

BOUNDED APPROXIMATION BY POLYNOMIALS WHOSE ZEROS LIE ON A CIRCLE⁽¹⁾

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1. **Introduction.** Let C be a rectifiable Jordan curve with interior D . We say that a sequence of polynomials P_n converges boundedly to a function f in D , or f is boundedly approximated by P_n in D , if P_n converges to f throughout D and $\sup \{|P_n(z)| : z \in D\}$ is bounded as a function of n . A polynomial whose zeros lie on C will be called a C -polynomial. It is obvious that the limit function of a boundedly convergent sequence of C -polynomials in D is a bounded zero free holomorphic function in D , unless it is identically zero. In this paper, we will present a proof of the somewhat unexpected converse for the case when C is a circle, as announced in [1].

MAIN THEOREM. *Every bounded zero free holomorphic function in the open unit disc can be boundedly approximated there by polynomials whose zeros lie on the unit circumference.*

More generally, suppose that C is any rectifiable Jordan curve, so smooth that its parametric representation in terms of arc length has a Hölder continuous derivative. The methods developed in this paper can be extended and modified to prove the possibility of bounded approximation by C -polynomials of functions f , defined and zero free in $\text{clos } D$, such that the derivative f' , relative to $\text{clos } D$, exists and is Hölder continuous throughout $\text{clos } D$. In particular, any function f holomorphic and zero free on $\text{clos } D$ can be boundedly approximated by C -polynomials [1], [2]. However, for arbitrary Jordan curves C , the problem of bounded approximation by C -polynomials is open, even when f is holomorphic and zero free in $\text{clos } D$.

It should be mentioned that a weaker kind of approximation by C -polynomials was studied by G. R. MacLane [7]. He proved that if C is a rectifiable Jordan curve with interior D and f is holomorphic and zero free in D , then there exists a sequence of C -polynomials which converges to f uniformly on every compact subset of D . This result was later extended by J. Korevaar and his students [5], [6], [8] to other domains D . Very recently, Professor Korevaar and the author considered the case where C is the disjoint union of two or more Jordan curves [3]. It is interesting to note how the approximation problem by C -polynomials breaks down in some situations.

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Throughout the rest of this paper, C will denote the unit circle and D , the open unit disc.

2. **Approximation of $S_k(z) = \prod_{m=1}^k (1 - ze^{-i\theta_m})^{\alpha_m}$.** We first construct a sequence of C -polynomials P_n which converges to the function

$$(2.1) \quad S_k(z) = \prod_{m=1}^k (1 - ze^{-i\theta_m})^{\alpha_m}$$

in D , where $0 \leq \theta_1 < \dots < \theta_k < 2\pi$ and $0 \leq \alpha_m < 1$, $m=1, \dots, k$.

The proof of the uniform boundedness of P_n on D will be included in §4. We use the same construction as indicated in [1]. Set $\sum_{m=1}^k \alpha_m = \alpha$. For all n so large that $2\pi/(n+k-\alpha) < \min(\theta_{m+1} - \theta_m)$, where $\theta_{k+1} = 2\pi + \theta_1$, we define numbers $t_j = t_j(n)$ and $\theta_m^*(n)$, $j=1, \dots, n$ and $m=1, \dots, k$, by the following procedure

$$t_1 = \theta_1 + \frac{(2-\alpha_1)\pi}{n+k-\alpha}, t_2 = t_1 + \frac{2\pi}{n+k-\alpha}, \dots, t_{j_1} = t_{j_1-1} + \frac{2\pi}{n+k-\alpha},$$

where j_1 is determined by the inequality

$$t_{j_1} < \theta_2 \leq t_{j_1} + \frac{2\pi}{n+k-\alpha}, \dots, t_{j_1+1} = t_{j_1} + \frac{2(2-\alpha_2)\pi}{n+k-\alpha},$$

$$t_{j_1+2} = t_{j_1+1} + \frac{2\pi}{n+k-\alpha}, \dots, t_{j_2} = t_{j_2-1} + \frac{2\pi}{n+k-\alpha},$$

where j_2 is determined by the inequality $t_{j_2} < \theta_3 \leq t_{j_2} + 2\pi/(n+k-\alpha)$;

$$t_{j_k-1+1} = t_{j_k-1} + 2(2-\alpha_k)\pi/(n+k-\alpha),$$

$$t_{j_k-1+2} = t_{j_k-1+1} + 2\pi/(n+k-\alpha), \dots,$$

$$t_{j_k} = t_{j_k-1} + 2\pi/(n+k-\alpha);$$

here j_k is determined by the inequality

$$t_{j_k} < \theta_1 + 2\pi \leq t_{j_k} + 2\pi/(n+k-\alpha);$$

finally,

$$\theta_1^* = \theta_1, \quad \theta_l^* = \frac{1}{2}(t_{j_l-1} + t_{j_l-1+1}), \quad l = 2, \dots, k.$$

Hence,

$$(2.2) \quad |\theta_m - \theta_m^*| \leq \frac{(2-\alpha_m)\pi}{n+k-\alpha} < \frac{2\pi}{n} \quad \text{for } m = 1, \dots, k.$$

We now define

$$(2.3) \quad P_{n+k}(z, k) = \prod_{j=1}^n (1 - ze^{-it_j}) \prod_{m=1}^k (1 - ze^{-i\theta_m^*}).$$

We shall prove

THEOREM 2.1. *For every k and every $z \in D$, $P_{n+k}(z, k) \rightarrow S_k(z)$, as $n \rightarrow \infty$.*

To prove this, we need a lemma of the Riemann-Lebesgue type. For each n , let s_n be the following step function on $[0, 2\pi]$. The jumps of s_n only occur at t_1, \dots, t_n and $\theta_1^*, \dots, \theta_k^*$. Each jump at t_j is equal to 1 and each jump at θ_m^* is equal to $(1 - \alpha_m)$. These conditions do not determine s_n completely; however, we find it more convenient to state the additional conditions in terms of the related function,

$$(2.4) \quad v_n(t) = s_n(t) - (n+k-\alpha)t/2\pi.$$

We set $v_n(\theta_1^*+) = \frac{1}{2}(1 - \alpha_1)$ and define $v_n(\theta_m^*) = v_n(t_j) = 0$ for $m=1, \dots, k$ and $j=1, \dots, n$. Then we have the following

LEMMA 2.1. *If g is an integrable function on $(0, 2\pi)$, then*

$$(2.5) \quad \lim_{n \rightarrow \infty} \int_0^{2\pi} g(t)v_n(t) dt = 0.$$

The proof of this lemma is the same as the proof of the Riemann-Lebesgue lemma. We first prove it for the case of a step function, and then approximate g by step functions, cf. [9].

Now we can prove Theorem 2.1. For fixed $z \in D$, let g be the integrable function

$$g(t) = \frac{d}{dt} \log(1 - ze^{-it}) = \frac{d}{dt} \left\{ - \sum_{p=1}^{\infty} \frac{z^p}{p} e^{-ipt} \right\}.$$

Here, as everywhere else in this paper, we choose the principal values of the logarithms. Integration by parts gives

$$\begin{aligned} \int_0^{2\pi} g(t)v_n(t) dt &= \int_0^{2\pi} -\log(1 - ze^{-it}) dv_n(t) \\ &= - \sum_{j=1}^n \log(1 - ze^{-it_j}) - \sum_{m=1}^k (1 - \alpha_m) \log(1 - ze^{-i\theta_m^*}), \end{aligned}$$

and this tends to 0 by the lemma. Using (2.2), we see that

$$\sum_{j=1}^n \log(1 - ze^{-it_j}) + \sum_{m=1}^k \log(1 - ze^{-i\theta_m^*}) \rightarrow \sum_{m=1}^k \alpha_m \log(1 - ze^{-i\theta_m^*}).$$

This is equivalent to $P_{n+k}(z, k) \rightarrow S_k(z)$ for each k and each $z \in D$.

3. Bounded approximation by $S_k(z)$. The following lemma is a trivial consequence of Herglotz's theorem [4].

LEMMA 3.1. *Let f be holomorphic and zero free in D , $f(0)=1$ and $\sup |f(z)| = M < +\infty$. Then there exists a nondecreasing real-valued function v in $[0, 2\pi]$, with $v(0)=0$ and $v(2\pi)=2 \log M$, such that*

$$(3.1) \quad f(z) = \exp \left\{ \int_0^{2\pi} \frac{-ze^{-it}}{1 - ze^{-it}} dv(t) \right\} \text{ for each } z \text{ in } D.$$

The above lemma indicates the importance of the following function,

$$f(z) = \exp \{-z/(1-z)\}$$

which is zero free, holomorphic and bounded in D . The first thing we do is to study the bounded approximation of $\exp \{-z/(1-z)\}$ by expressions of the form

$$(3.2) \quad S_k(z) = \prod_{m=1}^k (1 - ze^{-i2\pi m/k})^{\alpha_{k,m}}$$

throughout D , where $\alpha_{k,m} \geq 0$ for $m=1, \dots, k$ and

$$(3.3) \quad \max_{1 \leq m \leq k} \alpha_{k,m} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

We establish the following

THEOREM 3.1. *Let $f(z) = \exp \{-z/(1-z)\}$ and $\varepsilon > 0$ be given. There exist functions S_k , $k=1, 2, \dots$, of the form indicated in (3.2) and (3.3), such that S_k converges to f throughout D and*

$$(3.4) \quad \max_{|z| \leq 1} |S_k(z)| \leq e^{1/2 + \varepsilon} \text{ for all } k.$$

Proof. Let k and p be positive integers with $k > 2p$. We define

$$(3.5) \quad \alpha_{k,m}^{(p)} = \sum_{j=1}^p \frac{2j}{k} \left(1 - \frac{j}{p}\right) \left(1 + \cos \frac{2\pi jm}{k}\right)$$

where $m=1, \dots, k$. Clearly, each $\alpha_{k,m}^{(p)} \geq 0$ and

$$(3.6) \quad \max_{1 \leq m \leq k} \alpha_{k,m}^{(p)} \leq \frac{p^2}{k}.$$

For each z in D ,

$$(3.7) \quad \begin{aligned} & \sum_{m=1}^k \left(1 + \cos \frac{2\pi jm}{k}\right) \log (1 - ze^{-i2\pi m/k}) \\ &= - \sum_{\nu=1}^{\infty} \sum_{m=1}^k \frac{z^\nu}{\nu} e^{-i2\pi \nu m/k} \left(1 + \cos \frac{2\pi jm}{k}\right) \\ &= - \sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu} \left\{ \sum_{m=1}^k e^{-i2\pi \nu m/k} + \sum_{m=1}^k \cos \frac{2\pi \nu m}{k} \cdot \cos \frac{2\pi jm}{k} \right. \\ & \quad \left. - i \sum_{m=1}^k \sin \frac{2\pi \nu m}{k} \cdot \cos \frac{2\pi jm}{k} \right\}. \end{aligned}$$

Here,

$$(3.8) \quad \begin{aligned} \sum_{m=1}^k e^{-i2\pi \nu m/k} &= k \text{ if } k|\nu, \\ &= 0 \text{ if } k \nmid \nu. \end{aligned}$$

It follows that

$$\begin{aligned}
 (3.9) \quad \sum_{m=1}^k \cos \frac{2\pi\nu m}{k} \cdot \cos \frac{2\pi jm}{k} &= \frac{1}{2} \sum_{m=1}^k \left\{ \cos \frac{2\pi(\nu+j)m}{k} + \cos \frac{2\pi(\nu-j)m}{k} \right\} \\
 &= (1/2)k \quad \text{if } k|(\nu+j) \text{ or } k|(\nu-j), \\
 &= 0 \quad \text{otherwise.}
 \end{aligned}$$

(k cannot divide both $\nu+j$ and $\nu-j$ since $2j \leq 2p \leq k$.) Similarly,

$$(3.10) \quad \sum_{m=1}^k \sin \frac{2\pi\nu m}{k} \cos \frac{2\pi jm}{k} = \frac{1}{2} \sum_{m=1}^k \left\{ \sin \frac{2\pi(\nu+j)m}{k} + \sin \frac{2\pi(\nu-j)m}{k} \right\} = 0.$$

Note that $k|(\nu+j)$ if and only if $\nu = \mu k - j$ where $\mu = 1, 2, \dots$ and $k|(\nu-j)$ if and only if $\nu = \mu k + j$ where $\mu = 0, 1, 2, \dots$. Hence, substituting (3.8), (3.9) and (3.10) into (3.7) and combining with (3.5), we get

$$\begin{aligned}
 (3.11) \quad \sum_{m=1}^k \alpha_{k,m}^{(p)} \log(1 - ze^{-i2\pi m/k}) &= - \sum_{j=1}^p \frac{2j}{k} \left(1 - \frac{j}{p}\right) \\
 &\quad \times \left\{ \frac{k}{2} \cdot \frac{z^j}{j} + \sum_{\mu=1}^{\infty} \frac{z^{\mu k}}{\mu k} \cdot k + \sum_{\mu=1}^{\infty} \frac{z^{\mu k + j}}{\mu k + j} \cdot \frac{k}{2} + \sum_{\mu=1}^{\infty} \frac{z^{\mu k - j}}{\mu k - j} \cdot \frac{k}{2} \right\}.
 \end{aligned}$$

To estimate this expression, we compare the terms with denominators $\mu k \pm j$ with similar terms with denominators μk . For each z in D , we get, remembering $p < \frac{1}{2}k$,

$$\begin{aligned}
 &\left| \sum_{j=1}^p \frac{2j}{k} \left(1 - \frac{j}{p}\right) \left\{ \sum_{\mu=1}^{\infty} \frac{z^{\mu k \pm j}}{\mu k \pm j} \cdot \frac{k}{2} - \sum_{\mu=1}^{\infty} \frac{z^{\mu k}}{\mu k} \cdot \frac{k}{2} \right\} \right| \\
 &\leq \sum_{j=1}^p j \left(1 - \frac{j}{p}\right) \sum_{\mu=1}^{\infty} \frac{j}{(\mu k - j)\mu k} \\
 &\leq \sum_{j=1}^p j^2 \left(1 - \frac{j}{p}\right) \sum_{\mu=1}^{\infty} \frac{2}{\mu^2 k^2} < \frac{\pi^2}{36} \cdot \frac{p^3}{k^2}.
 \end{aligned}$$

Hence, uniformly for $z \in D$,

$$\begin{aligned}
 (3.12) \quad \sum_{m=1}^k \alpha_{k,m}^{(p)} \log(1 - ze^{-i2\pi m/k}) &= - \sum_{j=1}^p \left(1 - \frac{j}{p}\right) z^j + \sum_{j=1}^p \frac{2j}{k} \left(1 - \frac{j}{p}\right) \\
 &\quad \times \left\{ - \sum_{\mu=1}^{\infty} \frac{z^{\mu k}}{\mu} - \frac{z^j}{2} \sum_{\mu=1}^{\infty} \frac{z^{\mu k}}{\mu} - \frac{z^{-j}}{2} \sum_{\mu=1}^{\infty} \frac{z^{\mu k}}{\mu} \right\} + O(p^3/k^2) \\
 &= - \sum_{j=1}^p \left(1 - \frac{j}{p}\right) z^j + \sum_{j=1}^p \frac{2j}{k} \left(1 - \frac{j}{p}\right) \\
 &\quad \times \left(1 + \frac{z^j + z^{-j}}{2}\right) \log(1 - z^k) + O(p^3/k^2).
 \end{aligned}$$

By the maximum principle for harmonic functions,

$$\begin{aligned} & \max_{|z| \leq 1} \sum_{m=1}^k \alpha_{k,m}^{(p)} \log |1 - ze^{-i2\pi m/k}| \\ & \leq \max_{0 \leq \theta \leq 2\pi} \left\{ \sum_{j=1}^p -\left(1 - \frac{j}{p}\right) \cos j\theta \right\} \\ & \quad + \max_{0 \leq \theta \leq 2\pi} \left\{ \sum_{j=1}^p \frac{2j}{k} \left(1 - \frac{j}{p}\right) (1 + \cos j\theta) \log |1 - e^{ik\theta}| \right\} \\ & \quad + O(p^3/k^2) = I_1 + I_2 + O(p^3/k^2), \end{aligned}$$

say. Since $\log |1 - e^{ik\theta}| \leq \log 2$,

$$I_2 \leq 4 \log 2 \sum_{j=1}^p \frac{j}{k} \left(1 - \frac{j}{p}\right) < \frac{p^2}{k}.$$

Also, it is well known that

$$-\sum_{j=1}^p \left(1 - \frac{j}{p}\right) \cos j\theta = \frac{1}{2} - F_p(\theta)$$

when $F_p(\theta)$ denotes the Fejér kernel of order p . Since the Fejér kernels are non-negative, we see that $I_1 \leq \frac{1}{2}$ for all p . Hence,

$$(3.13) \quad \max_{|z| \leq 1} \prod_{m=1}^k |1 - ze^{-i2\pi m/k}| \alpha_{k,m}^{(p)} \leq \exp \left[\frac{1}{2} + O(p^2/k) \right].$$

Furthermore, from (3.12) we see that for each $z \in D$,

$$\sum_{m=1}^k \alpha_{k,m}^{(p)} \log (1 - ze^{-i2\pi m/k}) = \sum_{j=1}^p -\left(1 - \frac{j}{p}\right) z^j + O\left(\frac{p^2}{k} |z|^{k-p}\right) + O\left(\frac{p^3}{k^2}\right).$$

We now let k tend to ∞ through the positive integers, and let p tend to ∞ in such a way that $p^2/k \rightarrow 0$. The series $-\sum_{j=1}^{\infty} z^j$ converges to $-z/(1-z)$ in D ; it follows that the Cesàro means

$$-\sum_{j=1}^p \left(1 - \frac{j}{p}\right) z^j$$

of the partial sums also converge to $-z/(1-z)$. Thus for $z \in D$,

$$\sum_{m=1}^k \alpha_{k,m}^{(p)} \log (1 - ze^{-i2\pi m/k})$$

converges to $-z/(1-z)$. We conclude, by (3.6), that from the numbers $\alpha_{k,m}^{(p)}$, we can form a family of finite sequences $\{\alpha_{k,m}\}$, $m=1, \dots, k$, $k=1, 2, \dots$ which satisfies the conditions of the theorem.

We are now ready to prove a more general result, namely

THEOREM 3.2. *Let f be any zero free holomorphic function in D , $f(0)=1$ and $\sup |f(z)|=M < +\infty$. Then for any $\varepsilon > 0$, there exists a sequence of functions S_k of the form indicated in (3.2) and (3.3), such that S_k converges to f throughout D , and*

$$\max_{|z| \leq 1} |S_k(z)| \leq M^{1+\varepsilon} \text{ for all } k.$$

Proof. By Lemma 3.1, we can find a nondecreasing real-valued function $\nu(t)$ on $[0, 2\pi]$ with $\nu(0)=0, \nu(2\pi)=2 \log M$ such that

$$f(z) = \exp \left\{ \int_0^{2\pi} \frac{-ze^{-it}}{1-ze^{-it}} d\nu(t) \right\}$$

for $z \in D$. Thus, setting

$$\beta_{n,j} = \nu(2\pi j/n) - \nu(2\pi(j-1)/n),$$

where $j=1, \dots, n$, so that $\beta_{n,j} \geq 0$ and $\sum_{j=1}^n \beta_{n,j} = 2 \log M$, we have

$$f(z) = \lim_{n \rightarrow \infty} f_n(z),$$

where

$$f_n(z) = \prod_{j=1}^n \exp \left\{ \beta_{n,j} \frac{-ze^{-i2\pi j/n}}{1-ze^{-i2\pi j/n}} \right\}.$$

It is clear that

$$(3.14) \quad \sup_{|z| < 1} |f_n(z)| \leq M$$

for all n . By Theorem 3.1, we can choose finite sequences $\gamma_{m,p}, p=1, \dots, m, m=1, 2, \dots$, such that $\gamma_{m,p} \geq 0$ and $\max_p \gamma_{m,p} \rightarrow 0$,

$$S_m(w) = \prod_{p=1}^m (1 - we^{-i2\pi p/m})^{\gamma_{m,p}} \rightarrow \exp \left\{ \frac{-w}{1-w} \right\}$$

for $w \in D$ and

$$\max_{|w| \leq 1} |S_m(w)| \leq e^{(1+\varepsilon)/2}$$

for all m . In particular, for fixed integers n, j and $z \in D$, the sequence

$$S_m(ze^{-i2\pi j/n})^{\beta_{n,j}}$$

converges to

$$\exp \left\{ -\beta_{n,j} ze^{-i2\pi j/n} / (1 - ze^{-i2\pi j/n}) \right\}.$$

We now take our integers m of the form $kn, k=1, 2, \dots$, and define

$$S_k(z, n) = \prod_{j=1}^n S_{kn}(ze^{-i2\pi j/n})^{\beta_{n,j}} = \prod_{p=1}^n (1 - ze^{-i2\pi p/n})^{\alpha_{kn,p}}$$

where

$$\alpha_{kn,p} = \sum_{j+q=p \pmod{kn}} \gamma_{kn,q} \beta_{n,j}$$

for $p=1, \dots, kn$. Clearly, $\alpha_{kn,p} \geq 0$ and

$$\max_p \alpha_{kn,p} \leq \sum_{j=1}^n \beta_{n,j} \cdot \max_q \gamma_{kn,q} = 2 \log M \cdot \max_q \gamma_{kn,q}$$

which tends to 0 as k tends to ∞ . Furthermore, for fixed n and $z \in D$,

$$S_k(z, n) \rightarrow f_n(z), \text{ as } k \rightarrow \infty;$$

and for all k , and all n ,

$$(3.15) \quad \max_{|z| \leq 1} |S_k(z, n)| \leq \prod_{j=1}^n \{e^{(1+\varepsilon)/2}\}^{\beta_{n,j}} = M^{1+\varepsilon}.$$

Now let $A_K = \{z : |z| \leq 1 - 1/K\}$. By (3.14) and the Stieltjes-Osgood theorem, f_n converges uniformly to f on A_K ; and by (3.15), and fixed n , $S_k(z, n)$ converges uniformly to f_n . Hence, for a fixed integer K , we determine $n = n_K$ so large that

$$\max_{z \in A_K} |f_n(z) - f(z)| < \frac{1}{K},$$

and determine $k = k_K$ so large that for $n = n_K$

$$\max_{z \in A_K} |S_k(z, n) - f_n(z)| < \frac{1}{K}.$$

Setting $S_K(z) = S_{k_K}(z, n_K)$, we complete the proof of the theorem.

4. Bounded approximation by C-polynomials. We observe the following

LEMMA 4.1. For $0 < \alpha \leq \pi/4$ and $|z| < 1$, we have

$$(4.1) \quad \log |1 - z| - \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \log |1 - ze^{-it}| dt < 3.$$

Proof. By the maximum principle it is sufficient to consider $z = e^{i\theta}$, and by symmetry, we assume that $0 < \theta \leq \pi$. We first note that

$$\begin{aligned} \log |1 - e^{i\theta}| - \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \log |1 - e^{i(\theta-t)}| dt &= \log \sin \frac{\theta}{2} - \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \log \left| \sin \frac{\theta-t}{2} \right| dt \\ &\leq \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \log \left| \frac{\theta}{\theta-t} \right| dt + 2 \\ &= \frac{\theta}{2\alpha} \int_{-\alpha/\theta}^{\alpha/\theta} \log \frac{1}{|1-u|} du + 2. \end{aligned}$$

The function

$$h(x) = \frac{1}{2x} \int_{-x}^x \log \frac{1}{|1-u|} du$$

is continuous on $(0, \infty)$, $h(0+) = 0$, and $h(x) \rightarrow -\infty$ as $x \rightarrow +\infty$. Hence, $h(x)$ must be bounded above on $(0, \infty)$. Actually, it is bounded above by 1. This completes the proof of the lemma.

LEMMA 4.2. For each $m=2, 3, \dots$, let $\tau_k = \tau_k(m)$, $k=1, \dots, m$, be points on $[0, 2\pi]$ such that $0 < \tau_1 < \dots < \tau_m \leq 2\pi$, and, setting $\tau_{m+1} = \tau_1 + 2\pi$,

$$(4.2) \quad \max_{1 \leq k \leq m} (\tau_{k+1} - \tau_k) / \min_{1 \leq k \leq m} (\tau_{k+1} - \tau_k) \leq A.$$

Also let $\omega_k = \omega_k(m)$ and $\alpha_k = \alpha_k(m)$ be defined as

$$\omega_k = \frac{1}{2}(\tau_{k+1} + \tau_k), \quad \alpha_k = m(\tau_{k+1} - \tau_k)/2\pi.$$

Finally, let

$$(4.3) \quad R_m(z) = \prod_{k=1}^m (1 - ze^{-i\omega_k})^{\alpha_k}, \quad z \in D.$$

Then for all large m ,

$$(4.4) \quad \max_{|z| \leq 1} |R_m(z)| \leq \exp(19A^3).$$

Proof. For convenience, set $\frac{1}{2}(\tau_{k+1} - \tau_k) = \Delta_k = \Delta_k(m)$, and let n_1 denote the positive integer $[\sqrt{2A^2}] + 2$. Without loss of generality, we can assume that the maximum of $|R_m(z)|$ is attained at $e^{i\theta}$, where $\tau_m - 2\pi \leq \theta \leq \tau_1$. Since

$$\int_0^{2\pi} \log |1 - e^{it}| dt = 0,$$

we may write

$$\log |R_m(e^{i\theta})| = \sum_{k=1}^m \frac{m}{2\pi} \int_{-\Delta_k}^{\Delta_k} \log \left| \frac{1 - e^{i(\theta - \omega_k)}}{1 - e^{i(\theta - \omega_k - t)}} \right| dt.$$

We split the sum into two parts $\sum' + \sum''$, where \sum' denotes the sum over $1 \leq k \leq n_1 - 1$ and $m - n_1 + 1 \leq k \leq m$, \sum'' denotes the sum over $n_1 \leq k \leq m - n_1$.

By Lemma 4.1, we can conclude that for all large m (those for which $\Delta_k(m) \leq \pi/4$),

$$\sum' = \sum' \alpha_k \left\{ \frac{1}{2\Delta_k} \int_{-\Delta_k}^{\Delta_k} \log \left| \frac{1 - e^{i(\theta - \omega_k)}}{1 - e^{i(\theta - \omega_k - t)}} \right| dt \right\} \leq 3 \sum' \alpha_k.$$

By (4.2), we see that

$$\min_{1 \leq k \leq m} (\tau_{k+1} - \tau_k) \geq \frac{1}{A} \cdot \frac{2\pi}{m}; \quad \max_{1 \leq k \leq m} (\tau_{k+1} - \tau_k) \leq A \cdot \frac{2\pi}{m}.$$

Hence, since $A \geq 1$,

$$(4.5) \quad \begin{aligned} \sum' &\leq 3 \cdot \max \alpha_k \cdot \text{number of terms in } \sum' \\ &\leq 3 \cdot \frac{m}{\pi} A \frac{2\pi}{m} (2n_1 - 1) < 18A^3. \end{aligned}$$

We now turn to \sum'' . We can write

$$\begin{aligned} \sum'' &= \frac{m}{2\pi} \sum_{k=n_1}^{m-n_1} \int_0^{\Delta_k} \log \left| \frac{(1 - e^{i(\theta - \omega_k)})^2}{(1 - e^{i(\theta - \omega_k)})(1 - e^{i(\theta - \omega_k + t)})} \right| dt \\ &= \frac{m}{2\pi} \sum_{k=n_1}^{m-n_1} \int_0^{\Delta_k} -\log \frac{(1 - \cos(\theta - \omega_k - t))^{1/2} (1 - \cos(\theta - \omega_k + t))^{1/2}}{1 - \cos(\theta - \omega_k)} dt. \end{aligned}$$

Using the trigonometric identity

$$(1 - \cos(\alpha - \beta))(1 - \cos(\alpha + \beta)) = (\cos \alpha - \cos \beta)^2,$$

we obtain

$$\begin{aligned} \sum'' &= \frac{m}{2\pi} \sum_{k=n_1}^{m-n_1} \int_0^{\Delta_k} -\log \left(1 - \frac{1 - \cos t}{1 - \cos(\theta - \omega_k)} \right) dt \\ (4.6) \quad &= \frac{m}{2\pi} \sum_{k=n_1}^{m-n_1} \int_0^{\Delta_k} -\log \left(1 - \frac{\sin^2(t/2)}{\sin^2((\theta - \omega_k)/2)} \right) dt. \end{aligned}$$

Setting

$$\begin{aligned} B_1 &= \{k : n_1 \leq k \leq m - n_1, 0 \leq \omega_k - \theta \leq \pi\}, \\ B_2 &= \{k : n_1 \leq k \leq m - n_1, \pi \leq \omega_k - \theta \leq 2\pi\}, \end{aligned}$$

we see that for $k \in B_1$,

$$\sin((\omega_k - \theta)/2) \geq (\omega_k - \theta)/\pi,$$

and for $k \in B_2$,

$$\sin((\omega_k - \theta)/2) \geq 2 - (\omega_k - \theta)/\pi = [(2\pi + \theta) - \omega_k]/\pi.$$

Hence, for $0 \leq t \leq \Delta_k$ and $k \in B_1$,

$$\begin{aligned} 0 &\leq \frac{\sin^2(t/2)}{\sin^2((\theta - \omega_k)/2)} \leq \frac{(\Delta_k/2)^2}{[(\omega_k - \theta)/\pi]^2} \leq \frac{\pi^2}{4} \frac{\Delta_k^2}{(\tau_k - \tau_1)^2} \\ &\leq \frac{\pi^2}{4^2} \cdot \frac{A^4}{(n_1 - 1)^2} < \frac{1}{2}, \end{aligned}$$

by the definition of n_1 . The same inequality holds for $0 \leq t \leq \Delta_k$ and $k \in B_2$. Thus by the inequality

$$-\log(1 - x) \leq x/(1 - x) \leq 2x, \quad \text{for } 0 \leq x \leq \frac{1}{2},$$

and by (4.6), we get

$$\begin{aligned} \sum'' &\leq \frac{m}{\pi} \sum_{k=n_1}^{m-n_1} \int_0^{\Delta_k} \frac{\sin^2(t/2)}{\sin^2((\theta - \omega_k)/2)} dt \\ &\leq \frac{m}{\pi} \sum_{k \in B_1} \int_0^{\Delta_k} \frac{\pi^2}{4(\omega_k - \theta)^2} t^2 dt \\ &\quad + \frac{m}{\pi} \sum_{k \in B_2} \int_0^{\Delta_k} \frac{\pi^2}{4[(2\pi + \theta) - \omega_k]^2} t^2 dt \\ &\leq \frac{\pi}{12} m \sum_{k \in B_1} \frac{\Delta_k^3}{(\tau_k - \tau_1)^2} + \frac{\pi}{12} m \sum_{k \in B_2} \frac{\Delta_k^3}{(\tau_m - \tau_{k+1})^2} < A^3. \end{aligned}$$

Combining this with (4.5), we obtain (4.4).

Now we have enough machinery to prove the main theorem, namely

THEOREM 4.1. *Let f be any zero free holomorphic function in D , $f(0)=1$ and $\sup |f|=M < +\infty$. Then there exists a sequence of C -polynomials*

$$P_n(z) = \prod_{j=1}^n (1 - ze^{-it_{n,j}})$$

which converges to f in D , and is such that for an arbitrary $\epsilon > 0$,

$$(4.7) \quad \max_{|z| \leq 1} |P_n(z)| \leq e^{2\theta} \cdot M^{1+\epsilon} \quad \text{for all } n.$$

Proof. By Theorem 3.2, we can find a sequence of functions S_k of the form indicated in (3.2) and (3.3) which converges to f in D and is such that for all k ,

$$(4.8) \quad \max_{|z| \leq 1} |S_k(z)| \leq M^{1+\epsilon}.$$

By (3.3) it may be assumed that for all k

$$\max_{1 \leq m \leq k} a_{k,m} \leq \epsilon_1$$

where ϵ_1 is an arbitrarily fixed number between 0 and 1. By §2, we can find C -polynomials

$$P_{n+k}(z, k) = \prod_{j=1}^n (1 - ze^{-it_j}) \prod_{m=1}^k (1 - ze^{-i\theta_m^*}),$$

taking $\theta_m = 2\pi m/k$ in §2, such that P_{n+k} converges to S_k for any fixed k .

To get a suitable bound of P_{n+k} , we will apply the above lemma, using the present t_j and θ_m^* as points ω_j . The intervals (τ_j, τ_{j+1}) in the lemma will have the points t_j and θ_m^* as midpoints. Thus, the interval (τ_j, τ_{j+1}) with midpoint t_j will have length $2\pi/(n+k-\alpha)$; the interval (τ_j, τ_{j+1}) with midpoint θ_m^* will have length $(1-\alpha_{k,m})2\pi/(n+k-\alpha)$. Since $p=n+k$, the exponent of $(1 - z \exp(-it_j))$ in $R_p(z) = R_{n+k}(z)$ will be

$$\frac{n+k}{2\pi} \cdot \frac{2\pi}{n+k-\alpha} = \frac{n+k}{n+k-\alpha},$$

and the exponent of $(1 - ze^{-i\theta_m^*})$ will be

$$\frac{n+k}{2\pi} (1-\alpha_{k,m}) \frac{2\pi}{n+k-\alpha} = \frac{(1-\alpha_{k,m})(n+k)}{n+k-\alpha}.$$

It follows that we can take $A=1/(1-\epsilon_1)$ in the lemma. Hence, for large n ,

$$\max_{|z| \leq 1} |R_{n+k}(z)| \leq \exp [19(1-\epsilon_1)^{-3}].$$

That is,

$$\max_{|z| \leq 1} |P_{n+k}(z, k)| \leq \exp [19(1-\epsilon_1)^{-3}] \cdot \max_{|z| \leq 1} \prod_{m=1}^k |1 - ze^{-i\theta_m^*}|^{\alpha_{k,m}}.$$

But $\theta_m^* \rightarrow \theta_m = 2\pi m/k$ and the maximum norm is continuous. Hence, for an arbitrary $\varepsilon_2 > 0$, we have, for large k ,

$$\max_{|z| \leq 1} \prod_{m=1}^k |1 - ze^{-i\theta_m^*}|^{\alpha_{k,m}} \leq (1 + \varepsilon_2) \max_{|z| \leq 1} |S_k(z)|.$$

Taking ε_1 and ε_2 sufficiently small, we can conclude that for any k and sufficiently large n ,

$$\max_{|z| \leq 1} |P_{n+k}(z, k)| \leq e^{20} M^{1+\varepsilon}.$$

Since S_k also satisfies a uniform boundedness condition (4.8), the same argument as in the proof of Theorem 3.2 implies that there is a sequence of C -polynomials

$$P_n(z) = P_{n+k}(z, k), \quad n = n(k),$$

which satisfies (4.7) and which converges to f throughout D .

The bound in (4.7) can be improved just a little if we no longer require that the approximating polynomials have the value 1 at $z=0$. Indeed, let ε_n be a sequence of positive real numbers tending to 0; then there is a sequence of C -polynomials P_n satisfying (4.7) with ε replaced by ε_n . Multiplying P_n by $M^{-\varepsilon_n}$, we get a sequence of C -polynomials Q_n , which converges to f and satisfies

$$(4.9) \quad \max_{|z| \leq 1} |Q_n(z)| \leq e^{20} \cdot \sup_{|z| < 1} |f(z)|.$$

That is, we obtain the following

COROLLARY 4.1. *A zero free bounded holomorphic function f in D can be boundedly approximated by C -polynomials Q_n in D , which satisfy the inequality (4.9) for all n .*

REFERENCES

1. C. Chui, *Bounded approximation by polynomials with restricted zeros*, Bull. Amer. Math. Soc. **73** (1967), 967–972.
2. ———, *Bounded approximation by polynomials with restricted zeros*, Ph.D. Thesis, Univ. of Wisconsin, Madison, 1967.
3. J. Korevaar and C. Chui, *Potentials of families of unit masses on disjoint Jordan curves*, Proc. Conference on Approximation Theory and Abstract Spaces, Oberwolfach, Germany, (to appear).
4. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962, p. 67.
5. J. Korevaar, *Asymptotically neutral distributions of electrons and polynomial approximation*. Ann. of Math. **80** (1964), 403–410.
6. ———, “Limits of polynomials whose zeros lie in a given set,” in *Entire functions and related parts of analysis*, (1966 Summer Institute), Proc. Sympos. Pure Math., Vol. 11, Amer. Math. Soc., Providence, R. I., 1968.
7. G. R. MacLane, *Polynomials with zeros on a rectifiable Jordan curve*, Duke Math. J. **16** (1949), 461–477.
8. M. D. Thompson, *Approximation by polynomials whose zeros lie on a curve*, Duke Math. J. **31** (1964), 255–265.
9. A. Zygmund, *Trigonometric series*. I, Cambridge Univ. Press, New York, 1959, p. 49.

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