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# BOUNDED COHOMOLOGY OF CERTAIN GROUPS OF HOMEOMORPHISMS

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ABSTRACT. We consider the condition when bounded cohomology injects into ordinary cohomology and prove the vanishing of bounded cohomology of the group of all compactly supported homeomorphisms of $\mathbf{R}^n$.

**Introduction.** In this note we consider relations among bounded cohomology, ordinary real cohomology and $l^1$ homology of spaces or groups. In particular we present a necessary and sufficient condition under which bounded cohomology injects into ordinary cohomology and by using it prove the vanishing of bounded cohomology and $l^1$ homology of $\mathrm{Homeo}_K\mathbf{R}^n$, the group of all homeomorphisms of $\mathbf{R}^n$ with compact support. We also determine the second bounded cohomology of $\mathrm{SL}_2\mathbf{R}$.

**1. Bounded cohomology.** Let us quickly review the theory of bounded cohomology developed by Gromov [**2**] (see also Brooks [**1**] and Mitsumatsu [**5**]). Let $X$ be a topological space and let $\mathscr{C}_*(X) = \{C_q(X), \partial_q\}$ be the singular chain complex of $X$ with real coefficients. Define a norm on $C_q(X)$ by $\|\sum_{i=1}^n a_i\sigma_i\| = \sum_{i=1}^n |a_i|$. The differentials $\partial_q$ are then bounded linear operators.

Let $\mathscr{C}_*^{l_1}(X) = \{C_q^{l_1}(X), \partial_q\}$ be the norm completion of $\mathscr{C}_*(X)$. Thus $C_q^{l_1}(X) = \{\sum_{i=1}^\infty a_i\sigma_i | \sum_{i=1}^\infty |a_i| < \infty\}$ is a Banach space. Passing to the dual Banach spaces, we obtain a cochain complex $\mathscr{C}_b^*(X) = \{C_b^q(X), \delta_q\}$. It is a subcomplex of the ordinary singular cochain complex consisting of bounded cochains. The homology of $\mathscr{C}_*^{l_1}(X)$, denoted by $H_*^{l_1}(X)$, is called $l_1$ homology of $X$ and the cohomology of $\mathscr{C}_b^*(X)$, denoted by $H_b^*(X)$, is called bounded cohomology of $X$. The inclusions induce homomorphisms $H_*(X) \to H_*^{l_1}(X)$ and $H_b^*(X) \to H^*(X)$.

Since the image of a bounded operator is not necessarily a closed subspace, it may happen that the pseudonorms induced on $H_*^{l_1}(X)$ or $H_b^*(X)$ are not norms. Following Mitsumatsu [**5**], we define $\overline{H}_*^{l_1}(X)$ (resp. $\overline{H}_b^*(X)$) to be the quotient of $H_*^{l_1}(X)$ (resp. $H_b^*(X)$) by the subspace of pseudonorm zero. In other words, $\overline{H}_q^{l_1}(X) = Z_q^{l_1}(X)/\overline{B}_q^{l_1}(X)$ and $\overline{H}_b^q(X) = Z_b^q(X)/\overline{B}_b^q(X)$, where $Z$ or $B$ denotes the spaces of (co)cycles or (co)boundaries of the corresponding complex and $\overline{B}$ denotes the closure of $B$. Notice that $\overline{H}_q^{l_1}(X)$ and $\overline{H}_b^q(X)$ are Banach spaces. There is a

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surjective homomorphism $\overline{H}_b^q(X) \to (\overline{H}_q^{l_1}(X))'$, where $'$ denotes the dual Banach space.

Now for a group $G$, starting from the real chain complex $\mathscr{C}_*(G) = \{C_q(G), \delta_q\}$, similar constructions as above are made, yielding $l^1$ homology $H_*^{l_1}(G)$ and bounded cohomology $H_b^*(G)$ of $G$.

**2. Uniform boundary condition.** A chain complex is called *normed* if each chain group is a normed linear space over $\mathbf{R}$ and each differential is a bounded linear operator.

DEFINITION 2.1. A normed chain complex $\mathscr{C}_*$ is said to satisfy *$q$ uniform boundary condition* ($q$-UBC, for short) if there exists a number $K > 0$ such that for any boundary $z \in B_q$, there is a chain $c \in C_{q+1}$ satisfying $\partial c = z$ and $\|c\| \leqslant K\|z\|$.

DEFINITION 2.2. (i) A topological space or a group is said to satisfy $q$-UBC if its ordinary chain complex satisfies $q$-UBC.

(ii) It is said to satisfy $q$-UBC$^{l_1}$ if its $l^1$ chain complex satisfies $q$-UBC.

THEOREM 2.3. *For spaces or groups, the following conditions are equivalent*:

(i) $q$-UBC$^{l_1}$.

(ii) $B_q^{l_1}$ *is closed in* $C_q^{l_1}$.

(iii) $\overline{H}_q^{l_1} = H_q^{l_1}$.

(iv) $\overline{H}_b^{q+1} = H_b^{q+1}$.

(v) *The surjective homomorphism* $H_b^{q+1} \to (\overline{H}_{q+1}^{l_1})'$ *is injective.*

PROOF. (i) is clearly equivalent to that the bijection $C_{q+1}^{l_1}/Z_{q+1}^{l_1} \to B_q^{l_1}$ has a bounded inverse, that is, they are homeomorphic. Since $C_{q+1}^{l_1}/Z_{q+1}^{l_1}$ is a Banach space, it follows that $B_q^{l_1}$ is also a Banach space and hence (ii) holds. Conversely if (ii) is satisfied, then $B_q^{l_1}$ is a Banach space and the open mapping theorem (cf. [**11**]) applied to the above bijection implies (i).

(ii) $\Leftrightarrow$ (iii) is clear.

(iii) $\Leftrightarrow$ (iv) is a direct consequence of the closed range theorem.

(i) $\Rightarrow$ (v). Take $f \in Z_b^{q+1}$ so that $f(z) = 0$ for all $z \in Z_{q+1}^{l_1}$. By (i), the map $B_q^{l_1} \leftarrow C_{q+1}^{l_1}/Z_{q+1}^{l_1} \xrightarrow{f} \mathbf{R}$ is bounded and thus by the Hahn-Banach theorem has an extension to $C_q^{l_1}$. This shows (v).

(v) $\Rightarrow$ (iv). It suffices to show $\operatorname{Ker}(H_b^{q+1} \to \overline{H}_b^{q+1}) \subset \operatorname{Ker}(H_b^{q+1} \to (\overline{H}_{q+1}^{l_1})')$. Let $f \in \overline{B}_b^{q+1}$. Then $f = \lim_{i\to\infty} f_i$ ($f_i \in B_b^{q+1}$). Thus for any $z \in Z_{q+1}^{l_1}$, we have $f(z) = \lim_{i\to\infty} f_i(z) = 0$. This completes the proof. Q.E.D.

COROLLARY 2.4. (i) *If* $\overline{H}_q^{l_1} = H_q^{l_1}$ *and* $\overline{H}_{q+1}^{l_1} = 0$, *then* $H_b^{q+1} = 0$.

(ii) *If* $\overline{H}_b^{q+1} = H_b^{q+1}$ *and* $\overline{H}_b^q = 0$, *then* $H_q^{l_1} = 0$.

(iii) *The reduced bounded cohomology* $\tilde{H}_b^*$ *vanishes if and only if the reduced* $l^1$ *homology* $\tilde{H}_*^{l_1}$ *also vanishes.*

In Brooks [**1**] and Gromov [**2**], it is shown that the reduced bounded cohomology vanishes for spaces with amenable $\pi_1$. This, combined with the above corollary, gives

COROLLARY 2.5. *If* $\pi_1(X)$ *is amenable*, *then* $\tilde{H}_*^{l_1}(X) = 0$.

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REMARK 2.6. It seems plausible that the $l^1$ homology of a space depends only on its fundamental group. But we do not have a proof.

COROLLARY 2.7. *For any space or a group, we have*
(i) $H_1^{l_1} = H_b^1 = 0$.
(ii) $H_b^{-2} = H_b^2$. *Equivalently*, $H_b^2$ *is a Banach space*.

PROOF. First we deal with a group $G$. In [5], Mitsumatsu constructed for $g \in g$,

$$S(g) = \sum_{k=1}^{\infty} \frac{1}{2^k}(g^k, g^k) \in C_2^{l_1}(G).$$

Clearly, $\|S(g)\| = 1$ and $\partial S(g) = g$, showing that $H_1^{l_1}(G) = 0$ and that $G$ satisfies 1-UBC$^{l_1}$. From this follows the corollary. Notice that 0-UBC$^{l_1}$ is always satisfied.

For spaces, according to Gromov [2], $H_b^2(X)$ is isometric to $H_b^2(\pi_1(X))$. This shows (ii). Also it is known that $H_b^1(X) = 0$ [2]. Thus (i) follows from Corollary 2.4. Q.E.D.

We have shown that 1-UBC$^{l_1}$ is always true. It would be interesting to determine whether $q$-UBC$^{l_1}$ always holds or not.

Next, we investigate $q$-UBC for spaces or groups.

THEOREM 2.8. *The following conditions are equivalent*:
(i) $q$-UBC.
(ii) $q$-UBC$^{l_1}$ *and* $Z_{q+1}$ *is dense in* $Z_{q+1}^{l_1}$.
(iii) *The homomorphism* $H_b^{q+1} \to H^{q+1}$ *is injective*.

PROOF. (i) $\Rightarrow$ (ii). We first prove $q$-UBC$^{l_1}$. Clearly $B_q$ is dense in $B_q^{l_1}$. Further a standard argument shows that for any $z \in B_q^{l_1}$, there exist $z_i \in B_q$ such that $\sum_{i=1}^{\infty} z_i = z$ and $\sum_{i=1}^{\infty}\|z_i\| \leqslant (1+\varepsilon)\|z\|$. Now by $q$-UBC, one can choose $c_i \in C_{q+1}$ such that $\partial c_i = z_i$ and $\|c_i\| \leqslant K\|z_i\|$. Let $c = \sum_{i=1}^{\infty} c_i \in C_{q+1}^{l_1}$. Then $\partial c = z$ and $\|c\| \leqslant (1+\varepsilon)K\|z\|$.

Next we prove that $Z_{q+1}$ is dense in $Z_{q+1}^{l_1}$. Take $z \in Z_{q+1}^{l_1}$ and let $z = \lim_{i\to\infty} c_i$ ($c_i \in C_{q+1}$). Choose an element $d_i \in C_{q+1}$ such that $\partial d_i = -\partial c_i$ and $\|d_i\| \leqslant K\|\partial c_i\|$. Then we have

$$\|\partial c_i\| = \|\partial(c_i - z)\| \leqslant \|\partial\|\,\|c_i - z\| = (q+2)\|c_i - z\|.$$

Hence $\|d_i\| \leqslant (q+2)K\|c_i - z\|$. Now $c_i + d_i \in Z_{q+1}$ and $c_i + d_i \to z$.

(ii) $\Rightarrow$ (i). This is left to the reader.

(ii) $\Rightarrow$ (iii). Take $f \in Z_b^{q+1}$ such that $[f] = 0$ in $H^{q+1}$, that is, $f(z) = 0$ for any $z \in Z_{q+1}$. Then $f(z) = 0$ for any $z \in Z_{q+1}^{l_1}$. Thus $f$ induces a bounded map $\bar{f}$: $C_{q+1}^{l_1}/Z_{q+1}^{l_1} \to \mathbf{R}$. Now $q$-UBC$^{l_1}$ implies that the bijection $C_{q+1}^{l_1}/Z_{q+1}^{l_1} \to B_q^{l_1}$ has a bounded inverse. Compose it with $\bar{f}$ and extend to the whole of $C_q^{l_1}$ by the Hahn-Banach theorem. This shows (iii).

(iii) $\Rightarrow$ (ii). Notice that the map $H_b^{q+1} \to H^{q+1}$ is a composite of maps $H_b^{q+1} \to \overline{H}_b^{q+1} \to H^{q+1}$. Injectivity of the first map implies $q$-UBC$^{l_1}$ by Theorem 2.3. Next, clearly (iii) implies that the image of the map $H_{q+1} \to \overline{H}_{q+1}^{l_1}$ is dense. From this the denseness of $Z_{q+1}$ follows easily. Q.E.D.

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DEFINITION 2.9. A group $G$ is said to be *uniformly perfect* if for some $N > 0$, any $g \in G$ can be represented as a product of at most $N$ commutators.

LEMMA 2.10. *If a group $G$ is uniformly perfect, then it satisfies* 1-UBC.

PROOF. Notice that for $f_i, g, h \in G$,

$$\partial\big((f_1, f_2) + (f_1 f_2, f_3) + \cdots + (f_1 f_2 \cdots f_{N-1}, f_N)\big) \\ = (f_1) + (f_2) + \cdots + (f_N) - (f_1 f_2 \cdots f_N)$$

and

$$\partial\big(([g, h], h) + (ghg^{-1}, g) - (g, h)\big) = ([g, h])$$

where ( ) denotes a chain and [ ] a commutator. This shows for all $f \in G$ there exists $c \in C_2(G)$ such that $\partial c = (f)$ and $\|c\| \leqslant 4N - 1$. That is, $G$ satisfies 1-UBC. Q.E.D.

COROLLARY 2.11. *If $G$ is uniformly perfect, then the map $H_b^2(G) \to H^2(G)$ is injective.*

As applications, we shall compute $H_b^2$ for some groups.

EXAMPLE 2.12. The group of all orientation preserving homeomorphisms of $S^1$, denoted by $\mathrm{Homeo}_+(S^1)$, is uniformly perfect. In fact any element is a product of two homeomorphisms with compact support, which are commutators by Mather [**4**]. Thus $H_b^2$ injects into $H^2$. Now it is a consequence of Thurston's general result [**8**] that $H^*(\mathrm{Homeo}_+(S^1); \mathbf{Z}) \cong \mathbf{Z}[\chi]$, where $\chi \in H^2$ is the Euler class. Now $H_b^*$ is mapped onto real cohomology, because $\chi$ can be represented by a bounded cocycle (see Morita [**6**]). Hence we have $H_b^2(\mathrm{Homeo}_+(S^1)) \cong \mathbf{R}$.

EXAMPLE 2.13. Sah and Wagoner [**7**] have calculated second homology of certain Lie groups (considered to be discrete groups). Combined with our result, this gives information about $H_b^2$. For example, $\mathrm{SL}_2\mathbf{R}$ is uniformly perfect (see Wood [**10**]) and $H_2(\mathrm{SL}_2\mathbf{R}; \mathbf{Z})$ is isomorphic to $\mathbf{Z} \oplus A$, where $A$ is a certain $\mathbf{Q}$-vector space. $\mathbf{Z}$ is detected by the "volume class" $\in H^2(\mathrm{SL}_2\mathbf{R})$, which is a bounded cohomology class. Any element of $A$ is supported on a torus (see [**7**] and Tsuboi [**9**]). From these, we can conclude $H_b^2(\mathrm{SL}_2\mathbf{R}) \cong H_b^2(\mathrm{PSL}_2\mathbf{R}) \cong \mathbf{R}$.

**3. Bounded cohomology of** $\mathrm{Homeo}_K(\mathbf{R}^n)$. In this section we prove the vanishing of bounded cohomology and $l^1$ homology of $\mathrm{Homeo}_K(\mathbf{R}^n)$, the group of all homeomorphisms of $\mathbf{R}^n$ with compact support.

THEOREM 3.1. *For $q > 0$,*

$$H_b^q(\mathrm{Homeo}_K(\mathbf{R}^n)) = H_q^{l_1}(\mathrm{Homeo}_K(\mathbf{R}^n)) = 0.$$

Our argument is a refinement of Mather's proof of the acyclicity of $\mathrm{Homeo}_K(\mathbf{R}^n)$. In the sequel we follow Mather [**4**]. We write $G = \mathrm{Homeo}_K(\mathbf{R}^n)$ and $G^i = \{g \in G\colon \operatorname{supp} g \subset \operatorname{Int} iD^n\}$ $(i = 1, 2, 3)$, where $D^n$ is the unit ball. Inclusions are denoted by $\iota^1\colon G^1 \to G^2$, $\iota^2\colon G^2 \to G^3$, $\bar{\iota} = \iota^2\iota^1$ and $\iota\colon G^1 \to G$. Let $C_q$ and $C_q^i$ be the chain complex of $G$ and $G^i$. $Z_q$ and $Z_q^i$ (resp. $B_q$ and $B_q^i$) denote the cycle group (resp.

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boundary group) of the corresponding complexes. In fact, $Z_q = B_q$ and $Z_q^i = B_q^i$ by the acyclicity of the group.

We shall prove inductively the existence of bounded linear operators $S_q\colon B_q^1 \to C_{q+1}$ such that $\partial_{q+1} S_q = \iota_*$. Let us show first that this suffices for our purpose. We have only to show $q$-UBC for $G$, because the acyclicity of $G$, together with Theorem 2.8 and Corollary 2.4 yields Theorem 3.1. Take $z \in B_q^1$. Choose $\varphi \in G$ such that $\varphi$ is the identity on supp $z$ and maps supp $S_q(z)$ into Int $D^n$. Define $I_\varphi\colon G \to G$ by $I_\varphi(g) = \varphi g \varphi^{-1}$. Then we have $I_{\varphi *}(S_q(z)) \in C_{q+1}^1$, $\|I_{\varphi *}(S_q(z))\| = \|S_q(z)\| \leqslant \|S_q\| \, \|z\|$ and $\partial(I_{\varphi *}(S_q(z))) = I_{\varphi *}(\partial S_q(z)) = I_{\varphi *}(z) = z$. This proves $q$-UBC for $G^1$, hence for $G$.

$S_1$ is constructed in an elementary fashion as follows. Choose $k \in G$ such that $k(3D^n) \cap 3D^n = \varnothing$ and that $k^i(3D^n)$ tends to one point as $i \to \infty$. Define $\psi_1\colon G^3 \to G$ by $\psi_1(g) = \sum_{i=1}^{\infty} k^i g k^{-i}$ and let $\psi_0(g) = k^{-1}\psi_1(g)k$. Then for any $g \in G^3$, supp $g \cap$ supp $\psi_1(g) = \varnothing$ and $\psi_0(g) = g\psi_1(g)$. The restrictions of $\psi_i$ to $G^1$ are denoted by the same letter. Now we define a bounded linear map $S_1\colon B_1^1 \to C_2$ by

$$S_1(g) = (k, g) - (\psi_1(g), k) + (kg, \psi_1(g)).$$

Direct computation shows $\partial_2 S_1 = \iota_*$.

Next we assume there exist $S_j^1\colon B_j^1 \to C_{j+1}^2$ and $S_j^2\colon B_j^2 \to C_{j+1}^3$ for $0 \leqslant j \leqslant q-1$ and construct $S_q\colon B_j^1 \to C_{j+1}$. Let $\alpha\colon C_*(G \times G) \to C_* \otimes C_*$ (resp. $\beta\colon C_* \otimes C_* \to C_*(G \times G)$) be the Alexander-Whitney map (resp. Eilenberg-Mac Lane map) (see Mac Lane [**3**]). Those maps for $G^i$ are also denoted by the same letters. They are functorial and if we give a norm to $C_* \otimes C_*$ in a canonical manner, they are bounded linear.

For each $z \in B_q^1$, define $D(z) = \alpha\Delta_* z - (z \otimes 1 + 1 \otimes z)$, where $\Delta\colon G^1 \to G^1 \times G^1$ is the diagonal map. Then $D(z) \in Z_q'(C^1 \otimes C^1) = Z_q(C^1 \otimes C^1) \cap \sum_{i=1}^{q-1} C_i^1 \otimes C_{q-}^1$. $D$ is bounded linear. Now let $Z^1$ and $\overline{B}^1$ be the chain complexes (with trivial differential) defined by $(Z^1)_q = Z_q^1$ and $(\overline{B}^1)_q = B_{q-1}^1$. Then we have the following commutative diagram, whose horizontal sequences are all exact.

$$\begin{array}{ccccccccc}
0 & \to & (C^1 \otimes Z^1)_{q+1} & \to & (C^1 \otimes C^1)_{q+1} & \overset{1\otimes\partial}{\to} & (C^1 \otimes \overline{B}^1)_{q+1} & \to & 0 \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
0 & \to & (C^1 \otimes Z^1)_{q} & \to & (C^1 \otimes C^1)_{q} & \overset{1\otimes\partial}{\to} & (C^1 \otimes \overline{B}^1)_{q} & \to & 0 \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
0 & \to & (C^1 \otimes Z^1)_{q-1} & \to & (C^1 \otimes F^1)_{q-1} & \overset{1\otimes\partial}{\to} & (C^1 \otimes \overline{B}^1)_{q-1} & \to & 0
\end{array}$$

Analogous diagrams are considered for $G^2$ and $G^3$. They are combined by $\iota_*^1$ and $\iota_*^2$. Now because $\partial(D(z)) = 0$, $(1 \otimes \partial)D(z)$ is contained in $Z_q'(C^1 \otimes \overline{B}^1)$. Notice that $Z_q'(C \otimes \overline{B}^1) = (Z^1 \otimes \overline{B}^1)_q'$. Hence by the induction assumption we can consider

$$(S^1 \otimes S^1)(1 \otimes \partial)D(z) = (S^1 \otimes S^1\partial)D(z) \in (C^2 \otimes C^2)_{q+1}.$$

Let $u = (\iota_*^1 \otimes \iota_*^1)D(z) - \partial(S^1 \otimes S^1\partial)D(z) \in (C^2 \otimes C^2)_q$. Direct computation shows that $(1 \otimes \partial)u = 0$. Thus we have $u \in (C^2 \otimes Z^2)_q'$. Also we have $\partial u = 0$. That is,

$$u \in Z_q'(C^2 \otimes Z^2) = (Z^2 \otimes Z^2)_q'.$$

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Therefore by the induction assumption, we can consider

$$(S^2 \otimes (\iota_*^2 - S^2\partial))u \in (C^3 \otimes C^3)_{q+1}.$$

Again direct computation shows that

$$\partial\big(S^2 \otimes (\iota_*^2 - S^2\partial)\big)u = \big(\iota_*^2 \otimes \iota_*^2\big)u.$$

Hence we have $(\bar{\iota}_* \otimes \bar{\iota}_*)D(z) = \partial(ED(z))$, where

$$E = \big(\iota_*^2 S^1 \otimes \iota_*^2 S^1\partial\big) + \big(S^2 \otimes (\iota_*^2 - S^2\partial)\big)\big(\iota_*^1 - \partial(S^1 \otimes S^1\partial)\big).$$

$E\colon Z_q'(C^1 \otimes C^1) \to (C^3 \otimes C^3)_{q+1}$ is a bounded linear map. We can now write

$$(\bar{\iota}_* \otimes \bar{\iota}_*)\alpha\Delta_* z = (\bar{\iota}_* \otimes \bar{\iota}_*)z \otimes 1 + \partial\big(ED(z)\big) + (\bar{\iota}_* \otimes \bar{\iota}_*)(1 \otimes z).$$

Let $\eta\colon G^3 \otimes G^3 \to G$ be the homomorphism given by $\eta(g, h) = g\psi_1(h)$. Applying $\eta_*\beta$, we get

$$\eta_*(\bar{\iota} \times \bar{\iota})_*\beta\alpha\Delta_* z = \iota_* z + \eta_*\beta\partial\big(ED(z)\big) + \chi_{1*} z.$$

As is well known, $\beta\alpha$ is chain homotopic to the identity. Namely there is a linear map $\Phi\colon C_*(G^1 \times G^1) \to C_{*+1}(G^1 \times G^1)$ such that $\beta\alpha - \mathrm{id} = \partial\Phi + \Phi\partial$. It is easy to show that we can choose $\Phi$ as a bounded linear map. Now $\beta\alpha\Delta_* z = \Delta_* z + \partial\Phi(\Delta_* z)$. Because $\eta\Delta = \psi_0$, we have

$$\psi_{0*} z + \eta_*(\bar{\iota} \times \bar{\iota})_*\partial\Phi\Delta_* z = \iota_* z + \eta_*\beta\partial ED(z) = \psi_{1*} z.$$

Two homomorphisms $\psi_0$ and $\psi_1$ are conjugate and thus $\psi_{0*}$ is chain homotopic to $\psi_{1*}$. Namely there is a linear map $\Theta\colon C_*^1 \to C_{*+1}$ such that $\psi_{1*} - \psi_{0*} = \partial\Theta + \Theta\partial$. Here we can also choose $\Theta$ to be bounded linear. Finally we obtain $\iota_* = \partial_{q+1}S_q$, where

$$S_q = \eta_*(\bar{\iota} \times \bar{\iota})_*\Phi\Delta_* - \Theta - \eta_*\beta ED.$$

This completes the proof.

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