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Bounded elements of C*-inductive locally convex spaces

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Abstract The notion of bounded element of C*-inductive locally convex spaces (or C*-inductive partial *-algebras) is introduced and discussed in two ways: The first one takes into account the inductive structure provided by certain families of C*-algebras; the second one is linked to the natural order of these spaces. A particular attention is devoted to the relevant instance provided by the space of continuous linear maps acting in a rigged Hilbert space.

Keywords Bounded elements · Inductive limit of C*-algebras · Partial *-algebras

Mathematics Subject Classification 47L60 · 47L40

1 Introduction

Some locally convex spaces exhibit an interesting feature: They contain a large number of C*-algebras that often contribute to their topological structure, in the sense that these spaces can be thought as *generalized* inductive limits of C*-algebras. These objects were called *C*-inductive locally convex spaces* in [8] and their structure was examined in detail, also taking in mind that they arise naturally when one considers the operators acting in the *joint topological limit* of an inductive family of Hilbert spaces as described in [9]. Indeed, a typical instance of this structure is obtained by considering the space $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^{\times})$ of operators acting in the rigged Hilbert space canonically associated with an O*-algebra of unbounded operators acting on a dense domain $\mathcal{D}$ of Hilbert space $\mathcal{H}$. In [8], a series of features of this structure was studied giving a particular attention to the order structure, positive linear functionals

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and representation theory. The space $\mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$ contains a subspace isomorphic to the *-algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators in $\mathcal{H}$ whose elements can be in natural way considered as the *bounded elements* of $\mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$. The notion of bounded element of a locally convex *-algebra $\mathfrak{A}$ was first introduced by Allan [1] with the aim of developing a spectral theory for topological *-algebras: An element x of the topological *-algebra $\mathfrak{A}[\tau]$ is *Allan bounded* if there exists $\lambda \neq 0$ such that the set $\{(\lambda^{-1}x)^n;\ n = 1, 2, \dots\}$ is a bounded subset of $\mathfrak{A}[\tau]$. This definition was suggested by the successful spectral analysis for closed operators in Hilbert space $\mathcal{H}$: A complex number λ is in the resolvent set $\rho(T)$ of a closed operator T if $T - \lambda I$ has an inverse in the *-algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators.

There are, however, several other possibilities for defining bounded elements. For instance, one may say that x is bounded if $\pi(x)$ is a bounded operator, for every (continuous, in a certain sense) *-representation π defined on a dense domain $\mathcal{D}_\pi$ of some Hilbert space $\mathcal{H}_\pi$. This could be a reasonable definition in itself, provided that $\mathfrak{A}$ possesses sufficiently many *-representations in Hilbert space.

Moreover some attempts to extend this notion to the larger setup of locally convex quasi *-algebras [10,17–20] or locally convex partial *-algebras [2,5,6] have been done. But in these cases, Allan's notion cannot be adopted, since powers of a given element x need not be defined.

In the case of *-algebras, bounded elements in purely algebraic terms have been considered by Vidav [22] and Schmüdgen [15] with respect to some (positive) wedge.

The aim of this paper is to extend the notion of bounded element to the case of C*-inductive locally convex spaces $\mathfrak{A}$ with defining family of C*-algebras $\{\mathfrak{B}_\alpha; \alpha \in \mathbb{F}\}$($\mathbb{F}$ is an index set directed upward). There are also in this case several possibilities: The first one consists in taking elements that have *representatives* in every C*-algebra $\mathfrak{B}_\alpha$ of the family whose norms are uniformly bounded; the second one consists into taking into account the order structure of $\mathfrak{A}$, in the same spirit of the quoted papers of Vidav and Schmüdgen.

The paper is organized as follows. After some preliminaries (Sect. 2), we study, in Sect. 3, how *bounded elements* of $\mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$ can be derived from its C*-inductive structure and from its order structure. We show that these two notions are equivalent and that an element X is bounded if and only if X maps $\mathcal{D}$ into $\mathcal{H}$ and $\overline{X} \in \mathfrak{B}(\mathcal{H})$. Finally, in Sect. 4, we consider the same problem for abstract C*-inductive locally convex spaces and give conditions for some of the characterizations proved for $\mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$ maintain their validity. Some of these results are then specialized to the case where $\mathfrak{A}$ is a C*-inductive locally convex partial *-algebra.

2 Notations and preliminaries

For general aspects of the theory of partial *-algebras and of their representations, we refer to the monograph [3]. For the convenience of the reader, however, we repeat here the essential definitions.

A partial *-algebra $\mathfrak{A}$ is a complex vector space with conjugate linear involution * and a distributive partial multiplication $\cdot$, defined on a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$, satisfying the property that $(x, y) \in \Gamma$ if, and only if, $(y^*, x^*) \in \Gamma$ and $(x \cdot y)^* = y^* \cdot x^*$. From now on, we will write simply xy instead of $x \cdot y$ whenever $(x, y) \in \Gamma$. For every $y \in \mathfrak{A}$, the set of left (resp. right) multipliers of y is denoted by $L(y)$ (resp. $R(y)$), i.e., $L(y) = \{x \in \mathfrak{A} : (x, y) \in \Gamma\}$, (resp. $R(y) = \{x \in \mathfrak{A} : (y, x) \in \Gamma\}$). We denote by $L\mathfrak{A}$ (resp. $R\mathfrak{A}$) the space of universal left (resp. right) multipliers of $\mathfrak{A}$. In general, a partial *-algebra is not associative.

The *unit* of partial *-algebra $\mathfrak{A}$, if any, is an element $e \in \mathfrak{A}$ such that $e = e^*$, $e \in R\mathfrak{A} \cap L\mathfrak{A}$ and $xe = ex = x$, for every $x \in \mathfrak{A}$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $D(X)=\mathcal{D}$, $D(X^*)\supseteq\mathcal{D}$. The map $X\to X^\dagger = X^*_{\restriction\mathcal{D}}$ defines an involution on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, which can be made into a partial *-algebra with respect to the *weak* multiplication [3]; however, this fact will not be used in this paper.

Let $\mathcal{L}^\dagger(\mathcal{D})$ be the subspace of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to the usual operations. A *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$, containing the identity I of $\mathcal{D}$, is called an O*-algebra.

Let $\mathfrak{M}$ be an O*-algebra. The *graph topology* $t_{\mathfrak{M}}$ on $\mathcal{D}$ is the locally convex topology defined by the family $\{\|\cdot\|_A\}_{A\in\mathfrak{M}}$, where

$$\|\xi\|_A=\sqrt{\|\xi\|^2+\|A\xi\|^2}=\|(I+A^*\overline{A})^{1/2}\xi\|,\quad \xi\in\mathcal{D}.$$

For $A=0$, the null operator of $\mathcal{L}^\dagger(\mathcal{D})$, $\|\cdot\|_0$ is exactly the norm of $\mathcal{H}$, thus we will omit the 0 in the notation of the norm.

The topology $t_{\mathfrak{M}}$ is finer than the norm topology, unless $\mathfrak{M}$ does consist of bounded operators only.

If $\mathfrak{M}$ is an O*-algebra, we write $A\preceq B$ if $\|A\xi\|\le\|B\xi\|$, for every $\xi\in\mathcal{D}$. Then, $\mathfrak{M}$ is directed upward with respect to this order relation.

If $A\in\mathfrak{M}$, we denote by $\mathcal{H}_A$ the Hilbert space obtained by endowing $D(\overline{A})$ with the graph norm $\|\cdot\|_A$.

If $A,B\in\mathfrak{M}$ and $A\preceq B$, then $U_{BA}=(I+B^*\overline{B})^{-1/2}(I+A^*\overline{A})^{1/2}$ is a contractive map of $\mathcal{H}_A$ into $\mathcal{H}_B$; i.e., $\|U_{BA}\xi\|_B\le\|\xi\|_A$, for every $\xi\in\mathcal{H}_A$.

If the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ is complete, then $\mathfrak{M}$ is said to be *closed*.

If $\mathfrak{M}=\mathcal{L}^\dagger(\mathcal{D})$ then the corresponding graph topology denoted by $t_\dagger$ instead of $t_{\mathcal{L}^\dagger(\mathcal{D})}$.

As is known, a locally convex topology t on $\mathcal{D}$ is finer than the topology induced by the Hilbert norm defines, in standard fashion, a *rigged Hilbert space* (RHS)

$$\mathcal{D}[t]\hookrightarrow\mathcal{H}\hookrightarrow\mathcal{D}^\times[t^\times],$$

where $\mathcal{D}^\times$ is the vector space of all continuous conjugate linear functionals on $\mathcal{D}[t]$, i.e., the conjugate dual of $\mathcal{D}[t]$, endowed with the *strong dual topology* $t^\times=\beta(\mathcal{D}^\times,\mathcal{D})$, and $\hookrightarrow$ denotes a continuous embedding with dense range. The Hilbert space $\mathcal{H}$ is identified (by considering the form which puts $\mathcal{D}$ and $\mathcal{D}^\times$ into conjugate duality as an extension of the inner product of $\mathcal{D}$) with a dense subspace of $\mathcal{D}^\times[t^\times]$.

Let $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ denote the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$. In $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$, an involution $X\mapsto X^\dagger$ can be introduced by the equality

$$\langle X\xi\,|\eta\rangle=\overline{\langle X^\dagger\eta\,|\xi\rangle},\quad\forall\xi,\eta\in\mathcal{D}.$$

Hence, $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ is a *-invariant vector space.

To every $X\in\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$, there corresponds a separately continuous sesquilinear form θ_X on $\mathcal{D}\times\mathcal{D}$ defined by

$$\theta_X(\xi,\eta)=\langle X\xi\,|\eta\rangle,\quad\xi,\ \eta\in\mathcal{D}.$$

The vector space of all *jointly* continuous sesquilinear forms on $\mathcal{D}\times\mathcal{D}$ will be denoted with $\mathsf{B}(\mathcal{D},\mathcal{D})$. We denote by $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ the subspace of all $X\in\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ such that $\theta_X\in\mathsf{B}(\mathcal{D},\mathcal{D})$ and by $\mathfrak{L}^\dagger(\mathcal{D})$ the *-algebra consisting of all operators of $\mathcal{L}^\dagger(\mathcal{D})$, which together with their adjoints are continuous from $\mathcal{D}[t]$ into $\mathcal{D}[t]$. If $t=t_\dagger$, then $\mathfrak{L}^\dagger(\mathcal{D})=\mathcal{L}^\dagger(\mathcal{D})$. We will refer to the rigged Hilbert space defined by endowing $\mathcal{D}$ with the topology $t_\dagger$ as to the *canonical* rigged Hilbert space defined by $\mathcal{L}^\dagger(\mathcal{D})$ on $\mathcal{D}$. In this case $(\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times), \mathcal{L}^\dagger(\mathcal{D}))$ is a quasi *-algebra [3].

The spaces $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ and $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ have been studied at length by several authors (see, e.g., [11–13,21]) and several pathologies concerning their multiplicative structure have been considered (see also [3,4] and references therein). Recently some spectral properties of operators of these classes have also been studied [7].

3 Bounded elements of $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$

The inductive structure of $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$, with $\mathcal{D}$ endowed with the graph topology $t_\dagger$, has been discussed in [8, Section 5]. To keep the paper reasonably self-contained, we sum the main features up.

By the definition itself, $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ if, and only if, there exists $\gamma_X > 0$ and $A \in \mathcal{L}^\dagger(\mathcal{D})$ such that

$$|\theta_X(\xi, \eta)| = |\langle X\xi | \eta\rangle| \leq \gamma_X \|\xi\|_A \|\eta\|_A, \quad \forall \xi, \eta \in \mathcal{D}. \tag{1}$$

Conversely, if $\theta \in \mathsf{B}(\mathcal{D}, \mathcal{D})$, there exists a unique $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ such that $\theta = \theta_X$.

Thus, the map

$$\mathbb{I} : X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times) \mapsto \theta_X \in \mathsf{B}(\mathcal{D}, \mathcal{D})$$

is an isomorphism of vector spaces and $\mathbb{I}(\theta^*) = X^\dagger$, where $\theta^*(\xi, \eta) = \overline{\theta(\eta, \xi)}$, for every $\xi, \eta \in \mathcal{D}$.

We denote by $\mathsf{B}^A(\mathcal{D}, \mathcal{D})$ the subspace of $\mathsf{B}(\mathcal{D}, \mathcal{D})$ consisting of all $\theta \in \mathsf{B}(\mathcal{D}, \mathcal{D})$ such that (1) holds for fixed $A \in \mathcal{L}^\dagger(\mathcal{D})$.

If $\theta \in \mathsf{B}^A(\mathcal{D}, \mathcal{D})$, it extends to a bounded sesquilinear form on $\mathcal{H}_A \times \mathcal{H}_A$ (we use the same symbol for this extension). Hence, there exists a unique operator $X_A^\theta \in \mathfrak{B}(\mathcal{H}_A)$ such that

$$\theta(\xi, \eta) = \left\langle X_A^\theta \xi | \eta \right\rangle_A, \quad \forall \xi, \eta \in \mathcal{H}_A.$$

On the other hand, if $X_A \in \mathfrak{B}(\mathcal{H}_A)$, then the sesquilinear form θ_{X_A} defined by

$$\theta_{X_A}(\xi, \eta) = \langle X_A \xi | \eta\rangle_A, \quad \xi, \eta \in \mathcal{D},$$

is an element of $\mathsf{B}^A(\mathcal{D}, \mathcal{D})$ and the map

$$\Phi_A : X_A \in \mathfrak{B}(\mathcal{H}_A) \to \theta_{X_A} \in \mathsf{B}^A(\mathcal{D}, \mathcal{D})$$

is a *-isomorphism of vector spaces with involution.

If $B \succeq A$, then, for $\xi, \eta \in \mathcal{D}$,

$$|\theta_{X_A}(\xi, \eta)| = |\langle X_A \xi | \eta\rangle_A| \leq \|X_A\|_{A,A} \|\xi\|_A \|\eta\|_A \leq \|X_A\|_{A,A} \|\xi\|_B \|\eta\|_B,$$

where $\|\cdot\|_{A,A}$ denotes the operator norm in $\mathfrak{B}(\mathcal{H}_A)$. Hence, there exists a unique $X_B \in \mathfrak{B}(\mathcal{H}_B)$ such that

$$\langle X_A \xi | \eta\rangle_A = \langle X_B \xi | \eta\rangle_B, \quad \forall \xi, \eta \in \mathcal{D}.$$

So it is natural to define

$$J_{BA}(X_A) = X_B, \quad \forall X_A \in \mathfrak{B}(\mathcal{H}_A).$$

It is easily seen that $J_{BA} = \Phi_B^{-1} \Phi_A$.

The space $\mathfrak{L}_{\mathsf{B}}^{A}(\mathcal{D},\mathcal{D}^{\times}) := \mathbb{I}^{-1}\mathsf{B}^{A}(\mathcal{D},\mathcal{D})$ is a Banach space, with norm

$$\|X\|^{A} := \sup_{\|\xi\|_A, \|\eta\|_A \leq 1} |\theta_X(\xi,\eta)|$$

and $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})$ can be endowed with the inductive topology τ_{ind} defined by the family of subspaces $\{\mathfrak{L}_{\mathsf{B}}^{A}(\mathcal{D},\mathcal{D}^{\times});\ A \in \mathcal{L}^{\dagger}(\mathcal{D})\}$ as in [16, Section 1.2.III].

In conclusion,

$$X_A \in \mathfrak{B}(\mathcal{H}_A) \leftrightarrow \theta_{X_A} \in \mathsf{B}^{A}(\mathcal{D},\mathcal{D}) \leftrightarrow X \in \mathfrak{L}_{\mathsf{B}}^{A}(\mathcal{D},\mathcal{D}^{\times})$$

are isometric *-isomorphisms of Banach spaces.

Hence, to every $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})$ one can associate the net $\{X_B;\ B \in \mathcal{L}^{\dagger}(\mathcal{D});\ B \succeq A\}$ of its representatives in each of the spaces $\mathcal{H}_B$.

Definition 3.1 We say that $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})$ is a *bounded element* of $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})$ if X has a representative X_A in every $\mathfrak{B}(\mathcal{H}_A)$ and

$$\|X\|_b := \sup_{A \in \mathcal{L}^{\dagger}(\mathcal{D})} \|X_A\|_{A,A} < +\infty.$$

The space $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})_b$ of all bounded elements of $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})$ is a Banach space with norm $\|\cdot\|_b$.

Proposition 3.2 *$\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})_b$ is *-isomorphic (as Banach space) to a C*-algebra of operators.*

Proof Let $\mathcal{H}_{\oplus}$ denote the Hilbert space direct sum of the $\mathcal{H}_A$, $A \in \mathcal{L}^{\dagger}(\mathcal{D})$; i.e.,

$$\begin{aligned}\mathcal{H}_{\oplus} &:= \bigoplus_{A \in \mathcal{L}^{\dagger}(\mathcal{D})} \mathcal{H}_A \\ &= \left\{\xi_{\oplus} = (\xi_A);\ \xi_A \in \mathcal{H}_A, \forall A \in \mathcal{L}^{\dagger}(\mathcal{D}) \text{ and } \sum_A \|\xi_A\|_A^2 < +\infty\right\}.\end{aligned}$$

If $\{X_A\}_{A \in \mathcal{L}^{\dagger}(\mathcal{D})}$ is a net of operators $X_A \in \mathfrak{B}(\mathcal{H}_A)$, $A \in \mathcal{L}^{\dagger}(\mathcal{D})$, we define $X_{\oplus}\xi_{\oplus} = \{X_A\xi_A\}$ provided that $\sum_A \|X_A\xi_A\|^2 < +\infty$, $\xi_A \in \mathcal{H}_A$.

The operator $X_{\oplus} = \{X_A\}$ is bounded if and only if $\sup_A \|X_A\|_{A,A} < +\infty$. The space constructed in this way is $\prod_A \mathfrak{B}(\mathcal{H}_A) = \mathfrak{B}(\mathcal{H}_{\oplus})$. To every $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})_b$, we can associate the net $\{X_A\}$ which we have defined above. Clearly, $\{X_A\} \in \mathfrak{B}(\mathcal{H}_{\oplus})$. It is easily seen that the map

$$\tau : X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})_b \mapsto \{X_A\} \in \mathfrak{B}(\mathcal{H}_{\oplus})$$

is isometric. Thus, the statement is proved. □

Remark 3.3 An element $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^{\times})$ having a representative X_A for every $A \in \mathcal{L}^{\dagger}(\mathcal{D})$ need not be bounded in the sense of Definition 3.1. The spaces $\{\mathcal{H}_A;\ A \in \mathcal{L}^{\dagger}(\mathcal{D})\}$, together with their conjugate duals, make $D^{\times}$ into an indexed PIP-space [4, Chap. 2]. In that language, operators having representatives in every $\mathcal{H}_A$ are called totally regular operators. For more details on their behavior see [4, Section 3.3.3] where also a C*-agebra corresponding to our bounded elements has been studied.

Our next goal is to characterize bounded elements of $\mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$ in several different ways. For doing this, we need to consider the natural order structure of $\mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$.

We say that $X \in \mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$ is *positive*, and write $X \geq 0$, if $\langle X\xi | \xi \rangle \geq 0$, for every $\xi \in \mathcal{D}$.

It is easy to see that if X is positive, then it is *symmetric*; i.e., $X = X^\dagger$.

Proposition 3.4 *The following conditions are equivalent.*

(i) $X \geq 0$.
(ii) There exists $A \in \mathcal{L}^\dagger(\mathcal{D})$ *such that* $X_B \geq 0, \quad \forall B \succeq A$.

Proof (i)⇒(ii): Since $X \in \mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$, there exists $A \in \mathcal{L}^\dagger(\mathcal{D})$ and $\gamma > 0$ such that

$$|\langle X\xi | \eta \rangle| \leq \gamma \|\xi\|_B \|\eta\|_B, \quad B \succeq A.$$

If $X \geq 0$, then, for every $\xi \in \mathcal{D}$,

$$\langle X_B\xi | \xi \rangle_B = \langle X\xi | \xi \rangle \geq 0, \quad \forall B \succeq A.$$

Since $\mathcal{D}$ is dense in $\mathcal{H}_B$, we have $\langle X_B\xi | \xi \rangle_B \geq 0,\ \forall \xi \in \mathcal{H}_B$.

(ii)⇒(i): Let $X_B \geq 0$ for every $B \succeq A$. Then, for every $\xi \in \mathcal{D}$, $\langle X\xi | \xi \rangle = \langle X_B\xi | \xi \rangle_B \geq 0$. □

Theorem 3.5 *Let* $X \in \mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$. *The following statements are equivalent.*

(i) $X : \mathcal{D} \to \mathcal{H}$ *and* $\overline{X} \in \mathcal{B}(\mathcal{H})$.
(ii) $X \in \mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)_b$.
(iii) There exists $\lambda > 0$ *such that*

$$-\lambda I \leq \Re(X) \leq \lambda I, \quad -\lambda I \leq \Im(X) \leq \lambda I$$

where $\Re(X) = \frac{X+X^\dagger}{2}$ *and* $\Im(X) = \frac{X-X^\dagger}{2i}$.

Proof (i)⇒(ii): If $X : \mathcal{D} \to \mathcal{H}$ and X is bounded, then, for every $A \in \mathcal{L}^\dagger(\mathcal{D})$,

$$|\langle X\xi | \eta \rangle| \leq \|\overline{X}\| \|\xi\| \|\eta\| \leq \|\overline{X}\| \|\xi\|_A \|\eta\|_A. \tag{2}$$

This means that X has a bounded representative X_A in every $\mathcal{B}(\mathcal{H}_A)$. By (2), $\|X_A\|_{A,A} \leq \|\overline{X}\|$, for every $A \in \mathcal{L}^\dagger(\mathcal{D})$, so $\sup_{A\in\mathcal{L}^\dagger(\mathcal{D})} \|X_A\|_{A,A} < +\infty$.

(ii)⇒(i) Let $X \in \mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)_b$. Then, for every $A \in \mathcal{L}^\dagger(\mathcal{D})$

$$|\langle X\xi | \eta \rangle| \leq \|X_A\|_{A,A} \|\xi\|_A \|\eta\|_A, \quad \forall \xi, \eta \in \mathcal{D}.$$

In particular, for $A = 0$,

$$|\langle X\xi | \eta \rangle| \leq \|X_0\| \|\xi\| \|\eta\|, \quad \forall \xi, \eta \in \mathcal{D}. \tag{3}$$

By (3), for every $\xi \in \mathcal{D}$, $F(\eta) = \langle X\xi | \eta \rangle$ is a bounded conjugate linear functional on $\mathcal{D}$, so by Riesz's lemma $X\xi \in \mathcal{H}$. It is finally easily seen that $\overline{X} \in \mathcal{B}(\mathcal{H})$.

(iii)⇒(i) Suppose first that $X = X^\dagger$. Note that the operator X satisfies the following: $0 \leq \frac{X+\lambda I}{2\lambda} \leq I$; so $\frac{X+\lambda I}{2\lambda}$ is a positive operator and $\left\langle \frac{X+\lambda I}{2\lambda}\xi | \xi \right\rangle \leq \langle \xi | \xi \rangle$, $\forall \xi \in \mathcal{D}$; this implies that

$$\left|\left\langle \frac{X+\lambda I}{2\lambda}\xi | \eta \right\rangle\right| \leq \|\xi\| \, \|\eta\|, \quad \forall \xi, \eta \in \mathcal{D} \tag{4}$$

and by Riesz's lemma there exists $\zeta \in \mathcal{H}$ such that

$$\left\langle \frac{X+\lambda I}{2\lambda}\xi \,|\eta \right\rangle = \langle \zeta \,|\eta \rangle\,, \quad \forall \xi, \eta \in \mathcal{D} \tag{5}$$

and then $\frac{X+\lambda I}{2\lambda}\xi \in \mathcal{H}$. This implies that $X\xi \in \mathcal{H}$ too. Moreover, X has a representative for every $A \in \mathcal{L}^\dagger(\mathcal{D})$. Indeed,

$$|\langle X\xi \,|\eta \rangle| \leq \gamma \|\xi\| \|\eta\| \leq \gamma \|\xi\|_A \|\eta\|_A \quad \forall A \in \mathcal{L}^\dagger(\mathcal{D}),$$

where $\gamma > 0$. From (4), it follows that X is bounded and $\overline{X} \in \mathcal{B}(\mathcal{H})$. In the very same way, one can prove the boundedness of X if $X^\dagger = -X$. The result for a general X follows easily.

(i)⇒ (iii): This is a standard result of the C*-algebras theory. □

4 Bounded elements of C*-inductive locally convex spaces

The results obtained in Sect. 3 have an abstract generalization to locally convex spaces that are inductive limits of C*-algebras in a generalized sense. These spaces were called *C*-inductive locally convex spaces* in [8]. We begin with recalling the basic definitions.

Let $\mathfrak{A}$ be a vector space over $\mathbb{C}$. Let $\mathbb{F}$ be a set of indices directed upward and consider, for every $\alpha \in \mathbb{F}$, a space $\mathfrak{A}_\alpha \subset \mathfrak{A}$ such that:

(I.1) $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$, if $\alpha \leq \beta$;
(I.2) $\mathfrak{A} = \bigcup_{\alpha \in \mathbb{F}} \mathfrak{A}_\alpha$;
(I.3) $\forall \alpha \in \mathbb{F}$, there exists a C*-algebra $\mathfrak{B}_\alpha$ (with unit e_α and norm $\|\cdot\|_\alpha$) and an isomorphism of vector spaces $\phi_\alpha : \mathfrak{B}_\alpha \to \mathfrak{A}_\alpha$ which makes of $\mathfrak{A}_\alpha$ a Banach space under the norm $\|x\|^\alpha := \|x_\alpha\|_\alpha$, if $x \in \mathfrak{A}_\alpha$, $x = \phi_\alpha(x_\alpha)$;
(I.4) $x_\alpha \in \mathfrak{B}_\alpha^+ \Rightarrow x_\beta = (\phi_\beta^{-1}\phi_\alpha)(x_\alpha) \in \mathfrak{B}_\beta^+$, for every $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$.

We put $j_{\beta\alpha} = \phi_\beta^{-1}\phi_\alpha$, if $\alpha, \beta \in \mathbb{F},\ \beta \geq \alpha$.

If $x \in \mathfrak{A}$, there exists $\alpha \in \mathbb{F}$ such that $x \in \mathfrak{A}_\alpha$ and, for every $\beta \geq \alpha$, a unique $x_\beta \in \mathfrak{B}_\beta$ such that $x = \phi_\beta(x_\beta)$.

Then, we put

$$j_{\beta\alpha}(x_\alpha) := x_\beta \text{ if } \alpha \leq \beta.$$

By (I.4), it follows easily that $j_{\beta\alpha}$ preserves the involution; i.e., $j_{\beta\alpha}(x_\alpha^*) = (j_{\beta\alpha}(x_\alpha))^*$.

Remark 4.1 From the previous discussion, it follows that to every $x \in \mathfrak{A}$ there corresponds a family of *representatives* $\{x_\beta;\ x_\beta \in \mathfrak{B}_\beta,\ \beta \geq \alpha\}$. We write, for short, $x = (x_\beta)$. If $x = (x_\beta)$, $y = (y_\beta)$ and $x_\beta = y_\beta$, for every β larger than a certain $\gamma \in \mathbb{F}$, then $x = y$. With this identification, the mentioned correspondence is one-to-one.

The family $\{\mathfrak{B}_\alpha, j_{\beta\alpha}, \beta \geq \alpha\}$ is a *directed system of C*-algebras*, in the sense that:

(J.1) for every $\alpha, \beta \in \mathbb{F}$, with $\beta \geq \alpha$, $j_{\beta\alpha} : \mathfrak{B}_\alpha \to \mathfrak{B}_\beta$ is a linear and injective map; $j_{\alpha\alpha}$ is the identity of $\mathfrak{B}_\alpha$,
(J.2) for every $\alpha, \beta \in \mathbb{F}$, with $\alpha \leq \beta$, $\phi_\alpha = \phi_\beta j_{\beta\alpha}$,
(J.3) $j_{\gamma\beta} j_{\beta\alpha} = j_{\gamma\alpha}, \alpha \leq \beta \leq \gamma$.

We assume that, in addition, the $j_{\beta\alpha}$s are Schwarz maps (see, e.g., [14]); i.e.,
(sch) $j_{\beta\alpha}(x_\alpha)^* j_{\beta\alpha}(x_\alpha) \leq j_{\beta\alpha}(x_\alpha^* x_\alpha), \quad \forall x_\alpha \in \mathfrak{B}_\alpha,\ \alpha \leq \beta.$

For every $\alpha, \beta \in \mathbb{F}$, with $\alpha \leq \beta$, $j_{\beta\alpha}$ is continuous [14] and, moreover,

$$\|j_{\beta\alpha}(x_\alpha)\|_\beta \leq \|x_\alpha\|_\alpha, \quad \forall x_\alpha \in \mathfrak{B}_\alpha.$$

An involution in $\mathfrak{A}$ is defined as follows. Let $x \in \mathfrak{A}$. Then $x \in \mathfrak{A}_\alpha$, for some $\alpha \in \mathbb{F}$, i.e., $x = \phi_\alpha(x_\alpha)$, for a unique $x_\alpha \in \mathfrak{B}_\alpha$. Put $x^* := \phi_\alpha(x_\alpha^*)$. Then if $\beta \geq \alpha$, we have

$$\phi_\beta^{-1}(x^*) = \phi_\beta^{-1}(\phi_\alpha(x_\alpha^*)) = j_{\beta\alpha}(x_\alpha^*) = (j_{\beta\alpha}(x_\alpha))^* = x_\beta^*.$$

It is easily seen that the map $x \mapsto x^*$ is an involution in $\mathfrak{A}$. Moreover, by the definition itself, it follows that every map ϕ_α *preserves the involution*; i.e., $\phi_\alpha(x_\alpha^*) = (\phi_\alpha(x_\alpha))^*$, for all $x_\alpha \in \mathfrak{B}_\alpha$, $\alpha \in \mathbb{F}$.

Definition 4.2 Let $\mathfrak{A}$ be a vector space with involution * and $\mathbb{F}$ a directed (upward) set.

- A *defining system* for $\mathfrak{A}$ consists of a family $\{\{\mathfrak{B}_\alpha, \phi_\alpha\}, \alpha \in \mathbb{F}\}$, where, for every $\alpha \in \mathbb{F}$, $\mathfrak{B}_\alpha$ is a C*-algebra and ϕ_α is a linear injective map of $\mathfrak{B}_\alpha$ into $\mathfrak{A}$, satisfying the above conditions (I.1)–(I.4) and (sch), with $\mathfrak{A}_\alpha = \phi_\alpha(\mathfrak{B}_\alpha)$, $\alpha \in \mathbb{F}$.
- If $\mathfrak{A}$ is endowed with the locally convex inductive topology τ_{ind} generated by the family $\{\{\mathfrak{B}_\alpha, \phi_\alpha\}, \alpha \in \mathbb{F}\}$, then we say that $\mathfrak{A}$ is a *C*-inductive locally convex space*.

We notice that the involution is automatically continuous in $\mathfrak{A}[\tau_{\text{ind}}]$.
A C*-inductive locally convex space has a natural positive cone.
An element $x \in \mathfrak{A}$ is called *positive* if there exists $\gamma \in \mathbb{F}$ such that $\phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha^+$, $\forall \alpha \geq \gamma$.
We denote by $\mathfrak{A}^+$ the set of all positive elements of $\mathfrak{A}$.
Then,

(i) Every positive element $x \in \mathfrak{A}$ is hermitian; i.e., $x \in \mathfrak{A}_h := \{y \in \mathfrak{A} : y^* = y\}$.
(ii) $\mathfrak{A}^+$ is a non empty convex pointed cone; i.e., $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$.
(iii) If $\alpha \in \mathbb{F}$ and $x_\alpha \in \mathfrak{B}_\alpha^+$, $\phi_\alpha(x_\alpha)$ is positive.

Moreover, every hermitian element $x = x^*$ is the difference of two positive elements, i.e., there exist $x^+, x^- \in \mathfrak{A}^+$ such that $x = x^+ - x^-$.

A linear functional ω is said to be *positive* if $\omega(x) \geq 0$ for every $x = (x_\alpha) \in \mathfrak{A}^+$. As shown in [8, Prop. 3.9, 3.10], ω is positive if, and only if, $\omega_\alpha(x_\alpha) := \omega(\phi_\alpha(x_\alpha)) \geq 0$ for every $\alpha \in \mathbb{F}$. We write, in this case, $\omega = \underrightarrow{\lim}\,\omega_\alpha$.

4.1 Bounded elements

Definition 4.3 Let $\mathfrak{A}$ be a C*-inductive locally convex space. An element $x = (x_\alpha) \in \mathfrak{A}$, with $x_\alpha \in \mathfrak{B}_\alpha$, is called *bounded* if $x \in \mathfrak{A}_\alpha$, for every $\alpha \in \mathbb{F}$ and $\sup_{\alpha \in \mathbb{F}} \|x_\alpha\|_\alpha < \infty$. The set of bounded elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_b$.

Proposition 4.4 *The set $\mathfrak{A}_b$ is a Banach space under the norm $\|x\|_b := \sup_{\alpha \in \mathbb{F}} \|x_\alpha\|_\alpha$.*

Proof We only prove the completeness. Let $\{x_n\}$ be a Cauchy sequence in $\mathfrak{A}_b$. Then, for every $\alpha \in \mathbb{F}$ the sequence $\{x_n^\alpha\}$, with $x_n^\alpha := (x_n)_\alpha$, is Cauchy in $\mathfrak{B}_\alpha$, so it converges to some $x_\alpha \in \mathfrak{B}_\alpha$. Since the $j_{\beta\alpha}$'s are continuous, one easily proves that the family $\{x_\alpha\}$ defines an element $x = (x_\alpha)$ of $\mathfrak{A}$. From the Cauchy condition, for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\sup_{\alpha \in \mathbb{F}} \|x_n^\alpha - x_m^\alpha\|_\alpha < \epsilon \tag{6}$$

If $m > n_\epsilon$,

$$\|x_\alpha\|_\alpha \leq \|x_\alpha - x_m^\alpha\|_\alpha + \|x_m^\alpha\|_\alpha \leq \epsilon + \|x_m^\alpha\|_\alpha.$$

Hence,

$$\sup_{\alpha\in\mathbb{F}} \|x_\alpha\|_\alpha \leq \epsilon + \sup_{\alpha\in\mathbb{F}} \|x_m^\alpha\|_\alpha < \infty.$$

Thus $x \in \mathfrak{A}_b$.

Fix now $n > n_\epsilon$ and let $m \to \infty$ in (6). Then,

$$\sup_{\alpha\in\mathbb{F}} \|x_n^\alpha - x_\alpha\|_\alpha \leq \epsilon.$$

This proves that $x_n \to x$. □

In what follows, we will consider *-representations of a C*-inductive locally convex space. We recall the basic definitions.

Let $\mathbb{F}$ be a set directed upward by $\leq$. A family $\{\mathcal{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$, where each $\mathcal{H}_\alpha$ is a Hilbert space (with inner product $\langle\cdot|\cdot\rangle_{(\alpha)}$ and norm $\|\cdot\|_{(\alpha)}$) and, for every $\alpha, \beta \in \mathbb{F}$, with $\beta \geq \alpha$, $U_{\beta\alpha}$ is a linear map from $\mathcal{H}_\alpha$ into $\mathcal{H}_\beta$, is called a *directed contractive system of Hilbert spaces* if the following conditions are satisfied

(i) $U_{\beta\alpha}$ is injective;
(ii) $\|U_{\beta\alpha}\xi_\alpha\|_{(\beta)} \leq \|\xi_\alpha\|_{(\alpha)}, \quad \forall \xi_\alpha \in \mathcal{H}_\alpha$;
(iii) $U_{\alpha\alpha} = I_\alpha$, the identity of $\mathcal{H}_\alpha$;
(iv) $U_{\gamma\alpha} = U_{\gamma\beta}U_{\beta\alpha}, \alpha \leq \beta \leq \gamma$.

A directed contractive system of Hilbert spaces defines a conjugate dual pair $(\mathcal{D}^\times, \mathcal{D})$ which is called the *joint topological limit* [9] of the directed contractive system $\{\mathcal{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ of Hilbert spaces.

Definition 4.5 Let $\mathfrak{A}$ be the C*-inductive locally convex space defined by the system $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\}, \alpha \in \mathbb{F}\}$ as in Definition 4.2.

For each $\alpha \in \mathbb{F}$, let π_α be a *-representation of $\mathfrak{B}_\alpha$ in Hilbert space $\mathcal{H}_\alpha$. The collection $\pi := \{\pi_\alpha\}$ is said to be a *-representation of $\mathfrak{A}$ if

(i) for every $\alpha, \beta \in \mathbb{F}$, there exists a linear map $U_{\beta\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$ such that the family $\{\mathcal{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ is a directed contractive system of Hilbert spaces;
(ii) the following equality holds

$$\pi_\beta(j_{\beta\alpha}(x_\alpha)) = U_{\beta\alpha}\pi_\alpha(x_\alpha)U_{\beta\alpha}^*, \quad \forall x_\alpha \in \mathfrak{B}_\alpha, \ \beta \geq \alpha. \tag{7}$$

In this case, we write $\pi(x) = \underrightarrow{\lim}\,\pi_\alpha(x_\alpha)$ for every $x = (x_\alpha) \in \mathfrak{A}$ or, for short, $\pi = \underrightarrow{\lim}\,\pi_\alpha$.

The *-representation π is said to be *faithful* if $x \in \mathfrak{A}^+$ and $\pi(x) = 0$ imply $x = 0$ (of course, $\pi(x) = 0$ means that there exists $\gamma \in \mathbb{F}$ such that $\pi_\alpha(x_\alpha) = 0$, for $\alpha \geq \gamma$).

Remark 4.6 With this definition (which is formally different from that given in [8] but fully equivalent), $\pi(x)$, $x \in \mathfrak{A}$, is not an operator but rather a collection of operators. But as shown in [8], $\pi(x)$ can be regarded as an operator acting on the joint topological limit $(\mathcal{D}^\times, \mathcal{D})$ of $\{\mathcal{H}_\alpha, U_{\beta\alpha}, \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$. The corresponding space of operators was denoted by $\mathsf{L}_\mathsf{B}(\mathcal{D}, \mathcal{D}^\times)$; it behaves in the very same way as the space $\mathfrak{L}_\mathsf{B}(\mathcal{D}, \mathcal{D}^\times)$ studied in Sect. 3 and reduces to it when the family of Hilbert spaces is exactly $\{\mathcal{H}_A;\ A \in \mathcal{L}^\dagger(\mathcal{D})\}$. The main difference consists in the fact that the $\mathcal{H}_\alpha$'s need not be all subspaces of a certain Hilbert space $\mathcal{H}$.

Lemma 4.7 *Let $\pi = \underrightarrow{\lim}\,\pi_\alpha$ be a faithful *-representation of $\mathfrak{A}$. Then, for every $\alpha \in \mathbb{F}$, π_α is a faithful *-representation of $\mathfrak{B}_\alpha$.*

Proof Let $x_\alpha \in \mathfrak{B}_\alpha^+$ with $\pi_\alpha(x_\alpha) = 0$. Let $x \in \mathfrak{A}$ be the unique element of $\mathfrak{A}$ such that $x = \phi_\alpha(x_\alpha)$. Then $\pi_\beta(x_\beta) = \pi_\beta(j_{\beta\alpha}(x_\alpha)) = U_{\beta\alpha}\pi_\alpha(x_\alpha)U^*_{\beta\alpha} = 0$. Hence $\pi(x) = 0$, and therefore, $x = 0$. Thus there exists $\overline{\gamma} \in \mathbb{F}$ such that $x_\gamma = 0$, for $\gamma \geq \overline{\gamma}$. Let $\beta \geq \alpha, \overline{\gamma}$. Then $0 = x_\beta = j_{\beta\alpha}(x_\alpha)$. Hence, by the injectivity of $j_{\beta\alpha}$, $x_\alpha = 0$. □

As shown in [8, Proposition 3.16], if a C*-inductive locally convex space $\mathfrak{A}$ fulfills the following conditions

(r_1) if $x_\alpha \in \mathfrak{B}_\alpha$ and $j_{\beta\alpha}(x_\alpha) \geq 0$ for some $\beta \geq \alpha$, then $x_\alpha \geq 0$;
(r_2) $e_\beta \in j_{\beta\alpha}(\mathfrak{B}_\alpha), \quad \forall \alpha, \beta \in \mathbb{F}, \beta \geq \alpha$;
(r_3) every positive linear functional $\omega = \underrightarrow{\lim}\,\omega_\alpha$ on $\mathfrak{A}$ satisfies the following property

- if $\alpha \in \mathbb{F}$ and $\omega_\beta(j_{\beta\alpha}(x^*_\alpha)j_{\beta\alpha}(x_\alpha)) = 0$, for some $\beta > \alpha$ and $x_\alpha \in \mathfrak{B}_\alpha$, then $\omega_\alpha(x^*_\alpha x_\alpha) = 0$;

then, $\mathfrak{A}$ admits a faithful representation. The conditions (r_1), (r_2), in fact, guarantee that $\mathfrak{A}$ possesses sufficiently many positive linear functionals, in the sense that for every $x \in \mathfrak{A}^+$, $x \neq 0$ there exists a positive linear functional ω on $\mathfrak{A}$ such that $\omega(x) > 0$ [8, Theorem 3.14].

Theorem 4.8 *Let $\mathfrak{A}$ be a C*-inductive locally convex space and $x = (x_\alpha) \in \mathfrak{A}$. The following statements hold.*

*(i) If $x \in \mathfrak{A}_b$, then, for every *-representation $\pi = \underrightarrow{\lim}\,\pi_\alpha$ of $\mathfrak{A}$, one has*

$$\sup_{\alpha\in\mathbb{F}} \|\pi_\alpha(x_\alpha)\|_{\alpha\alpha} < \infty,$$

where $\|\cdot\|_{\alpha\alpha}$ denotes the norm of $\mathfrak{B}(\mathcal{H}_\alpha)$.

*(ii) Conversely, if $\mathfrak{A}$ admits a faithful *-representation $\pi^f = \underrightarrow{\lim}\,\pi^f_\alpha$ and*

$$\sup_{\alpha\in\mathbb{F}} \|\pi^f_\alpha(x_\alpha)\|_{\alpha\alpha} < \infty,$$

then $x \in \mathfrak{A}_b$.

Proof (i): For every $\alpha \in \mathbb{F}$, π_α is a *-representation of the C*-algebra $\mathfrak{B}_\alpha$. Hence

$$\|\pi_\alpha(x_\alpha)\|_{\alpha\alpha} \leq \|x_\alpha\|_\alpha.$$

Thus if $x \in \mathfrak{A}_b$ the statement follows immediately from the definition.

(ii): Let $\pi^f(x) = \underrightarrow{\lim}\,\pi^f_\alpha(x_\alpha)$. Then, by Lemma 4.7, for every $\alpha \in \mathbb{F}$, π^f_α is a faithful representation of $\mathfrak{B}_\alpha$. The *-representation π^f_α is an isometric isomorphism of C*-algebras, for all $\alpha \in \mathbb{F}$; hence

$$\sup_{\alpha\in\mathbb{F}} \|x_\alpha\|_\alpha = \sup_{\alpha\in\mathbb{F}} \|\pi^f_\alpha(x_\alpha)\|_{\alpha\alpha} < \infty.$$

This proves that x is a bounded element of $\mathfrak{A}$.

□

4.2 Order bounded elements

Let $\mathfrak{A}$ be a C*-inductive locally convex space. If $x \in \mathfrak{A}$, we put

$$\Re(x) = \frac{x + x^*}{2} \quad \text{and} \quad \Im(x) = \frac{x - x^*}{2i}.$$

Both $\Re(x)$ and $\Im(x)$ are symmetric elements of $\mathfrak{A}$.

Assume that $\mathfrak{A}$ has an element $u = u^*$ such that $\|u_\alpha\|_\alpha \leq 1$, for every $\alpha \in \mathbb{F}$, and there exists $\gamma \in \mathbb{F}$ such that $u_\beta = j_{\beta\gamma}(e_\gamma) \forall \beta \geq \gamma$, ($e_\gamma$ is the unit of $\mathfrak{B}_\gamma$). For shortness, we call the element u a *pre-unit* of $\mathfrak{A}$.

Remark 4.9 The pre-unit $u \in \mathfrak{A}$, if any, is unique. Indeed, let suppose there is another $v \in \mathfrak{A}$ satisfying the same properties as u. Then,

$$\exists \gamma, \gamma' \in \mathbb{F};\; u_\beta = j_{\beta\gamma}(e_\gamma),\; v_{\beta'} = j_{\beta'\gamma'}(e_{\gamma'}), \quad \forall \beta \geq \gamma, \beta' \geq \gamma'$$

so, if $\delta \geq \gamma, \gamma'$, one has $u_\lambda = v_\lambda, \forall \lambda \geq \delta$. The statement then follows from Remark 4.1.

Definition 4.10 Let $\mathfrak{A}$ be a C*-inductive locally convex space with pre-unit u. We say that $x \in \mathfrak{A}$ is *order bounded* (with respect to u) if there exists $\lambda > 0$ such that

$$-\lambda u \leq \Re(x) \leq \lambda u \qquad -\lambda u \leq \Im(x) \leq \lambda u.$$

Theorem 4.11 *Let $\mathfrak{A}$ be a C*-inductive locally convex space satisfying condition* (r_1). *Assume that $\mathfrak{A}$ has a pre-unit u.*

Then, $x \in \mathfrak{A}_b$ if, and only if, x has a representative for every $\alpha \in \mathbb{F}$ (i.e., for every $\alpha \in \mathbb{F}$, there exists $x_\alpha \in \mathfrak{B}_\alpha$ such that $x = \phi_\alpha(x_\alpha)$) and x is order bounded with respect u.

Proof Let us assume that $x = x^* \in \mathfrak{A}_b$. Then, x has a representative x_α, with $x_\alpha^* = x_\alpha$, in every $\mathfrak{B}_\alpha$ and $\lambda := \sup_{\alpha \in \mathbb{F}} \|x_\alpha\|_\alpha < \infty$. Hence, we have

$$-\lambda e_\alpha \leq x_\alpha \leq \lambda e_\alpha, \quad \forall \alpha \in \mathbb{F},$$

where e_α denotes the unit of $\mathfrak{B}_\alpha$. By the definition of u, there exists $\gamma \in \mathbb{F}$ such that $u_\beta = j_{\beta\gamma}(e_\gamma)$ for $\beta \geq \gamma$. Hence, taking into account that the maps $j_{\beta\alpha}$ preserve the order, we have

$$-\lambda u_\beta \leq x_\beta \leq \lambda u_\beta, \quad \forall \beta \geq \gamma.$$

This implies that $-\lambda u \leq x \leq \lambda u$.

Now, let us suppose that for some $\lambda > 0$, $-\lambda u \leq x \leq \lambda u$. Then, there exists $\gamma \in \mathbb{F}$ such that

$$-\lambda u_\beta \leq x_\beta \leq \lambda u_\beta, \qquad \forall \beta \geq \gamma. \tag{8}$$

Let now $\alpha \in \mathbb{F}$. Then, there is $\delta \geq \alpha, \gamma$ such that (8) holds. Hence, using (r_1), we conclude that

$$-\lambda u_\alpha \leq x_\alpha \leq \lambda u_\alpha, \quad \forall \alpha \in \mathbb{F}.$$

This implies that $\|x_\alpha\|_\alpha \leq \lambda$, for every $\alpha \in \mathbb{F}$. Thus, $x \in \mathfrak{A}_b$. □

From the proof of the previous theorem, it follows easily that

Proposition 4.12 *Assume that the assumptions of Theorem 4.11 hold and let $x = x^* \in \mathfrak{A}_b$. Put*

$$p(x) = \inf\{\lambda > 0;\; -\lambda u \leq x \leq \lambda u\}.$$

Then, $p(x) = \|x\|_b$.

5 C*-inductive partial *-algebras

As shown in [8], a partial multiplication in $\mathfrak{A}$ can be defined by a family $w = \{w_\alpha\}$, $w_\alpha \in \mathfrak{B}_\alpha$. Let $w = \{w_\alpha\}$ be a family of elements, such that each $w_\alpha \in \mathfrak{B}_\alpha^+$ and $j_{\beta\alpha}(w_\alpha) = w_\beta$, for all $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$.

Let $x, y \in \mathfrak{A}$. The partial multiplication $x \cdot y$ is defined by the conditions:

$$\exists \gamma \in \mathbb{F} : \phi_\beta(\phi_\beta^{-1}(x) w_\beta \phi_\beta^{-1}(y)) = \phi_{\beta'}(\phi_{\beta'}^{-1}(x) w_{\beta'} \phi_{\beta'}^{-1}(y)), \ \forall \beta, \beta' \geq \gamma$$
$$x \cdot y = \phi_\beta(\phi_\beta^{-1}(x) w_\beta \phi_\beta^{-1}(y)), \quad \beta \geq \gamma.$$

Then, $\mathfrak{A}$ is an *associative* partial *-algebra with respect to the usual operations and the above-defined multiplication (see [3, Section 2.1.1] for the definitions) and we will call it a *C*-inductive partial *-algebra*.

The partial *-algebra $\mathfrak{A}$ has a unit e (that is, an element e which is a left- and right universal multiplier such that $x \cdot e = e \cdot x = x$, for every $x \in \mathfrak{A}$) if, and only if, every element w_α of the family $\{w_\alpha\}$ defining the multiplication is invertible and

$$j_{\beta\alpha}(w_\alpha^{-1}) = w_\beta^{-1}, \ \forall \alpha, \ \beta \in \mathbb{F}, \ \beta \geq \alpha. \tag{9}$$

In this case, $e = \phi_\alpha(w_\alpha^{-1})$, independently of $\alpha \in \mathbb{F}$.

The element e is called a *bounded unit* if it is a bounded element of $\mathfrak{A}$ and $\|e\|_b = 1$.

Proposition 5.1 *Let $\mathfrak{A}$ be a C*-inductive partial *-algebra with the multiplication defined by a family $\{w_\alpha\}$. Assume that $e = (w_\alpha^{-1})$ is a bounded unit of $\mathfrak{A}$. Then $\mathfrak{A}_b$ is a Banach partial *-algebra; that is, $\mathfrak{A}_b[\|\cdot\|_b]$ is a Banach space with isometric involution * and there exists $C \geq 1$ such that the following inequality holds*

$$\|x \cdot y\|_b \leq C \|x\|_b \|y\|_b, \quad \forall x, y \in \mathfrak{A}_b \quad \textit{with } x \cdot y \textit{ well-defined.} \tag{10}$$

Remark 5.2 The constant C in (10) can be taken equal to 1 if $w_\alpha^{-1} = e_\alpha$, for each $\alpha \in \mathbb{F}$, where e_α is the unit of the C*-algebra $\mathfrak{B}_\alpha$. Under the same assumption, the norm of $\mathfrak{A}_b$ satisfies the C*-property, which in our case reads

$$\|x^* \cdot x\|_b = \|x\|_b^2, \quad \forall x \in \mathfrak{A}_b \quad \text{with } x^* \cdot x \text{ well-defined.}$$

This is no longer true in the general case.

Remark 5.3 In Example 5.3 of [8], two of us tried to construct a family $\{W_A \in \mathfrak{B}(\mathcal{H}_A); \ A \in \mathcal{L}^\dagger(\mathcal{D})\}$ so that the partial multiplication defined in $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ by the method mentioned above would reproduce the quasi *-algebra structure of $(\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times), \mathcal{L}^\dagger(\mathcal{D}))$ (see Sect. 2). Unfortunately, the conclusion of that discussion is uncorrect (see [8, Erratum/Addendum] for more details).

Let $\mathfrak{A}$ be a C*-inductive partial *-algebra with the multiplication defined by a family $\{w_\alpha\}$ as above. The spaces $R\mathfrak{A}$ and $L\mathfrak{A}$ of the right-, respectively, left universal multipliers (with respect to w) of $\mathfrak{A}$ are algebras. Hence, $\mathfrak{A}_0 := L\mathfrak{A} \cap R\mathfrak{A}$ is a *-algebra and, thus,

(i) $(\mathfrak{A}, \mathfrak{A}_0)$ is a quasi *-algebra.
(ii) If $\mathfrak{A}$ is endowed with τ_{ind}, then the maps $x \mapsto x^*$, $x \mapsto a \cdot x$, $x \mapsto x \cdot b$, $a, b \in \mathfrak{A}_0$ are continuous.

It is easily seen from the very definition that if $a \in R\mathfrak{A}$ and $x \in \mathfrak{A}^+$, then $a^*xa \in \mathfrak{A}^+$. Hence, if $\mathcal{P}(\mathfrak{A})$ denotes the family of all positive linear functionals on $\mathfrak{A}$, we have in particular $\omega(a^*xa) \geq 0$, for every $\omega \in \mathcal{P}(\mathfrak{A})$.

Theorem 5.4 *Let $\mathfrak{A}$ be a C*-inductive partial *-algebra with the multiplication defined by a family $\{w_\alpha\}$ and with pre-unit u. Assume, moreover, that the following condition* (P) *holds:*
(P) $y \in \mathfrak{A}, \omega(a^*ya) \geq 0, \forall \omega \in \mathcal{P}(\mathfrak{A})$ *and* $a \in R\mathfrak{A} \Rightarrow y \in \mathfrak{A}^+$;
then, for $x \in \mathfrak{A}$, the following conditions are equivalent.

(i) x *is order bounded with respect to* u.
(ii) *There exists* $\gamma_x > 0$ *such that*

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*ua), \quad \forall \omega \in \mathcal{P}(\mathfrak{A}), \quad \forall a \in R\mathfrak{A}.$$

(iii) *There exists* $\gamma_x > 0$ *such that*

$$|\omega(b^*xa)|^2 \leq \gamma_x \omega(a^*ua)\omega(b^*ub), \quad \forall \omega \in \mathcal{P}(\mathfrak{A}), \ \forall a, b \in R\mathfrak{A}.$$

Proof It is sufficient to consider the case $x = x^*$;

(i)⇒(ii): Let $\omega \in \mathcal{P}(\mathfrak{A})$. By the hypothesis, $-\gamma u \leq x \leq \gamma u$, for some $\gamma > 0$; then $\omega(\gamma u - x) \geq 0$ and $\omega(a^*(\gamma u - x)a) \geq 0, \forall a \in R\mathfrak{A}$. On the other hand, similarly, one can show that $\omega(a^*(x - \gamma u)a) \geq 0$.

(ii)⇒(i): Assume now that u is a pre-unit and there exists $\gamma_x > 0$ such that

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*ua), \quad \forall \omega \in \mathcal{P}(\mathfrak{A}), \quad a \in R\mathfrak{A}.$$

Then

$$\gamma_x \omega(a^*ua) \pm \omega(a^*xa) \geq 0 \Rightarrow \omega(a^*(\gamma_x u \pm x)a) \geq 0, \quad \forall \omega \in \mathcal{P}(\mathfrak{A}), a \in R\mathfrak{A}.$$

So, by (P), $\gamma_x u \pm x \geq 0$.

(i)⇒(iii): By the assumption, there exists $\gamma > 0$ such that $-\gamma u \leq x \leq \gamma u$. Let $\omega \in \mathcal{P}(\mathfrak{A})$. Then, the linear functional ω_a on $\mathfrak{A}$, defined by $\omega_a(x) := \omega(a^*xa)$, is positive. Hence, if $x = x^*$

$$-\gamma \omega_a(u) \leq \omega_a(x) \leq \gamma \omega_a(u);$$

i.e.,

$$|\omega(a^*xa)| \leq \gamma \omega(a^*ua).$$

Now, let $x \in \mathfrak{A}^+, a, b \in R\mathfrak{A}$. Let us define $\Omega^x_\omega(a, b) := \omega(b^*xa)$. Then, it is easily checked that Ω^x_ω is a positive sesquilinear form on $R\mathfrak{A} \times R\mathfrak{A}$. Using the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} |\omega(b^*xa)| &\leq \omega(a^*xa)^{1/2}\omega(b^*xb)^{1/2} \\ &\leq \gamma \omega(a^*ua)^{1/2}\omega(b^*ub)^{1/2}. \end{aligned}$$

The extension to arbitrary $x \in \mathfrak{A}$ goes through as in the proof of Proposition 4.3 of [8].

(iii)⇒(ii) It is trivial. □

The previous proof shows that if $x = x^* \in \mathfrak{A}$ is order bounded with respect to u then

$$p(x) \leq \sup\{|\omega(b^*xa)|; \omega \in \mathcal{P}(\mathfrak{A}); \ a, b \in R\mathfrak{A}; \omega(a^*ua) = \omega(b^*ub) = 1\}.$$

where $p(x)$ is the quantity defined in Proposition 4.12.

The following statement is an easy consequence of Proposition 4.12 and Theorem 5.4.

Theorem 5.5 *Let $\mathfrak{A}$ be a C*-inductive partial *-algebra with the multiplication defined by a family $\{w_\alpha\}$ and pre-unit u. Assume that conditions* (r_1) *and* (P) *are satisfied. For an element $x \in \mathfrak{A}$, having a representative in every $\mathfrak{B}_\alpha, \alpha \in \mathbb{F}$, the following statements are equivalent.*

(i) $x \in \mathfrak{A}_b$.
(ii) *x is order bounded with respect to u.*
(iii) *For every* $\omega \in \mathcal{P}(\mathfrak{A})$

$$|\omega(b^* x a)|^2 \leq \gamma_x \omega(a^* u a)\omega(b^* u b), \quad \forall a, b \in R\mathfrak{A}.$$

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