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BOUNDED EXPONENTIAL SUMS

EMMANUEL LESIGNE AND KARL PETERSEN

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1. Introduction

T. Kamae has asked (personal communication) whether it is possible to find a sequence (a_k) of ± 1's such that the sums

$$\sum_{k=m}^{m+n} a_k e^{-ik\theta}$$

stay bounded (for all integers m and n with $n \geq 0$) for all $\theta \in [-\pi, \pi)$ (with the bound possibly depending on θ). We show that there is no such sequence. In fact, the only such real-valued sequences must be "essentially zero" in a sense explained below.

This conclusion is reached by adopting a dynamical viewpoint, applying the Spectral Theorem, and showing that every nonzero element of L^2 must have nonzero mean power at some frequency. This latter observation is equivalent to the triviality of the intersection of all the spaces of "twisted coboundaries" for a unitary operator.

2. Results

Suppose that $a=(a_k) \in \boldsymbol{R}^{\boldsymbol{Z}}$ is a doubly infinite sequence with the property that

$$|\sum_{k=m}^{m+n} a_k e^{-ik\theta}| \leq c(\theta) < \infty \qquad \text{for all } m \in \boldsymbol{Z}, \text{ all } n \geq 0, \text{ and all } \theta \in [-\pi, \pi).$$

Taking $n=\theta=0$, we see that a is bounded and so takes values in a compact interval I. Let X denote the closure of the orbit of a under the shift transformation σ in the compact metric space $I^{\boldsymbol{Z}}$. Let μ be a shift-invariant Borel probability measure on X.

Given $x \in X$ and a block $B=b_0 \cdots b_n$ which appears in x, we can find a block $D=d_0 \cdots d_n$ in a such that $|b_i - d_i| < 1/(n+1)$ for $i=0, \cdots, n$. Consequently

$$|\sum_{k=0}^{n} b_k e^{-ik\theta}| \leq c(\theta)+1 \qquad \text{for all } \theta .$$

If $Tg = g \circ \sigma$ for $g \in L^2(X, \mu)$ and $f(x) = \pi_0 x = x_0$ for $x \in X$, we have then that

$$\|\sum_{k=0}^{n} T^k f e^{-ik\theta}\|_2 \leq c(\theta)+1 \qquad \text{for all } \theta \text{ and all } n\geq 0\,.$$

We will see that this is impossible unless $f=0$ in L^2. Since this cannot happen for a sequence a which assumes only finitely many values, all nonzero, the original question will be settled. For general sequences, the conclusion is that boundedness against all θ is possible only if projection onto the central coordinate is 0 a.e. with respect to every invariant measure on the orbit closure X of the sequence; that is, the only invariant probability measure on X is concentrated on the fixed point 0^∞. In this case we say that the sequence (a_k) is *essentially zero.*

Theorem. *Let H be a Hilbert space and $T\colon H\to H$ a unitary operator. For each $n=1, 2, \cdots, \theta\in[-\pi, \pi)$, and $f\in H$ let*

$$S_n^\theta f = \sum_{k=0}^{n-1} e^{-ik\theta}\, T^k f\,.$$

If $\sup_n \|S_n^\theta f\|<\infty$ for all θ, then $f=0$.

Proof. Applying the Spectral Theorem with common notations and conventions, we may write

$$Tf = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda) f\,,$$

$$S_n^\theta f = \int_{-\pi}^{\pi} \sum_{k=0}^{n-1} e^{ik(\lambda-\theta)} dE(\lambda) f = \int_{-\pi}^{\pi} \frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\, dE(\lambda) f\,,$$

and

$$\|S_n^\theta f\|^2 = \int_{-\pi}^{\pi} \left|\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\|Ef\|^2(\lambda)\,.$$

The following Lemma will show that such expressions cannot stay bounded for any positive measure (such as $\nu=\|E(\ )f\|^2$ if $f\neq 0$ a.e.), thereby completing the proof.

Lemma. *There is a constant $C>0$ such that if ν is a positive measure on $[-\pi, \pi)$, n is a positive integer, $\varepsilon>0$, and*

$$A_n(\theta) = \frac{1}{n}\int_{-\pi}^{\pi} \left|\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\nu(\lambda)\,,$$

then $\nu\{\theta\in[-\pi, \pi)\colon A_n(\theta)<\varepsilon\}<\dfrac{\varepsilon}{C}$.

Proof. Let C_1 and C_2 be positive constants such that $|\alpha|<\pi$ implies that $C_1|\alpha|\leq|1-e^{i\alpha}|\leq C_2|\alpha|$. Then

$$A_n(\theta)\geq \frac{1}{n}\int_{\theta-(\pi/n)}^{\theta+(\pi/n)} \left|\frac{1-e^{in(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\nu(\lambda) \geq \left[\frac{C_1}{C_2}\right]^2 n\nu\left(\theta-\frac{\pi}{n}, \theta+\frac{\pi}{n}\right).$$

Let $\varepsilon>0$ and $n>0$, and let $\delta=\delta(\varepsilon)=\nu\{\theta: A_n(\theta)<\varepsilon\}$. Suppose that $\delta>0$, since otherwise we are finished. Choose a compact set $K\subset\{\theta: A_n(\theta)<\varepsilon\}$ with $\nu(K)>\delta/2$. There are $\theta_1, \cdots, \theta_p\in K$ such that the intervals $\left(\theta_i-\frac{\pi}{n}, \theta_i+\frac{\pi}{n}\right)$ cover K and no more than two of them intersect at any point. Since the union of these intervals is contained in $(-2\pi, 2\pi)$, it follows that $p\frac{2\pi}{n}\leq 8\pi$, and hence $p\leq 4n$. Therefore

$$\nu(K)\leq\sum_{i=1}^{p}\nu\left(\theta_i-\frac{\pi}{n}, \theta_i+\frac{\pi}{n}\right)\leq p\left[\frac{C_2}{C_1}\right]^2\frac{1}{n}\varepsilon\leq 4\left[\frac{C_2}{C_1}\right]^2\varepsilon ,$$

and $\delta<8(C_2/C_1)^2\varepsilon$, proving the Lemma and hence also the Theorem.

Corollary 1. *If ν is a positive measure on $[-\pi, \pi)$ and (n_j) is an increasing sequence of positive integers, then*

$$\limsup_{j\to\infty}\frac{1}{n_j}\int_{-\pi}^{\pi}\left|\frac{1-e^{in_j(\lambda-\theta)}}{1-e^{i(\lambda-\theta)}}\right|^2 d\nu(\lambda)>0 \qquad \textit{for } \nu\textit{-almost all } \theta .$$

Proof. For each $\varepsilon>0$, $\{\theta: \limsup A_{n_j}(\theta)=0\}\subset\{\theta: A_{n_j}(\theta)<\varepsilon$ for all large enough $j\}$, a set of measure less than ε/C by the Lemma.

Corollary 2. *Let H be a Hilbert space, $T: H\to H$ a unitary operator, and $0\neq f\in H$. Then there exists a frequency θ at which the "mean power" of f, defined by*

$$\bar{P}(\theta) = \limsup_{n\to\infty}\frac{1}{n}\,||\sum_{k=0}^{n-1}e^{-ik\theta}T^k f||^2 ,$$

is positive.

Corollary 3. *Let H be a Hilbert space and $T: H\to H$ a unitary operator. For each $\theta\in[-\pi, \pi)$ let*

$$\mathcal{B}_\theta = \{e^{i\theta}g-Tg: g\in H\}$$

be the space of "θ-twisted coboundaries" for T. Then $\bigcap_{\theta\in[\pi,\pi)}\mathcal{B}_\theta=\{0\}$.

Proof. If $f\in\mathcal{B}_\theta$, then $\{||S_n^\theta f||: n=1, 2, \cdots\}$ is bounded.

REMARK. As in [2], by considering fixed points of the operator $V_f^\theta g=e^{-i\theta}(f+Tg)$, one can show that in fact $f\in\mathcal{B}_\theta$ if and only if $\{||S_n^\theta f||: n=1, 2,\cdots\}$ is bounded. For further developments in this direction, see [1].

Corollary 4. *As in* [3], *define the "spectral notch" subshift $\sum(r, \theta)$ corresponding to $r>0$ and $\theta\in[-\pi, \pi)$ to be the set of all those $x\in\{-1, 1\}^{\mathbf{Z}}$ for which*

$$|\sum_{k=m}^{m+n} x_k e^{-ik\theta}| < r \qquad \textit{for all } m \in \boldsymbol{Z} \textit{ and all } n \geq 0\,.$$

Then $\bigcap_{\theta \in [-\pi, \pi)} \bigcup_{r > 0} \Sigma(r, \theta) = \phi$.

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Emmanuel Lesigne
Université de Bretagne Occidentale
Faculté des Sciences
Av. Le Gorgeu, 29287 Brest Cédex, France

Karl Petersen
CB 3250, Phillips Hall
Dept. of Mathematics, U.N.C.
Chapel Hill, NC 27599, USA