varepsilon + 1/s'}$.

From iii) and iv) above this implies $\tau < \frac{1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon}$.

Second. Choose θ so that $\frac{1/s + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} < \theta < \frac{(\beta + 1/s) + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon}$.

Since

$$\theta < \frac{(\beta + 1/s) + \tau(\varepsilon + 1/s')}{\beta + 1 + \varepsilon} < \frac{\beta + 1/s + \varepsilon}{\beta + 1 + \varepsilon} < 1,$$

we see (since the intervals defining possible choices for τ and θ are non-empty) that a suitable choice for τ and θ is possible with $\eta > \beta + 1$.

To verify an estimate of the form:

$$(y^\beta M_s(\varphi; y))^s \leq C \sum_j \frac{2^{ja} y^b}{(y + 2^j)^{a+b}} \sum_t (\lambda_{tj})^s$$

with $a, b > 0$ it is indeed sufficient to consider three subcases: $0 < r \leq s$. Then $0 < r/s \leq 1$, so

$$(N_{sr}^\beta(\varphi))^r \leq C \sum_j \int_0^\infty \left(\frac{t^b}{(1+t)^{a+b}} \right)^{r/s} \frac{dt}{t} \left(\sum_t (\lambda_{tj})^s \right)^{r/s} = C \|\lambda\|_s^r.$$

$r = \infty$. Observe that

$$\sum_j \frac{2^{ja} y^b}{(y + 2^j)^{a+b}} \leq \sum_{2^j \leq y} \left(\frac{2^j}{y} \right)^a + \sum_{2^j > y} \left(\frac{y}{2^j} \right)^b \leq C,$$

where $C > 0$ does not depend on y . Consequently,

$$(y^\beta M_s(\varphi; y))^s \leq C \sup_j \sum_t (\lambda_{tj})^s$$

for all $y > 0$, and so

$$(N_{s\infty}^\beta(\varphi))^s \leq C \|\lambda\|_{s\infty}^s.$$

$1 < s \leq r < \infty$. Then $1 \leq r/s < \infty$, so

$$\begin{aligned}
(N_{sr}^\beta(\varphi))^s &\leq C \left[\int_0^\infty \left\{ \sum_j \frac{2^{ja} y^b}{(y + 2^j)^{a+b}} \sum_l (\lambda_{lj})^s \right\}^{r/s} \frac{dy}{y} \right]^{s/r} \\
&\leq C \left[\sum_k \left\{ \sum_j \frac{2^{ja} 2^{kb}}{(2^j + 2^k)^{a+b}} \sum_l (\lambda_{lj})^s \right\}^{r/s} \right]^{s/r} \\
&\leq C \left[\sum_k \left\{ \sum_{j \geq k} 2^{(k-j)b} \sum_l (\lambda_{lj})^s \right\}^{r/s} \right]^{s/r} \\
&\quad + C \left[\sum_k \left\{ \sum_{j < k} 2^{(j-k)a} \sum_l (\lambda_{lj})^s \right\}^{r/s} \right]^{s/r} \\
&\leq C \|\lambda\|_{sr}^s.
\end{aligned}$$

This completes the proof of the lemma.

(5.9) **REMARK.** Note that we have shown that the series defining φ converges absolutely and uniformly, not only in compact subsets of \mathbb{R}_+^2 , but in each proper infinite strip of the form, $y \geq y_0$, $|x| \leq x_0$, where $x_0, y_0 > 0$. We will use this remark in § 7 where we consider extensions of the A_{sr}^β and \mathcal{A}_{sr}^β spaces to the case where $\beta > -1/s$.

6. – The representation theorems.

First we need a lemma for harmonic functions that acts as a replacement for the monotonicity of $M_s(f; y)$ in y , when f is holomorphic (which was established in the proof of (4.1)).

(6.1) **LEMMA.** *If $f \in \mathcal{A}_{sr}^\beta$ then*

$$(6.2) \quad \left\{ \int_{2^{j-1}}^{2^{j+2}} M_s(f; y)^s \frac{dy}{y} \right\}^{1/s} < C \left\{ \int_{2^{j-2}}^{2^{j+3}} M_s(f; y)^r \frac{dy}{y} \right\}^{1/r},$$

for all $j \in \mathbb{Z}$ where $C > 0$ is a constant that depends only on r and s .

PROOF. There are several cases where the result is trivial. Thus, if $s \leq r < \infty$ we can use Hölder's inequality with index r/s , and if $s \geq 1$ we can use the well known monotonicity of the mean $M_s(f; y)$. In any case we

established in (2.4) and the calculation (2.5) the following inequality if $f \in \mathcal{A}_{sr}^\beta$:

$$M_s(f; y) \leq C \left\{ \int_{y/2}^{\frac{3}{2}y} M_s(f; \eta)^r \frac{d\eta}{\eta} \right\}^{1/r}.$$

Then (6.2) is a trivial consequence of this inequality.

We consider a partition of \mathbf{R}_+^2 constructed as follows. Divide \mathbf{R}_+^2 into squares Q_{lj} with vertices $l2^j + i2^j$, $(l+1)2^j + i2^j$, $l2^j + i2^{j+1}$ and $(l+1)2^j + i2^{j+1}$. Then divide each square Q_{lj} into M^2 equal squares Q_{lj}^k , $k = 1, 2, \dots, M^2$, each of side length $2^j/M$.

(6.3) LEMMA. *If f is harmonic on \mathbf{R}_+^2 then $N_{sr}^\beta(f)$ is equivalent to*

$$\| \{ 2^{j(\beta+1/s)} \sup_{z \in Q_{lj}} |f(z)| \} \|_{sr}.$$

PROOF. For the sake of simplicity we will write the proof for the cases where $s, r \neq \infty$. Adjustments for the exceptional cases are trivial.

$$\begin{aligned} N_{sr}^\beta(f) &= \left\{ \sum_j \int_{2^j}^{2^{j+1}} \left(y^\beta \left(\sum_l \int_{2^j}^{(l+1)2^j} |f(x+iy)|^s dx \right)^{1/s} \right)^r \frac{dy}{y} \right\}^{1/r} \\ &\leq C \left\{ \sum_j 2^{j\beta r} \int_{2^j}^{2^{j+1}} \left(\sum_l 2^j \sup_{z \in Q_{lj}} |f(z)|^s \right)^{r/s} \frac{dy}{y} \right\}^{1/r} \\ &\leq C \left\{ \sum_j \left(2^{j(\beta+1/s)} \left(\sum_l \sup_{z \in Q_{lj}} |f(z)|^s \right)^{1/s} \right)^r \right\}^{1/r} = C \| 2^{j(\beta+1/s)} \sup_{z \in Q_{lj}} |f(z)| \|_{sr}. \end{aligned}$$

We now assume that f is harmonic and $N_{sr}^\beta(f) < \infty$. From (2.1) it follows that if $z \in Q_{lj}$ and D_z is the disk with center z and of radius 2^{j-1} then

$$|f(z)|^s \leq C 2^{-2j} \int_{D_z} |f(\zeta)|^s d\zeta.$$

Consequently,

$$(6.4) \quad \sup_{z \in Q_{lj}} |f(z)|^s \leq C 2^{-2j} \int_{\bar{Q}_{lj}} |f(\zeta)|^s d\zeta,$$

where $\bar{Q}_{lj} = \{ \zeta : d(\zeta, Q_{lj}) \leq 2^{j-1} \}$. From (6.2) and (6.4) we have,

$$\left(\sum_l \sup_{z \in Q_{lj}} |f(z)|^s \right)^{1/s} \leq C 2^{-2j/s} \left\{ \sum_l \int_{\bar{Q}_{lj}} |f(\zeta)|^s d\zeta \right\}^{1/s}$$

$$\begin{aligned} &\leq C 2^{-i/s} \left\{ \int_{2^{j-1}}^{2^{j+2}} M_s(f; y)^s \frac{dy}{y} \right\}^{1/s} \\ &\leq C 2^{-i/s} \left\{ \int_{2^{j-2}}^{2^{j+3}} M_s(f; y)^r \frac{dy}{y} \right\}^{1/r}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\{ \sum_j \left(2^{j(\beta+1/s)} \left(\sum_l \sup_{z \in Q_{lj}} |f(z)|^s \right)^{1/s} \right)^r \right\}^{1/r} \\ &\leq C \left\{ \sum_j 2^{j\beta r} \int_{2^{j-2}}^{2^{j+3}} M_s(f; y)^r \frac{dy}{y} \right\}^{1/r} \leq C \left\{ \int_0^\infty (y^\beta M_s(f; y))^r \frac{dy}{y} \right\}^{1/r}. \end{aligned}$$

Except for minor adjustments in case r or $s = \infty$ the proof is complete.

PROOF OF (1.5). We choose $\{\zeta_{lj}^k\}$ to be a collection of points in \mathbb{R}_+^2 such that ζ_{lj}^k is *any* point in Q_{lj}^k . For part i) of the proof M is any positive integer. For part ii) M must be chosen large enough. To prove part i) just observe that

$$\begin{aligned} \left| \sum_{k=1}^{M^2} \lambda_{lj}^k \frac{(\operatorname{Im} \zeta_{lj})^{\eta-(\beta+1/s)}}{(z - \bar{\zeta}_{lj}^k)^\eta} \right| &\leq C \sum_{k=1}^{M^2} |\lambda_{lj}^k| \frac{2^{j(\eta-(\beta+1/s))}}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta/2}} \\ &\leq C_M \left\{ \sum_{k=1}^{M^2} |\lambda_{lj}^k|^s \right\}^{1/s} \frac{2^{j(\eta-(\beta+1/s))}}{((x - l2^j)^2 + (y + 2^j)^2)^{\eta/2}}. \end{aligned}$$

One then just applies (5.1) with $\lambda_{lj} = \left\{ \sum_{k=1}^{M^2} |\lambda_{lj}^k|^s \right\}^{1/s}$. It follows from the uniform convergence on compact subsets of \mathbb{R}_+^2 that f is holomorphic and then $N_{sr}^\beta(f) \leq C \|\lambda\|_{sr}$.

In order to prove the converse we will construct a sequence of functions $\{f_n\}$ in A_{sr}^β such that

$$\begin{aligned} &\text{i) } f_n(z) = \sum_{lk} c_{lj}^{kn} \frac{(\operatorname{Im} \zeta_{lj}^k)^{\eta-(\beta+1/s)}}{(z - \bar{\zeta}_{lj}^k)^\eta} \\ (6.4') \quad &\text{ii) } \|c^n\|_{sr} \leq C 2^{-n} N_{sr}^\beta(f) \quad \text{and so} \quad N_{sr}^\beta(f_n) \leq C 2^{-n} N_{sr}^\beta(f) \\ &\text{iii) } N_{rs}^\beta\left(f - \sum_1^n f_m\right) \leq 2^{-n} N_{sr}^\beta(f) \end{aligned}$$

where we have set $c^n = \{c_{lj}^{kn}\}$. The result then follows easily. Note that the homogeneity of $\|\cdot\|_{sr}$ is $h = \min\{1, s, r\}$, so that $\|\lambda + \mu\|_{sr} \leq \{\|\lambda\|_{sr}^h + \|\mu\|_{sr}^h\}^{1/h}$.

+ $\|\mu\|_{sr}^h\}^{1/h}$. If we now let $\lambda_{ij}^k = \sum_{n=1}^{\infty} c_{ij}^{kn}$ it follows from (6.4') ii) that $\|\lambda\|_{sr} \leq CN_{sr}^{\beta}(f)$.

Thus, by the first part of the theorem

$$g(z) = \sum_{ijk} \lambda_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta - (\beta + 1/s)}}{(z - \bar{\zeta}_{ij}^k)^{\eta}}$$

is a function in A_{sr}^{β} . But then we see that

$$N_{sr}^{\beta}\left(g - \sum_1^n f_m\right) \leq C \left\| \lambda - \sum_1^n c^m \right\|_{sr} \leq C 2^{-n} N_{sr}^{\beta}(f).$$

Thus, for each positive integer n ,

$$N_{sr}^{\beta}(f - g) \leq \left\{ N_{sr}^{\beta}\left(f - \sum_1^n f_m\right)^h + N_{sr}^{\beta}\left(g - \sum_1^n f_m\right)^h \right\}^{1/h} \leq (1 + c^h)^{1/h} N_{sr}^{\beta}(f) 2^{-n}.$$

Thus, $N_{sr}^{\beta}(f - g) = 0$ and so $f = g$.

The existence of a sequence $\{f_n\}$ follows from the corresponding properties of an operator S . Given $f \in A_{sr}^{\beta}$ there is a function $Sf \in A_{sr}^{\beta}$ such that

$$(6.5) \quad \begin{aligned} \text{i) } Sf(z) &= \sum_{ijk} c_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta - (\beta + 1/s)}}{(z - \bar{\zeta}_{ij}^k)^{\eta}}; \\ \text{ii) } \|c\|_{sr} &\leq CN_{sr}^{\beta}(f) \quad \text{and so} \quad N_{sr}^{\beta}(Sf) \leq CN_{sr}^{\beta}(f); \\ \text{iii) } N_{sr}^{\beta}(f - Sf) &\leq \frac{1}{2} N_{sr}^{\beta}(f). \end{aligned}$$

One then just lets $f_1 = Sf$ and $f_n = S\left(f - \sum_1^{n-1} f_m\right)$ for $n > 1$.

The construction of the operator S is due to Coifman and Rochberg. For $f \in A_{sr}^{\beta}$ we just let $Sf(z)$ be the Riemann sum of f corresponding to the partition $\{Q_{ij}^k\}$ and the selection of points $\{\zeta_{ij}^k\}$. That is,

$$(6.6) \quad Sf(z) = \sum_{ijk} f(\zeta_{ij}^k) \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta - 1}}{(z - \bar{\zeta}_{ij}^k)^{\eta}} |Q_{ij}^k|$$

(we will write out the details only for the cases $r, s \neq \infty$).

It follows from (5.1) that

$$N_{sr}^{\beta}(Sf) \leq C \left\{ \sum_j \left(\sum_i \left(\sum_k |f(\zeta_{ij}^k)| 2^{j(\beta + 1/s - 2)} |Q_{ij}^k| \right)^s \right)^{r/s} \right\}^{1/s}.$$

But,

$$\begin{aligned} & \sum_{k=1}^{M^2} |f(\zeta_{ij}^k)| 2^{j(\beta+1/s-2)} |Q_{ij}^k| \\ & \leq M^{-2} 2^{j(\beta+1/s)} \sum_{k=1}^{M^2} |f(\zeta_{ij}^k)| \leq C_M 2^{j(\beta+1/s)} \left\{ \sum_k |f(\zeta_{ij}^k)|^s \right\}^{1/s} \\ & \leq C_M 2^{j(\beta+1/s)} \sup_{\zeta \in Q_{ij}} |f(\zeta)|. \end{aligned}$$

It now follows from (6.3) that $N_{sr}^\beta(Sf) \leq C_M N_{sr}^\beta(f)$. Then using (5.1) we see that $Sf \in A_{sr}^\beta$ and parts i) and ii) of (6.5) are verified. We now need to estimate $N_{sr}^\beta(f - Sf)$. Here we use the reproducing kernel whose existence was established in (4.4). We have,

$$f(z) - Sf(z) = \sum_{ij} \int_{Q_{ij}^k} \left[f(\zeta) \frac{(\operatorname{Im} \zeta)^{\eta-2}}{(z - \bar{\zeta})^\eta} - f(\zeta_{ij}^k) \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta-2}}{(z - \bar{\zeta}_{ij}^k)^\eta} \right] d\zeta.$$

Thus,

$$\begin{aligned} |f(z) - Sf(z)| & \leq \sum_{ij} |Q_{ij}^k| \dim Q_{ij}^k \sup_{\zeta \in Q_{ij}^k} \left| \nabla_\zeta f(\zeta) \frac{(\operatorname{Im} \zeta)^{\eta-2}}{(z - \bar{\zeta})^\eta} \right| \\ & \leq \frac{C}{M} \sum_{ij} 2^{3j} \sup_{\zeta \in Q_{ij}} \left| \nabla_\zeta f(\zeta) \frac{(\operatorname{Im} \zeta)^{\eta-2}}{(z - \bar{\zeta})^\eta} \right|. \end{aligned}$$

The strategy is now clear. The sum in the last term does not depend on M . So all we need to do is to get control of that term and then just take M large enough.

It is easy to see that $|f'(\zeta)| \leq C 2^{-j} \sup_{|\omega - z| = 2^{j-1}} |f(\omega)|$, using either the Cauchy integral formula or the Poisson integral formula. Thus, if $\bar{Q}_{ij} = \{\zeta : d(\zeta, Q_{ij}) \leq 2^{j-1}\}$ we see that

$$\sup_{\zeta \in \bar{Q}_{ij}} |f'(\zeta)| \leq C 2^{-j} \sup_{\zeta \in Q_{ij}} |f(\zeta)|.$$

An easy calculation now shows that

$$\sup_{\zeta \in \bar{Q}_{ij}} \left| \nabla_\zeta f(\zeta) \frac{(\operatorname{Im} \zeta)^{\eta-2}}{(z - \bar{\zeta})^\eta} \right| \leq C \frac{2^{j(\eta-3)}}{|z - l2^j + i2^j|^\eta} \sup_{\zeta \in Q_{ij}} |f(\zeta)|.$$

From (5.1) and (6.3) we have,

$$N_{sr}^\beta(f - Sf) \leq \frac{C}{M} \left[\sum_j \left\{ 2^{j(\beta+1/s)} \left(\sum_i \sup_{\zeta \in Q_{ij}} |f(\zeta)|^s \right)^{1/s} \right\} \right]^{1/r}$$

$$\begin{aligned} &\leq \frac{C}{M} \left[\sum_j \left\{ 2^{j(\beta+1/s)} \left(\sum_l \sup_{\zeta \in Q_{l,j}} |f(\zeta)|^s \right)^{1/s} \right\}^r \right]^{1/r} \\ &\leq \frac{C}{M} N_{sr}^\beta(f). \end{aligned}$$

The constant C in this last relation does not depend on M , so we may choose M so large that $C/M < \frac{1}{2}$ and the proof is complete.

PROOF OF (1.10). The proof of (1.5) was set up so that it carries over directly for the spaces of harmonic functions. One just uses the reproducing kernel from (3.1) instead of (4.4) and when one makes estimates on the gradient of the kernel toward the end of the proof note simply that $(\partial^\alpha / \partial y^\alpha) P(x - x_0, y + y_0)$ is the real part of a constant times $((x + iy) - (x_0 + iy_0))^{\alpha+1}$ and use the estimates for the holomorphic kernels.

An immediate consequence of the remarks in the proof of (1.10) above is the following corollary:

(6.7) COROLLARY. *A real valued harmonic function is in \mathcal{A}_{sr}^β if and only if it is the real part of a holomorphic function in A_{sr}^β with an equivalent norm.*

PROOF. Observe simply that in the representation theorem for harmonic functions that are real valued, the coefficients $\{\lambda_{ij}^k\}$ are also real valued.

REMARK. It follows from this corollary that if $f \in \mathcal{A}_{sr}^\beta$ then $M_s(f; y)$ may not be monotone in y but it is certainly equivalent to a monotone function. If $1 \leq s \leq \infty$ it is well known that it is monotone (just use the semi-group property of the Poisson integral operator) and if f is holomorphic we showed in the proof of (4.1) that it was monotone. But if $0 < s < 1$ and f is harmonic we only have the equivalence with a monotone function.

(6.8) COROLLARY. *Derivation of order α is an isomorphism of A_{sr}^β onto $A_{sr}^{\beta+\alpha}$.*

PROOF. The result follows from Theorem (1.5) and the part of Lemma (5.1) concerning uniform convergence on compacta.

Similar results can be obtained for fractional derivations defined naturally on the kernels and then extended to functions in A_{sr}^β and \mathcal{A}_{sr}^β .

REMARK. It is easy to check that if s and r are both finite, the representing series for f in A_{sr}^β or \mathcal{A}_{sr}^β converges to f in the topology of A_{sr}^β (resp. \mathcal{A}_{sr}^β). If at least one of the coefficients is infinity then only the appropriate weak convergence holds. It is possible to define closed subspaces of these limiting spaces where strong convergence also holds; one simply adds the appropriate « $o(1)$ » condition on the behaviour at infinity.

7. - Extensions to negative values of β .

Because of Corollary (6.8) one can define A_{sr}^β for non-positive values of β as the space of holomorphic functions on \mathbb{R}_+^2 whose derivative of order κ is in $A^{\beta+\kappa}$, as long as $\beta + \kappa > 0$. One should say, in this case, that A_{sr}^β consists of equivalence classes of holomorphic functions, modulo polynomials of degree less than κ . On the other hand it is always possible, if $\beta + 1/s > 0$, to find a unique representative in any such equivalence class that tends to zero as $y \rightarrow \infty$.

(7.1) LEMMA. *Let f be a holomorphic function on \mathbb{R}_+^2 such that $f^{(\kappa)} \in A_{sr}^\gamma$, $\gamma > 0$. If $\gamma + 1/s > \kappa$, there is a unique holomorphic function g on \mathbb{R}_+^2 such that $g^{(\kappa)} = f^{(\kappa)}$ and $\lim_{y \rightarrow \infty} g(x + iy) = 0$ for every $x \in \mathbb{R}$.*

PROOF. Let

$$g(x + iy) = \frac{(-1)^\kappa}{(\kappa - 1)!} \int_y^\infty f^{(\kappa)}(x + it)(y - t)^{\kappa-1} dt.$$

Since $f^{(\kappa)} \in A_{sr}^\gamma \subset A_{\infty\infty}^{\gamma+1/s}$, $|f^{(\kappa)}(x + iy)| \leq Cy^{-(\gamma+1/s)}$ and so the integral converges absolutely ($\gamma + 1/s > \kappa$). Thus, $g(x + iy) \leq Cy^{\kappa-(\gamma+1/s)} = o(1)$ as $y \rightarrow \infty$. Uniqueness is trivial.

DEFINITION. *For $-1/s < \beta \leq 0$ we define A_{sr}^β to be the space of holomorphic functions f on \mathbb{R}_+^2 such that $f(x + iy) = o(1)$ as $y \rightarrow \infty$ for all $x \in \mathbb{R}$ and such that for some $\kappa > -\beta$, $f^{(\kappa)} \in A_{sr}^{\beta+\kappa}$.*

We give it the topology induced by the « norm » $N_{sr}^\beta(f^{(\kappa)})$ and it follows from (6.8) that while the norm depends on κ , for any two acceptable choices of κ the norms are equivalent and the spaces of functions are the same.

(7.2) THEOREM. *The representation theorem (1.5) extends to the spaces A_{sr}^β , $-1/s < \beta \leq 0$ (with the same restrictions on η).*

PROOF. Suppose $f \in A_{sr}^\beta$. Then there is a $\kappa > -\beta$ such that if $\eta > \max\{\beta + \kappa + 1/s, \beta + \kappa + 1\}$ then (by (1.5))

$$f^{(\kappa)}(z) = \sum_{ijk} \lambda_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta-(\beta+\kappa+1/s)}}{(z - \bar{\zeta}_{ij}^k)^\eta}.$$

Consider the series

$$\frac{(-1)^\kappa}{(\eta - 1) \dots (\eta - \kappa)} \sum_{ijk} \lambda_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{(\eta-\kappa)-(\beta+1/s)}}{(z - \bar{\zeta}_{ij}^k)^{(\eta-\kappa)}}.$$

Using the remark (5.9) following the technical lemma we see that both series converge uniformly on proper infinite strips so the sum of the second series represents a holomorphic function vanishing as $y \rightarrow \infty$, and whose k -th derivative is $f^{(k)}$. It follows from (7.1) that this holomorphic function is f .

8. - Duality.

In this section we will characterize the spaces of continuous linear functionals on the A_{sr}^β spaces for finite values of s and r . We will use the representation theorem (1.5).

Fix $\eta > \max\{\beta + 1/s, \beta + 1\}$, $\beta > 0$ and denote $f_\zeta(z) = (z - \bar{\zeta})^{-\eta}$, $z \in \mathbb{R}_+^2$. It is easy to check that $f_\zeta \in A_{sr}^\beta$ so that if L is a continuous linear functional on A_{sr}^β we can define $F'_\eta(\zeta) = \overline{L(f_\zeta)}$.

In the rest of this section we let $\sigma = \max\{1/s, 1\}$.

(8.1) LEMMA. *The function F_η is in $A_{s'r'}^{\eta-(\beta+\sigma)}$, and $N_{s'r'}^{\eta-(\beta+\sigma)}(F_\eta) \leq C_\eta \|L\|$, where $\|L\|$ is the norm of L as a continuous linear functional on A_{sr}^β .*

PROOF. By the dominated convergence theorem we can see that

$$\lim_{h \rightarrow 0} N_{sr}^\beta \left(\frac{f_{\zeta+h} - f_\zeta}{h} - \frac{\eta}{(\cdot - \bar{\zeta})^{\eta+1}} \right) = 0.$$

This implies that $F'_\eta(\zeta) = \eta F_{\eta+1}(\zeta)$, but more importantly, it implies that $F'_\eta(\zeta)$ is holomorphic.

We apply (6.3) and obtain,

$$\begin{aligned} N_{s'r'}^{\eta-\beta-\sigma}(F_\eta) &\leq C \left\{ \sum_j \left(2^{j(\eta-(\beta+1/s))} \left(\sum_{lk} \sup_{\zeta \in Q_{ij}^k} |F'_\eta(\zeta)|^{s'} \right)^{r'} \right)^{1/r'} \right\} \\ &= C \sup_{\|\lambda\|_{sr} \leq 1} \left| \sum_{ijk} \lambda_{ij}^k 2^{j(\eta-(\beta+1/s))} F'_\eta(\zeta_{ij}^k) \right|, \end{aligned}$$

where the $\{\zeta_{ij}^k\}$ are the points in Q_{ij}^k where the maximum is achieved. Several comments are in order. For the first step we note that $\eta - \beta - \sigma + 1/s' = \eta - (\beta + 1/s)$ under convention (1.15). For the second step we need to observe that even when s' or $r' = \infty$ and $0 < s < 1$ or $0 < r < 1$, an elementary computation shows that equality holds.

Since L is continuous on A_{sr}^β we have,

$$\begin{aligned} N_{s'r'}^{\eta-\beta-\sigma}(F_\eta) &\leq C \sup_{\|\lambda\|_{sr} \leq 1} \left| L \left(\sum_{ijk} \lambda_{ij}^k \frac{(\operatorname{Im} \zeta_{ij}^k)^{\eta-(\beta+1/s)}}{(z - \bar{\zeta}_{ij}^k)^\eta} \right) \right| \\ &\leq C \|L\|, \end{aligned}$$

where $\|L\|$ is norm of L as a continuous linear functional on A_{sr}^β . The last step requires the observation that in (1.7) the constant C does not depend on the choice of the representatives $\{\zeta_{ij}^k\}$. This completes the proof of the lemma.

(8.2) **THEOREM.** *The continuous linear functionals on A_{sr}^β , $0 < s, r < \infty$, are in one-to-one correspondence with the functions $F \in A_{s'r'}^{\eta, \beta - \sigma}$, by means of the duality:*

$$(8.3) \quad L(f) = \int_0^\infty \int_{-\infty}^{+\infty} y^{\eta-1} f(x+iy) \overline{F(x+iy)} \frac{dy}{y}$$

which is to say that L is a continuous linear functional on A_{sr}^β if and only if it can be represented uniquely in the form (8.3) with $F \in A_{s'r'}^{\eta, \beta - \sigma}$ (s', r' and σ as defined prior to the statement of (8.1)) and the norm of F in $A_{s'r'}^{\eta, \beta - \sigma}$ is equivalent to the norm of L as a continuous linear functional.

PROOF. Suppose $F \in A_{s'r'}^{\eta, \beta - \sigma}$. The first step is to show that the integral in (8.3) converges absolutely if f is in A_{sr}^β . Suppose that $s \geq 1, r \geq 1$. Then $F \in A_{s'r'}^{\eta, \beta - 1}$, and we get $|L(f)| \leq N_{s'r'}^{\eta, \beta - 1}(F) N_{sr}^\beta(f)$ by a double application of Hölder's inequality. For the other cases we just use the continuous inclusions of Proposition (2.2). Thus if $s < 1, r \geq 1$, $F \in A_{\infty r'}^{\eta, \beta - 1/s}$ and so defines a continuous linear functional on $A_{1r}^{\beta + 1/s - 1} \supset A_{sr}^\beta$. The other two cases go the same way. Consequently, if $F \in A_{s'r'}^{\eta, \beta - \sigma}$ then $|L(f)| \leq CN_{s'r'}^{\eta, \beta - \sigma}(F) N_{sr}^\beta(f)$ where C may depend on the indices but not on F or f .

For the converse, suppose L is a continuous linear functional on A_{sr}^β . Then by Lemma (8.1) there is an associated function F_η ($\eta > \max\{\beta + 1, \beta + 1/s\}$) where $\overline{F_\eta(\zeta)} = L(f_\zeta)$, $F_\eta \in A_{s'r'}^{\eta, \beta - \sigma}$ and $N_{s'r'}^{\eta, \beta - \sigma}(F_\eta) \leq C\|L\|$. Using the reproducing formula of Lemma (4.4) we have

$$F_\eta(\zeta) = c_\eta \int_0^\infty \int_{-\infty}^{+\infty} F_\eta(x+iy) \frac{y^{\eta-2}}{(\zeta - (x-iy))^\eta} dx dy$$

since $\eta > (\eta - \beta - \sigma) + 1$. We take complex conjugates and write this

$$L(f_\zeta) = c_\eta \int_0^\infty \int_{-\infty}^{+\infty} y^{\eta-1} f_\zeta(x+iy) \overline{F_\eta(x+iy)} \frac{dy}{y}.$$

The formula then extends to all $f \in A_{sr}^\beta$ by means of the representation theorem, (1.5). It is now clear that the representing function is unique,

since the difference of two such functions would represent the zero linear functional which by what we have proven would have norm zero in $A_{s,r}^{\eta-\beta-\sigma}$ and by continuity considerations this implies that the difference is zero.

REMARK. The results of this section as well as the previous one for β negative can all be extended to the spaces of harmonic functions.

9. - Boundary values of functions in \mathcal{A}_{sr}^β .

Let $\beta > 0$, $0 < s, r \leq \infty$ and $f \in \mathcal{A}_{sr}^\beta$. For $y > 0$, we denote by $f_y(x)$ the function $f(x + iy)$. We prove in this section that one can give a meaning to $f_0 = \lim_{y \rightarrow 0} f_y$ and that one can obtain f from f_0 by means of the Poisson integral.

We start with the definition of the oscillation spaces $MO_{s,r}^\alpha$, where $1 < s, r \leq \infty$ and $\alpha + 1/s > 0$.

Let t be a positive number and $\mathfrak{J} = \{I_i\}$ be a covering of the line with intervals I_i , such that $|I_i| = t$ and no point belongs to more than three of the intervals I_i . In this case we say that \mathfrak{J} is an admissible covering of length $|\mathfrak{J}| = t$. Given a locally integrable function g on the line, an interval I and an integer $m \geq 0$, let $g_I^{(m)}(x)$ be the unique polynomial of degree less than or equal to m such that

$$(9.1) \quad \int_I (g(x) - g_I^{(m)}(x)) x^k dx = 0, \quad k = 0, 1, \dots, m.$$

Define

$$(9.2) \quad \text{osc}_{s,m}(g, \mathfrak{J}) = \left[\sum_i \left(\frac{1}{|I_i|} \int_{I_i} |g(x) - g_{I_i}^{(m)}(x)| dx \right)^s \right]^{1/s}$$

and

$$(9.3) \quad \omega_{s,m}(g, t) = \sup \{ \text{osc}_{s,m}(g, \mathfrak{J}) \mid |\mathfrak{J}| = t \}.$$

DEFINITION. Let $1 < s, r \leq \infty$ and $\alpha > -1/s$. MO_{sr}^α is the space of equivalence classes of locally integrable functions g , modulo polynomials of degree less than or equal to $[\alpha + 1/s]$, such that

$$(9.4) \quad MO_{sr}^\alpha(g) = \left[\int_0^\infty (t^{-\alpha} \omega_{s,m}(g, t))^r \frac{dt}{t} \right]^{1/r} < \infty,$$

where $m = m(\alpha, s) = [\alpha + 1/s]$.

We need two other descriptions of MO_{sr}^x , which will be useful in the sequel (see (9.9) and (9.12) below). Given an interval I centered at x , we denote by I_ϱ , $\varrho > 0$, the interval obtained from I by a dilation of a factor ϱ about x .

(9.5) **LEMMA.** *Let I be an interval, I^* another interval containing I and contained in I_4 and $\varrho > 1$. Then for a locally integrable function g and an integer $m \geq 0$,*

$$\sup_{x \in I_\varrho} |g_I^{(m)}(x) - g_{I^*}^{(m)}(x)| \leq C \varrho^m \frac{1}{|I^*|} \int_I |g(x) - g_{I^*}^{(m)}(x)| dx.$$

PROOF. We can assume that I is centered at 0. Let $\{\psi_j(x)\}$, $0 \leq j \leq m$ be the polynomials with the properties:

$$(9.6) \quad \begin{aligned} & \text{i) } \deg \psi_j = j; \\ & \text{ii) } \frac{1}{|I|} \int_I \psi_j(x) x^k dx = \delta_{jk}. \end{aligned}$$

Therefore

$$(9.7) \quad g_I^{(m)}(x) = \sum_0^m \left(\frac{1}{|I|} \int_I g(t) \psi_j(t) dt \right) x^j.$$

Also let $g_{I^*}^{(m)}(x) = \sum_0^m c_j x^j$. We have that

$$\begin{aligned} |g_I^{(m)}(x) - g_{I^*}^{(m)}(x)| &= \left| \sum_0^m \left(\frac{1}{|I|} \int_I g(t) \psi_j(t) dt \right) x^j - \sum_0^m c_j x^j \right| \\ &\leq \sum_0^m |x|^j \left| \frac{1}{|I|} \int_I \left(g(t) - \sum_0^m c_k t^k \right) \psi_j(t) dt \right| \\ &\leq \left(\sum_0^m |x|^j \sup_{t \in I} |\psi_j(t)| \right) \frac{1}{|I|} \int_I |g(t) - g_{I^*}^{(m)}(t)| dt \\ &\leq 4 \left(\sum_0^m |x|^j \sup_{t \in I} |\psi_j(t)| \right) \frac{1}{|I^*|} \int_{I^*} |g(t) - g_{I^*}^{(m)}(t)| dt. \end{aligned}$$

Now we evaluate

$$(9.8) \quad \left(\sup_{x \in I_\varrho} |x|^j \right) \left(\sup_{t \in I} |\psi_j(t)| \right) = \varrho^j \left(\frac{|I|}{2} \right)^j \sup_{t \in I} |\psi_j(t)|.$$

Observe that if I_1 and I_2 are two intervals centered at 0 and $\{\psi_j^1(x)\}$,

$\{\psi_j^2(x)\}$ the corresponding dual bases to the monomials, then

$$\psi(x) = \left(\frac{|I_1|}{|I_2|}\right)^j \psi_j^1\left(\frac{|I_1|}{|I_2|}x\right)$$

because conditions (9.6) completely determine the polynomials.

Therefore the quantity in (9.8) equals ϱ^j times a constant c_j independent of I . This gives

$$\sup_{x \in I_\varrho} |g_I^{(m)}(x) - g_{I^*}^{(m)}(x)| \leq 4 \left(\sum_0^m c_j \varrho^j\right) \frac{1}{|I^*|} \int_{I^*} |g(t) - g_{I^*}^{(m)}(t)| dt$$

which proves the Lemma, since $\varrho > 1$.

Let I_{ij} be the interval $[(l-1)2^j, (l+2)2^j]$, $j, l \in \mathbb{Z}$. For each fixed j , the covering $\mathfrak{J}_j = \{I_{ij}\}$ is an admissible covering of length $3 \cdot 2^j$.

(9.9) LEMMA. *Let g be a locally integrable function on the line. Then $g \in MO_{sr}^\alpha$ if and only if the quantity*

$$(9.10) \quad \left[\sum_j (2^{-j\alpha} \text{osc}_{s,m}(g, \mathfrak{J}_j))^r \right]^{1/r}, \quad m = \left[\alpha + \frac{1}{s} \right]$$

is finite. In this case (9.10) provides a norm on MO_{sr}^α equivalent to (9.4).

PROOF. Let $\mathfrak{J} = \{J_\lambda\}$ be an admissible covering of length t , $2^j < t \leq 2^{j+1}$. Then any interval J_λ is contained in either one or two of the intervals in \mathfrak{J}_j and no more than six of the intervals in \mathfrak{J} are contained in the same interval I_{ij} . On the other hand, if $J_\lambda \subset I_{ij}$ then $I_{ij} \subset (J_\lambda)_s$. In this case one has

$$\begin{aligned} \frac{1}{|J_\lambda|} \int_{J_\lambda} |g(x) - g_{J_\lambda}^{(m)}(x)| dx &\leq \frac{5}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)| dx \\ &\leq \frac{5}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)| dx + 5 \sup_{x \in (J_\lambda)_s} |g_{I_{ij}}^{(m)}(x) - g_\lambda^{(m)}(x)| \\ &\leq \frac{c}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)| dx, \end{aligned}$$

by Lemma (9.5). Therefore

$$\text{osc}_{s,m}(g, \mathfrak{J}) \leq C \text{osc}_{s,m}(g, \mathfrak{J}_j)$$

and

$$(9.11) \quad \sup_{2^j < t \leq 2^{j+1}} \omega_{s,m}(g, t) \leq C \operatorname{osc}_{s,m}(g, \mathfrak{J}_j).$$

This shows that

$$\begin{aligned} \int_0^\infty (t^{-\alpha} \omega_{s,m}(g, t))^r \frac{dt}{t} &= \sum_j \int_{2^j}^{2^{j+1}} (t^{-\alpha} \omega_{s,m}(g, t))^r \frac{dt}{t} \\ &\leq C \sum_j (2^{-j\alpha} \operatorname{osc}_{s,m}(g, \mathfrak{J}_j))^r. \end{aligned}$$

The converse estimate makes use of the same considerations and is even simpler. For $2^{j+1}/3 \leq t < 2^{j+2}/3$, let $\mathfrak{J}_t = \{I_{tl}\}$, where $I_{tl} = [(l-1)t, (l+2)t]$ so that \mathfrak{J}_t is an admissible covering if length $3t$. The same argument as before shows that

$$\operatorname{osc}_{s,m}(g, \mathfrak{J}_j) \leq C \operatorname{osc}_{s,m}(g, \mathfrak{J}_t) \leq C \omega_{s,m}(g, t)$$

which gives

$$\begin{aligned} \sum_j (2^{-j\alpha} \operatorname{osc}_{s,m}(g, \mathfrak{J}_j))^r &\leq C \sum_j 2^{-j\alpha r} \int_{2^{j+1}}^{2^{j+2}} \omega_{s,m}(g, t)^r \frac{dt}{t} \\ &\leq C \sum_j \int_{2^{j+1}}^{2^{j+2}} (t^{-\alpha} \omega_{s,m}(g, t))^r \frac{dt}{t} \\ &= CMO_{sr}^\alpha(g)^r. \end{aligned}$$

We will also need the following characterization of MO_{sr}^α , which extends the characterization of BMO given by Fefferman and Stein [5], page 142.

(9.12) LEMMA. *Let g be a locally integrable function on the line. Then $g \in MO_{sr}^\alpha$ if and only if given a number $\eta > \alpha + 1 + 1/s$, the quantity*

$$(9.13) \quad \left\{ \sum_j \left[\sum_t \left(\int_{-\infty}^{+\infty} |g(x) - g_{I_{tl}}^{(m)}(x)| \frac{2^{j(\eta-\alpha-1)}}{(2^j + |x - l2^j|)^\eta} dx \right)^s \right]^{r/s} \right\}^{1/r}, \quad m = \left[\alpha + \frac{1}{s} \right]$$

is finite. In this case (9.13) provides a norm on MO_{sr}^α which is equivalent to (9.4).

PROOF. Let I be an interval and I_{2^k} be that obtained by dilation of a

factor 2^k around its center. Using Lemma (9.5) one obtains:

$$\begin{aligned} \frac{1}{|I_{2^k}|} \int_{I_{2^k}} |g(x) - g_{I_{2^k}}^{(m)}(x)| dx &\leq \frac{1}{|I_{2^k}|} \int_{I_{2^k}} |g(x) - g_{I_{2^k}}^{(m)}(x)| dx + \sum_{h=1}^k \sup_{x \in I_{2^h}} |g_{I_{2^h}}^{(m)}(x) - g_{I_{2^{h-1}}}^{(m)}(x)| \\ &\leq C \sum_{h=0}^k 2^{m(k-h)} \left(\frac{1}{|I_{2^h}|} \int_{I_{2^h}} |g(x) - g_{I_{2^h}}^{(m)}(x)| dx \right). \end{aligned}$$

If $I_{lj} = [(l-1)2^j, (l+2)2^j]$, we denote by I_{lj}^k the interval $(I_{lj})_{2^k}$. We also set

$$(9.14) \quad c_{lj} = \int_{-\infty}^{+\infty} |g(x) - g_{I_{lj}}^{(m)}(x)| \frac{2^{j(\eta-\alpha-1)}}{(2^j + |x - l2^j|)^\eta} dx.$$

We have (calling $I_{lj}^{-1} = \emptyset$)

$$\begin{aligned} c_{lj} &= \sum_{k=0}^{\infty} \int_{I_{lj}^k \setminus I_{lj}^{k-1}} |g(x) - g_{I_{lj}}^{(m)}(x)| \frac{2^{j(\eta-\alpha-1)}}{(2^j + |x - l2^j|)^\eta} dx \\ &\leq C \sum_{k=0}^{\infty} \frac{2^{j(\eta-\alpha-1)}}{2^{(j+k)\eta}} \int_{I_{lj}^k} |g(x) - g_{I_{lj}}^{(m)}(x)| dx \\ &\leq C 2^{-j\alpha} \sum_{k=0}^{\infty} 2^{-k(\eta-1)} \sum_{h=0}^k 2^{m(k-h)} \frac{1}{|I_{lj}^h|} \int_{I_{lj}^h} |g(x) - g_{I_{lj}^h}^{(m)}(x)| dx \\ &\leq C 2^{-j\alpha} \sum_{h=0}^{\infty} 2^{-mh} \left(\frac{1}{|I_{lj}^h|} \int_{I_{lj}^h} |g(x) - g_{I_{lj}^h}^{(m)}(x)| dx \right) \sum_{k=h}^{\infty} 2^{-k(\eta-m-1)}. \\ &= C 2^{-j\alpha} \sum_{h=0}^{\infty} 2^{-h(\eta-1)} \left(\frac{1}{|I_{lj}^h|} \int_{I_{lj}^h} |g(x) - g_{I_{lj}^h}^{(m)}(x)| dx \right), \end{aligned}$$

since $\eta - m - 1$ is positive after our assumptions on η and m .

Observe now that for a fixed h any point on the line belongs to not more than $3 \cdot 2^h$ intervals in $\{I_{lj}^h\}_{l,j}$. Therefore the collection $\{I_{lj}^h\}_{l,j}$ is the union of 2^h admissible coverings of the line of length $3 \cdot 2^{j+h}$. Summing over l , we then obtain that

$$\begin{aligned} \left(\sum_l c_{lj}^s \right)^{1/s} &\leq C 2^{-j\alpha} \sum_{h=0}^{\infty} 2^{-h(\eta-1)} \left[\sum_l \left(\frac{1}{|I_{lj}^h|} \int_{I_{lj}^h} |g(x) - g_{I_{lj}^h}^{(m)}(x)| dx \right)^s \right]^{1/s} \\ &\leq C 2^{-j\alpha} \sum_{h=0}^{\infty} 2^{-h(\eta-1-1/s)} \omega_{s,m}(g, 2^{j+h}). \end{aligned}$$

Summing now over j ,

$$\begin{aligned} \left(\sum_j \left(\sum_i c_{ij}^s \right)^{r/s} \right)^{1/r} &\leq C \sum_{h=0}^{\infty} 2^{-h(\eta-1-1/s)} \left[\sum_j (2^{-j\alpha} \omega_{s,m}(g, 2^{j+h}))^r \right]^{1/r} \\ &= C \left[\sum_j (2^{-j\alpha} \omega_{s,m}(g, 2^j))^r \right]^{1/r} \sum_{h=0}^{\infty} 2^{-h(\eta-\alpha-1-1/s)} \\ &\leq C \left[\sum_j (2^{-j\alpha} \text{osc}_{s,m}(g, \mathcal{J}_j))^r \right]^{1/r}, \end{aligned}$$

by (9.11). Lemma (9.9) implies at this point that the expression (9.13) is majorized by a constant times $MO_{sr}^\alpha(g)$.

The majorization in the other direction is trivial, since by (9.14)

$$c_{ij} \geq C 2^{-j\alpha} \frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)| dx.$$

We are now in a position to prove the existence of boundary values for functions in \mathcal{A}_{sr}^β .

(9.15) **THEOREM.** *Let f be in \mathcal{A}_{sr}^β , $\beta > 0$, $0 < s, r \leq \infty$. Let s' and r' be the conjugate exponents of s and r (in the sense of (1.15)). For any fixed $y > 0$, $f_y(x) = f(x, y)$ determines a continuous linear functional on $MO_{s'r'}^{\beta+1/s-1}$ and the limit $\lim_{y \rightarrow 0} \langle f_y, g \rangle = \langle f_0, g \rangle$ exists for every $g \in MO_{s'r'}^{\beta+1/s-1}$ and defines a continuous linear functional f_0 on the same space.*

PROOF. We use the representation theorem (1.10). The function f can be decomposed as a sum

$$(9.16) \quad f(x) = \sum_{il} \lambda_{il} \varphi_{il}(x, y);$$

in this expression the coefficients λ_{il} satisfy the condition

$$(9.17) \quad \|\lambda\|_{sr} = \left(\sum_j \left(\sum_i |\lambda_{ij}|^s \right)^{r/s} \right)^{1/r} \sim N_{sr}^\beta(f)$$

and the functions φ_{il} are obtained as convex combinations of M^2 functions of the form

$$(y_{il}^k)^{m+1} \frac{\partial^{m+1} P}{\partial y^{m+1}} (x - x_{il}^k, y + y_{il}^k),$$

where $P(x, y) = (1/\pi)(y/(x^2 + y^2))$ is the Poisson kernel for the upper half-plane and for given j and l , $x_{ij}^k + iy_{ij}^k$ are M^2 fixed points in the square Q_{ij} with vertices $l2^j + i2^j$, $(l + 1)2^j + i2^j$, $l2^j + i2^{j+1}$ and $(l + 1)2^j + i2^{j+1}$. Also let $m = \max \{[\beta + 1/s - 1], [\beta]\}$.

What is relevant for us is that

$$(9.18) \quad \int \varphi_{ij}(x, y)x^k dx = 0 \quad \text{for } k = 0, \dots, m \text{ and } y \geq 0$$

and that

$$(9.19) \quad |\varphi_{ij}(x, y)| \leq C \frac{2^{j(m+2-\beta-1/s)}}{(2^j + y + |x - l2^j|)^{m+2}}.$$

Using these properties of φ_{ij} and Lemma (9.12), one can see that f_ν defines a continuous linear functional on $MO_{s'r}^{\beta+1/s-1}$ by the formula

$$(9.20) \quad \begin{aligned} \langle f_\nu, g \rangle &= \sum_{ij} \lambda_{ij} \int_{-\infty}^{+\infty} \varphi_{ij}(x, y)g(x) dx \\ &= \sum_{ij} \lambda_{ij} \int_{-\infty}^{+\infty} \varphi_{ij}(x, y)(g(x) - g_{I_{ij}}^{(m)}(x)) dx \end{aligned}$$

where the integral and the sum in the last member are absolutely convergent. Also

$$(9.21) \quad |\langle f_\nu, g \rangle| \leq CN_{sr}^\beta(f) MO_{s'r'}^{\beta+1/s-1}(g).$$

The estimate (9.19) yields

$$\sup_{y>0} |\varphi_{ij}(x, y)| \leq C \frac{2^{j(m+2-\beta-1/s)}}{(2^j + |x - l2^j|)^{m+2}}$$

which allows us to use the dominated convergence theorem to pass to the limit for $y \rightarrow 0$ in (9.20):

$$(9.22) \quad \lim_{\nu \rightarrow 0} \langle f_\nu, g \rangle = \sum_{ij} \lambda_{ij} \int_{-\infty}^{+\infty} \varphi_{ij}(x)(g(x) - g_{I_{ij}}^{(m)}(x)) dx = \langle f_0, g \rangle.$$

Formula (9.22) defines the boundary functional f_0 .

We prove now that f is the Poisson integral of f_0 .

(9.23) LEMMA. *The Poisson kernel $P_\nu(x) = (1/\pi)(y/(x^2 + y^2))$ is in MO_{sr}^α for every $s, r, \alpha, 1 \leq s, r < \infty, \alpha > -1/s$.*

PROOF. We may restrict ourselves to $y = 1$ and call $P(x) = P_1(x)$. For any given interval I and any integer $m \geq 0$ we obtain two estimates:

$$(9.24) \quad \frac{1}{|I|} \int_I |P(x) - P_I^{(m)}(x)| dx \leq C \frac{1}{|I|} \int_I P(x) dx$$

because (9.7) gives, for any function g and any interval J centered at 0, that

$$\begin{aligned} \frac{1}{|J|} \int_J |g(x) - g_J^{(m)}(x)| &= \frac{1}{|J|} \int_J \left| g(x) - \sum_0^m \left(\frac{1}{|J|} \int_J g(t) \psi_j(t) dt \right) x^j \right| dx \\ &= \frac{1}{|J|} \int_J \left| \sum_0^m \left(\frac{1}{|J|} \int_J (g(x) - g(t)) \psi_j(t) dt \right) x^j \right| dx \\ &\leq \sum_0^m \left(\sup_{t \in J} |\psi_j(t)| \right) \left(\sup_{x \in J} |x|^j \right) \frac{1}{|J|^2} \int_J \int_J |g(x) - g(t)| dx dt \\ &\leq C \frac{1}{|J|} \int_J |g(x)| dx \end{aligned}$$

(see the discussion with respect to (9.8)).

The second estimate we need is

$$(9.25) \quad \frac{1}{|I|} \int_I |P(x) - P_I^{(m)}(x)| dx \leq C |I|^m \int_I \left| \frac{d^m P}{dx^m} \right| (x)$$

which can be proved as follows: let $T(x)$ be the Taylor polynomial of order m obtained by the expansion of P from the left end point x_0 of I ; arguing as above,

$$\begin{aligned} \frac{1}{|I|} \int_I |P(x) - P_I^{(m)}(x)| dx &\leq C \frac{1}{|I|} \int_I |P(x) - T(x)| dx \\ &= C \frac{1}{|I|} \int_I \left| \int_{x_0}^x (x-t)^m \frac{d^{m+1} P}{dt^{m+1}}(t) dt \right| dx \end{aligned}$$

which gives (9.25).

From (9.24) one obtains

$$(9.26) \quad \begin{aligned} \text{osc}_{s,m}(P, \mathfrak{J}_j) &\leq C \left[\sum_I \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} P(x) dx \right)^s \right]^{1/s} \\ &\leq C \left(\sum_I \frac{1}{|I_{ij}|} \int_{I_{ij}} P(x)^s dx \right)^{1/s} \\ &\leq C 2^{-j/s} \end{aligned}$$

and from (9.25)

$$(9.27) \quad \text{osc}_{s,m}(P, \mathfrak{J}_j) \leq C \left[\sum_j \left(|I_{ij}|^{m+1} \frac{1}{|I_{ij}|} \int_{I_{ij}} \left| \frac{d^{m+1}P}{dx^{m+1}}(x) \right| dx \right)^s \right]^{1/s} \\ < C 2^{j(m+1-1/s)}.$$

Because of Lemma (9.9), for $m = [\alpha + 1/s]$,

$$MO_{sr}^\alpha(P) \leq C \left[\sum_{j < 0} 2^{j(m+1-1/s-\alpha)} + \sum_{j \geq 0} 2^{-j(1/s+\alpha)} \right]$$

which is finite.

(9.28) **THEOREM.** *Let $f \in \mathcal{A}_{sr}^\beta$ and f_0 be its boundary functional on $MO_{s'r'}^{\beta+1/s-1}$. Then $f(x, y) = f_0 * P_\nu(x)$ and therefore the correspondence between f and f_0 is one-to-one.*

PROOF. From Lemma (9.23), $P_\nu \in MO_{s'r'}^{\beta+1/s-1}$ and therefore the convolution $f_0 * P_\nu$ is a well defined continuous function of x . By (9.22)

$$f_0 * P_\nu(x) = \sum_j \lambda_{ij} \varphi_{ij}(x + iy) \\ = f(x + iy)$$

since φ_{ij} is a bounded harmonic function on \mathbb{R}_+^2 and Poisson integral formula holds.

10. - The spaces H_{sr}^p .

In this section we characterize the functionals f_0 which arise as boundary values of functions in \mathcal{A}_{sr}^β among all the linear functionals on $MO_{s'r'}^{\beta+1/s-1}$.

DEFINITION. *Let $0 < p < s \leq \infty$, $0 < r \leq \infty$. An H_{sr}^p -atom is a function $a(x)$ supported on an interval I and such that*

$$(10.1) \quad |a(x)| \leq \frac{1}{|I|^{1/p}}$$

and

$$(10.2) \quad \int_I a(x) x^j = 0 \quad \text{for } j = 0, \dots, \max \left\{ \left[\frac{1}{p} - 1 \right], \left[\frac{1}{p} - \frac{1}{s} \right] \right\}.$$

This definition extends to the case $0 < p < \infty$ the one given by Coifman and Weiss [4].

DEFINITION. $H_{s,r}^p$ is the space of continuous linear functionals f on $MO_{s',r'}^{1/p-1}$ which can be represented as sums

$$(10.3) \quad f(x) = \sum_{ij} \lambda_{ij} a_{ij}(x)$$

where a_{ij} is an H_{sr}^p -atom supported on $I_{ij} = [(l-1)2^i, (l+2)2^i]$ and

$$(10.4) \quad \|\lambda\|_{rs} = \left(\sum_j \left(\sum_i |\lambda_{ij}|^s \right)^{r/s} \right)^{1/r} < \infty.$$

The norm of f in H_{sr}^p is given by

$$(10.5) \quad H_{sr}^p(f) = \inf \{ \|\lambda\|_{sr} \mid f = \sum \lambda_{ij} a_{ij} \text{ as in (10.3)} \}.$$

Some comments are in order:

i) Formula (10.3) defines a linear functional by setting

$$(10.6) \quad \langle f, g \rangle = \sum_{ij} \lambda_{ij} \int_{I_{ij}} a_{ij}(x) g(x) dx = \sum_{ij} \lambda_{ij} \int_{I_{ij}} a_{ij}(x) (g(x) - g_{I_{ij}}^{(m)}(x)) dx,$$

where $g \in MO_{s',r'}^{1/p-1}$ and $m = [1/p - 1 + 1/s'] = \max \{ [1/p - 1], [1/p - 1/s] \}$.

In fact, by (10.1) it is easy to check that

$$(10.7) \quad |\langle f, g \rangle| \leq C \|\lambda\|_{sr} MO_{s',r'}^{1/p-1}(g),$$

which yields, by (10.5)

$$(10.8) \quad |\langle f, g \rangle| \leq C H_{sr}^p(f) MO_{s',r'}^{1/p-1}(g).$$

ii) H_{sr}^p is a linear space: if $f_1, f_2 \in H_{sr}^p$,

$$f_1(x) = \sum_{ij} \lambda_{ij} a_{ij}(x), \quad f_2(x) = \sum_{ij} \mu_{ij} b_{ij}(x)$$

as in (10.3), then

$$f_1(x) + f_2(x) = \sum_{ij} (|\lambda_{ij}| + |\mu_{ij}|) c_{ij}(x)$$

where c_{ij} is an H_{st}^p -atom supported on I_{ij} ,

$$c_{ij}(x) = \frac{\lambda_{ij}a_{ij}(x) + \mu_{ij}b_{ij}(x)}{|\lambda_{ij}| + |\mu_{ij}|}.$$

iii) H_{sr}^p is a complete metric space. For $s, r \geq 1$ it is a Banach space; in general, if $h = \min \{s, r, 1\}$

$$H_{sr}^p(f_1 + f_2)^h \leq H_{sr}^p(f_1)^h + H_{sr}^p(f_2)^h.$$

(10.9) THEOREM. Let f be in \mathcal{A}_{sr}^β and let f_0 be its boundary functional (on $MO_{sr}^{\beta+1/s-1}$). Then $f_0 \in H_{sr}^p$, $1/p = \beta + 1/s$, and $H_{sr}^p(f_0) \leq CN_{sr}^\beta(f)$.

PROOF. Consider one of the functions $\varphi_{ij}(x)$ introduced in the proof of Theorem (9.15). Because of (9.18) and (9.19) they are, in the terminology of Taibleson and Weiss [14], $(p, \infty, [1/p - 1 + 1/s'], [1/p + 1/s'])$ -molecules. Using Theorem (2.9) in [14] (the proof extends to $p > 1$), one sees that φ_{ij} can be decomposed, as a function, as

$$(10.10) \quad \varphi_{ij}(x) = \sum_{k=0}^{\infty} c_{ij}^k a_{ij}^k(x)$$

where $a_{ij}^k(x)$ is an H_{sr}^p -atom supported on $I_{ij}^k = (I_{ij})_{2^k}$, and

$$(10.11) \quad |c_{ij}^k| \leq C2^{-k(m-1/p)},$$

where $m = [1/p + 1 + 1/s']$.

Now let $f \in \mathcal{A}_{sr}^\beta$ and write, as in (9.16),

$$f(x, y) = \sum_{ij} \lambda_{ij} \varphi_{ij}(x, y),$$

so that, by Theorem (9.15),

$$f_0(x) = \sum_{ij} \lambda_{ij} \varphi_{ij}(x).$$

We show first that the formal substitution suggested by (10.10)

$$(10.12) \quad f_0(x) = \sum_{ij} \lambda_{ij} \left(\sum_{k=0}^{\infty} c_{ij}^k a_{ij}^k(x) \right)$$

gives a decomposition of f_0 as a linear functional, and that the sum in the

second term in (10.12) can be rearranged as a linear functional. A proper rearrangement of the sum will provide the atomic decomposition.

For any $g \in MO_{s,r}^{\beta+1/s-1}$ the sum

$$(10.13) \quad \sum_{I_j^k} |\lambda_{I_j}| \sum_{k=0}^{\infty} |c_{I_j^k}^k| \left| \int a_{I_j^k}^k(x) g(x) dx \right|$$

is convergent. In fact, (10.13) can be majorized by

$$\begin{aligned} & C \sum_{I_j^k} |\lambda_{I_j}| \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \int_{I_j^k} |a_{I_j^k}^k(x)| |g(x) - g_{I_j^k}^{(m)}(x)| dx \\ & \leq C \sum_{I_j^k} |\lambda_{I_j}| \sum_{k=0}^{\infty} \frac{2^{-k(m-1/p)}}{|I_j^k|^{1/p-1}} \frac{1}{|I_j^k|} \int |g(x) - g_{I_j^k}^{(m)}(x)| dx \\ & = C \sum_{k=0}^{\infty} 2^{-k(m+1)} \sum_{I_j^k} |\lambda_{I_j}| 2^{-j(1/p-1)} \frac{1}{|I_j^k|} \int |g(x) - g_{I_j^k}^{(m)}(x)| dx \\ & \leq C \sum_{k=0}^{\infty} 2^{-k(m+1)} \sum_j 2^{-j(1/p-1)} \left(\sum_I |\lambda_{I_j}|^s \right)^{1/s} \left[\sum_I \left(\frac{1}{|I_j^k|} \int |g(x) - g_{I_j^k}^{(m)}(x)| dx \right)^{s'} \right]^{1/s'}. \end{aligned}$$

As we observed in the proof of Lemma (9.12), for any given k , $\{I_{ij}^k\}_{i,j}$ is the union of 2^k admissible coverings of length $3 \cdot 2^{j+k}$, so that (10.13) is less than

$$\begin{aligned} & C \sum_{k=0}^{\infty} 2^{-k(m+1)} \sum_j 2^{-j(1/p-1)} \left(\sum_I |\lambda_{I_j}|^s \right)^{1/s} 2^{k/s'} \omega_{s',m}(g, 3 \cdot 2^{j+k}) \\ & \leq C \sum_{k=0}^{\infty} 2^{-k(m+1-1/s')} \|\lambda\|_{sr} \left[\sum_j (2^{-j(1/p-1)} \omega_{s',m}(g, 3 \cdot 2^{j+k}))^{r'} \right]^{1/r'} \\ & \leq C \|\lambda\|_{sr} \left[\sum_j (2^{-j(1/p-1)} \text{osc}_{s',m}(g, 2^j))^r \right]^{1/r'} \sum_{k=0}^{\infty} 2^{-k(m+2-1/p-1/s')} \\ & \leq C \|\lambda\|_{sr} MO_{s,r}^{1/p-1}(g). \end{aligned}$$

We used in the last two steps (9.11), Lemma (9.9) and the fact that $m = [1/p - 1 + 1/s']$.

We can now regroup the summands in (10.12): given the interval $I_{\mu\nu} = [(\mu-1)2^\nu, (\mu+2)2^\nu]$, consider a partition $\{\mathfrak{J}_{\mu\nu}\}_{\mu\nu}$ of the family of intervals $\{I_{ij}^k\}_{i,j,k}$ such that $\mathfrak{J}_{\mu\nu}$ consists of intervals of length $\frac{3}{2} 2^\nu$ contained in $I_{\mu\nu}$. We can write

$$(10.14) \quad f_0(x) = \sum_{\mu,\nu} \sum_{I_{ij}^k \in \mathfrak{J}_{\mu\nu}} \lambda_{I_j} c_{I_j^k}^k a_{I_j^k}^k(x).$$

The function

$$\alpha_{\mu\nu}(x) = \sum_{I_{ij}^k \in \mathfrak{J}_{\mu\nu}} \lambda_{ij} \sigma_{ij}^k \alpha_{ij}^k(x)$$

is supported on $I_{\mu\nu}$ and

$$|\alpha_{\mu\nu}(x)| \leq \sum_{I_{ij}^k \in \mathfrak{J}_{\mu\nu}} |\lambda_{ij}| 2^{-k(m-1/p)} \left(\frac{3}{2} 2^\nu\right)^{-1/p},$$

so that the function

$$(10.15) \quad b_{\mu\nu}(x) = \left(\sum_{I_{ij}^k \in \mathfrak{J}_{\mu\nu}} |\lambda_{ij}| 2^{-k(m-1/p)} \right)^{-1} \alpha_{\mu\nu}(x)$$

is an H_{sr}^p -atom supported on $I_{\mu\nu}$. Therefore (10.14) becomes

$$(10.16) \quad f_0(x) = \sum_{\mu, \nu} \left(\sum_{I_{ij}^k \in \mathfrak{J}_{\mu\nu}} |\lambda_{ij}| 2^{-k(m-1/p)} \right) b_{\mu\nu}(x) = \sum_{\mu, \nu} \sigma_{\mu\nu} b_{\mu\nu}(x).$$

We compute now $\|\sigma\|_{sr}$. Observe that, in order to have $I_{ij}^k \in \mathfrak{J}_{\mu\nu}$, it must be so that $j+k = \nu-1$ and $(\mu-1)2^{k+1} \leq l \leq (\mu+2)2^{k+1}$. We have to consider four cases:

1st case: $s, r \geq 1$. We have

$$(10.17) \quad \begin{aligned} \left(\sum_{\mu} |\sigma_{\mu\nu}|^s \right)^{1/s} &\leq \left[\sum_{\mu} \left(\sum_{k=0}^{\infty} 2^{-k(m-1/p)} \sum_{l=(\mu-1)2^{k+1}}^{(\mu+2)2^{k+1}} |\lambda_{l, \nu-k-1}| \right)^s \right]^{1/s} \\ &\leq \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \left[\sum_{\mu} \left(\sum_{l=(\mu-1)2^{k+1}}^{(\mu+2)2^{k+1}} |\lambda_{l, \nu-k-1}| \right)^s \right]^{1/s} \\ &\leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p-1/s')} \left(\sum_{\mu} \sum_{l=(\mu-1)2^{k+1}}^{(\mu+2)2^{k+1}} |\lambda_{l, \nu-k-1}|^s \right)^{1/s} \\ &\leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p-1/s')} \|\lambda(\nu-k-1)\|_s, \end{aligned}$$

where we set $\|\lambda(j)\|_s = \left(\sum_l |\lambda_{lj}|^s \right)^{1/s}$.

It follows that

$$(10.18) \quad \begin{aligned} \|\sigma\|_{sr} &\leq C \left[\sum_{\nu} \left(\sum_{k=0}^{\infty} 2^{-k(m-1/p-1/s')} \|\lambda(\nu-k-1)\|_s \right)^r \right]^{1/r} \\ &\leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p-1/s')} \|\lambda\|_{sr} = C \|\lambda\|_{sr}. \end{aligned}$$

This implies that $H_{sr}^p(f_0) \leq CN_{sr}^\beta(f)$.

2nd case: $0 < r < 1 \leq s$. Instead of (10.18), one estimates

$$\begin{aligned} \|\sigma\|_{sr}^r &\leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p-1/s')r} \|\lambda\|_{sr}^r \\ &= C \|\lambda\|_{sr}^r \end{aligned}$$

and obtains the same conclusion.

3rd case: $0 < s < 1, r \geq s$. Instead of (10.17) one evaluates

$$\begin{aligned} (10.19) \quad \sum_{\mu} |\sigma_{\mu\nu}|^s &\leq \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \sum_{\mu} \left(\sum_{l=(\mu-1)2^{k+1}}^{(\mu+2)2^{k+1}} |\lambda_{l, \nu-k-1}| \right)^s \\ &\leq \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \sum_{\mu} \sum_{l=(\mu-1)2^{k+1}}^{(\mu+2)2^{k+1}} |\lambda_{l, \nu-k-1}|^s \\ &\leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \|\lambda(\nu-k-1)\|_s^s. \end{aligned}$$

Hence,

$$\|\sigma\|_{sr}^s = \left[\sum_{\nu} \left(\sum_{\mu} |\sigma_{\mu\nu}|^s \right)^{r/s} \right]^{s/r} \leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \left(\sum_{\nu} \|\lambda(\nu-k-1)\|_s^s \right)^{s/r} = C \|\lambda\|_{sr}^s.$$

4th case: $0 < r < s < 1$. From (10.19) one derives

$$\begin{aligned} \|\sigma\|_{sr}^r &= \sum_{\nu} \left(\sum_{\mu} |\sigma_{\mu\nu}|^s \right)^{r/s} \\ &\leq C \sum_{k=0}^{\infty} 2^{-k(m-1/p)} \sum_{\nu} \|\lambda(\nu-k-1)\|_s^r \\ &= C \|\lambda\|_{sr}^r. \end{aligned}$$

We prove now that the Poisson integral of an element in H_{sr}^p is in \mathcal{A}_{sr}^{β} , $1/p = \beta + 1/s$. This will provide the identification between the two spaces.

(10.20) LEMMA. *Suppose $a(x)$ is an H_{sr}^p -atom supported on $[-r, r]$, $r > 0$. There is a constant $c > 0$ such that, if $a(x, y)$ is the Poisson integral of $a(x)$, then*

$$(10.21) \quad |a(x, y)| \leq \begin{cases} C \frac{r^{m+2-1/p}}{|x + i(y+r)|^{m+2}} & \text{for } |x| \geq 2r \text{ or } |y| \geq r \\ C r^{-1/p} & \text{for } |x| < 2r \text{ and } |y| < r \end{cases} \left(m = \left[\frac{1}{p} - 1 - \frac{1}{s'} \right] \right).$$

PROOF. The second estimate follows immediately from the fact that $|a(x)| \leq (2r)^{-1/p}$ and that $\int P_\nu(x) dx = 1$.

To prove the first estimate, it is convenient to assume that a is real valued (which does not cause a loss of generality) and to take the Cauchy integral of a , recalling that the Poisson kernel is essentially the real part of the Cauchy kernel. Let therefore, for $z = x + iy$

$$A(z) = \frac{i}{\pi} \int a(t) \frac{1}{z-t} dt.$$

Since

$$\frac{1}{z-t} = \frac{1}{z} + \frac{t}{z^2} + \dots + \frac{t^m}{z^{m+1}} + \frac{t^{m+1}}{z^{m+1}(z-t)},$$

$$A(z) = \frac{i}{\pi} \frac{1}{z^{m+1}} \int a(t) \frac{t^{m+1}}{z-t} dt.$$

Consequently,

$$(10.22) \quad |A(z)| \leq C \frac{r^{m+2-1/p}}{|z|^{m+1}} \sup_{|t| < r} \frac{1}{|z-t|}.$$

If $|x| \geq 2r$ or $|y| \geq r$, $|z-t| \geq \frac{1}{2}|z+ir|$ for any $t \in [-r, r]$ so that

$$|A(z)| \leq C \frac{r^{m+2-1/p}}{|z+ir|^{m+1}}.$$

By taking the real part of $A(z)$, one obtains (10.21).

(10.23) THEOREM. Let $f \in H_{sr}^p$ and Pf be its Poisson integral. Then $Pf \in \mathcal{A}_{sr}^\beta$, $\beta = 1/p - 1/s > 0$, and $N_{sr}^\beta(Pf) \leq CH_{sr}^p(f)$.

PROOF. Suppose $f(x) = \sum_{ij} \lambda_{ij} a_{ij}(x)$ as in (10.3). Then

$$(10.24) \quad Pf(x, y) = f * P_\nu(x) = \sum_{ij} \lambda_{ij} a_{ij}(x, y)$$

(10.21) implies that

$$|a_{ij}(x, y)| \leq C \frac{2^{j(m+2-\beta-1/s)}}{((x-l2^j)^2 + (y+2^j)^2)^{(m+2)/2}}.$$

We can therefore apply Lemma (5.1), to have

$$N_{sr}^\beta(Pf) \leq C \|\lambda\|_r$$

which yields $N_{sr}^\beta(Pf) \leq CH_{sr}^p(f)$. Also, the same Lemma asserts that the convergence of the series in (10.24) is uniform on compact sets in \mathbb{R}_+^2 . Since the functions $a_{ij}(x, y)$ are harmonic, Pf is also harmonic. Then $Pf \in \mathcal{A}_{sr}^\beta$.

REMARK. Because of the definition of H_{sr}^p spaces, as linear functionals on different spaces, one can not compare two of them directly in terms of inclusion. But, if $H_{s_1r_1}^{p_1}$ and $H_{s_2r_2}^{p_2}$ are given, one can consider the intersection X of the corresponding MO -spaces, and the elements of the two spaces as linear functionals on X . What is important is that the restrictions of these functionals to X completely determines them. In fact, this follows from Lemma (9.23) and Theorem (9.28), namely that a functional is determined by its restriction to the subset $\{P_y | y > 0\}$. One obtains the following inclusions:

(10.25) THEOREM. a) Let $p < s_1 \leq s_2$ and $r_1 \leq r_2$: Then $H_{s_1r_1}^p$ is contained in $H_{s_2r_2}^p$.

b) If $p < s$ and $p < r$, then the Hardy space H^p is contained in H_{sr}^p :

The proof follows immediately from the definition of H_{sr}^p and the atomic characterization of Hardy spaces [4].

Because of the identification of H_{sr}^p with \mathcal{A}_{sr}^β , $\beta = 1/p - 1/s$, Theorem (10.25) a) is a restatement of Proposition (2.2) and b) is Flett's theorem [6].

11. - Extensions.

In the theory of Hardy spaces one shows that one can modify (10.1) and (10.2) in the definition of an atom ([4] and [14]) and still obtain the same atomic space. In the same way, the definitions of the Campanato-Morrey spaces [1] and of BMO [8] also allow the possibility of taking other means, or higher order polynomials. The same can be done with the spaces with which we are dealing.

DEFINITIONS. a) Let $1 < s, r < \infty$, $1 < q < \infty$, $\alpha > \max\{-1/s, -1/q\}$ and $m \geq m(\alpha, s) = [\alpha + 1/s]$. The space $MO_{sr}^{\alpha q m}$ consists of the equivalence classes, modulo polynomials of degree at most m , of those locally integrable functions g for which

$$(11.1) \quad MO_{sr}^{\alpha q m}(g) = \left[\int_0^\infty (t^{-\alpha} \omega_{s, \alpha, m}(g, t))^q \frac{dt}{t} \right]^{1/r}$$

is finite, where

$$(11.2) \quad \omega_{s,q,m}(g, t) = \sup \{ \text{osc}_{s,q,m}(g, \mathfrak{J}) \mid |\mathfrak{J}| = t \}$$

and, for $\mathfrak{J} = \{I_i\}$,

$$(11.3) \quad \text{osc}_{s,q,m}(g, \mathfrak{J}) = \left[\sum_i \frac{1}{|I_i|} \int_{I_i} |g(x) - g_{I_i}^{(m)}(x)|^q dx \right]^{s/q}.$$

b) A (p, q, m) -atom is a function $a(x)$ supported on an interval I such that

$$(11.4) \quad \left(\frac{1}{|I|} \int_I |a(x)|^q dx \right)^{1/q} \leq \frac{1}{|I|^{1/p}}$$

$$(11.5) \quad \int a(x)x^j dx = 0 \quad \text{for } j = 0, \dots, m.$$

c) For $0 < p < s < \infty$, $0 < r < \infty$, $q > p$, $q \geq 1$ and $m \geq [1/p - 1 + 1/s']$, $H_{sr}^{p,q,m}$ is the space of continuous linear functionals f on $MO_{sr}^{1/p-1, q', m}$ which can be written as

$$(11.6) \quad f(x) = \sum_{ij} \lambda_{ij} a_{ij}(x)$$

where a_{ij} is a (p, q, m) -atom supported on $I_{ij} = [(l-1)2^j, (l+2)2^j]$, and $\|\lambda\|_{sr} < \infty$.

One can easily modify the proofs of Lemma (9.9) and Lemma (9.12) to obtain the following lemma:

(11.7) LEMMA. The following are equivalent, for a locally integrable function g :

$$(11.8) \quad \begin{aligned} & \text{i) } g \in MO_{sr}^{\alpha, q, m} \\ & \text{ii) } \left[\sum_j (2^{-j\alpha} \text{osc}_{s,q,m}(g, \mathfrak{J}_j))^r \right]^{1/r} < \infty \\ & \text{iii) if } \eta > \max \left\{ m + \frac{1}{q}, \alpha + \frac{1}{q} + \frac{1}{\lambda} \right\}, \end{aligned}$$

$$(11.9) \quad \left\{ \sum_j \left[\sum_i \left(\frac{1}{|I_{ij}|} \int_{-\infty}^{+\infty} \left(|g(x) - g_{I_{ij}}^{(m)}(x)| \frac{2^{j(\eta-\alpha)}}{(2^j + |x - l2^j|)^\eta} \right)^q dx \right)^{s/q} \right]^{1/r} \right\} < \infty.$$

For $g \in MO_{sr}^{\alpha, q, m}$, (11.8) and (11.9) provide norms equivalent to $MO_{sr}^{\alpha, q, m}(g)$.

The extension of Lemma (9.23) is the following

(11.10) LEMMA. *The Poisson kernel $P_y(x) = (1/\pi)(y/(x^2 + y^2))$ is in $MO_{sr}^{\alpha, q, m}$ for any choice of admissible exponents as given in Definition a).*

PROOF. One can obtain, as in the proof of Lemma (9.3), the substitutes for (9.24) and (9.25); namely,

$$(11.11) \quad \left(\frac{1}{|I|} \int_I |P(x) - P_I^{(m)}(x)|^q dx \right)^{1/q} \leq C \left(\frac{1}{|I|} \int_I P(x)^q dx \right)^{1/q}$$

and

$$(11.12) \quad \left(\frac{1}{|I|} \int_I |P(x) - P_I^{(m)}(x)|^q dx \right)^{1/q} \leq C |I|^{m+1} \left(\frac{1}{|I|} \int_I \left| \frac{d^m P}{dx^m}(x) \right|^q dx \right)^{1/q},$$

valid for any interval I . Now one has to obtain the corresponding estimates to (9.26) and (9.27) for $\text{osc}_{s, q, m}$.

If $q \leq s$ one obtains exactly the same estimates, using Hölder's inequality.

For $q > s$, one obtains by interpolation:

$$(11.13) \quad \text{osc}_{s, q, m}(g, \mathfrak{J}_j) \leq \text{osc}_{s, s, m}(g, \mathfrak{J}_j)^{s/q} \text{osc}_{s, \infty, m}(g, \mathfrak{J}_j)^{1-s/q}.$$

Using (11.11) when $j \geq 0$,

$$\begin{aligned} \text{osc}_{s, \infty, m}(g, \mathfrak{J}_j) &= \left[\sum_I \sup_{x \in I_j} P(x)^s \right]^{1/s} \\ &\leq \left[\sum_I \frac{1}{(1 + l^2 2^{2j})^s} \right]^{1/s} \\ &\leq C. \end{aligned}$$

Hence for $j \geq 0$ and $q > s$, using also (9.26), we have

$$(11.14) \quad \text{osc}_{s, q, m}(g, \mathfrak{J}_j) \leq C 2^{-j/q}.$$

For $j < 0$, one proceeds as before using (11.12) and (9.27). In particular, observe that, since $1/(1 + x^2) = \text{Im } 1/(x - i)$,

$$\begin{aligned} (11.15) \quad \left| \frac{d^m P}{dx^m}(x) \right| &= C \left| \text{Im} \frac{1}{(x - i)^{m+1}} \right| \\ &= C \left| \frac{\text{Im} [(x + i)^{m+1} - x^{m+1}]}{(1 + x^2)^{m+1}} \right| \\ &\leq \frac{C}{(1 + x^2)^{(m+2)/2}}; \end{aligned}$$

so that

$$\begin{aligned} \text{osc}_{s,\infty,m}(g, \mathfrak{J}_j) &\leq C2^{j(m+1)} \left[\sum_t \frac{1}{(1 + t^2 2^{2j})^{s(m+2)/2}} \right]^{1/s} \\ &\leq C2^{j(m+1)} \left[\sum_{|t| > 2^{-j}} 1 + 2 \sum_{2^{-j}}^{\infty} \frac{1}{2^{js(m+2)} t^{s(m+2)}} \right]^{1/s} \\ &\leq C2^{j(m+1-1/s)}. \end{aligned}$$

Therefore, for $q > s$ and $j < 0$,

$$(11.16) \quad \text{osc}_{s,q,m}(g, \mathfrak{J}_j) \leq C2^{j(m+1-1/s)},$$

and the proof now continues as in Lemma (9.23).

Using Lemmas (11.7) and (11.10), one can extend Theorems (9.15) and (9.28) as follows:

(11.17) **THEOREM.** *Let $f \in \mathcal{A}_{sr}^\beta$, $0 < s, r \leq \infty, \beta > 0$. Then $\lim_{y \rightarrow 0} f_y = f_0$ exists as a continuous linear functional on $MO_{s'r'}^{\beta+1/s-1,a,m}$ for every q such that $1/q > 1 - \beta - 1/s$ and every $m \geq \max \{[\beta + 1/s - 1], [\beta]\}$. Also $f(x + iy) = f_0 * P_y(x)$.*

One simply has to observe that the functions φ_{ν} , which appear in the proof of Theorem (9.15) can be chosen in such a way that (9.18) and (9.19) hold for any given integer $m \geq \max \{[\beta + 1/s - 1], [\beta]\}$, just by taking k large enough in Theorem (1.10). Then the proof goes as in Theorems (9.15) and (9.28).

At this point one obtains the exact analogue of Theorem (10.9) with the space $H_{sr}^p = H_{sr}^{p,\infty,[1/p-1+1/s']}$ replaced by $H_{sr}^{p,q,m}$, by means of the decomposition of a molecule into atoms given by Taibleson and Weiss [14].

More delicate estimates are necessary to show that the Poisson integral of an element of $H_{sr}^{p,q,m}$ is in \mathcal{A}_{sr}^β , $\beta = 1/p - 1/s$, and we give them in detail.

(11.18) **LEMMA.** *Suppose $a(x)$ is a (p, q, m) -atom supported on the interval $[-r, r]$, $r > 0$. Then there is a constant C , depending only on p, q and m , so that if $a(x, y)$ is the Poisson integral of a ,*

$$(11.19) \quad |a(x, y)| \leq \begin{cases} C \frac{r^{m+2-1/p}}{|x + i(y+r)|^{m+2}} & \text{for } |x| \geq 2r \text{ or } |y| \geq r \\ C \frac{r^{1/q-1/p}}{y^{1/q}} & \text{for } |x| \leq 2r \text{ and } |y| \leq r. \end{cases}$$

Furthermore, if $0 < t \leq q$

$$(11.20) \quad \int_{|x| \leq 2r} |a(x + iy)|^t dx \leq Cr^{1-t/p}.$$

PROOF. The first estimate in (11.19) can be proved as in Lemma (10.20). The second estimate is proved using Hölder's inequality:

$$|a(x + iy)| \leq \|a\|_q \|P_y\|_q \leq Cr^{1/\alpha-1/p} y^{-1/\alpha}.$$

To obtain (11.20), note that $\|P_y\|_1 = 1$, so

$$\begin{aligned} \int_{|x| \leq 2r} |a(x + iy)|^t dx &\leq \left(\int_{|x| \leq 2r} |a(x + iy)|^q dx \right)^{t/q} (2r)^{(1/t-1/q)t} \\ &\leq C \|a\|_q^t r^{(1/t-1/q)t} \\ &\leq Cr^{(1/t-1/p)t}. \end{aligned}$$

At this point we can prove

(11.21) THEOREM. Let $f \in H_{s,r}^{p,q,m}$ and Pf be its Poisson integral. Then $Pf \in \mathcal{A}_{sr}^\beta$, $\beta = 1/p - 1/s$, and $N_{sr}^\beta(Pf) \leq CH_{sr}^{p,q,m}(f)$.

PROOF. Suppose $f(x) = \sum_{ij} \lambda_{ij} a_{ij}(x)$ as in (11.6). Then

$$(11.22) \quad Pf(x, y) = f * P_y(x) = \sum_{ij} \lambda_{ij} a_{ij}(x, y)$$

where $a_{ij}(x, y)$ is the Poisson integral of $a_{ij}(x)$.

Lemma (11.18) can be applied to a_{ij} and restated in the following way: it is possible to decompose $a_{ij}(x, y)$ into the sum

$$(11.23) \quad a_{ij}(x, y) = a_{ij}^{(1)}(x, y) + a_{ij}^{(2)}(x, y)$$

where

$$(11.24) \quad |a_{ij}^{(1)}(x, y)| \leq C \frac{2^{j(m+2-1/p)}}{((x-l2^j)^2 + (y+2^j)^2)^{(m+2)/2}}$$

for all $(x, y) \in \mathbb{R}_+^2$ and $a_{ij}^{(2)}$ is supported in the « window »

$$W_{ij} = \{(x, y) \mid |x - l2^j| \leq 2^{j+2}, 0 < y \leq 2^j\}$$

and satisfies the two conditions

$$(11.25) \quad |a_{ij}^{(2)}(x, y)| \leq C 2^{j(1/q-1/p)} y^{-1/q},$$

and for $0 < t \leq q$

$$(11.26) \quad \int_{-\infty}^{+\infty} |a_{ij}^{(2)}(x, y)|^t dx \leq C 2^{j(1-t/p)}.$$

To obtain such a decomposition, one can take as $a_{ij}^{(2)}$ the function a_{ij} times the characteristic function of W_{ij} . One can now apply Lemma (5.1) to show that the sum

$$Pf^{(1)}(x, y) = \sum_{ij} \lambda_{ij} a_{ij}^{(1)}(x, y)$$

converges absolutely and uniformly on compact sets and that

$$N_{sr}^\beta(Pf^{(1)}) \leq C \|\lambda\|_{sr}.$$

To complete the proof we need the same result for

$$Pf^{(2)}(x, y) = \sum_{ij} \lambda_{ij} a_{ij}^{(2)}(x, y).$$

Once we have proved that, the uniform convergence will provide the harmonicity of Pf . Also, by the definition of the « norm » in $H_{sr}^{p,q,m}$ we will obtain

$$N_{sr}^\beta(Pf) \leq CH_{sr}^{p,q,m}(f).$$

The estimates for $Pf^{(2)}$ follow from a variant of Lemma (5.1). Since a detailed proof will require almost as much as the proof of the lemma itself, we will only give a series of hints that will allow the interested reader to carry out all the details, vis-à-vis with the proof of Lemma (5.1).

We observe that for $q = \infty$ we have no need for such a proof, and in fact the proof in this case is the same as in Theorem (10.23).

The estimate (11.25) suffices to show the uniform and absolute convergence. It also allows us to obtain the estimate for $N_{sr}^\beta(Pf^{(2)})$ when $s = \infty$. For other values of s , (11.26) is also required.

For absolute and uniform convergence, in the case $0 < s \leq 1$, the sum is split into three terms with respect to the summation index j . The first term disappears if the parameter J_1 is chosen small enough, and the other two can be controlled in the same way as in Lemma (5.1), using the fact that $1/p - 1/q > 0$.

For the case $1 < s < \infty$, one controls the first two terms as for the case $0 < s \leq 1$, and the third term vanishes if $K > 5$.

For the norm estimate, in the case $0 < s \leq 1$, one has

$$|Pf^{(2)}(x, y)|^s \leq \sum_{2^j \geq y} \sum_t |\lambda_{tj}|^s |a_{tj}^{(2)}(x, y)|^s$$

so that

$$(y^\beta M_s(f^{(2)}; y))^s \leq y^{\beta s} \sum_{2^j \geq y} \sum_t |\lambda_{tj}|^s \int_{-\infty}^{+\infty} |a_{tj}^{(2)}(x, y)|^s dx,$$

and taking $t = s$ in (11.25), one obtains

$$(y^\beta M_s(f^{(2)}; y))^s \leq C \sum_{2^j \geq y} \left(\frac{y}{2^j}\right)^{\beta s} \sum_{tj} |\lambda_{tj}|^s;$$

and the proof continues essentially as in Lemma (5.1).

Finally, if $1 < s < \infty$, we consider a parameter θ , $0 < \theta < 1$ (which will be determined later) and write

$$|f^{(2)}(x, y)|^s \leq \left[\sum_{tj} |\lambda_{tj}|^s |a_{tj}^{(2)}(x, y)|^{0s} \right] \left[\sum_{tj} |a_{tj}^{(2)}(x, y)|^{(1-\theta)s} \right]^{s/s'}.$$

Now,

$$\begin{aligned} \left[\sum_{tj} |a_{tj}^{(2)}(x, y)|^{(1-\theta)s} \right]^{s/s'} &\leq C \left[\sum_{2^j \geq y} \left(\frac{2^{j(1/q-1/p)}}{y^{1/q}} \right)^{(1-\theta)s'} \right]^{s/s'} \\ &\leq C y^{-(1/p)s(1-\theta)} \\ &\leq C y^{-(\beta s+1)(1-\theta)}. \end{aligned}$$

Thus,

$$(y^\beta M_s(f^{(2)}; y))^s \leq C \sum_{2^j \geq y} \frac{y^{\beta s}}{y^{(\beta s+1)(1-\theta)}} \sum_t |\lambda_{tj}|^s \int_{-\infty}^{+\infty} |a_{tj}^{(2)}(x, y)|^{s\theta} dx,$$

and taking $t = \theta s$ in (11.26) (which assumes $0 < \theta s \leq q$),

$$(y^\beta M_s(f^{(2)}; y))^s \leq C \sum_{2^j \geq y} y^{\beta s \theta + \theta - 1} 2^{j(1-s\theta/p)} \sum_t |\lambda_{tj}|^s = C \sum_{2^j \geq y} \left(\frac{y}{2^j}\right)^{s\theta/p-1} \sum_t |\lambda_{tj}|^s.$$

The rest of the proof goes through if we have $s\theta/p - 1 > 0$. Therefore the two conditions we have to impose on θ are

$$\frac{p}{s} < \theta \leq \min \left\{ \frac{q}{s}, 1 \right\}.$$

Since $p < q$ and $p < s$, such a choice for θ in $(0, 1)$ is always possible. The proof of the theorem is therefore complete.

We can apply to the spaces $H_{sr}^{p, q, m}$ the observations we made in the Remark at the end of Section 10.

In particular, for fixed p, s and r , we can say that if $q_1 \geq q_2$ and $m_1 \geq m_2$, then H_{sr}^{p, q_1, m_1} is contained in H_{sr}^{p, q_2, m_2} (we want to recall that we defined these spaces only for some admissible values of the exponents).

This inclusion has a meaning if we look at the elements in these spaces as continuous linear functionals on $MO_{s', r'}^{1/p-1} = MO_{s', r'}^{1/p-1, 1, [1/p-1+1/s']}$. This space is contained in both $MO_{s', r'}^{1/p-1, q_1, m_1}$ and $MO_{s', r'}^{1/p-1, q_2, m_2}$, but contains the Poisson kernels, so that, by Theorems (11.17) and (11.21), an element of $H_{sr}^{p, q, m}$ is completely determined by its restriction to $MO_{s', r'}^{1/p-1}$.

On the other hand, comparing Theorems (10.9), (10.23), (11.17) and (11.21), we obtain the following corollary:

(11.27) COROLLARY. *For any $q > p$, $q \geq 1$ and $m \geq [1/p - 1 + 1/s]$, $H_{sr}^{p, q, m} = H_{sr}^p$.*

REMARK. The same statement is true for atomic Hardy spaces [4], and can be obtained without Poisson integral, by decomposing the atoms. We cannot use this proof here, since we would lose control of the supports of the atoms, which is of extreme importance when $s \neq r$ and when $s = r > 1$.

12. - The dual space of H_{sr}^p .

We show in this section that for $s, r < \infty$ the dual space of $H_{sr}^{p, q, m}$ is $MO_{s', r'}^{1/p-1, q', m}$. Since we have proved that the former spaces do not depend on q or m (Corollary (11.27)), we will have obtained the result that the latter spaces do not depend on q' or m , as far as these exponents are in the range in which the spaces have been defined.

When talking of inclusions or equalities of different mean oscillation spaces, we have to keep in mind that the elements of these spaces are equivalence classes of functions, modulo polynomials of a certain maximum degree, which depends on the space. Therefore identification of two elements from two different spaces may mean inclusion of an equivalence class into the other. Between any pair of $MO_{sr}^{p, q, m}$ spaces for which only the q changes or the m changes, there is a natural inclusion with respect to the equivalence classes. It will not be difficult to check that this inclusion is a one-to-one onto map that corresponds to the identification of the

spaces as dual objects of the $H_{sr}^{p,q,m}$ spaces. The argument follows the same lines as in [14].

The natural duality between an element of $H_{sr}^{p,q,m}$ and an element in $MO_{s',r'}^{1/p-1,a',m}$ is already included in the definition of $H_{sr}^{p,q,m}$: if $f \in H_{sr}^{p,q,m}$ is a sum of (p, q, m) -atoms,

$$f(x) = \sum_{I_j} \lambda_{I_j} a_{I_j}(x)$$

and $g \in MO_{s',r'}^{1/p-1,a',m}$, then

$$(12.1) \quad \langle f, g \rangle = \sum_{I_j} \lambda_{I_j} \int_{-\infty}^{+\infty} a_{I_j}(x) g(x) dx = \sum_{I_j} \lambda_{I_j} \int_{-\infty}^{+\infty} a_{I_j}(x) (g(x) - g_{I_j}^{(m)}(x)) dx.$$

It follows from the definition of $f \in H_{sr}^{p,q,m}$ as a linear functional (with a particular decomposition) that the value of $\langle f, g \rangle$ does not depend on the specific atomic decomposition of f .

(12.2) **THEOREM.** *Assume $s, r < \infty$. Given $g \in MO_{s',r'}^{1/p-1,a',m}$, the map $f \rightarrow \langle f, g \rangle$, defined as in (12.1), defines a continuous linear functional on $H_{sr}^{p,q,m}$. Conversely, given a continuous linear functional L on $H_{sr}^{p,q,m}$ there is a unique g in $MO_{s',r'}^{1/p-1,a',m}$ such that $Lf = \langle f, g \rangle$, and the norm of L as a linear functional is equivalent to the norm of g .*

PROOF. The first statement is trivial (see also (10.8)).

In order to prove the second part of the theorem, assume first that $q < \infty$. If a is a (p, q, m) -atom supported on I_{I_j} , we must have $|La| \leq \|L\|$, where $\|L\|$ denotes the operator norm of L . This means that L induces a continuous linear functional on the closed subspace of $L^q(I_{I_j}, dx)$ consisting of functions φ such that

$$(12.3) \quad \int_{I_{I_j}} \varphi(x) x^k dx = 0, \quad k = 0, \dots, m.$$

If we use the same letter L for this functional, there is a function $g_{I_j} \in L^q(I_{I_j}, dx)$ such that

$$(12.4) \quad L\varphi = \int_{I_{I_j}} \varphi(x) g_{I_j}(x) dx.$$

This function g_{I_j} is unique modulo polynomials of degree at most m . It is possible to define a locally integrable function g on the line such that the restriction of g to each I_{I_j} represents the functional L as in (12.4).

Now, for each I_{ij} , pick a (p, q, m) -atom a_{ij} such that

$$(12.5) \quad \int a_{ij}(x)g(x) dx \geq C \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)|^{q'} dx \right)^{1/q'} |I_{ij}|^{1-1/p}$$

where C is a positive constant. One obtains (12.5) as follows: for φ in $L^q(I_{ij}, (1/|I_{ij}|) dx)$ satisfying (12.3),

$$\left| \frac{1}{|I_{ij}|} \int_{I_{ij}} \varphi(x)g(x) dx \right| \leq \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |\varphi(x)|^q dx \right)^{1/q} \inf \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - P(x)|^{q'} dx \right)^{1/q'}$$

where the infimum is taken over all the polynomials P of degree less than or equal to m . Also, it is possible to pick up a particular φ such that

$$(12.6) \quad \frac{1}{|I_{ij}|} \int_{I_{ij}} \varphi(x)g(x) dx \geq \frac{1}{2} \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |\varphi(x)|^q dx \right)^{1/q} \inf \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - P(x)|^{q'} dx \right)^{1/q'}$$

Arguing as in the proof of formula (9.24), one can show that for any polynomial P of degree less than or equal to m

$$(12.7) \quad \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)|^{q'} dx \right)^{1/q'} \leq C \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - P(x)|^{q'} dx \right)^{1/q'}$$

Normalizing φ so that it becomes a (p, q, m) -atom supported on I_{ij} and using (12.7), formula (12.6) immediately gives (12.5).

If now $\{\lambda_{ij}\}$ is a finite sequence of positive numbers,

$$(12.8) \quad L\left(\sum_{ij} \lambda_{ij} a_{ij}\right) = \sum_{ij} \lambda_{ij} \int a_{ij}(x)g(x) dx \geq C \sum_{ij} \lambda_{ij} |I_{ij}|^{1-1/p} \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)|^{q'} dx \right)^{1/q'}$$

We choose now a particular sequence $\{\lambda_{ij}\}$: for any j fix a finite sequence $\{\mu_{ij}\}$ of positive numbers, such that

$$(12.9) \quad \sum_i \mu_{ij} \left(\frac{1}{|I_{ij}|} \int_{I_{ij}} |g(x) - g_{I_{ij}}^{(m)}(x)|^{q'} dx \right)^{1/q'} > \frac{1}{2} \text{osc}_{s', a', m}(g, \mathfrak{J}_j)$$

and $(\sum_i \mu_{ij}^s)^{1/s} = 1$. (This is possible since $s < \infty$).

Next we choose a finite sequence $\{v_j\}$ of positive numbers such that

$$(12.10) \quad \sum_j v_j 2^{-j(1/p-1)} \operatorname{osc}_{s',a',m}(g, \mathfrak{J}_j) \\ > \frac{1}{2} \left(\sum_j v_j^r \right)^{1/r} \left[\sum_j (2^{-j(1/p-1)} \operatorname{osc}_{s',a',m}(g, \mathfrak{J}_j))^{r'} \right]^{1/r'} \geq C \left(\sum_j v_j^r \right)^{1/r} MO_{s',r'}^{1/p-1,a',m}(g).$$

Let $\lambda_{ij} = v_j \mu_{ij}$. We put together (12.8), (12.9) and (12.10), and we have

$$(12.11) \quad L \left(\sum_{ij} \lambda_{ij} a_{ij} \right) \geq C \sum_j v_j 2^{j(1-1/p)} \sum_l \mu_{lj} \left(\frac{1}{|I_{lj}|} \int_{I_{lj}} |g(x) - g_{I_{lj}}^{(m)}(x)|^{a'} dx \right)^{1/a'} \\ \geq C \sum_j v_j 2^{j(1-1/p)} \operatorname{osc}_{s',a',m}(g, \mathfrak{J}_j) \\ \geq C \left(\sum_j v_j^r \right)^{1/r} MO_{s',r'}^{1/p-1,a',m}(g) \\ = C \|\lambda\|_{sr} MO_{s',r'}^{1/p-1,a',m}(g) \\ \geq CH_{s,r}^{p,q,m} \left(\sum \lambda_{ij} a_{ij} \right) MO_{s',r'}^{1/p-1,a',m}(g).$$

This shows that $g \in MO_{s',r'}^{1/p-1,a',m}$.

To deal with the case $q = \infty$, we use the result for the case $q < \infty$, together with Corollary (11.27): we take an exponent q larger than both p and 1, but finite. Then we know that $H_{s,r}^{p,\infty,m} = H_{s,r}^{p,q,m}$, so that if L is a continuous linear functional on $H_{s,r}^{p,\infty,m}$, there is a function g in $MO_{s',r'}^{1/p-1,a',m}$ which represents L .

The proof is complete if we observe that $MO_{s',r'}^{1/p-1,a',m}$ is contained continuously in $MO_{s',r'}^{1/p-1,1,m}$.

REMARK. It is not difficult to prove that the norm of L as a functional on $H_{s,r}^{p,q,m}$ in terms of the representing function $g \in MO_{s',r'}^{1/p-1,a',m}$ is given by the expression

$$(12.12) \quad \left\{ \sum_j \left[2^{-j(1/p-1)} \left(\sum_l \inf \left(\frac{1}{|I_{lj}|} \int_{I_{lj}} |g(x) - P(x)|^{a'} dx \right)^{s'/a'} \right)^{1/s'} r' \right] \right\}^{1/r'}$$

where the infima are taken over all polynomials of degree less than or equal to m .

REMARK. If either s or r is infinity, one can consider the closed subspace of $H_{s,r}^{p,q,m}$ built up with the same atoms, but with sequences $\{\lambda_{ij}\}$ satisfying a $o(1)$ -condition in the appropriate index. It can be shown that these spaces do not depend on q or m , if these exponents are in the admissible range, and that their dual space is $MO_{s',r'}^{1/p-1,a',m}$.

The last remark allows us to state the following theorem also for s and r equal to one:

(12.13) THEOREM. For fixed α, s, r , the spaces $MO_{sr}^{\alpha, a, m}$ all coincide, provided $1/q + \alpha > 0$ and $m \geq [\alpha + 1/s]$.

This is an obvious consequence of Theorem (12.2).

We want to make a connection between Theorem (12.2) and the duality Theorem (8.2), or, speaking more properly, with its analogue for the \mathcal{A}_{sr}^β spaces. For convenience, we state that result here explicitly.

(12.14) THEOREM. Let m be a positive integer, $m > \max\{\beta + 1/s - 1, \beta\}$. The continuous linear functionals on \mathcal{A}_{sr}^β , $0 < s, r < \infty$, are in one-to-one correspondence with the functions in $\mathcal{A}_{s'r'}^{m-\beta-1/s+1}$, if $0 < s \leq 1$, or $\mathcal{A}_{s'r'}^{m-\beta}$ if $1 < s < \infty$, by means of the duality

$$(12.15) \quad Lf = \int_0^\infty \int_{-\infty}^{+\infty} y^m f(x + iy)g(x + iy) \frac{dy}{y}.$$

The norm of L as a linear functional is equivalent to the norm of g in $\mathcal{A}_{s'r'}^{m-\beta-1/s+1}$, or $\mathcal{A}_{s'r'}^{m-\beta}$ respectively.

This theorem, together with Theorem (12.2) and the fact that H_{sr}^p and \mathcal{A}_{sr}^β can be identified, for $\beta = 1/p - 1/s$, by means of the Poisson integral, shows that there is a one-to-one correspondence between $MO_{s'r'}^{1/p-1}$ and $\mathcal{A}_{s'r'}^{m-\beta-1/s+1}$ (resp. $\mathcal{A}_{s'r'}^{m-\beta}$). We give an explicit description of this correspondence.

(12.16) LEMMA. Let $1 \leq s, r < \infty$ and g be in MO_{sr}^α . Let κ be an integer, $\kappa \geq [\alpha + 1/s + 1]$. The convolution of g with the derivative of order κ in y of the Poisson kernel $P_\nu(x)$,

$$g^{(\kappa)}(x, y) = g * \left(\frac{\partial^\kappa P_\nu}{\partial y^\kappa} \right) (x)$$

is in $\mathcal{A}_{sr}^{\kappa-\alpha-1/s}$ and $N_{sr}^{\kappa-\alpha-1/s}(g^{(\kappa)}) \leq CMO_{sr}^\alpha(g)$.

PROOF. First of all observe that $g^{(\kappa)}$ is well defined, since $\partial^\kappa P_\nu / \partial y^\kappa$, as a function of x , is in $H_{s'r'}^{(\alpha+1)^{-1}}$.

Consider the partition $\{Q_{lj}\}$ of \mathbb{R}_+^2 , where Q_{lj} is the square with vertices $l2^j + i2^j$, $(l+1)2^j + i2^j$, $l2^j + i2^{j+1}$, $(l+1)2^j + i2^{j+1}$, and assume that

$x + iy \in Q_{l^j}$. We have

$$(12.17) \quad g^{(\kappa)}(x, y) = \int_{-\infty}^{+\infty} g(t) \frac{\partial^\kappa P}{\partial y^\kappa}(x-t, y) dt = \int_{-\infty}^{+\infty} (g(t) - g_{l^j}^{(m)}(t)) \frac{\partial^\kappa P}{\partial y^\kappa}(x-t, y) dt$$

where $m = [\alpha + 1/s]$, since $\partial^\kappa P / \partial y^\kappa$ has vanishing moments up to the order $\kappa - 1$.

It follows from (12.17) that

$$(12.18) \quad \sup_{x+iy \in Q_{l^j}} |g^{(\kappa)}(x, y)| \leq \int_{-\infty}^{+\infty} |g(t) - g_{l^j}^{(m)}(t)| \sup_{x+iy \in Q_{l^j}} \left| \frac{\partial^\kappa P}{\partial y^\kappa}(x-t, y) \right| dt.$$

The estimate we need at this point is

$$(12.19) \quad \left| \frac{\partial^\kappa P}{\partial y^\kappa}(x, y) \right| \leq C \frac{y}{(|x| + y)^{\kappa+2}}$$

which can be obtained as we obtained (11.15). Therefore

$$\sup_{x+iy \in Q_{l^j}} |g^{(\kappa)}(x, y)| \leq C \int_{-\infty}^{+\infty} |g(t) - g_{l^j}^{(m)}(t)| \frac{2^j}{(|t - l2^j| + 2^j)^{\kappa+2}} dt.$$

Using Lemma (9.12) and Lemma (6.3), we have

$$\begin{aligned} N_{rs}^{\kappa-\alpha-1/s}(g^{(\kappa)}) &\leq C \left\| \left\{ 2^{j(\kappa-\alpha)} \sup_{x+iy \in Q_{l^j}} |g^{(\kappa)}(x, y)| \right\} \right\|_{sr} \\ &\leq C \left\| \left\{ \int_{-\infty}^{+\infty} |g(t) - g_{l^j}^{(m)}(t)| \frac{2^{j(\kappa-\alpha+1)}}{(|t - l2^j| + 2^j)^{\kappa+2}} dt \right\} \right\|_{sr} \\ &\leq CMO_{sr}^\kappa(g). \end{aligned}$$

This immediately gives us the following identity:

(12.20) LEMMA. Let $f \in H_{sr}^p$ and $g \in MO_{s',r'}^{1/p-1}$. Then

$$(12.21) \quad \langle f, g \rangle = C_\kappa \int_0^\infty \int_{-\infty}^{+\infty} y^\kappa f(x, y) \frac{\partial^\kappa g}{\partial y^\kappa}(x, y) dx \frac{dy}{y}$$

for every $\kappa \geq [1/p + 1/s']$, and the integral converges absolutely.

PROOF. The last statement is a consequence of Theorem (8.2), Theorem (10.23) and Lemma (12.16).

In order to prove the identity (12.21) write

$$(12.22) \quad f(x, y) = \sum_{ik} \lambda_{ik}^k (\eta_{ik}^k)^{\kappa+1-\beta-1/s} \frac{\partial^\kappa P}{\partial y^\kappa} (x - \xi_{ik}^k, y + \eta_{ik}^k)$$

as in Theorem (1.10). Since

$$(12.23) \quad \langle f, g \rangle = \sum_{ik} \lambda_{ik}^k (\eta_{ik}^k)^{\kappa+1-\beta-1/s} \int_{-\infty}^{+\infty} \frac{\partial^\kappa P}{\partial y^\kappa} (x - \xi_{ik}^k, \eta_{ik}^k) g(x) dx,$$

and one can interchange the sum in (12.22) with the integral in (12.21), it is enough to prove (12.21) when $f(x) = (\partial^\kappa P / \partial y^\kappa)(x - \xi, \eta)$. In this case

$$\begin{aligned} \langle f, g \rangle &= \int_{-\infty}^{+\infty} g(x) \frac{\partial^\kappa P}{\partial y^\kappa} (x - \xi, \eta) dx = \frac{\partial^\kappa g}{\partial y^\kappa} (\xi, \eta) \\ &= C_\kappa \int_0^\infty \int_{-\infty}^{+\infty} f(x + iy) y^\kappa \frac{\partial^\kappa P}{\partial y^\kappa} (x - \xi, y + \eta) dx \frac{dy}{y}, \end{aligned}$$

by Theorem (3.1).

The final result is the following:

(12.24) THEOREM. *The linear map T_κ which assigns to every $g \in MO_{sr}^\alpha$ the function $g^{(\kappa)} \in \mathcal{A}_{sr}^{\kappa-\alpha-1/s}$, $\kappa \geq [\alpha + 1/s + 1]$, is an isomorphism between the two spaces.*

PROOF. By (12.23), if $g^{(\kappa)} = 0$, then $g = 0$. To show that T_κ is onto, let $\psi \in \mathcal{A}_{sr}^{\kappa-\alpha-1/s}$. Take any $f \in H_{s',r'}^{(\alpha+1)^{-1}}$, so that the Poisson integral of f is in $\mathcal{A}_{s',r'}^{\alpha+1/s}$. The map

$$f \rightarrow c_\kappa \int_0^\infty \int_{-\infty}^{+\infty} y^\kappa f(x + iy) \psi(x + iy) dx \frac{dy}{y}$$

is a continuous linear functional on $H_{s',r'}^{(\alpha+1)^{-1}}$, so there is $g \in MO_{sr}^\alpha$ such that

$$\langle f, g \rangle = c_\kappa \int_0^\infty \int_{-\infty}^{+\infty} y^\kappa f(x + iy) \psi(x + iy) dx \frac{dy}{y}.$$

By Lemma (12.20) and the fact that T_κ is one-to-one it follows that $\psi = T_\kappa g$.

The description of the homogeneous Besov spaces given by B. H. Qui [11] provides the following consequence:

(12.25) COROLLARY. *The space MO_{sr}^α coincides with the homogeneous Besov-Lipschitz space $A_{sr}^{\alpha+1/s}$.*

REFERENCES

- [1] S. CAMPANATO, *Proprietà di hólderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa, (3) **17** (1963), pp. 175-188.
- [2] R. R. COIFMAN, *A real variable characterization of H^p* , Studia Math., **51** (1974), pp. 269-274.
- [3] R. R. COIFMAN - R. ROCHBERG, *Representation theorems for holomorphic and harmonic functions in L^p* , Astérisque, **77** (1980), pp. 12-66.
- [4] R. R. COIFMAN - G. WEISS, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83** (1977), pp. 569-645.
- [5] C. FEFFERMAN - E. M. STEIN, *H^p spaces of several variables*, Acta Math., **129** (1972), pp. 137-193.
- [6] T. M. FLETT, *On the rate of growth of mean values of holomorphic and harmonic functions*, Proc. London Math. Soc., **20** (1970), pp. 749-768.
- [7] C. S. HERZ, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech., **18** (1968), pp. 283-324.
- [8] F. JOHN - L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), pp. 415-426.
- [9] R. JOHNSON, *Temperatures, Riesz potentials, and the Lipschitz spaces of Herz*, Proc. London Math. Soc., **27** (1973), pp. 290-316.
- [10] R. LATTE, *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math., **62** (1978), pp. 93-101.
- [11] B. H. QUI, *Hermonic functions, Riesz potentials and the Lipschitz spaces of Herz*, Hiroshima Math. J., **9** (1979), pp. 245-295.
- [12] E. M. STEIN - G. WEISS, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, 1971.
- [13] M. STOLL, *Mean value theorems for harmonic and holomorphic functions on bounded symmetric domains*, J. Reine Angew. Math., **290** (1977), pp. 191-198.
- [14] M. TAIBLESON - G. WEISS, *The molecular characterization of certain Hardy spaces*, Astérisque 77 (1980), pp. 68-149.

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino

Department of Mathematics
Washington University
Saint Louis, Missouri 63130