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Ginzburg-Landau Vortices for Thin Ferromagnetic Films

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1 Introduction

In this paper, we consider ferromagnetic bodies, represented by a bounded, open domain $\Omega \subset \mathbb{R}^3$. The magnetization of Ω is described by a vector field $m : \Omega \to \mathbb{R}^3$ which satisfies the saturation constraint $|m| = 1$ almost everywhere. In the absence of an external magnetic field, and with the contribution of a crystalline anisotropy neglected, the energy of this configuration, as derived in the theory of micromagnetics, is given by the expression

$$\mathcal{E}(m) = \frac{\epsilon^2}{2} \int_{\Omega} |\nabla m|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx, \tag{1.1}$$

where $u \in H^1(\mathbb{R}^3)$ is determined by the static Maxwell equations, written in the form

$$\Delta u = \operatorname{div} m \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

Here m is extended by 0 outside of Ω.

The first term on the right-hand side of (1.1), called the exchange energy, penalizes spatial variations of m. It models the tendency for parallel alignment of the magnetization vectors of the underlying atomic structure. The parameter ϵ is a material constant. The second term is the so-called magnetostatic energy. It corresponds to the energy of the magnetic field induced by m. For more details, see, for example, Hubert and Schäfer [15].

Our aim is to study minimizers of $\mathcal{E}$ for ferromagnetic samples in the shape of very thin films. That is, we assume that Ω is of the form $\Omega = \Omega' \times (0, \delta)$ for a small

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number $\delta > 0$. We want to find the limiting behaviour for this variational problem in a special asymptotic regime, defined by certain relations between the thickness δ of the film, the length scale L of the cross section Ω', and the parameter ϵ. Namely, we study the limit $\delta/L \to 0$ under the condition $\epsilon^2/L\delta = 1$. With respect to polynomial order, this is the border case of the situation studied by DeSimone, Kohn, Müller, and Otto [8]. In that paper, the limiting behaviour for $\delta/L \to 0$ and $(\epsilon^2/L\delta)\log(L/\delta) \to 0$ was established.

For simplicity, we set $L = 1$ in the rest of the paper. Thus the condition above yields $\delta = \epsilon^2$. We assume that $\Omega' \subset \mathbb{R}^2$ is a bounded, open, simply connected domain with smooth boundary. For $0 < \epsilon \leq 1$, we define $\Omega_\epsilon = \Omega' \times (0, \epsilon^2)$. For a vector field $m \in L^2(\Omega_\epsilon, \mathbb{R}^3)$, we denote by $u_\epsilon(m)$ the unique distributional solution of equation (1.2) for $\Omega = \Omega_\epsilon$ in the space $H^1(\mathbb{R}^3)$. That is, $u_\epsilon(m) \in H^1(\mathbb{R}^3)$ is to satisfy

$$\int_{\mathbb{R}^3} \nabla u_\epsilon(m) \cdot \nabla \phi \, dx = \int_{\Omega_\epsilon} m \cdot \nabla \phi \, dx \tag{1.3}$$

for all $\phi \in C_0^\infty(\mathbb{R}^3)$. For $k \geq 1$, let $\mathbb{S}^k$ denote the unit sphere in $\mathbb{R}^{k+1}$. Divide $\mathcal{E}$ by ϵ^4 to obtain the functionals

$$E_\epsilon(m) = \frac{1}{2\epsilon^2}\left(\int_{\Omega_\epsilon} |\nabla m|^2 dx + \frac{1}{\epsilon^2}\int_{\mathbb{R}^3} \left|\nabla u_\epsilon(m)\right|^2 dx\right) \tag{1.4}$$

on the spaces

$$H^1\left(\Omega_\epsilon, \mathbb{S}^2\right) = \left\{m \in H^1\left(\Omega_\epsilon, \mathbb{R}^3\right) : |m| = 1 \text{ almost everywhere}\right\}. \tag{1.5}$$

Note that one of the properties of the magnetostatic energy is that it favours a magnetization which is tangential on the boundary $\partial\Omega_\epsilon$. Thus for minimizers of E_ϵ, the third component of m tends to be small on the surfaces $\Omega' \times \{0, \epsilon^2\}$ (cf. Section 2).

We now consider the limit $\epsilon \searrow 0$. We first note that we have necessarily

$$\lim_{\epsilon \searrow 0} E_\epsilon\left(m_\epsilon\right) = \infty \tag{1.6}$$

for any choice of $m_\epsilon \in H^1(\Omega_\epsilon, \mathbb{S}^2)$. Indeed, suppose this were not true. Then one could find a sequence $\epsilon_k \searrow 0$ such that the maps

$$\bar{m}_k(x') = \frac{1}{\epsilon_k^2}\int_0^{\epsilon_k^2} m_{\epsilon_k}(x', s)\, ds, \quad x' \in \Omega', \tag{1.7}$$

would converge weakly in $H^1(\Omega', \mathbb{R}^3)$. For the limit map $m \in H^1(\Omega', \mathbb{S}^2)$, write $m = (m', m^3)$, where $m' \in H^1(\Omega', \mathbb{R}^2)$ and $m^3 \in H^1(\Omega')$. Then it must satisfy $|m'| = 1$ and $m^3 = 0$

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almost everywhere in Ω', and $m' \cdot \nu' = 0$ almost everywhere on $\partial\Omega'$, where ν' is the outer normal vector to $\partial\Omega'$. (The arguments to prove this are given in the proof of Proposition 4.2.) But there is no map in $H^1(\Omega', \mathbb{R}^3)$ with these properties, hence (1.6) holds true. This rules out the "naive" approach of trying to establish weak H^1-convergence for minimizers of E_ϵ, or even Γ-convergence of the functionals.

What kind of limiting behaviour can one expect instead for $\epsilon \searrow 0$? Consider for the moment a simplification of E_ϵ. Assume that the magnetization $m = (m', m^3)$ is independent of the third argument, and model the penalization of m^3 by the L^2-norm (instead of the magnetostatic energy). Owing to the constraint $|m| = 1$ almost everywhere, this leads to the functionals

$$F_\epsilon(m) = \frac{1}{2}\int_{\Omega'} \left(|\nabla' m|^2 + \frac{1 - |m'|^2}{\epsilon^2} \right) dx', \quad m = \left(m', m^3\right) \in H^1\left(\Omega', \mathbb{S}^2\right), \tag{1.8}$$

where $\nabla' = (\partial/\partial x^1, \partial/\partial x^2)$. This, on the other hand, is reminiscent of the Ginzburg-Landau functionals

$$I_\epsilon(f) = \frac{1}{2}\int_{\Omega'} \left(|\nabla' f|^2 + \frac{1}{2\epsilon^2}\left(|f|^2 - 1\right)^2 \right) dx', \quad f \in H^1\left(\Omega', \mathbb{R}^2\right). \tag{1.9}$$

The limiting problem for $\epsilon \searrow 0$ for minimizers of I_ϵ was first studied by Bethuel, Brezis, and Hélein [3, 4], and by many other authors since then. One of the main results (which was proven in [4] for star-shaped domains, and extended by Struwe [25, 26] to arbitrary bounded domains with smooth boundaries) can be summarized as follows. Suppose that for $0 < \epsilon \leq 1$, certain maps $f_\epsilon \in H^1(\Omega', \mathbb{R}^2)$ are given, which minimize I_ϵ for fixed Dirichlet boundary data $g : \partial\Omega' \to \mathbb{S}^1$. Then there exist finitely many points $x'_1, \ldots, x'_N \in \Omega'$ (their number depending on the topological degree of g) and a sequence $\epsilon_k \searrow 0$, such that the sequence $\{f_{\epsilon_k}\}$ converges in $C^\infty_{loc}(\Omega' \backslash \{x'_1, \ldots, x'_N\}, \mathbb{R}^2)$ to a harmonic map $f : \Omega \backslash \{x'_1, \ldots, x'_N\} \to \mathbb{S}^1$. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we can write f in the form

$$f(z) = \left(\prod_{j=1}^{N} \frac{z - z_j}{|z - z_j|} \right) e^{i\theta(z)}, \tag{1.10}$$

or the complex conjugate of this, where $z_j = x_j^1 + i x_j^2$ for $x'_j = (x_j^1, x_j^2)$. The function θ satisfies $\Delta'\theta = 0$ in Ω', where Δ' is the Laplace operator in $\mathbb{R}^2$. This (and more) has been generalized to the corresponding problem for the functionals F_ϵ by André and Shafrir [1] and Hang and Lin [11].

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Our aim is to prove a similar result for minimizers of E_ϵ. For technical reasons, we impose Dirichlet boundary data on $\partial\Omega' \times (0, \epsilon^2)$. It turns out (cf. Proposition 4.2) that only two choices for the boundary data are reasonable, namely,

$$m = \left(-\nu^2, \nu^1, 0\right) \quad \text{on } \partial\Omega' \times \left(0, \epsilon^2\right) \tag{1.11}$$

(where we write $\nu' = (\nu^1, \nu^2)$ for the normal vector to $\partial\Omega'$), and the same with ν' replaced by $-\nu'$. Moreover, the second case is reduced to the first one by reflection. Thus, we define $\bar{H}^1(\Omega_\epsilon, \mathbb{S}^2)$ to be the space of all maps $m \in H^1(\Omega_\epsilon, \mathbb{S}^2)$ satisfying (1.11), and consider only maps therein. For every $\epsilon \in (0, 1]$, we fix a map m_ϵ which minimizes E_ϵ in $\bar{H}^1(\Omega_\epsilon, \mathbb{S}^2)$.

The Euler-Lagrange equation for this variational problem is

$$\epsilon^2 \left(\Delta m_\epsilon + \left|\nabla m_\epsilon\right|^2 m_\epsilon\right) - \nabla u_\epsilon\left(m_\epsilon\right) + \left(m_\epsilon \cdot \nabla u_\epsilon\left(m_\epsilon\right)\right) m_\epsilon = 0 \quad \text{in } \Omega_\epsilon, \tag{1.12}$$

and we have the homogeneous Neumann boundary conditions

$$\frac{\partial m_\epsilon}{\partial x^3} = 0 \quad \text{on } \Omega' \times \left\{0, \epsilon^2\right\}. \tag{1.13}$$

There exists another form of (1.12) which will prove useful. Namely, denoting by $\wedge$ the exterior product $\wedge : \mathbb{R}^n \times \mathbb{R}^n \to \Lambda_2 \mathbb{R}^n$, it is easily checked that (1.12) is equivalent to

$$\epsilon^2 \operatorname{div}\left(m_\epsilon \wedge \nabla m_\epsilon\right) = m_\epsilon \wedge \nabla u_\epsilon\left(m_\epsilon\right) \quad \text{in } \Omega_\epsilon. \tag{1.14}$$

Both (1.12) and (1.14) are to be understood in the distribution sense.

Before we state our first main result, we define the operator which is to play the role of a limit of $\epsilon^{-2} u_\epsilon$ for $\epsilon \searrow 0$. Suppose $m' \in W^{1,4/3}(\Omega', \mathbb{R}^2)$ is a map with the property $m' \cdot \nu' = 0$ almost everywhere on $\partial\Omega'$. Then for any $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} \left| \int_{\Omega'} m'(x') \cdot \nabla' \phi(x', 0)\, dx' \right| &= \left| \int_{\Omega'} \operatorname{div}' m'(x') \left(\phi(x', 0) - \fint_{\Omega'} \phi(y', 0)\, dy' \right) dx' \right| \\ &\leq C \| \operatorname{div}' m' \|_{L^{4/3}(\Omega')} \| \nabla \phi \|_{L^2(\mathbb{R}^3)} \end{aligned} \tag{1.15}$$

for a constant $C = C(\Omega')$, owing to the continuity of the trace operator $T : H^1(\Omega_1) \to L^4(\Omega')$, which is given by $Tv(x') = v(x', 0)$. (Here div' denotes the divergence in $\mathbb{R}^2$.) Hence there exists a unique function $u(m') \in H^1_{loc}(\mathbb{R}^3)$ with $\|\nabla u(m')\|_{L^2(\mathbb{R}^3)} + \|u(m')\|_{L^6(\mathbb{R}^3)} < \infty$,

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such that

$$\int_{\Omega'} m'(x')\cdot\nabla'\phi(x',0)dx' = \int_{\mathbb{R}^3} \nabla u(m')\cdot\nabla\phi\, dx \tag{1.16}$$

for every $\phi \in C_0^\infty(\mathbb{R}^3)$. We furthermore define $u'(m') = Tu(m')$. By standard results from the theory of singular integrals (see [24]), it follows that $u'(m') \in W^{1,4/3}_{\text{loc}}(\Omega')$.

We have the following result (cf. [4, 11, 25, 26]).

Theorem 1.1. (i) There exist a sequence $\epsilon_k \searrow 0$ and a point $x_0' \in \Omega'$, such that the maps $\bar{m}_k$, defined as in (1.7), converge weakly in $H^1_{\text{loc}}(\overline{\Omega'}\backslash\{x_0'\},\mathbb{R}^3)$, and weakly in $W^{1,p}(\Omega',\mathbb{R}^3)$ for any $p < 2$, to a map of the form $\bar{m} = (m', 0)$ with $|m'| = 1$ almost everywhere.

(ii) The limit map m' satisfies the equations

$$\operatorname{div}'(m'\wedge\nabla'm') = m'\wedge\nabla'u'(m') \quad \text{in } \Omega', \tag{1.17}$$

$$\Delta'm' + |\nabla'm'|^2m' - \nabla'u'(m') + \big(m'\cdot\nabla'u'(m')\big)m' = 0 \quad \text{in } \Omega'\backslash\{x_0'\} \tag{1.18}$$

in the distribution sense.

(iii) If $\mathbb{R}^2$ is identified with $\mathbb{C}$, then m' is of the form

$$m'(z) = \frac{z-z_0}{|z-z_0|}e^{i\theta(z)}, \quad z\in\Omega'\backslash\{z_0\}, \tag{1.19}$$

where $z_0 = x_0^1 + ix_0^2$ for $x_0' = (x_0^1, x_0^2)$ and $\theta : \Omega' \to \mathbb{R}$ is a solution of

$$\Delta'\theta = m'\wedge\nabla'u'(m') \quad \text{in } \Omega'. \tag{1.20}$$

□

The proof of Theorem 1.1 will follow roughly the outline of the arguments in [4], and it will also use some arguments from [11]. The problem considered here has a few additional difficulties however. For instance, the nonlinear constraint $|m| = 1$ almost everywhere generates nonlinearities in the Euler-Lagrange equation which involve first derivatives. It has been shown in [11] how this problem itself may be overcome; but in conjunction with the fact that the Ω_ϵ's are three-dimensional domains, the situation is even more difficult. We cannot expect that minimizers of E_ϵ are smooth here (cf. Brezis, Coron, and Lieb [6], and Lin [17]), and in particular we do not have certain pointwise estimates for the gradient, as we have in two dimensions. For variational problems of this kind, regularity can usually be obtained only if the energy is small. But we have seen in (1.6) that this is not the case if ϵ becomes small. What we will prove instead is that suitable estimates for the gradients hold except in small, controllable sets.

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Another difference to the situation of [11] is the fact that the functionals E_ϵ contain the nonlocal operator u_ϵ. However, it turns out that this only causes minor difficulties for this problem.

The result of Theorem 1.1 has the disadvantage that it requires Dirichlet boundary data on $\partial\Omega' \times (0, \epsilon^2)$. It would be more natural to consider minimizers of E_ϵ among all maps in $H^1(\Omega_\epsilon, \mathbb{S}^2)$. However, we need the boundary conditions for technical reasons. To obtain an idea of the thin film limiting behaviour for free boundary data, we consider a model problem, based on a generalization of the Ginzburg-Landau functionals I_ϵ, in Section 5. We will find a similar result as Theorem 1.1, but instead of one vortex in the interior of the domain Ω', we will rather have two "half-vortices" at the boundary.

Vortices at the boundary have also been studied by Kurzke [16] for a slightly different model (with the Ginzburg-Landau penalizing term replaced by a constraint). Similar results as those presented in Section 5 are proven in Kurzke's work, among other things.

Notation. As we have already done above, we will systematically mark objects belonging to $\mathbb{R}^2$ with a prime to distinguish them clearly from their three-dimensional equivalents.

For $x_0' \in \mathbb{R}^2$ and $r > 0$, we write $B_r'(x_0')$ for the open ball in $\mathbb{R}^2$ with centre x_0' and radius r. Moreover, we define $D_r'(x_0') = \Omega' \cap B_r'(x_0')$ and $D_{r,\epsilon}(x_0') = D_r'(x_0') \times (0, \epsilon^2)$ for $\epsilon \in (0, 1]$.

2 Preliminaries

In this section, we will prove certain estimates that will be needed later. In particular, we will find an upper bound for the terms in $E_\epsilon(m_\epsilon)$ of the type as expected from the theory of [3, 4]. Moreover, we will obtain certain relations between the magnetostatic energy and the L^2-norm of the third component of the magnetization.

Lemma 2.1. Suppose that c is the smallest constant satisfying the inequality

$$\left\|v(\cdot, 0)\right\|_{L^4(\Omega')} \leq c\|\nabla v\|_{L^2(\mathbb{R}^3)} \tag{2.1}$$

for all $v \in H^1(\mathbb{R}^3)$. (Such a constant exists by the trace theorem for Sobolev spaces.) Then for any $\epsilon \in (0, 1]$, any map $m = (m', m^3) \in \bar{H}^1(\Omega_\epsilon, \mathbb{S}^2)$ satisfies the inequality

$$\left\|\nabla u_\epsilon(m)\right\|_{L^2(\mathbb{R}^3)} \leq c\Big(4\sqrt{\epsilon}\|\nabla m\|_{L^{4/3}(\Omega_\epsilon)} + 2\left\|m^3(\cdot, 0)\right\|_{L^{4/3}(\Omega')}\Big). \tag{2.2}$$

□

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Proof. Note that

$$\begin{aligned}
\int_{\mathbb{R}^3} \left|\nabla u_\epsilon(m)\right|^2 dx
&= \int_{\Omega_\epsilon} m \cdot \nabla u_\epsilon(m) dx \\
&= \int_{\Omega' \times \{\epsilon^2\}} m^3 u_\epsilon(m) dx' - \int_{\Omega' \times \{0\}} m^3 u_\epsilon(m) dx' - \int_{\Omega_\epsilon} \operatorname{div} m u_\epsilon(m) dx \qquad (2.3)\\
&\le c \left\|\nabla u_\epsilon(m)\right\|_{L^2(\mathbb{R}^3)} \Big(\left\|m^3(\cdot, 0)\right\|_{L^{4/3}(\Omega')} + \left\|m^3(\cdot, \epsilon^2)\right\|_{L^{4/3}(\Omega')} \\
&\qquad + 3 \int_0^{\epsilon^2} \left\|\nabla m(\cdot, s)\right\|_{L^{4/3}(\Omega')} ds \Big).
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\epsilon^2} \left\|\nabla m(\cdot, s)\right\|_{L^{4/3}(\Omega')} ds \le \sqrt{\epsilon} \|\nabla m\|_{L^{4/3}(\Omega_\epsilon)}, \\
&\left\|m^3(\cdot, \epsilon^2) - m^3(\cdot, 0)\right\|_{L^{4/3}(\Omega')}^{4/3} = \int_{\Omega'} \left| \int_0^{\epsilon^2} \frac{\partial m^3}{\partial x^3}(x', s) ds \right|^{4/3} dx' \qquad (2.4)\\
&\qquad \le \epsilon^{2/3} \|\nabla m\|_{L^{4/3}(\Omega_\epsilon)}^{4/3}
\end{aligned}$$

by the Hölder inequality. The claim now follows immediately. ■

Lemma 2.2. There exists a constant C, depending only on Ω', such that

$$E_\epsilon\left(m_\epsilon\right) \le C - \pi \log \epsilon \qquad (2.5)$$

for any $\epsilon \in (0, 1]$. □

Proof. Since m_ϵ is E_ϵ-minimizing, it suffices to construct any map which satisfies the inequality. We assume for simplicity that the closed unit ball $\overline{B_1'(0)}$ is contained in Ω'. (Otherwise we scale and translate everything.)

Choose a map $n_0 \in H^1(B_1'(0), \mathbb{S}^2)$ with

$$n_0\left(x^1, x^2\right) = \left(-x^2, x^1, 0\right) \quad \text{on } \partial B_1'(0), \qquad (2.6)$$

and another map $n_1' \in H^1(\Omega' \backslash B_1'(0), \mathbb{S}^1)$ such that $n_1' = (-\nu^2, \nu^1)$ on $\partial\Omega'$ and $n_1'(x^1, x^2) = (-x^2, x^1)$ on $\partial B_1'(0)$. Now define

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$$n_\epsilon(x^1,x^2,x^3) = \begin{cases} (n_1'(x^1,x^2),0), & \text{if } (x^1,x^2) \in \Omega'\backslash B_1'(0), \\ \dfrac{(-x^2,x^1,0)}{\sqrt{(x^1)^2+(x^2)^2}}, & \text{if } (x^1,x^2) \in B_1'(0)\backslash B_\epsilon'(0), \\ n_0\left(\dfrac{x^1}{\epsilon},\dfrac{x^2}{\epsilon}\right), & \text{if } (x^1,x^2) \in B_\epsilon'(0). \end{cases} \tag{2.7}$$

It is readily checked that

$$\begin{aligned} &\int_{\Omega_\epsilon} |\nabla n_\epsilon|^2 dx \le (C_1 - 2\pi \log \epsilon)\epsilon^2, \\ &\int_{\Omega_\epsilon} |\nabla n_\epsilon|^{4/3} dx \le C_2 \epsilon^2, \end{aligned} \tag{2.8}$$

for constants C_1 and C_2 which depend only on Ω' and the choice of n_0 and n_1'. Write $n_\epsilon = (n_\epsilon', n_\epsilon^3)$. Then Lemma 2.1 implies that

$$\|\nabla u_\epsilon(n_\epsilon',0)\|_{L^2(\mathbb{R}^3)} \le C_3\epsilon^2, \tag{2.9}$$

where $C_3 = C_3(\Omega', n_0, n_1')$. Finally, we have

$$\begin{aligned} \|\nabla u_\epsilon(n_\epsilon) - \nabla u_\epsilon(n_\epsilon',0)\|_{L^2(\mathbb{R}^3)} &= \|\nabla u_\epsilon(0,n_\epsilon^3)\|_{L^2(\mathbb{R}^3)} \\ &\le \|(0,n_\epsilon^3)\|_{L^2(\Omega_\epsilon)} \\ &\le \sqrt{\pi}\epsilon^2, \end{aligned} \tag{2.10}$$

because n_ϵ^3 is supported in $\overline{D_{\epsilon,\epsilon}(0)}$. Combining these inequalities, the lemma is proven. ■

Lemma 2.3. For $\epsilon \in (0,1]$, suppose that $m = (m', m^3) \in H^1(\Omega_\epsilon, \mathbb{R}^3)$ is a map which satisfies $m^3 = 0$ almost everywhere on $\partial\Omega' \times (0,\epsilon^2)$. Then

$$\int_{\Omega_\epsilon} (m^3)^2 dx \le (1+\epsilon^2)\epsilon^2 \int_{\Omega_\epsilon} |\nabla m^3|^2 dx + 2\int_{\mathbb{R}^3} |\nabla u_\epsilon(m)|^2 dx + \epsilon^4|\Omega'|. \tag{2.11}$$

□

Proof. The basic idea for the following argument is due to Gioia and James [10].

Define the function $\phi \in H^1(\mathbb{R}^3)$ by $\phi \equiv 0$ in $(\mathbb{R}^2\backslash\Omega') \times \mathbb{R}$, and

$$\phi(x',x^3) = \begin{cases} 0, & \text{if } x^3 \le 0, \\ \displaystyle\int_0^{x^3} m^3(x',s)ds, & \text{if } 0 < x^3 \le \epsilon^2, \\ \left(2 - \dfrac{x^3}{\epsilon^2}\right)\displaystyle\int_0^{\epsilon^2} m^3(x',s)ds, & \text{if } \epsilon^2 < x^3 \le 2\epsilon^2, \\ 0, & \text{if } x^3 > 2\epsilon^2, \end{cases} \tag{2.12}$$

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for $x' \in \Omega'$. Then we have

$$\left|\nabla'\phi(x', x^3)\right| \le \int_0^{\epsilon^2} \left|\nabla' m^3(x', s)\right| ds \tag{2.13}$$

in $\Omega' \times (0, 2\epsilon^2)$, and $\nabla'\phi = 0$ elsewhere. Thus

$$\int_{\Omega_\epsilon} |\nabla'\phi|^2 dx \le \epsilon^4 \int_{\Omega_\epsilon} \left|\nabla m^3\right|^2 dx. \tag{2.14}$$

Furthermore, since $\partial\phi/\partial x^3 = m^3$ in $\Omega' \times (0, \epsilon^2)$,

$$\frac{\partial\phi}{\partial x^3}(x', x^3) = -\frac{1}{\epsilon^2}\int_0^{\epsilon^2} m^3(x', s) ds \quad \text{in } \Omega' \times (\epsilon^2, 2\epsilon^2), \tag{2.15}$$

and $\partial\phi/\partial x^3 = 0$ elsewhere, we have

$$\int_{\mathbb{R}^3} |\nabla\phi|^2 dx \le 2\int_{\Omega_\epsilon} \left(\epsilon^4\left|\nabla m^3\right|^2 + \left(m^3\right)^2\right) dx. \tag{2.16}$$

Testing (1.3) with ϕ yields

$$\begin{aligned}\int_{\Omega_\epsilon} \left(m^3\right)^2 dx &= \int_{\Omega_\epsilon} m \cdot \nabla\phi \, dx - \int_{\Omega_\epsilon} m' \cdot \nabla'\phi \, dx \\ &= \int_{\mathbb{R}^3} \nabla u_\epsilon(m) \cdot \nabla\phi \, dx - \int_{\Omega_\epsilon} m' \cdot \nabla'\phi \, dx \\ &\le \int_{\mathbb{R}^3} \left|\nabla u_\epsilon(m)\right|^2 dx + \frac{1}{2}\int_{\Omega_\epsilon} \left(\epsilon^4\left|\nabla m^3\right|^2 + \left(m^3\right)^2\right) dx \\ &\quad + \frac{\epsilon^4}{2}|\Omega'| + \frac{\epsilon^2}{2}\int_{\Omega_\epsilon} \left|\nabla m^3\right|^2 dx,\end{aligned} \tag{2.17}$$

and the lemma follows. ■

Lemma 2.4. There exists a constant C, depending only on Ω', such that for $0 < \epsilon \le 1$, the inequality

$$\frac{1}{\epsilon^2}\int_{\Omega_\epsilon} \left(\left|\nabla m_\epsilon^3\right|^2 + \left|\frac{\partial m_\epsilon}{\partial x^3}\right|^2 + \frac{\left(m_\epsilon^3\right)^2}{\epsilon^2}\right) dx + \frac{1}{\epsilon^4}\int_{\mathbb{R}^3} \left|\nabla u_\epsilon(m_\epsilon)\right|^2 dx \le C \tag{2.18}$$

is satisfied. □

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Proof. We combine an argument from [11] with Lemmas 2.2 and 2.3.

For almost every $x^3 \in (0, \epsilon^2)$, we have

$$\begin{aligned}
\int_{\Omega'} &\left(|\nabla' m_\epsilon'(x', x^3)|^2 + \frac{1}{4\epsilon^2} \left(m_\epsilon^3(x', x^3)\right)^2 \right) dx' \\
&= \int_{\Omega'} \left(|\nabla' m_\epsilon'(x', x^3)|^2 + \frac{1}{4\epsilon^2} \left(1 - |m_\epsilon'(x', x^3)|^2\right) \right) dx' \\
&\geq \int_{\Omega'} \left(|\nabla' m_\epsilon'(x', x^3)|^2 + \frac{1}{4\epsilon^2} \left(1 - |m_\epsilon'(x', x^3)|^2\right)^2 \right) dx' \\
&\geq -2\pi \log \epsilon - C_1
\end{aligned} \tag{2.19}$$

for a constant $C_1 = C_1(\Omega')$. For the last step, we have used results from [25, 26]. Together with Lemma 2.2, this yields

$$\begin{aligned}
\frac{1}{\epsilon^2} \int_{\Omega_\epsilon} &\left(|\nabla m_\epsilon^3|^2 + \left| \frac{\partial m_\epsilon}{\partial x^3} \right|^2 - \frac{(m_\epsilon^3)^2}{4\epsilon^2} \right) dx + \frac{1}{\epsilon^4} \int_{\mathbb{R}^3} |\nabla u_\epsilon(m_\epsilon)|^2 dx \\
&= 2E_\epsilon(m_\epsilon) - \frac{1}{\epsilon^2} \int_{\Omega_\epsilon} \left(|\nabla' m_\epsilon'|^2 + \frac{1}{4\epsilon^2} (m_\epsilon^3)^2 \right) dx \\
&\leq C_2 = C_2(\Omega').
\end{aligned} \tag{2.20}$$

Finally, we use Lemma 2.3 to end the proof. ■

3 Regularity and a gradient estimate

We now want to find a pointwise estimate for ∇m_ϵ of the form $|\nabla m_\epsilon| \leq C/\epsilon$ for an appropriate constant C. Such estimates were used, for example, in [3] or [11]. As pointed out in the introduction, however, this can only be expected to be true under additional assumptions.

First, we observe that regularity and a gradient estimate are implied by a small energy condition.

Lemma 3.1. There exist constants $\epsilon_0, \lambda_0 > 0$, depending only on Ω', such that for any $\epsilon \in (0, \epsilon_0]$ and for any $x_0' \in \Omega'$ and $r \geq \epsilon^2$ with the property

$$\frac{1}{\epsilon^2} \int_{D_{r,\epsilon}(x_0')} \left(|\nabla m_\epsilon|^2 + \frac{r^2}{\epsilon^4} |\nabla u_\epsilon(m_\epsilon)|^2 \right) dx \leq \lambda_0, \tag{3.1}$$

the map m_ϵ is smooth in $D_{r/2,\epsilon}(x_0')$ and ∇m_ϵ is continuous in $\overline{D_{r/2,\epsilon}(x_0')}$. □

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Proof. In the case $B'_r(x'_0) \subset \Omega'$, the Hölder continuity of m_ϵ in $\overline{D_{r/2,\epsilon}(x'_0)}$ was proven in [18, Proposition 2.1]. Higher regularity then follows by well-known arguments (cf. Borchers and Garber [5], and Simon [23]). If $B'_r(x'_0) \not\subset \Omega'$, it is not difficult to modify the arguments so that they prove the claim also in this situation (combining them, e.g., with methods from Schoen and Uhlenbeck [22]).

For an alternative proof, different arguments to prove regularity for minima of functionals of the form of E_ϵ can be found in papers of Hardt and Kinderlehrer [12] and Carbou [7]. (If they are to be applied here, however, they first have to be adapted to the situation of thin films.) All of these arguments use well-known methods from the regularity theory for harmonic maps (cf. Schoen and Uhlenbeck [21, 22], Hélein [13, 14], Evans [9], and Bethuel [2]). ■

Lemma 3.2. There exist numbers $\epsilon_1, \lambda_1, c_1 > 0$, depending only on Ω', with the following property. Suppose that for $\epsilon \in (0, \epsilon_1]$, there exist a point $x'_0 \in \Omega'$ and a radius $r \in [\epsilon^2, \epsilon]$, such that

$$\frac{1}{\epsilon^2} \int_{D_{2r,\epsilon}(x'_0)} \left(|\nabla m_\epsilon|^2 + \frac{r^2}{\epsilon^4} |\nabla u_\epsilon(m_\epsilon)|^2 \right) dx \le \lambda_1. \tag{3.2}$$

Then

$$\sup_{D_{r/2,\epsilon}(x'_0)} |\nabla m_\epsilon| \le \frac{c_1}{r}. \tag{3.3}$$

□

Proof. We use a modified version of an argument due to Schoen [20].

Assume the statement is false. Then there exist a sequence $\epsilon_k \searrow 0$ and minimizers $m_k \in \bar{H}^1(\Omega_{\epsilon_k}, \mathbb{S}^2)$ of E_{ϵ_k} such that

$$m_k \in C^1(\overline{D_{r_k,\epsilon_k}(x'_k)}, \mathbb{S}^2) \tag{3.4}$$

(cf. Lemma 3.1) for certain points $x'_k \in \Omega'$ and certain numbers $r_k \in [\epsilon_k^2, \epsilon_k]$, and

$$\frac{1}{\epsilon_k^2} \int_{D_{2r_k,\epsilon_k}(x'_k)} \left(|\nabla m_k|^2 + \frac{r_k^2}{\epsilon_k^4} |\nabla u_{\epsilon_k}(m_k)|^2 \right) dx =: \mu_k \longrightarrow 0, \tag{3.5}$$

but

$$\sup_{D_{r_k/2,\epsilon_k}(x'_k)} |\nabla m_k| > \frac{d_k}{r_k}, \tag{3.6}$$

where $d_k \to \infty$ for $k \to \infty$.

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For each k, set

$$\begin{aligned} \Phi_k(\sigma) &= \big(r_k - \sigma\big)^2 \sup_{D_{\sigma,\epsilon_k}(x_k')} \big|\nabla m_k\big|^2, \quad 0 < \sigma \le r_k, \\ \Phi_k(0) &= r_k^2 \sup_{0<s<\epsilon_k^2} \big|\nabla m_k\big(x_k', s\big)\big|^2. \end{aligned} \tag{3.7}$$

Choose $\rho_k \in [0, r_k)$ such that

$$\Phi_k\big(\rho_k\big) = \max_{0\le\sigma\le r_k} \Phi_k(\sigma). \tag{3.8}$$

Moreover, choose $y_k = (y_k', y_k^3) \in \overline{D_{\rho_k,\epsilon_k}(x_k')}$ with the property

$$\big|\nabla m_k\big(y_k\big)\big| = \sup_{D_{\rho_k,\epsilon_k}(x_k')} \big|\nabla m_k\big| \tag{3.9}$$

if $\rho_k > 0$, and $y_k = (y_k', y_k^3) \in \{x_k'\} \times [0, \epsilon_k^2]$ with

$$\big|\nabla m_k\big(y_k\big)\big| = \sup_{\{x_k'\}\times(0,\epsilon_k^2)} \big|\nabla m_k\big| \tag{3.10}$$

if $\rho_k = 0$. Set $e_k = |\nabla m_k(y_k)|$. Note that

$$d_k^2 < 4\Phi\left(\frac{r_k}{2}\right) \le 4\Phi_k\big(\rho_k\big) = 4\big(r_k - \rho_k\big)^2 e_k^2, \tag{3.11}$$

that is, $e_k^{-1} < 2((r_k - \rho_k)/d_k)$. The rescaled maps

$$\widehat{m}_k(x) = m_k\left(\frac{x}{e_k} + y_k\right) \tag{3.12}$$

are thus defined and smooth at least in the set

$$D_k = \big(B_{d_k/4}'(0) \cap \Omega_k'\big) \times \big(-e_k y_k^3, e_k\big(\epsilon_k^2 - y_k^3\big)\big), \tag{3.13}$$

where $\Omega_k' = e_k(\Omega' - y_k')$. Moreover, they have the properties

$$\big|\nabla \widehat{m}_k(0)\big| = 1, \tag{3.14}$$

$$\begin{aligned} \sup_{D_k} \big|\nabla \widehat{m}_k\big|^2 &\le e_k^{-2} \sup_{D_{(r_k+\rho_k)/2,\epsilon_k}(x_k')} \big|\nabla m_k\big|^2 \\ &\le \frac{4}{e_k^2\big(r_k - \rho_k\big)^2} \Phi_k\left(\frac{r_k + \rho_k}{2}\right) \\ &\le \frac{4}{e_k^2\big(r_k - \rho_k\big)^2} \Phi\big(\rho_k\big) = 4. \end{aligned} \tag{3.15}$$

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We have

$$\int_{\Omega_{\epsilon_k}\cap B_s(x_1)} |\nabla m_k|^2 dx \le C_1(\mu_k + r_k)\min\{s, \epsilon_k^2\} \tag{3.16}$$

for all $x_1 \in D_{r_k,\epsilon_k}(x_k')$ and $s \le r_k$, for a constant $C_1 = C_1(\Omega')$. This is proven in [18, Lemma 2.2] for the case $B_{2r_k}(x_k') \subset \Omega'$. If $B_{2r_k}(x_k')$ intersects the boundary of Ω', then one can use the same arguments, combined with methods from [22], to prove inequality (3.3). In particular, we have

$$\int_{D_k\cap B_1(0)} |\nabla \widehat{m}_k|^2 dx \le C_1(\mu_k + \epsilon_k)\min\{1, e_k\epsilon_k^2\}. \tag{3.17}$$

Remember that m_k satisfies the equation

$$\epsilon_k^2\left(\Delta m_k + |\nabla m_k|^2 m_k\right) = \nabla u_{\epsilon_k}(m_k) - \left(m_k \cdot \nabla u_{\epsilon_k}(m_k)\right)m_k \quad \text{in } \Omega_{\epsilon_k}. \tag{3.18}$$

Let $\widehat{v}_k \in H^1(\mathbb{R}^3)$ be the unique solutions of

$$\Delta \widehat{v}_k = \operatorname{div} \widehat{m}_k. \tag{3.19}$$

Then it follows that $\widehat{m}_k$ satisfies

$$e_k^2\epsilon_k^2\left(\Delta \widehat{m}_k + |\nabla \widehat{m}_k|^2 \widehat{m}_k\right) = \nabla \widehat{v}_k - \left(\widehat{m}_k \cdot \nabla \widehat{v}_k\right)\widehat{m}_k \quad \text{in } D_k. \tag{3.20}$$

Note that $e_k^2\epsilon_k^2 \ge d_k^2/4 \to \infty$ for $k \to \infty$.

By standard estimates, we have $\|\nabla \widehat{v}_k\|_{L^p(B_2(0))} \le C_2 = C_2(p)$ for any $p < \infty$. We conclude that there exist $C_3 = C_3(\Omega')$ and $\gamma = \gamma(\Omega') > 0$, such that

$$\|\nabla \widehat{m}_k\|_{C^{0,\gamma}(B_1(0)\cap D_k)} \le C_3. \tag{3.21}$$

But this is clearly a contradiction to (3.14) and (3.17). ∎

Lemma 3.2 is not yet good enough for our purpose. The next lemma will give an improvement.

Lemma 3.3. For every $C_0 > 0$, there exist numbers $\epsilon_2, \lambda_2, c_2 > 0$, depending only on C_0 and Ω', with the following property. Suppose that for $\epsilon \in (0, \epsilon_2]$, there is a point $x_0' \in \Omega'$, such that ∇m_ϵ is continuous in $\overline{D_{\epsilon,\epsilon}(x_0')}$ and satisfies

$$\sup_{D_{\epsilon,\epsilon}(x_0')} |\nabla m_\epsilon| \le \frac{C_0}{\epsilon^2}, \tag{3.22}$$

$$\frac{1}{\epsilon^2}\int_{D_{\epsilon,\epsilon}(x_0')} |\nabla m_\epsilon^3|^2 dx \le \lambda_2. \tag{3.23}$$

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Then

$$\sup_{D_{\epsilon/2,\epsilon}(x_0')} |\nabla m_\epsilon| \le \frac{c_2}{\epsilon}. \tag{3.24}$$

□

Proof. We use similar arguments as in the proof of Lemma 3.2, and we combine them with arguments due to Hang and Lin [11].

Assume that the statement is false. Then we construct the sequence $\{\widehat{m}_k\}$ as in the proof of Lemma 3.2. In this case, $\widehat{m}_k$ has the properties (3.14), (3.15), (3.21), and

$$\frac{1}{e_k \epsilon_k^2} \int_{D_k} |\nabla \widehat{m}_k^3|^2 dx = \mu_k \longrightarrow 0. \tag{3.25}$$

Furthermore, condition (3.22) guarantees that $e_k \epsilon_k^2 \le C_0$.

We choose a subsequence (without changing notation), such that both $e_k y_k^3$ and $e_k(\epsilon_k^2 - y_k^3)$ converge to a number in $[0, C_0]$. Assume first that $\lim_{k\to\infty} e_k y_k^3 = \lim_{k\to\infty} e_k(\epsilon_k^2 - y_k^3) = 0$. Define the maps

$$\bar{m}_k(x') = \frac{1}{\epsilon_k^2} \int_{-e_k y_k^3}^{e_k(\epsilon_k^2 - y_k^3)} \widehat{m}_k(x', s) ds, \quad x' \in \Omega_k'. \tag{3.26}$$

We may assume that Ω_k' converges to a set $\Sigma' \subset \mathbb{R}^2$ of the form

$$\Sigma' = \{x' \in \mathbb{R}^2 : a' \cdot x' < \alpha\}, \tag{3.27}$$

for some $a' = (a^1, a^2) \in \mathbb{S}^1$ and $0 \le \alpha \le \infty$. Moreover, by (3.21), we may assume that $\bar{m}_k$ converges to a map $\bar{m} : \Sigma' \to \mathbb{S}^2$ in the C^1-sense.

We want to show that $\bar{m}$ is a locally energy minimizing map for the Dirichlet energy, that is, for any ball $\overline{B_R'(x')} \subset \Sigma'$ and any map $\bar{n} \in H^1_{loc}(\Sigma', \mathbb{S}^2)$ with $\bar{n} = \bar{m}$ outside of $B_R'(x')$, we have

$$\int_{B_R'(x')} |\nabla' \bar{n}| dx' \ge \int_{B_R'(x')} |\nabla' \bar{m}| dx'. \tag{3.28}$$

To this end, suppose there existed such a map $\bar{n}$ which did not satisfy (3.28), that is,

$$\int_{B_R'(x')} |\nabla' \bar{n}| dx' \le \int_{B_R'(x')} |\nabla' \bar{m}| dx' - \sigma \tag{3.29}$$

for a positive number σ. Then clearly for any sufficiently large k, one could construct a map $n_k \in H^1(\Omega_{\epsilon_k}, \mathbb{S}^2)$ with $n_k = m_k$ outside of $D'_{R/e_k,\epsilon_k}(x'/e_k + y_k')$, such that

$$\int_{D'_{R/e_k,\epsilon_k}(x'/e_k+y_k')} |\nabla n_k|^2 dx \le \int_{D'_{R/e_k,\epsilon_k}(x'/e_k+y_k')} |\nabla m_k|^2 dx - \frac{\sigma \epsilon_k^2}{2}. \tag{3.30}$$

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Moreover,

$$\begin{aligned}\frac{1}{\epsilon_k^2}\|\nabla u_{\epsilon_k}(m_k)-\nabla u_{\epsilon_k}(n_k)\|_{L^2(\mathbb{R}^3)} &= \frac{1}{\epsilon_k^2}\|\nabla u_{\epsilon_k}(m_k-n_k)\|_{L^2(\mathbb{R}^3)} \\ &\le \frac{1}{\epsilon_k^2}\|m_k-n_k\|_{L^2(\Omega_{\epsilon_k})} \\ &\le \frac{\sqrt{2\pi}R}{e_k\epsilon_k} \le \frac{\sqrt{8\pi}R}{d_k} \longrightarrow 0.\end{aligned} \tag{3.31}$$

This would give a contradiction to the minimality of $E_{\epsilon_k}(m_k)$.

Hence $\bar{m}:\Sigma'\to\mathbb{S}^2$ is a locally energy minimizing map. It satisfies $\nabla\bar{m}^3=0$ and $|\nabla\bar{m}|\le 2$ in Σ', and $|\nabla\bar{m}(0)|=1$. If $\alpha<\infty$ in the representation (3.27) of Σ', then $\bar{m}\equiv(-a^2,a^1,0)$ on $\partial\Sigma'$. All this follows easily from the construction of $\bar{m}$ and inequalities (3.14), (3.15), and (3.25). It is readily concluded that $\bar{m}$ is of the form

$$\bar{m}(x')=\left(e^{i(b'\cdot x'+\beta)},0\right),\quad x'\in\Sigma', \tag{3.32}$$

for some $b'\in\mathbb{S}^1$ and $\beta\in\mathbb{R}$. But Hang and Lin [11] proved that this is not a locally energy minimizing map. Thus in this case, we have a contradiction.

If either $\lim_{k\to\infty} e_k y_k^3>0$ or $\lim_{k\to\infty} e_k(\epsilon_k^2-y_k^3)>0$, we use similar arguments. In this case, a subsequence of $\{\widehat{m}_k\}$ converges to a locally energy minimizing map $\widehat{m}:\Sigma'\times(s,t)\to\mathbb{S}^2$, where Σ' is as before and $s<t$. Moreover, $\partial\widehat{m}/\partial x^3=0$ on $\Sigma'\times\{s,t\}$. We conclude that

$$\widehat{m}(x',x^3)=\left(e^{i(b'\cdot x'+\beta)},0\right),\quad x'\in\Sigma',\ s<x^3<t, \tag{3.33}$$

as before. Again we can use the arguments of [11] to obtain a contradiction and thus conclude the proof. ■

4 Proof of Theorem 1.1

The following is the key lemma for the proof of Theorem 1.1. It will enable us to apply certain arguments from [4] and from [25, 26].

Lemma 4.1. There exist $\epsilon_3,\lambda_3,c_3>0$, depending only on Ω', such that the following holds true. For $\epsilon\in(0,\epsilon_3]$, suppose there exists $x_0'\in\Omega'$ with the property

$$\int_{D_{2\epsilon,\epsilon}(x_0')}\left(\frac{|\nabla m_\epsilon^3|^2}{\epsilon^2}-\frac{|\nabla m_\epsilon|^2}{\epsilon^2\log\epsilon}+\frac{(m_\epsilon^3)^2+|\nabla u_\epsilon(m_\epsilon)|^2}{\epsilon^4}\right)dx\le\lambda_3. \tag{4.1}$$

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Then m_ϵ is smooth in $D_{\epsilon/2,\epsilon}(x_0')$ with

$$\sup_{D_{\epsilon/2,\epsilon}(x_0')} \left|m_\epsilon^3\right| \le \frac{1}{2}, \tag{4.2}$$

$$\sup_{D_{\epsilon/2,\epsilon}(x_0')} \left|\nabla m_\epsilon\right| \le \frac{c_3}{\epsilon}. \tag{4.3}$$

□

Proof. Choose a number $\gamma \in (1,2)$. We can find a radius $r \in (\epsilon^2, \epsilon^\gamma)$ such that

$$r\int_{(\partial B_r'(x_0')\cap\Omega')\times(0,\epsilon^2)} \left|\nabla m_\epsilon\right|^2 do \le \frac{2\lambda_3\epsilon^2}{2-\gamma}. \tag{4.4}$$

(Otherwise we would have

$$\begin{aligned}\int_{D_{\epsilon,\epsilon}(x_0')} \left|\nabla m_\epsilon\right|^2 dx &\ge \frac{2\lambda_3\epsilon^2}{2-\gamma}\int_{\epsilon^2}^{\epsilon^\gamma}\frac{dr}{r} \\ &= -2\lambda_3\epsilon^2\log\epsilon,\end{aligned} \tag{4.5}$$

in contradiction to (4.1).) Moreover, there exists a number $s \in (0, \epsilon^2)$ such that

$$r\int_{(\partial B_r'(x_0')\cap\Omega')\times\{s\}} \left|\nabla m_\epsilon\right|^2 do' \le \frac{4\lambda_3}{2-\gamma}, \tag{4.6}$$

where do' indicates the arc length measure.

If $\epsilon \le \epsilon_3 \le r_0$ for a certain number r_0 which depends only on Ω', then $\partial B_r'(x_0')\cap\Omega'$ is connected. Hence for $x', y' \in \partial B_r'(x_0')\cap\Omega'$, we have in this case

$$\begin{aligned}\left|m_\epsilon(x',s) - m_\epsilon(y',s)\right| &\le \int_{(\partial D_r'(x_0')\cap\Omega')\times\{s\}} \left|\nabla m_\epsilon\right| do' \\ &\le \sqrt{\frac{8\pi\lambda_3}{2-\gamma}}.\end{aligned} \tag{4.7}$$

If $\lambda_3 \le (2-\gamma)/32\pi$, then the right-hand side is at most $1/2$. If ϵ_3 (and thus r) is also small enough, then $m_\epsilon(\partial D_r'(x_0')\times\{s\})$ is contained in a ball of radius 1. Then it is easy to construct a map $n_\epsilon \in H^1(D_r'(x_0')\times\{s\}, \mathbb{S}^2)$ with $n_\epsilon = m_\epsilon$ on $\partial D_r'(x_0')\times\{s\}$, and

$$\int_{D_r'(x_0')} \left|\nabla' n_\epsilon(x,s)\right|^2 dx' \le C_1\left(\lambda_3 + \epsilon_3\right) \tag{4.8}$$

for a constant $C_1 = C_1(\gamma, \Omega')$. If $B_r'(x_0') \subset \Omega'$, we extend n_ϵ to $D_{r,\epsilon}(x_0')$ by

$$n_\epsilon\left(x', x^3\right) = \begin{cases} n_\epsilon\left(\left(1 - \left|x^3 - s\right|/r\right)^{-1}x', s\right), & \text{if } \left|x^3 - s\right| \le r - |x'|, \\ m_\epsilon\left(rx'/|x'|, x^3 - r + |x'|\right), & \text{if } x^3 > r - |x'| + s, \\ m_\epsilon\left(rx'/|x'|, x^3 + r - |x'|\right), & \text{if } x^3 < s - r + |x'|, \end{cases} \tag{4.9}$$

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and to Ω_ϵ by $n_\epsilon = m_\epsilon$ outside of $D_{r,\epsilon}(x'_0)$. If $B'_r(x'_0) \not\subset \Omega'$, we construct a similar extension. In both cases, we thus find a map $n_\epsilon \in \bar{H}^1(\Omega_\epsilon, \mathbb{S}^2)$ with $n_\epsilon = m_\epsilon$ in $\Omega_\epsilon \backslash D_{r,\epsilon}(x'_0)$, and

$$\int_{D_{r,\epsilon}(x'_0)} |\nabla n_\epsilon|^2 dx \leq C_2(\lambda_3 + \epsilon_3)\epsilon^2 \tag{4.10}$$

for a constant $C_2 = C_2(\gamma, \Omega')$.

Note that

$$\begin{aligned} \|\nabla u_\epsilon(m_\epsilon) - \nabla u_\epsilon(n_\epsilon)\|_{L^2(\mathbb{R}^3)} &= \|\nabla u_\epsilon(m_\epsilon - n_\epsilon)\|_{L^2(\mathbb{R}^3)} \\ &\leq \|m_\epsilon - n_\epsilon\|_{L^2(D_{r,\epsilon}(x'_0))} \\ &\leq \sqrt{2\pi}\epsilon^{1+\gamma}. \end{aligned} \tag{4.11}$$

By the minimizing property of m_ϵ, we have

$$\begin{aligned} &\frac{1}{\epsilon^2}\int_{D_{r,\epsilon}(x'_0)} |\nabla m_\epsilon|^2 dx \\ &\quad \leq \frac{1}{\epsilon^2}\int_{D_{r,\epsilon}(x'_0)} |\nabla n_\epsilon|^2 dx + \frac{1}{\epsilon^4}\left(\|\nabla u_\epsilon(n_\epsilon)\|^2_{L^2(\mathbb{R}^3)} - \|\nabla u_\epsilon(m_\epsilon)\|^2_{L^2(\mathbb{R}^3)}\right) \\ &\quad \leq C_2(\lambda_3 + \epsilon_3) + C_3\epsilon_3^{\gamma-1} \end{aligned} \tag{4.12}$$

for a constant $C_3 = C_3(\Omega')$. For the last step, we have used Lemma 2.4 and inequality (4.11).

If λ_3 and ϵ_3 are sufficiently small, we can now apply Lemma 3.1, and find that m_ϵ is smooth in $\overline{D_{r/2,\epsilon}(x'_0)}$. Lemma 3.2 then even implies that $|\nabla m_\epsilon(x'_0)| \leq 2c_2/r$. Furthermore, we can apply the same arguments for any point $x' \in D_{\epsilon,\epsilon}(x'_0)$ instead of x'_0. Hence m_ϵ is even smooth in $\overline{D_{\epsilon,\epsilon}(x'_0)}$, and $|\nabla m_\epsilon| \leq 2c_2/\epsilon^2$ in this set.

Now, according to Lemma 3.3, we have (4.3) for a constant $c_3 = c_3(\Omega')$ provided that λ_3 and ϵ_3 are chosen appropriately. With this, inequality (4.2) follows easily from the inequality

$$\int_{D_{2\epsilon,\epsilon}(x'_0)} (m^3_\epsilon)^2 dx \leq \lambda_3\epsilon^4, \tag{4.13}$$

if λ_3 is sufficiently small. ■

For the proof of Theorem 1.1, we can now proceed as in [4].

For a fixed $\epsilon \in (0,1]$, cover Ω' with a collection of balls $\{B'_{\epsilon/2}(x'_i)\}_{1\leq i\leq I}$ with the properties $x'_i \in \Omega'$ and

$$B'_{\epsilon/8}(x'_i) \cap B'_{\epsilon/8}(x'_j) = \varnothing \quad \text{for } i \neq j. \tag{4.14}$$

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(For instance, a maximal collection of balls with centres in Ω', such that (4.14) holds, will do.) Consider all balls in this collection which satisfy

$$\int_{D_{2\epsilon,\epsilon}(x_i')} \left(\frac{|\nabla m_\epsilon^3|^2}{\epsilon^2} - \frac{|\nabla m_\epsilon|^2}{\epsilon^2 \log \epsilon} + \frac{(m_\epsilon^3)^2 + |\nabla u_\epsilon(m_\epsilon)|^2}{\epsilon^4} \right) dx > \lambda_3 \tag{4.15}$$

for the constant λ_3 from Lemma 4.1. By Lemmas 2.2 and 2.4, the number of such balls is bounded by a number J which depends only on Ω'. Using Lemma 4.1, we conclude that there exists a constant $R = R(\Omega)$, such that for any sufficiently small ϵ, we can construct a set of points $y_{\epsilon 1}', \ldots, y_{\epsilon J}' \in \Omega'$ with the properties

$$\begin{aligned} &|y_{\epsilon i}' - y_{\epsilon j}'| \geq 8R\epsilon \quad \text{or} \quad y_{\epsilon i} = y_{\epsilon j} \quad \text{for } 1 \leq i, j \leq J, \\ &|m_\epsilon^3| \leq \frac{1}{2}, \quad |\nabla m_\epsilon| \leq \frac{c_3}{\epsilon}, \quad \text{in } \Omega_\epsilon \backslash \left(\bigcup_{i=1}^{J} D_{R\epsilon,\epsilon}(y_{\epsilon i}') \right) \end{aligned} \tag{4.16}$$

for the constant c_3 from Lemma 4.1. Now we pick a sequence $\epsilon_k \searrow 0$, such that for every $i = 1, \ldots, J$, we have

$$y_{\epsilon_k i}' \longrightarrow y_i' \quad (k \longrightarrow \infty) \tag{4.17}$$

for a certain point $y_i' \in \overline{\Omega'}$. Choose $\rho > 0$ such that any two balls $B_\rho'(y_i')$ and $B_\rho'(y_j')$ are disjoint unless $y_i' = y_j'$. If k is sufficiently large, then

$$|m_{\epsilon_k}^3| \leq \frac{1}{2}, \quad |\nabla m_{\epsilon_k}| \leq \frac{c_3}{\epsilon_k}, \quad \text{in } \Omega_{\epsilon_k} \backslash \left(\bigcup_{i=1}^{J} D_{\rho,\epsilon_k}(y_i') \right). \tag{4.18}$$

In particular, for any sufficiently large k, the topological degree of the restriction of m_{ϵ_k} to $\partial D_\rho'(y_i') \times \{s\}$ is well defined for all $i = 1, \ldots, J$ and all $s \in (0, \epsilon_k^2)$, and is independent of s. Clearly it must be nonzero for at least one of the points y_i'. Without loss of generality, we may assume that this point is always the same; we denote it by x_0'. It follows from the arguments in the proofs of [4, Theorem V.2] or [25, Proposition 3.4] (cf. also Proposition 5.6) that

$$\frac{1}{\epsilon_k^2} \int_{D_{\rho,\epsilon_k}(x_0')} |\nabla m_{\epsilon_k}|^2 dx \geq 2\pi \log\left(\frac{\rho}{\epsilon_k}\right) - C_1 \tag{4.19}$$

for a constant C_1 which is independent of k and ρ, provided that k is sufficiently large.

Comparing this with Lemma 2.2, we obtain uniform estimates for

$$\frac{1}{\epsilon_k^2} \int_{\Omega' \backslash D_{\rho,\epsilon_k}(x_0')} |\nabla m_{\epsilon_k}|^2 dx \tag{4.20}$$

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for any $\rho > 0$, and for

$$\frac{1}{\epsilon_k^2}\int_{\Omega'} \left|\nabla m_{\epsilon_k}\right|^p dx \tag{4.21}$$

for any $p \in [1, 2)$. After passing to a subsequence once more, we find a map

$$\bar{m} \in H^1_{loc}(\overline{\Omega'}\backslash\{x_0'\}, \mathbb{S}^2) \cap \bigcap_{1\le p<2} W^{1,p}(\Omega', \mathbb{S}^2), \tag{4.22}$$

which is the limit of the maps $\bar{m}_k$ in the sense specified in Theorem 1.1. Now we use the following result.

Proposition 4.2. For $p > 4/3$ and for a sequence $\epsilon_k \searrow 0$, suppose that $m_k = (m_k^1, m_k^2, m_k^3) \in W^{1,p}(\Omega_{\epsilon_k}, \mathbb{S}^2)$ are distributional solutions of

$$\epsilon_k^2 \operatorname{div}\left(m_k \wedge \nabla m_k\right) = m_k \wedge \nabla u_{\epsilon_k}\left(m_k\right) \quad \text{in } \Omega_{\epsilon_k}, \tag{4.23}$$

satisfying the Neumann boundary conditions $\partial m_k/\partial x^3 = 0$ on $\Omega' \times \{0, \epsilon_k^2\}$. Define $\nu_k = \epsilon_k^{-2} u_{\epsilon_k}(m_k)$ and

$$\bar{m}_k(x') = \frac{1}{\epsilon_k^2}\int_0^{\epsilon_k^2} m_k(x', s)ds, \quad x' \in \Omega'. \tag{4.24}$$

Suppose that

$$\sup_{k\in\mathbb{N}}\left(\frac{1}{\epsilon_k^2}\int_{\Omega_{\epsilon_k}} \left|\nabla m_k\right|^p dx + \frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla \nu_k\right|^2 dx\right) < \infty. \tag{4.25}$$

Then there exist a map $\bar{m} = (m', 0) \in W^{1,p}(\Omega', \mathbb{S}^1 \times \{0\})$ and subsequences $\{\bar{m}_{k_j}\}$ and $\{\nu_{k_j}\}$ such that

$$\begin{aligned} &\bar{m}_{k_j} \rightharpoonup \bar{m} \quad \text{weakly in } W^{1,p}(\Omega', \mathbb{R}^3), \\ &\nabla \nu_{k_j} \rightharpoonup \nabla u(m') \quad \text{weakly in } L^2(\mathbb{R}^3, \mathbb{R}^3). \end{aligned} \tag{4.26}$$

The limit map satisfies $m' \cdot \nu' = 0$ almost everywhere on $\partial\Omega$, and equation (1.17) holds in the distribution sense. □

We postpone the proof and finish first the proof of Theorem 1.1. We now know that $\bar{m}$ is of the form $\bar{m} = (m', 0)$, where m' satisfies (1.17). Then (1.18) follows from (1.17). Moreover, we see that

$$\int_{\Omega'\backslash B_\rho(x_0')} |\nabla m'|^2 dx' \le -2\pi \log \rho + C_2 \tag{4.27}$$

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for a constant C_2 which is independent of ρ. We conclude that $x_0' \in \Omega'$, for otherwise we would have a contradiction to [4, Lemma VI.1]. This proves Theorem 1.1(i) and (ii).

For the proof of (iii), first note that $\bar{m}$ is smooth in $\Omega' \backslash \{x_0'\}$. This follows, for example, from [19, Theorem 1]. A simple generalization of those arguments proves that $\bar{m}$ is continuous in $\overline{\Omega'} \backslash \{x_0'\}$. In particular, there exists a continuous function $\theta : \Omega' \backslash \{x_0'\} \to \mathbb{R}$, such that m' has the representation (1.19), owing to the choice of the boundary data. We compute

$$m'(x') \wedge \nabla' m'(x') = \nabla' \theta(x') + \frac{\left(x_0^2 - x^2, x^1 - x_0^1\right)}{\left|x' - x_0'\right|^2}, \quad x' = \left(x^1, x^2\right) \in \Omega' \backslash \left\{x_0'\right\}. \tag{4.28}$$

The second term on the right-hand side is divergence free in Ω' in the distribution sense. Hence, θ is a distributional solution of (1.20). This completes the proof of Theorem 1.1.

Proof of Proposition 4.2. It is clear that there exist $\bar{m} = (m', m^3) \in W^{1,p}(\Omega', \mathbb{S}^2)$ and $v \in H^1_{loc}(\mathbb{R}^3)$, such that (4.26), for v instead of $u(m')$, hold for a certain subsequence. Since $|m_k| = 1$ almost everywhere, we may assume that $\bar{m}_{k_j} \to \bar{m}$ strongly in $L^q(\Omega', \mathbb{R}^3)$ for any $q < \infty$.

For any $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \nabla v_k \cdot \nabla \phi \, dx = \frac{1}{\epsilon_k^2} \int_{\Omega_{\epsilon_k}} m_k \cdot \nabla \phi \, dx. \tag{4.29}$$

In the limit, this yields

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla v \cdot \nabla \phi \, dx &= \int_{\Omega'} \bar{m}(x') \cdot \nabla \phi(x', 0) dx' \\ &= \int_{\Omega'} m^3(x') \frac{\partial \phi}{\partial x^3}(x, 0) dx' \\ &\quad - \int_{\Omega'} \operatorname{div}' m'(x') \phi(x', 0) dx' \\ &\quad + \int_{\partial \Omega'} \nu'(x') \cdot m'(x') \phi(x', 0) do'(x'). \end{aligned} \tag{4.30}$$

If the third component of $\bar{m}$ did not vanish or if the trace of $\bar{m}$ on $\partial\Omega'$ were not tangential to the boundary of Ω', it would be easy to construct a sequence of test functions such that the left-hand side of (4.30) would be bounded and the right-hand side would diverge. Thus we have $m^3 = 0$ almost everywhere in Ω', and $m' \cdot \nu' = 0$ almost everywhere on $\partial\Omega'$, as the proposition claims. Moreover, we see that $v = u(m')$.

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For $\psi' \in C_0^\infty(\Omega')$, set $\psi(x', x^3) = \psi'(x')$. Test (4.23) with ψ. An integration by parts in the third component yields

$$\int_{\Omega_{\epsilon_k}} \nabla\psi \cdot \left(m_k^1 \nabla m_k^2 - m_k^2 \nabla m_k^1\right) dx + \int_{\Omega_{\epsilon_k}} \psi v_k \left(\frac{\partial m_k^2}{\partial x^1} - \frac{\partial m_k^1}{\partial x^2}\right) dx - \int_{\Omega_{\epsilon_k}} v_k \left(m_k^1 \frac{\partial \psi}{\partial x^2} - m_k^2 \frac{\partial \psi}{\partial x^1}\right) dx = 0. \tag{4.31}$$

We have a continuous embedding $A : H^1(\mathbb{R}^3) \to C^{0,\alpha}([0,1], L^{p/(p-1)}(\Omega'))$ for $\alpha = 1/2 - 2/3p > 0$, given by the mapping

$$(Av)(t) = v(\cdot, t), \quad 0 \le t \le 1. \tag{4.32}$$

Moreover, the trace operator $H^1(\mathbb{R}^3) \to L^{p/(p-1)}(\Omega' \times \{0\})$ is compact, and we may hence assume that $v_{k_j}(\cdot, 0) \to u'(m')$ strongly in $L^{p/(p-1)}(\Omega')$. Hence (4.31) implies in the limit

$$\int_{\Omega'} \left(\nabla'\psi' \cdot (m' \wedge \nabla' m') + \psi' u'(m') \operatorname{curl}' m' + v' \nabla'\psi' \wedge m'\right) dx' = 0, \tag{4.33}$$

where curl' is the curl operator in $\mathbb{R}^2$. Now we can integrate by parts again and find that (1.17) holds true. ■

5 Free boundary data: a model problem

We would like to drop now the Dirichlet boundary conditions in Theorem 1.1, that is, to study the minimizers of E_ϵ among all maps in $H^1(\Omega_\epsilon, \mathbb{S}^2)$. The analysis is more difficult in this situation, however, therefore we consider only a simpler variational problem which may serve as a model for the more complex one.

We have already established certain connections between the magnetostatic energy and the L^2-norm of the third component of the magnetization in the previous sections. We may therefore regard the limiting problem for the functionals F_ϵ defined in the introduction as a model for the corresponding problem for E_ϵ under Dirichlet boundary conditions. The minimizers of F_ϵ, on the other hand, show a similar behaviour as those of the Ginzburg-Landau functionals I_ϵ.

For free boundary data, we need to penalize $m' \cdot \nu'$ on $\partial\Omega' \times (0, \epsilon^2)$ as well. For this purpose, we consider a boundary integral of the form

$$\int_{\partial\Omega' \times (0,\epsilon^2)} (m' \cdot \nu')^2 do. \tag{5.1}$$

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Throughout the rest of this section, we work in two dimensions. Therefore, we drop the prime marking two-dimensional objects. Hence from now on, Ω is a bounded, open, simply connected domain in $\mathbb{R}^2$ with smooth boundary, and $\nu = (\nu^1, \nu^2)$ denotes the outer normal vector to its boundary. We further set $\tau = (\tau^1, \tau^2) = (-\nu^2, \nu^1)$. For $x_0 \in \overline{\Omega}$ and $r > 0$, we denote $D_r(x_0) = \Omega \cap B_r(x_0)$ and $D_r^*(x_0) = \partial\Omega \cap B_r(x_0)$.

For a fixed $\alpha \in (0, 1]$ and for $0 < \epsilon \leq 1$, we consider the functionals

$$J_\epsilon(f) = \frac{1}{2}\int_\Omega \left(|\nabla f|^2 + \frac{1}{2\epsilon^2}\left(|f|^2 - 1\right)^2\right) dx + \frac{1}{2\epsilon^\alpha}\int_{\partial\Omega} (f \cdot \nu)^2 \, do \tag{5.2}$$

on $H^1(\Omega, \mathbb{R}^2)$. For any $\epsilon \in (0, 1]$, we fix a minimizer $f_\epsilon \in H^1(\Omega, \mathbb{R}^2)$ of J_ϵ.

Our aim is to prove a result similar to those in [4, 25, 26] for the functionals J_ϵ in order to obtain an idea of the limiting behaviour for minimizers of E_ϵ without restrictions on the boundary data.

Theorem 5.1. There exist a sequence $\epsilon_k \searrow 0$ and a set $\Sigma \subset \overline{\Omega}$, which is either of the form $\Sigma = \{x_0\}$ for a point $x_0 \in \Omega$, or $\Sigma = \{x_1, x_2\}$ for two different points $x_1, x_2 \in \partial\Omega$, such that $f_{\epsilon_k} \to f$ weakly in $H^1_{\mathrm{loc}}(\overline{\Omega}\backslash\Sigma, \mathbb{R}^2)$ and weakly in $W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p < 2$, where $f : \overline{\Omega}\backslash\Sigma \to \mathbb{S}^1$ is a harmonic map. The case $\Sigma = \{x_0\}$ can only occur if $\alpha = 1$. □

The proof of Theorem 5.1 will follow roughly the outline of the arguments in [25, 26]. First we need an estimate for the energy of f_ϵ.

Lemma 5.2. There exists a constant C, depending only on Ω, such that

$$J_\epsilon(f_\epsilon) \leq C - \alpha\pi \log \epsilon \tag{5.3}$$

for $0 < \epsilon \leq 1$. □

Proof. We assume for simplicity that $\partial\Omega$ contains two points x_1 and x_2, such that

$$\partial\Omega \cap B_1(x_i) = \{x \in B_1(x_i) : (x - x_i) \cdot \nu(x_i) = 0\}, \quad i = 1, 2, \tag{5.4}$$

and $B_2(x_1) \cap B_2(x_2) = \varnothing$. If this is not the case, we may map Ω onto a domain which has this property by a C^2-diffeomorphism. It is then easy to check that the following construction gives rise to a map which satisfies the estimate (5.3).

For $0 < \epsilon \leq 1$, set $x_{i\epsilon} = x_i + \epsilon^\alpha \nu(x_i)$, $i = 1, 2$. Define

$$g_\epsilon(x) = \begin{cases} \dfrac{x - x_{1\epsilon}}{|x - x_{1\epsilon}|}, & \text{if } x \in B_1(x_1), \\ \dfrac{x_{2\epsilon} - x}{|x_{2\epsilon} - x|}, & \text{if } x \in B_1(x_2). \end{cases} \tag{5.5}$$

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This map satisfies

$$\begin{aligned}\int_{\Omega\cap B_1(x_i)} |\nabla g_\epsilon|^2 dx &\le \pi \log\left(\frac{2}{\epsilon^\alpha}\right),\\ \int_{\partial\Omega\cap B_1(x_i)} (g_\epsilon\cdot\nu)^2 do &\le \int_{-1}^{1} \frac{\epsilon^{2\alpha}}{s^2+\epsilon^{2\alpha}} ds\\ &\le \epsilon^\alpha \int_{-\infty}^{\infty} \frac{ds}{s^2+1}\\ &= \pi\epsilon^\alpha\end{aligned} \tag{5.6}$$

for $i = 1, 2$. Obviously g_ϵ can be extended to Ω such that it satisfies (5.3). Hence also f_ϵ satisfies (5.3). ■

Lemma 5.3. The maps f_ϵ are smooth in Ω and satisfy $|f_\epsilon| \le 1$ and $|\nabla f_\epsilon| \le C/\epsilon$ for a constant C which depends only on Ω. □

Proof. The maps f_ϵ satisfy the equations

$$\Delta f_\epsilon = \frac{1}{\epsilon^2}\left(|f_\epsilon|^2 - 1\right) f_\epsilon \quad \text{in } \Omega, \tag{5.7}$$

with the boundary conditions

$$\frac{\partial f_\epsilon}{\partial \nu} = -\frac{1}{\epsilon^\alpha}(f_\epsilon\cdot\nu)\nu \quad \text{on } \partial\Omega. \tag{5.8}$$

The regularity thus follows from standard results in the theory of elliptic equations.

To prove $|f_\epsilon| \le 1$, we apply the maximum principle, similarly as in [3] or [25]. More precisely, for any fixed ϵ, we consider the function $g = |f_\epsilon|^2$ in the set $\Omega^+ = \{x \in \Omega : g(x) > 1\}$. We have

$$\begin{aligned}\Delta g &= \frac{2}{\epsilon^2}(g-1)g + 2|\nabla f_\epsilon|^2 \ge 0 \quad \text{in } \Omega^+,\\ \frac{\partial g}{\partial \nu} &= -\frac{2}{\epsilon^\alpha}(f_\epsilon\cdot\nu)^2 \le 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{5.9}$$

Hence g can take its maximum neither in Ω^+ nor on $\partial\Omega\cap\partial\Omega^+$ unless it is constant in Ω^+. It follows that $g \le 1$ and thus $|f_\epsilon| \le 1$.

For the gradient estimate, we first estimate the Dirichlet energy of f_ϵ on balls of radius ϵ. For a given point $x \in \Omega$, choose a cutoff function $\eta \in C_0^\infty(B_{2\epsilon}(x))$ with the

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properties $0 \le \eta \le 1, \eta \equiv 1$ in $B_\epsilon(x)$, and $|\nabla\eta| \le 2/\epsilon$. We have

$$\begin{aligned}\int_\Omega \eta^2 |\nabla f_\epsilon|^2 dx &= \frac{1}{\epsilon^2}\int_\Omega \eta^2\left(1-|f_\epsilon|^2\right)|f_\epsilon|^2 dx - 2\int_\Omega \eta \frac{\partial\eta}{\partial x^i} f_\epsilon \cdot \frac{\partial f_\epsilon}{\partial x^i} dx \\ &\quad - \frac{1}{\epsilon^\alpha}\int_{\partial\Omega} \eta^2 (f_\epsilon\cdot\nu)^2 do \\ &\le \frac{C_1}{2} + \frac{1}{2}\int_\Omega \eta^2|\nabla f_\epsilon|^2 dx\end{aligned} \tag{5.10}$$

for a constant $C_1 = C_1(\Omega)$. Here and in the following, we use the summation convention, that is, we sum over repeated indices from 1 to 2. We conclude that

$$\int_{B_\epsilon(x)} |\nabla f_\epsilon|^2 dx \le C_1. \tag{5.11}$$

We can now use a blowup argument similar to those in the proofs of Lemmas 3.2 and 3.3. If the estimate was not true, then we could find solutions $f_k \in C^\infty(\Omega, \mathbb{R}^2)$ of (5.7) and (5.8) for certain numbers $\epsilon_k \in (0,1]$, such that certain points $x_k \in \overline{\Omega}$ would exist with the property

$$e_k := |\nabla f_k(x_k)| = \sup_\Omega |\nabla f_k| > \frac{c_k}{\epsilon_k}, \tag{5.12}$$

where $c_k \to \infty$ for $k \to \infty$. Define

$$\widehat{f}_k(x) = f_k\left(\frac{2\sqrt{C_1}x}{e_k} + x_k\right) \tag{5.13}$$

so that $|\nabla\widehat{f}_k(0)| = 2\sqrt{C_1}$ and $|\nabla\widehat{f}_k| \le 2\sqrt{C_1}$ wherever $\widehat{f}_k$ is defined. We see that a subsequence of $\{\widehat{f}_k\}$ converges to a solution $\widehat{f} : \Sigma \to \mathbb{R}^2$ of Laplace's equation $\Delta\widehat{f} = 0$, with either $\Sigma = \mathbb{R}^2$ or $\Sigma = \{x \in \mathbb{R}^2 : a \cdot x > \alpha\}$ for some $a \in \mathbb{S}^1$ and some $\alpha \ge 0$. In the latter case, we have homogeneous Neumann boundary conditions for $\widehat{f}$ on $\partial\Sigma$. Furthermore, we have $|\nabla\widehat{f}(0)| = 2\sqrt{C_1}$, but also

$$\begin{aligned}|\nabla\widehat{f}(0)| &\le \frac{2}{\pi}\int_{\Sigma\cap B_1(0)} |\nabla\widehat{f}| dx \\ &\le \frac{2}{\sqrt{\pi}}\left(\int_{\Sigma\cap B_1(0)} |\nabla\widehat{f}|^2 dx\right)^{1/2} \\ &\le 2\sqrt{\frac{C_1}{\pi}}\end{aligned} \tag{5.14}$$

by the mean value theorem and the energy estimate above. Hence we have a contradiction, and the estimate is proven. ∎

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Lemma 5.4. There exist $C > 0$ and $r_0 > 0$, depending only on Ω, such that for $0 < \epsilon \le 1$, $x_0 \in \partial\Omega$, and $0 < r \le r_0$,

$$\begin{aligned}\frac{1}{2\epsilon^2}\int_{D_r(x_0)}\left(|f_\epsilon|^2-1\right)^2 dx + \frac{1}{\epsilon^\alpha}\int_{D_r^*(x_0)}(f_\epsilon\cdot\nu)^2 do \\ \le Cr\Bigg[\int_{D_r(x_0)}|\nabla f_\epsilon|^2 dx + \int_{\Omega\cap\partial B_r(x_0)}\left(|\nabla f_\epsilon|^2+\frac{1}{2\epsilon^2}\left(|f_\epsilon|^2-1\right)^2\right)do \\ + \frac{1}{\epsilon^\alpha}\sum_{x\in\partial\Omega\cap\partial B_r(x_0)}(f_\epsilon(x)\cdot\nu(x))^2+\frac{r}{\epsilon^\alpha}\Bigg].\end{aligned} \tag{5.15}$$

□

Proof. Let $\psi \in C^\infty(\Omega,\mathbb{R}^2)$ be a vector field which satisfies $\psi\cdot\nu = 0$ on $\partial\Omega$. Consider the 1-parameter family of diffeomorphisms $\Psi_t : \Omega \to \Omega$, obtained as the solution to

$$\frac{\partial\Psi_t}{\partial t} = \psi\circ\Psi_t, \qquad \Psi_0 = \mathrm{id}, \tag{5.16}$$

for t in a neighbourhood of 0. From the condition $d/dt|_{t=0}J_\epsilon(f_\epsilon\circ\Psi_t) = 0$, we derive by an integration by parts

$$\begin{aligned}0 = \int_\Omega\left[\frac{\partial\psi^i}{\partial x^j}\frac{\partial f_\epsilon}{\partial x^i}\cdot\frac{\partial f_\epsilon}{\partial x^j}-\frac{1}{2}\operatorname{div}\psi\left(|\nabla f_\epsilon|^2+\frac{1}{2\epsilon^2}\left(|f_\epsilon|^2-1\right)\right)\right]dx \\ -\frac{1}{\epsilon^\alpha}\int_{\partial\Omega}\left[\frac{1}{2}\tau^i\frac{\partial}{\partial x^i}(\psi\cdot\tau)(f_\epsilon\cdot\nu)^2+\kappa(\psi\cdot\tau)(f_\epsilon\cdot\nu)(f_\epsilon\cdot\tau)\right]do,\end{aligned} \tag{5.17}$$

where $\kappa = \tau^i(\partial\nu/\partial x_i)\cdot\tau$ is the curvature of $\partial\Omega$.

For an appropriate choice of r_0, there exists a vector field $\phi = (\phi^1,\phi^2) \in C^\infty(D_{r_0}(x_0),\mathbb{R}^2)$ with the properties

(i) $\phi\cdot\nu = 0$ on $D^*_{r_0}(x_0)$,
(ii) $|\phi(x)-(x-x_0)| \le C_1|x-x_0|^2$,
(iii) $|(\partial\phi^i/\partial x^j)(x)-\delta_{ij}| \le C_1|x-x_0|$,

for a constant $C_1 = C_1(\Omega)$. Choosing $\psi = \eta\phi$ as a test vector field in (5.17), where $\eta \in C_0^\infty(B_r(x_0))$, we see that

$$\begin{aligned}\frac{1}{4\epsilon^2}\int_\Omega\left(|f_\epsilon|^2-1\right)^2\eta\, dx + \frac{1}{2\epsilon^\alpha}\int_{\partial\Omega}(f_\epsilon\cdot\nu)^2\eta\, do \\ \le C_3 r\left[\int_\Omega\left(|\nabla f_\epsilon|^2+\frac{1}{2\epsilon^2}\left(|f_\epsilon|^2-1\right)^2\right)\eta\, dx+\frac{1}{\epsilon^\alpha}\int_{\partial\Omega}\left((f_\epsilon\cdot\nu)^2+|f_\epsilon\cdot\nu|\right)\eta\, do\right] \\ +\int_\Omega\left[\phi^i\frac{\partial\eta}{\partial x^j}\frac{\partial f_\epsilon}{\partial x^i}\cdot\frac{\partial f_\epsilon}{\partial x^j}-\frac{1}{2}\phi\cdot\nabla\eta\left(|\nabla f_\epsilon|^2+\frac{1}{2\epsilon^2}\left(|f_\epsilon|^2-1\right)^2\right)\right]dx \\ -\frac{1}{\epsilon^\alpha}\int_{\partial\Omega}(\nabla\eta\cdot\tau)(\phi\cdot\tau)(f_\epsilon\cdot\nu)^2 do,\end{aligned} \tag{5.18}$$

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where $C_3 = C_3(\Omega)$. Approximating the characteristic function of $B_r(x_0)$ by η, we conclude that (5.15) holds for a constant $C = C(\Omega)$ provided that $r_0 \le 1/4C_3$. ■

Now, choose two numbers β and γ with $3\alpha/4 \le \beta < \gamma < \alpha$.

Lemma 5.5. There exist constants $\epsilon_0, \lambda, C > 0$, depending only on Ω, β, and γ, such that for any $\epsilon \in (0, \epsilon_0]$ and any $x_0 \in \overline{\Omega}$, the condition

$$\int_{D_{\epsilon^\beta}(x_0)} \left(|\nabla f_\epsilon|^2 + \frac{1}{2\epsilon^2}\left(|f_\epsilon|^2 - 1\right)^2 \right) dx + \frac{1}{\epsilon^\alpha} \int_{D^*_{\epsilon^\beta}(x_0)} (f_\epsilon \cdot \nu)^2 \, do \le -\lambda \log \epsilon \tag{5.19}$$

implies $|f_\epsilon| \ge 1/2$ in $D_{\epsilon^\gamma}(x_0)$, $|f_\epsilon \cdot \nu| \le 1/4$ on $D^*_{\epsilon^\gamma}(x_0)$, and

$$\frac{1}{2\epsilon^2} \int_{D_{\epsilon^\gamma}(x_0)} \left(|f_\epsilon|^2 - 1\right)^2 dx + \frac{1}{\epsilon^\alpha} \int_{D^*_{\epsilon^\gamma}(x_0)} (f_\epsilon \cdot \nu)^2 \, do \le C\left(\lambda + \epsilon^{\alpha/2}\right). \tag{5.20}$$

□

Proof. For $B_{\epsilon^\beta}(x_0) \subset \Omega$, this is proven in [25]. In the other case, we assume for simplicity that $x_0 \in \partial\Omega$. The general case can be reduced to these two special cases.

There exists a radius $r \in (\epsilon^\gamma, \epsilon^\beta)$ with the property

$$r \int_{\Omega \cap \partial B_r(x_0)} \left(|\nabla f_\epsilon|^2 + \frac{1}{2\epsilon^2}\left(|f_\epsilon|^2 - 1\right)^2 \right) do + \frac{r}{\epsilon^\alpha} \sum_{x \in \partial\Omega \cap \partial B_r(x_0)} (f_\epsilon(x) \cdot \nu(x))^2 \le \frac{4\lambda}{\gamma - \beta}. \tag{5.21}$$

Estimate (5.20) then follows from Lemma 5.4.

Recall that $|\nabla f_\epsilon| \le C_1/\epsilon$ for a constant $C_1 = C_1(\Omega)$ by Lemma 5.3. Hence, if we had a point $x \in D_{\epsilon^\gamma}(x_0)$ with $|f_\epsilon(x)| < 1/2$, then we would conclude that $|f_\epsilon| \le 3/4$ in $D_{c\epsilon}(x)$ for $c = 1/4C_1$, and we would thus find a contradiction to (5.20) provided that λ and ϵ_0 are sufficiently small. Hence $|f_\epsilon| \ge 1/2$ in $D_{\epsilon^\gamma}(x_0)$.

We extend ν and τ to $\partial D_r(x_0)$ such that they are normal and tangential, respectively, to that boundary. If ϵ_0 is small enough, then $D_r(x_0)$ is strictly star-shaped in the sense that $(x - x_1) \cdot \nu(x) \ge r/4$ on $\partial D_r(x_0)$ for some point $x_1 \in D_r(x_0)$. Using the Pohožaev identity for solutions of (5.7) (cf. [4, 25]), we obtain

$$\begin{aligned} &\int_{\partial D_r(x_0)} (x - x_1) \cdot \nu \left|\frac{\partial f_\epsilon}{\partial \nu}\right|^2 do + \frac{1}{\epsilon^2} \int_{D_r(x_0)} \left(|f_\epsilon|^2 - 1\right)^2 dx \\ &\quad = \int_{\partial D_r(x_0)} \left((x - x_1) \cdot \nu \left|\frac{\partial f_\epsilon}{\partial \tau}\right|^2 - 2(x - x_1) \cdot \tau \frac{\partial f_\epsilon}{\partial \nu} \cdot \frac{\partial f_\epsilon}{\partial \tau} \right) do. \end{aligned} \tag{5.22}$$

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Note that we have proven (5.20) actually for the radius r instead of ϵ^γ. Combining the identity above with this version of (5.20) and with (5.21), we conclude that

$$\int_{D_r^*(x_0)} \left|\frac{\partial f_\epsilon}{\partial \tau}\right|^2 do \leq C_2 \left[\int_{D_r^*(x_0)} \left|\frac{\partial f_\epsilon}{\partial \nu}\right|^2 do + \lambda + \epsilon^{\alpha/2}\right] \tag{5.23}$$

for a constant $C_2 = C_2(\Omega, \beta, \gamma)$. By the boundary conditions (5.8), we even have

$$\begin{aligned} \int_{D_r^*(x_0)} \left|\frac{\partial f_\epsilon}{\partial \tau}\right|^2 do &\leq C_2 \left[\frac{1}{\epsilon^{2\alpha}} \int_{D_r^*(x_0)} \left(f_\epsilon \cdot \nu\right)^2 do + \lambda + \epsilon^{\alpha/2}\right] \\ &\leq \frac{C_3}{\epsilon^\alpha}, \end{aligned} \tag{5.24}$$

where $C_3 = C_3(\Omega, \beta, \gamma)$. Here we have used again the version of (5.20) for the radius r. For $x, y \in D_r^*(x_0)$, it follows that

$$\left|f_\epsilon(x) - f_\epsilon(y)\right| \leq C_4 \sqrt{|x - y|}\epsilon^{-\alpha/2}, \tag{5.25}$$

where $C_4 = C_4(\Omega, \beta, \gamma)$. The estimate for $|f_\epsilon \cdot \nu|$ on $D_{\epsilon^\gamma}^*(x_0)$ is now proven similarly as the one for $|f_\epsilon|$ in $D_{\epsilon^\gamma}(x_0)$. ■

Now choose a number $r_0 > 0$, such that for each $x_0 \in \overline{\Omega}$ and every $r \in (0, r_0]$, the sets $D_r(x_0)$ and $D_r^*(x_0)$ are connected. For $0 < r \leq R \leq r_0$ and $x_0 \in \overline{\Omega}$, define

$$\begin{aligned} A_{r,R}(x_0) &= \Omega \cap B_R(x_0) \backslash B_r(x_0), \\ A_{r,R}^*(x_0) &= \partial\Omega \cap B_R(x_0) \backslash B_r(x_0). \end{aligned} \tag{5.26}$$

Suppose that a continuous map $f : \overline{\Omega} \to \mathbb{R}^2$ is given such that $|f| \geq 1/2$ in $A_{r,R}(x_0)$ and $|f \cdot \nu| \leq 1/4$ on $A_{r,R}^*(x_0)$. These conditions imply in particular $|f \cdot \tau| \geq \sqrt{3}/4$ on $A_{r,R}^*(x_0)$. Hence the sign of $f \cdot \tau$ is constant on each connected component of $A_{r,R}^*(x_0)$ (of which there are exactly two). In the following, when we say that $f \cdot \tau$ changes sign in $D_r^*(x_0)$, we mean that it takes both signs on $A_{r,R}^*(x_0)$. If it does not change sign, we may extend the map $g = f|_{\Omega \cap \partial B_R(x_0)}$ to $\partial D_R(x_0)$ in such a way that $|g| \geq 1/2$ and $|g \cdot \nu| \leq 1/4$ hold also on $D_R^*(x_0)$. We say that g is topologically nontrivial if the topological degree of this extension (which maps $\partial D_R(x_0) \cong \mathbb{S}^1$ to $\mathbb{R}^2 \backslash B_{1/2}(0)$) is nonzero.

The following is a generalization of [26, Proposition 3.4′].

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Proposition 5.6. For $x_0 \in \overline{\Omega}$ and $0 < r < R \le r_0$, suppose that $f \in C^1(\overline{\Omega}, \mathbb{R}^2)$ satisfies $1/2 \le |f| \le 1$ in $A_{r,R}(x_0)$ and $|f \cdot \nu| \le 1/4$ on $A^*_{r,R}(x_0)$. Suppose furthermore that

$$\begin{aligned} &J_\epsilon(f) \le K(1 - \log \epsilon), \\ &\frac{1}{2\epsilon^2} \int_{D_{\epsilon^\beta}(x_0)} \left(|f|^2 - 1\right)^2 dx + \frac{1}{\epsilon^\alpha} \int_{D^*_{\epsilon^\beta}(x_0)} (f \cdot \nu)^2 do \le K, \end{aligned} \tag{5.27}$$

for some number K. There exists a constant C, depending only on Ω, β, and K, such that the following is true.

(i) Suppose $B_R(x_0) \subset \Omega$ and $r \ge \epsilon$. If the topological degree of the restriction of f to $\partial B_R(x_0)$ is not 0, then

$$\int_{A_{r,R}(x_0)} |\nabla f|^2 dx \ge 2\pi \log\left(\frac{R}{r}\right) - C. \tag{5.28}$$

(ii) Suppose $x_0 \in \partial\Omega$ and $r \ge \epsilon^\alpha$. If $f \cdot \tau$ changes sign in $D^*_r(x_0)$, then

$$\int_{A_{r,R}(x_0)} |\nabla f|^2 dx \ge \pi \log\left(\frac{R}{r}\right) - C. \tag{5.29}$$

(iii) Suppose $x_0 \in \partial\Omega$ and $r \ge \epsilon^\alpha$. If $f \cdot \tau$ does not change sign in $D^*_r(x_0)$ and if $f|_{\Omega \cap \partial B_R(x_0)}$ is topologically nontrivial, then

$$\int_{A_{r,R}(x_0)} |\nabla f|^2 dx \ge 4\pi \log\left(\frac{R}{r}\right) - C. \tag{5.30}$$

□

Proof. We only give a proof for (ii). Part (i) is proven in [25, 26], and the proof of (iii) is very similar to the proof of (ii). The following arguments are for the most part the same as in [25, 26].

We assume for simplicity that $x_0 = 0$ and $\nu(0) = (0, -1)$. Using polar coordinates $x = \rho e^{i\theta}$, we can write

$$f(x) = \sigma(x) e^{i(\theta + \phi(x))}, \tag{5.31}$$

where $\sigma, \phi \in C^1(A_{r,R}(0))$ with $1/2 \le \sigma \le 1$. We can choose ϕ such that either $|\phi(x)| \le C_1(|f(x) \cdot \nu(x)| + \rho)$ or $|\phi(x) - \pi| \le C_1(|f(x) \cdot \nu(x)| + \rho)$ on $A^*_{r,R}(0)$ for a constant $C_1 = C_1(\Omega)$.

Note that

$$\begin{aligned} |\nabla f|^2 &\ge \sigma^2 |\nabla\theta + \nabla\phi|^2 \\ &= \frac{\sigma^2}{\rho^2}\left(1 + 2\frac{\partial\phi}{\partial\theta}\right) + \sigma^2 |\nabla\phi|^2. \end{aligned} \tag{5.32}$$

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Furthermore,

$$\begin{aligned}\int_{A_{r,R}(0)} \frac{\sigma^2}{\rho^2}\,dx &= \int_{A_{r,R}(0)} \frac{1}{\rho^2}\,dx + \int_{A_{r,R}(0)} \frac{\sigma^2-1}{\rho^2}\,dx \\ &\geq \pi \log(R/r) - C_2 + \int_{A_{r,R}(0)} \frac{\sigma^2-1}{\rho^2}\,dx\end{aligned} \tag{5.33}$$

for a constant $C_2 = C_2(\Omega)$. Note that for every $\rho \in [r, R]$, we have

$$-\int_{\Omega\cap\partial B_\rho(0)} \frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\,d\theta \leq C_1\left(\sum_{x\in\partial\Omega\cap\partial B_\rho(0)} \left|f(x)\cdot\nu(x)\right| + 2\rho\right). \tag{5.34}$$

Thus

$$2\int_{A_{r,R}(0)} \frac{\sigma^2}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx \geq 2\int_{A_{r,R}(0)} \frac{\sigma^2-1}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx - 2C_1\int_{A^*_{r,R}(0)}\left(\frac{|f\cdot\nu|}{\rho}+1\right)do. \tag{5.35}$$

We write

$$\int_{A_{r,R}(0)} \frac{\sigma^2-1}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx = \int_{A_{r,\epsilon^\beta}(0)} \frac{\sigma^2-1}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx + \int_{A_{\epsilon^\beta,R}(0)} \frac{\sigma^2-1}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx \tag{5.36}$$

(provided that $r < \epsilon^\beta < R$; otherwise we consider only one of these terms) and we estimate

$$\begin{aligned}\left|\int_{A_{r,\epsilon^\beta}(0)} \frac{\sigma^2-1}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx\right| &\leq \frac{1}{\epsilon}\left(\int_{A_{r,\epsilon^\beta}(0)} \left(\sigma^2-1\right)^2 dx\right)^{1/2}\left(\int_{A_{r,\epsilon^\beta}(0)} |\nabla\phi|^2\,dx\right)^{1/2} \\ &\leq 4K + \frac{1}{8}\int_{A_{r,\epsilon^\beta}(0)} |\nabla\phi|^2\,dx, \\ \left|\int_{A_{\epsilon^\beta,R}(0)} \frac{\sigma^2-1}{\rho^2}\frac{\partial\phi}{\partial\theta}\,dx\right| &\leq \frac{1}{\epsilon^\beta}\left(\int_{A_{\epsilon^\beta,R}(0)} \left(\sigma^2-1\right)^2 dx\right)^{1/2}\left(\int_{A_{\epsilon^\beta,R}(0)} |\nabla\phi|^2\,dx\right)^{1/2} \\ &\leq 8K\epsilon^{2-2\beta}(1-\log\epsilon) + \frac{1}{8}\int_{A_{\epsilon^\beta,R}(0)} |\nabla\phi|^2\,dx.\end{aligned} \tag{5.37}$$

Similarly, we prove

$$\int_{A_{r,R}(0)} \frac{1-\sigma^2}{\rho^2}\,dx + \int_{A^*_{r,R}(0)}\left(\frac{|f\cdot\nu|}{\rho}+1\right)do \leq C_3(\Omega, K, \beta). \tag{5.38}$$

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Finally,

$$\int_{A_{r,R}(0)} \sigma^2 |\nabla\phi|^2 dx \geq \frac{1}{4}\int_{A_{r,R}(0)} |\nabla\phi|^2 dx. \tag{5.39}$$

To complete the proof, we only need to combine these estimates. ■

Proof of Theorem 5.1. For each $\epsilon \in (0,1]$, define the set

$$S_\epsilon = \left\{x \in \overline{\Omega} : \left|f_\epsilon(x)\right| < \frac{1}{2}\right\} \cup \left\{x \in \partial\Omega : \left|f_\epsilon \cdot \nu\right| > \frac{1}{4}\right\}. \tag{5.40}$$

Choose a maximal collection of balls $B_m = B_{\epsilon^\beta}(x_m)$, $m = 1, \ldots, M$, such that $x_m \in S_\epsilon$ and $B_{\epsilon^\beta/4}(x_l) \cap B_{\epsilon^\beta/4}(x_m) = \varnothing$ for $l \neq m$. Then obviously this collection covers S_ϵ. Moreover, Lemmas 5.2 and 5.5 imply that M is bounded by a number which is independent of ϵ. For each m, we use the arguments in the proof of Lemma 5.5 to show that

$$\frac{1}{2\epsilon^2}\int_{D_{2\epsilon^\beta}(x_m)} \left(\left|f_\epsilon\right|^2 - 1\right)^2 dx + \frac{1}{\epsilon^\alpha}\int_{D^*_{2\epsilon^\beta}(x_m)} (f_\epsilon \cdot \nu)^2 do \leq C_1, \tag{5.41}$$

where $C_1 = C_1(\Omega, \beta)$.

With the arguments from [4, 25, 26] (i.e., similarly as in the proof of Theorem 1.1), combined with the arguments from the proof of Lemma 5.5, we can now find numbers $R > 0$ and $N \in \mathbb{N}$, which are independent of ϵ, and points

$$y_{\epsilon 1}, \ldots, y_{\epsilon N} \in \Omega \cap \overline{\bigcup_{m=1}^N B_m} \tag{5.42}$$

such that

$$\begin{aligned} &\left|y_{\epsilon i} - y_{\epsilon j}\right| \geq 8R\epsilon^\alpha \quad \text{or} \quad y_{\epsilon i} = y_{\epsilon j} \quad \text{for } 1 \leq i,j \leq N, \\ &\left|f_\epsilon\right| \geq \frac{1}{2} \quad \text{in } \Omega \backslash \left(\bigcup_{i=1}^N B_{R\epsilon^\alpha}(y_{\epsilon i})\right), \\ &\left|f_\epsilon \cdot \nu\right| \leq \frac{1}{4} \quad \text{on } \partial\Omega \backslash \left(\bigcup_{i=1}^N B_{R\epsilon^\alpha}(y_{\epsilon i})\right), \end{aligned} \tag{5.43}$$

for any $\epsilon \in (0,1]$. Again we may pick $\epsilon_k \searrow 0$ such that $y_{\epsilon_k i} \to y_i$ for certain points $y_i \in \overline{\Omega}$. Choose $\rho > 0$ such that $B_\rho(y_i) \cap B_\rho(y_j) = \varnothing$ unless $y_i = y_j$, and $B_\rho(y_i) \subset \Omega$ unless $y_i \in \partial\Omega$. Now we may pick a subsequence (without changing notation) and relabel the points y_i such that either

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(i) $y_1 \in \Omega$ and $f_{\epsilon_k}|_{\partial B_\rho(y_1)}$ is topologically nontrivial, or

(ii) $y_2, y_3 \in \partial\Omega$, $y_2 \neq y_3$, and $f_{\epsilon_k} \cdot \tau$ changes sign in $D^*_\rho(y_2)$ and in $D^*_\rho(y_3)$.

(The conditions of (iii) in Proposition 5.6 cannot be satisfied for large k's because there is not enough energy.) Setting either $\Sigma = \{y_1\}$ or $\Sigma = \{y_2, y_3\}$, we conclude, using Proposition 5.6, that a subsequence of $\{f_{\epsilon_k}\}$ converges weakly in $H^1_{loc}(\overline{\Omega}\backslash\Sigma, \mathbb{R}^2)$ and weakly in $W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p < 2$. To see that the limit is a harmonic map from $\Omega\backslash\Sigma \to \mathbb{S}^1$, we use the form

$$\operatorname{div}\left(f_\epsilon \wedge \nabla f_\epsilon\right) = 0 \quad \text{in } \Omega \tag{5.44}$$

of (5.7). In order to prove that $\Sigma \subset \Omega$ can only happen for $\alpha = 1$, we repeat the arguments above with balls of radius ϵ instead of ϵ^α, and show thus that a vortex in the interior of Ω needs more energy than available for $\alpha < 1$. ∎

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