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BOUNDED AND PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECT IN A BANACH SPACE

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Abstract. Sufficient conditions are obtained for the existence of bounded and periodic solutions of linear and weakly non-linear differential equations with impulse effect in a Banach space on the axis or the semi-axis. The main results are new for equations in $\mathbf{R}^n$ as well.

1. Introduction. Differential equations with impulse effect describe the evolution of systems subject to perturbations of negligible duration. Systems with a finite number of degrees of freedom were considered in [1, 2] and a number of subsequent works by many authors.

In the present paper sufficient conditions are given for the existence of bounded and periodic solutions of linear and weakly non-linear differential equations with impulse effect in a Banach space on the axis or semi-axis. Moreover, the main results obtained are new for equations in $\mathbf{R}^n$ as well.

2. Preliminary notes. Let X be a complex Banach space, $L(X)$ be the set of all linear bounded operators $X \to X$. Consider the equation with impulse effect

$$\frac{dx}{dt} = Ax + F(t,x) + \sum_{j=-\infty}^{\infty} [Bx + H_j(x)]\delta(t - t_j). \tag{1}$$

Here δ is Dirac's delta-function; the points t_j are fixed so that

$$t_j < t_{j+1} \ \ (j \in \mathbf{Z}), \quad t_j \to \pm\infty \ \ (j \to \pm\infty);$$

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the function $F : (\mathbf{R}\backslash\{t_j\}) \times X \to X$ is continuously continuable in each layer $[t_j\,,\, t_{j+1}] \times X$ and satisfies there the local Lipschitz condition on x; the functions $H_j : X \to X$ are continuous; A, $B \in L(X)$ and there exists an operator $S \in L(X)$, such that

$$e^S = I + B \;\; (I = Id_x), \;\; AS = SA\,.$$

The solution $x(\cdot) : J \to X$ of equation (1) (we denote $x \in (1)_J$) will be assumed defined in some interval J, continuously differentiable except for the points t_j in which it may have discontinuities of first type, left continuous and moreover we assume that it satisfies the relation

$$\Delta x(t_j) := x(t_j) - x(t_j^-) = Bx(t_j^-) + H_j(x(t_j^-)).$$

Under the conditions formulated and an arbitrary initial condition $x(t^0) = x^0$, the solution exists and is unique in some right half-neighbourhood of the point t^0. For the solution which exists in the interval $[t^0, \infty)$ the definitions of stability and asymptotic stability by Lyapunov are extended in a natural way.

Denote by $CB(J, X)$ the Banach space of the bounded piecewise continuous functions $x : J \to X$ with possible discontinuities of first type in the points t_j and with the norm $|||x||| = |||x|||_J$. Along with equation (1) we consider the respective linear non-homogeneous equation

$$\frac{dx}{dt} = Ax + f(t) + \sum_{j=-\infty}^{\infty} (Bx + h_j)\delta(t - t_j) \tag{2}$$

($f \in CB(\mathbf{R}, X)$, $h_j \in X$) and homogeneous equation

$$\frac{dy}{dt} = Ay + \sum_{j=-\infty}^{\infty} By \cdot \delta(t - t_j)\,. \tag{3}$$

The solution of equation (2) with initial condition $x(t^0) = x^0$ exists and is unique for $t \in \mathbf{R}$ and the assertion about its stability (asymptotic stability) does not depend on the choice of t^0, x^0, f and $\{h_j\}$; that is why we shall speak of stability (asymptotic stability) of equation (2).

3. Main results.

3.1. Existence of a bounded solution of equation (2) on the axis. Denote by $\nu(t, \tau)$ for $t \le \tau$ the number of points t_j in the interval $(t, \tau]$; for $t > \tau$ we set $\nu(t, \tau) = -\nu(\tau, t)$.

Theorem 1. *Let the following conditions be fulfilled:*

(1) *Uniformly on $t \in \mathbf{R}$ there exists*

$$p := \lim_{T \to \infty} \frac{\nu(t, t+T)}{T} < \infty\,.$$

(2) *The spectrum $\sigma(\Lambda)$ of the operator $\Lambda := A + pS$ has no points on the imaginary axis.*

(3) $\sup_{j \in \mathbf{Z}} \|h_j\| < \infty\,.$

Then equation (2) has in $CB(\mathbf{R}, X)$ a unique solution $x(\cdot)$ such that

$$|||x||| \le C \max\{|||f||| \, , \sup_j \|h_j\|\} \, , \tag{4}$$

and C does not depend on f and $\{h_j\}$.

Proof: From condition (2) it follows that $\sigma(\Lambda) = \sigma_+(\Lambda) \cup \sigma_-(\Lambda)$ where $\sigma_+(\Lambda)$ $(\sigma_-(\Lambda))$ is the part of $\sigma(\Lambda)$ which lies in the right (left) half-plane. Accordingly, X splits up into a direct sum of the subspaces X_+ and X_- which generates projectors of the form

$$P_\pm = -\frac{1}{2\pi i} \oint_{\Gamma_\pm} R_\lambda \, d\lambda \, , \tag{5}$$

where $\Gamma_\pm$ are contours surrounding $\sigma_\pm(\Lambda)$ in the respective half-planes and R_λ is the resolvent of the operator Λ. Moreover, all operators $P_\pm$, A, S, Λ commute two by two.

Introduce the operator-valued function $G : \mathbf{R}^2 \to L(X)$ by the formula

$$G(t,\tau) = \begin{cases} -e^{(t-\tau)A - \nu(t,\tau)S} P_+, & -\infty < t \le \tau < \infty \\ e^{(t-\tau)A-\nu(t,\tau)S} P_-, & -\infty < \tau \le t < \infty. \end{cases}$$

It is continuous in the whole plane t, τ except for discontinuities of first type on the straight lines $t = t_j$. It is not difficult to prove the following properties of this function:

$$\begin{aligned} &\frac{dG(t,\tau)}{dt} = AG(t,\tau), && (\forall t \bar{\in} \{t_j\}, \ \tau \ne t); \\ &G(t_j^+, \tau) - G(t_j^-, \tau) = BG(t_j^-, \tau), && (\forall j \in \mathbf{Z}, \ \tau \ne t_j); \\ &G(t^+, t) - G(t^-, t) = I, && (\forall t \bar{\in} \{t_j\}), \end{aligned}$$

(here it is necessary to use the equality $P_+ + P_- = I$);

$$G(t_j^+, t_j) - G(t_j^-, t_j) = BG(t_j^-, t_j) + I, \qquad (\forall j \in \mathbf{Z});$$

$$\exists K > 0, \ \mu > 0 : \|G(t,\tau)\| \le K e^{-\mu|t-\tau|} \qquad (\forall t, \, \tau \in \mathbf{R}); \tag{6}$$

here the exponentially decreasing estimate of $\|e^{t\Lambda} P_\pm\|$ for $t \to \mp\infty$ is used which follows from (5) and the representation

$$(t-\tau)A + \nu(\tau,t)S = (t-\tau)\Lambda + \mathrm{o}(t-\tau)S \qquad (|t-\tau| \to \infty) \, .$$

The assertion of theorem 1 follows from the formula

$$x(t) = \int_{-\infty}^{\infty} G(t,\tau) f(\tau) \, d\tau + \sum_{j=-\infty}^{\infty} G(t,t_j) h_j \qquad (t \in \mathbf{R}) \, . \tag{7}$$

Last integral and series are convergent because of esimate (6) and in view of the following lemma:

Lemma 1. *Let the conditions of Theorem 1 hold. Then*

$$(7) \Longleftrightarrow \big(x \in CB(\mathbf{R}, X) \quad \textit{and} \quad x \in (2)_{\mathbf{R}}\big)\,.$$

Proof: For (7) the inclusion $x \in CB(\mathbf{R}, X)$ (and estimate (4)) follow from (6) and assertion $x \in (2)_{\mathbf{R}}$ follows from the representation

$$x(t) = \int_{-\infty}^{t} G(t,\tau)f(\tau)\,d\tau + \int_{t}^{\infty} G(t,\tau)f(\tau)\,d\tau + \sum_{j=-\infty}^{\infty} G(t,t_j)h_j\,,$$

and the properties of the operator-valued function G. Inversely, let $\tilde{x} \in CB(\mathbf{R}, X)$, $\tilde{x} \in (2)_{\mathbf{R}}$. We define x by formula (7) and denote $y = \tilde{x} - x$. Then $y \in CB(\mathbf{R}, X)$, $y \in (3)_{\mathbf{R}}$. It is not difficult to verify that

$$y(t) = e^{(t-t_0)A+\nu(t_0,t)S}y(t_0) \qquad (\forall t \in \mathbf{R}), \tag{8}$$

whence it follows that

$$\|e^{t\Lambda}y(t_0)\| = \Big\|\big\{\exp\big[p - \frac{\nu(t_0,t)}{t-t_0}tS + \frac{t_0\nu(t_0,t)}{t-t_0}S + t_0A\big]\big\}y(t)\Big\| = e^{o(t)} \;\; (t \to \pm\infty); \tag{9}$$

that is why $P_{\pm}y(t_0) = 0$; i.e, $y(t_0) = 0$, hence $y = 0$.

Lemma 1, and together with it, Theorem 1, is proved.

Corollary 1. *If the conditions of Theorem 1 hold and $\sigma_+(\Lambda) = \emptyset$, then equation (2) is asymptotically stable.*

In fact, it suffices to consider equation (3). We write down formula (8) in the form

$$y(t) = e^{(t-t_0)\Lambda}\, e^{[\nu(t_0,t)-p(t-t_0)]S}\, y(t_0)\,.$$

The assertion of Corollary 1 follows from the last formula and the fact that $\nu(t_0,t) - P(t - t_0) = o(t)$ for $t \to \infty$.

3.2. Existence of a periodic solution of periodic equation (2). We shall call equation (2) T-periodic ($T > 0$), if $f(t+T) \equiv f(t)$ and there exists a number $r \in \mathbf{N}$ such that

$$t_{j+r} = t_j + T, \qquad h_{j+r} = h_j \;\; (\forall j \in \mathbf{Z})\,. \tag{10}$$

Theorem 2. *If for the T-periodic equation (2)*

$$\frac{2k\pi i}{T}\bar{\in}\sigma(\Lambda) \qquad (\forall k \in \mathbf{Z};\;\; \Lambda = -A + pS,\;\; p = \frac{r}{T})\,,$$

then it has a unique T-periodic solution.

Proof: We choose $x(t_0)$ and by induction obtain

$$x(t) = e^{tA+kS}\Big[e^{-t_0A}x(t_0) + \sum_{j=0}^{k-1} e^{-jS}\int_{t_j}^{t_{j+1}} e^{-\tau A}f(\tau)\,d\tau$$
$$+ e^{-kS}\int_{t_k}^{t} e^{-\tau A}f(\tau)\,d\tau + \sum_{j=1}^{k} e^{-t_jA-jS}h_j\Big]$$
$$(t_k \le t < t_{k+1};\;\; k = 0, 1 \ldots)\,.$$

The condition $x(t+T) \equiv x(t)$ is equivalent to the relation $x(t_r) = x(t_0)$, i.e.,

$$e^{TA+rS}x(t_0) + e^{t_r A}\sum_{j=0}^{r-1} e^{(r-j)S}\int_{t_j}^{t_{j+1}} e^{-\tau A} f(\tau)\,d\tau + \sum_{j=1}^{r} e^{(t_r-t_j)A+(k-j)S}h_j = x(t_0)\,. \tag{11}$$

Since $e^{TA+rS} = e^{T\Lambda}$ and the operator $I - e^{T\Lambda}$ in view of the conditions of the theorem is invertible, then equation (11) is satisfied by a unique value of $x(t_0)$ which proves the theorem.

3.3. Existence of a family of bounded solutions of equation (2) on the semi-axis. Assume that $t_0 = 0$ and consider equation (2) on $R_+ := [0,\infty)$.

Theorem 3. *Let the following conditions be fulfilled:*

(1) *The spectrum $\sigma(A)$ has no points on the imaginary axis. Analogously to §3.1, we obtain $\sigma(A) = \sigma_+(A)\cup\sigma_-(A)$ and introduce the corresponding direct decomposition $X = X_1\dot{+}X_2$, the projectors $P_\pm$ and the operators $A_\pm := P_\pm A$ which satisfy for some $N_\pm$, $\mu_\pm > 0$ the estimates*

$$\|e^{A_-t}\| \le N_- e^{-\mu_- t}, \qquad \|e^{-A_+t}\| \le N_+ e^{-\mu_+ t} \ (\forall t \in \mathbf{R}_+),$$

(2) $SX_\pm \subseteq X_\pm$,

(3) $\theta := \lim_{j\to\infty} \inf(t_j - t_{j-1}) > 0$,

(4) $\|B_-\| < \frac{1}{N_-}e^{\mu_-\theta} - 1$, $\|(I+B_+)^{-1}\| < e^{\mu_+\theta}$, $\|B_+\| < \frac{1}{N_+}(1-e^{\mu_+\theta})$ *where* $B_\pm := P_\pm B$.

(5) $\sup_{j\in\mathbf{N}} \|h_j\| < \infty$.

Then $\forall x_0 \in X$, $\exists! x \in CB(\mathbf{R}_+, X) : (x \in (2)_{\mathbf{R}_+}, P_- x(0) = x_0)$.

Proof: The components $x_\pm(\cdot)$ of any $x \in (2)_{\mathbf{R}_+}$ satisfy the equations

$$\frac{dx_\pm}{dt} = A_\pm x_\pm + f_\pm(t) + \sum_{j=1}^{\infty}(B_\pm x_\pm + h_{j\pm})\delta(t-t_j) \quad (t \in \mathbf{R}_+), \tag{$2_\pm$}$$

considered respectively in the spaces $X_\pm$. Moreover,

$$\sigma(A_\pm) = \sigma_\pm(A), \quad f_\pm := P_\pm f, \quad h_\pm := P_\pm h_j\,.$$

Equation $(2)_-$ for any initial condition $x_-(0) = x_0 \in X_-$ has a solution $x_-(\cdot)$ satisfying the relation

$$x_-(t) = e^{tA_-}x_0 + \int_0^t e^{(t-\xi)A_-} f_-(\xi)\,d\xi + \sum_{0<t_j\le t} e^{(t-t_j)A_-}[B_- x_-(t_j-) + h_{j-}] \quad (t \in \mathbf{R}_+). \tag{12}$$

We shall prove that $x_- \in CB(\mathbf{R}_+, X_-)$. In fact, $\forall j \in \mathbf{N}$ we have

$$x_-(t_{j+1}) = (I + B_-)\big[e^{(t_{j+1}-t_j)A_-}x_-(t_j) + \int_{t_j}^{t_{j+1}} e^{(t_{j+1}-t_j)A_-}f_-(\tau)\,d\tau + h_{(j+1)^-}\big]$$

whence, $\forall\epsilon > 0$, for all j large enough we obtain

$$\|x_-(t_{j+1})\| \le (1 + \|B_-\|)N_- e^{-\mu_-(\theta-\epsilon)}\|x_-(t_j)\| + (1 + \|B_-\|)\frac{N_-}{\mu_-}|||f_-||| + \sup_j \|h_{j^-}\|\,.$$

Hence, by condition 4 of theorem 3 we obtain boundedness of the sequence $\{x_-(t_j)\}$, therefore of the solution $x_-(\cdot)$ too.

Equation $(2)_+$ may have in $CB(\mathbf{R}_+, X_+)$ no more than one solution. In fact, the difference $y_+(\cdot)$ of two such solutions can be represented in the form

$$y_+(t) = e^{tA_+}(I + B_+)^{\nu(0,t)}y_+(0)\,,$$

whence if $y_+ \in CB(\mathbf{R}_+, X)$, by condition 4 of theorem 3 we obtain that $\forall\epsilon > 0$ for $t \to \infty$

$$\|y_+(0)\| = \|e^{-tA_+}(I + B_+)^{-\nu(0,t)}y_+(t)\| = \mathrm{o}\big(e^{-\mu_+ t}\|(I + B_+)^{-1}\|^{t/(\theta-\epsilon)}\big) \to 0\,,$$

if ϵ is small enough, i.e., $y_\pm(t) \equiv 0$.

In order to prove the solvability of equation $(2)_+$ in $CB(\mathbf{R}_+, X_+)$ we consider the equation

$$x_+(t) = -\int_t^\infty e^{(t-\xi)A_+}f_+(\tau)\,d\tau - \sum_{t<t_j<\infty} e^{(t-t_j)A_+}[B_+x_+(t_{j^-}) + h_{j^+}] \quad (t \in \mathbf{R}_+), \quad (13)$$

which, provided that $x_+ \in CB(\mathbf{R}_+, X_+)$, is equivalent to $(2)_+$. Consider equation (13) on $[t_k, \infty)$, denote its right-hand side by $(Qx_+)(t)$ and obtain that $\forall\epsilon > 0$, $k \in \mathbf{N}$, x_+^1, $x_+^2 \in CB([t_k, \infty), X_+)$

$$\begin{aligned}|||Qx_+^1 - Qx_+^2||| &\le \sup_{t_k \le t < \infty} \sum_{t<t_j<\infty} N_+ e^{-\mu_+(t_j - t)}\|B_+\| \cdot |||x_+^1 - x_+^2||| \\ &\le \frac{N_+\|B_+\|}{1 - e^{-\mu_+(\theta-\epsilon)}}|||x_+^1 - x_+^2|||\,.\end{aligned}$$

From condition 4 of theorem 3 it follows that to equation (13) considered on $[t_k, \infty)$, we can apply the theorem of Banach of the contractive mappings. That is why it and, along with it equation (13), have in $CB(\mathbf{R}_+, X)$ a unique solution.

Adding the functions constructed $x_-(\cdot)$ and $x_+(\cdot)$, we obtain the assertion of theorem 3.

Remark 1. Introduce the operator-valued function $\mathcal{D} : \mathbf{R} \to L(x)$ by the formula

$$\mathcal{D}(t) = \begin{cases} P_- e^{tA_-}, & t \ge 0 \\ -P_+ e^{tA_+}, & t < 0. \end{cases}$$

Then from (12) and (13) we obtain that the constructed solution x satisfies the relation

$$x(t) = \mathcal{D}(t)x_0 + \int_0^\infty \mathcal{D}(t - \tau)f(\tau)\,d\tau + \sum_{j\in\mathbf{N}} \mathcal{D}(t - t_j)[Bx(t_j^-) + h_j] \quad (t \in \mathbf{R}_+)$$

which, provided that $x \in CB(\mathbf{R}_+, X)$, is equivalent to equation (2) with condition $P_- x(0) = x_0$.

3.4. Existence of a bounded and periodic solution of equation (1) on the axis. We shall write $(1) \in (M, q, \rho)$, if for $t \in \mathbf{R}$, $\|x\|$, $\|x_1\| \le \rho$, $j \in \mathbf{Z}$ the following inequalities hold:

$$\|F(t,x)\| + \|H_j(x)\| \le M,$$
$$\|F(t,x) - F(t,x_1)\| + \|H_j(x) - H_j(x_1)\| \le q\|x - x_1\| .$$

Theorem 4. *Let conditions 1 and 2 of Theorem 1 hold. Then* $\forall \rho > 0$, $\exists M > 0$, $q > 0$ *(*M*,* q *depend on* A*,* B*,* $\{t_j\}$*,* ρ*) such that*

$$(1) \in (M, q, \rho) \Rightarrow \exists! x(: \mathbf{R} \to X) : x \in (1), \qquad |||x|||_{\mathbf{R}} \le \rho .$$

Proof: From lemma 1 it follows that for $(1) \in (M, q, \rho)$, $|||x|||_{\mathbf{R}} \le \rho$ equation (1) is equivalent to the equation

$$x(t) = \int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau))\, d\tau + \sum_{j=-\infty}^{\infty} G(t, t_j) H_j(x(t_j^-)) \quad (t \in \mathbf{R}). \tag{15}$$

From estimate (6) and condition 1 of Theorem 1 it follows that if the number $\ell > 0$ is chosen so that $(1/\ell)\nu(t, t+\ell) < p + 1$ $(\forall t \in \mathbf{R})$, then for $\|x\| \le \rho$

$$\begin{aligned}
&\Big\| \int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau))\, d\tau + \sum_{j=-\infty}^{\infty} G(t, t_j) H_j(x(t_j^-)) \Big\| \\
&\le M\Big(\int_{-\infty}^{\infty} K e^{-\mu|t-\xi|}\, d\tau + \sum_{j=-\infty}^{\infty} K e^{-\mu|t-t_j|} \Big) \\
&\le MK\big(\frac{2}{\mu} + \frac{2(p+1)\ell}{1 - e^{-\mu\ell}}\big).
\end{aligned}$$

Thus, for

$$M \le p\big[K\big(\frac{2}{\mu} + \frac{2(p+1)\ell}{1 - e^{-\mu\ell}}\big)\big]^{-1},$$

the operator defined by the right-hand side of equation (15) maps the ball $|||x||| \le \rho$ into itself. In an analogous way it is verified that for q small enough this operator is contractive. This completes the proof of theorem 4.

Corollary 2. *If in the conditions of Theorem 4 it is given that* $F(t, 0) \equiv 0$ *and* $H_j(0) = 0$*, then the only solution of equation (1) for* $|||x||| \le \rho$ *is the solution* $x(t) \equiv 0$*.*

Corollary 3. *If in the conditions of Theorem 4 equation (1) is* T*-periodic (the respective definition is analogous to the one given in §3.2), then the solution of the equation considered in the theorem is* T*-periodic too.*

In fact, if $t \mapsto x(t)$ is this solution, then the function $t \mapsto x(t+T)$ is a solution of equation (1) too and $x(t+T) \equiv x(t)$.

3.5. Existence of a family of bounded solutions of equation (1) on the semi-axis. As in §3.3 we assume that $t_0 = 0$.

Theorem 5. *Let an operator A and points $\{t_j\}$ be given so that conditions 1 and 3 of Theorem 3 hold. Then* $\forall\rho > 0$, $\rho_1 \in (0, \rho\max\{N_-, N_+\})^{-1} \exists M, q, \beta > 0$: *if* B, F, $\{H_j\}$ *are given so that conditions 2 and 4 of Theorem 3 are fulfilled,* $\|B\| \leq \beta$ *and* $(1) \in (M, q, \rho)$, *then* $\forall x_0 \in X_- : \|x_0\| \leq \rho_1$, $\exists! x(: \mathbf{R}_+ \to X) : x \in (1)$, $\||x|\|_{\mathbf{R}_+} \leq \rho$, $P_- x(0) = x_0$.

Proof: From Remark 1 it follows that for $(1) \in (M, q, \rho)$, $\||x|\|_{\mathbf{R}_+} \leq \rho$ equation (1) with condition $P_- x(0) = x_0 \in X_-$ $(\|x_0\| \leq \rho)$ is equivalent to the equation

$$x(t) = \mathcal{D}(t)x_0 + \int_0^\infty \mathcal{D}(t-\tau)F(\tau, x(\tau))\,d\tau + \sum_{j\in\mathbf{N}} \mathcal{D}(t-t_j)[Bx(t_j^-) + H_j(x(t_j^-))] \quad (t \in \mathbf{R}_+).$$

For $\|x\|_{\mathbf{R}_+} \leq \rho$, $\|x_0\| \leq \rho_1$ $(\leq \rho)$, $(1) \in (M, q, \rho)$ the norm of the right-hand side of the above equation does not exceed

$$\max\{N_-, N_+\}\rho_1 + M\Big(\frac{N_-}{\mu_-} + \frac{N_+}{\mu_+}\Big) + (\|B\|_\rho + M)\Big(\frac{N_-}{1 - e^{-\mu_-\theta_1}} + \frac{N_+}{1 - e^{-\mu_+\theta_1}}\Big),$$

where $\theta_1 := \inf_{j\in\mathbf{N}}(t_j - t_{j-1})$. We choose β so that

$$\max\{N_-, N_+\}\rho_1 + \beta\rho\Big(\frac{N_-}{1 - e^{-\mu_-\theta_1}} + \frac{N_+}{1 - e^{-\mu_+\theta_1}}\Big) < \rho.$$

After that the proof of the theorem is completed in the same way as the proof of Theorem 4.

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