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BOUNDED MEASURABLE SIMULTANEOUS MONOTONE APPROXIMATION

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Let $X = [a, b]$ be a closed bounded real interval. Let B be the closed linear space of all bounded real valued functions defined on X, and let $M \subseteq B$ be the closed convex cone consisting of all monotone non-decreasing functions on X. For $f, g \in B$ and a fixed positive $w \in B$, we define the so-called best L_∞-simultaneous approximant of f and g to be an element $h^* \in M$ satisfying

$$\max\left(\|f - h^*\|_w, \|g - h^*\|_w\right) = d \leqslant \max\left(\|f - h\|_w, \|g - h\|_w\right),$$

for all $h \in M$, where

$$\|f\|_w = \sup_{a \leqslant x \leqslant b} w(x)|f(x)|.$$

We establish a duality result involving the value of d in terms of f, g and w only.

If in addition f, g and w are continuous, then some characterisation results are obtained.

1. INTRODUCTION

Let $X = [a, b]$ be a closed bounded interval of the real line. Let $B = B(X)$ be the linear space of all bounded real valued functions defined on X. Let $M = M(X) \subseteq B$ be the closed convex cone of monotone non-decreasing functions defined on X. Given a fixed $w \in B$, $w(x) \geqslant \delta > 0$ for all $x \in X$, define a weighted uniform norm $\|\cdot\|_w$ on B by

$$\|f\|_w = \sup\left(w(x)|f(x)| : x \in X\right). \tag{1}$$

The problem we are investigating in this paper is : Given f and g in B, find $h^* \in M$, if one exists, such that

$$d = \max\left(\|f - h^*\|_w, \|g - h^*\|_w\right) = \inf \max\left(\|f - h\|_w, \|g - h\|_w\right). \tag{2}$$

where the infimum is taken over all h in M. Such h^* is called a *best L_∞-simultaneous approximant* of f and g, abbreviated b.s.a. . Note that if $f \neq g$ then $d > 0$. Of course, when $w \equiv 1$, we have the usual well known uniform, or Tchebychev, norm.

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In [1] Ubhaya treated the case of the L_∞-approximation to a single function f by elements of M. He gave an explicit formula for computing d in terms of f and w only, where f is the function to be approximated with respect to the norm given by (1). He also characterised the set of all L_∞ approximants of f, and he established properties of this solution set and its behaviour on some parts of X. In addition , if f , w are continuous, $f \notin M$, he proved the existence of an infinitely differentiable function $h \in M$ which is a best L_∞-approximant of f .

Our main objective here is to generalise Ubhaya's results to the simultaneous approximation case. In Section 2 we start with the elimination of the trivial possibilities of values of d compared to the value of the distance between M and either of f or g alone. Then we generalise the duality results established in [1]. We also show the existence of a function $h^* \in M$ satisfying (2), and we give an explicit expression of the set of all such solutions which clearly forms a convex subset of M.

For simplicity, we supress w from the norm notation in (1) and (2).

2. Duality and Characterisation

Lemma 1. *Suppose that $f, g \in B \cap M$. Then $h^* = (f+g)/2$ is a best L_∞-simultaneous approximant of f and g.*

Proof: Suppose there exists $h \in M$ such that

$$\max(\|f-g\|, \|g-h\|) < \max(\|f-h^*\|, \|g-h^*\|) = \|f-g\|/2.$$

Then
$$\begin{aligned}\|f-g\| = \|f-h+h-g\| &\leqslant \|f-h\| + \|g-h\| \\ &< \|f-g\|/2 + \|f-g\|/2 = \|f-g\|.\end{aligned}$$

This is a contradiction! This establishes the Lemma. However it can be easily seen by an example that h^* is not unique in general. ∎

Remark 1. (i) When $f = g$ we end up with the single approximation case discussed in [1].

(ii) If $f \neq g$, and there exists an element $f_\infty \in M$ such that $\|g - f_\infty\| \leqslant \|f - f_\infty\|$ and f_∞ is a best L_∞-approximant of f , then clearly

$$\max(\|g-f_\infty\|, \|f-f_\infty\|) = \|f-f_\infty\| \leqslant \|f-h\| \leqslant \max(\|f-h\|, \|g-h\|)$$

for all $h \in M$ and hence f_∞ is a best L_∞-simultaneuos approximant of f and g .

To this end , we shall exclude for all practical purposes the three cases encountered above in Lemma 1 and Remark 1. With this assumption in mind we proceed to the next step.

Let $\triangle$ be the closed triangle given by

$$\triangle = \{(x,y) \in [a,b] \times [a,b] : x \leqslant y\}.$$

We also define the following

$$\begin{aligned}
u(x,y) &= w(x)w(y)/(w(x)+w(y)); \\
\theta_1 &= \sup\{u(x,y)(f(x)-g(y)) : (x,y) \in \triangle\}; \\
\theta_2 &= \sup\{u(x,y)(g(x)-f(y)) : (x,y) \in \triangle\}; \\
\theta &= \max\{\theta_1, \theta_2\}; \\
T_1 &= \{(x,y) \in \triangle : u(x,y)(f(x)-g(y)) = \theta\} \\
T_2 &= \{(x,y) \in \triangle : u(x,y)(g(x)-f(y)) = \theta\}; \\
T &= T_1 \cup T_2; \\
P &= \bigcup\{[x,y] : (x,y) \in T\}; \\
m(x,y) &= (w(x)f(x)+w(y)g(y))/(w(x)+w(y)), \qquad x,y \in X.
\end{aligned}$$

Finally define the functions $\underline{h}$ and $\overline{h}$ on $[a,b]$ by

$$\begin{aligned}
\underline{h}(x) &= \sup\{[f(z) \vee g(z) - \theta/w(z)] : z \in [a,x]\}, \\
\overline{h}(x) &= \inf\{[f(z) \wedge g(z) + \theta/w(z)] : z \in [x,b]\},
\end{aligned}$$

where $f \vee g = \max(f,g)$ and $f \wedge g = \min(f,g)$.

Remark 2. (i) In general $\theta_1 \neq \theta_2$. We assume here that $\theta_2 \leq \theta_1 = \theta$.

(ii) $T \neq \emptyset$. However T might consist of a single point (x,y) with $x \leq y$, hence P could consist of a single point $x \in [a,b]$.

(iii) $\underline{h}$ and $\overline{h}$ are both monotone non-decreasing .

(iv) $\theta = 0$ if and only if $f = g \in M$.

(v) If h^* is a best L_∞-simultaneous approximant of f and g, then $h^* + c$ is a best L_∞-simultaneous approximant of $f + c$ and $g + c$ where c is a constant. Therefore we may assume without loss of generality that both f and g are non-negative and so is h^* .

Example. Let $X = [0,1]$. Define f and g as follows: $f(0) = 3$, $f(1/3) = 0$, $f(2/3) = 5$, $f(1) = 4$ and the graph of f is linear between these points. Let $g(0) = 3$, $g(1/2) = 1$, $g(2/3) = 1$, $g(1) = 3$ and the graph of g is linear between these points. Let $w \equiv 1$. Then $\theta_1 = (5-1)/4 = 2 > \theta_2 = (3-0)/2 = 3/2$, $T = \{(2/3, 2/3)\}$ and $P = \{2/3\}$. Notice also that $\underline{h}(2/3) = \overline{h}(2/3) = m(2/3, 2/3) = 3$, and $\underline{h}(x) < \overline{h}(x)$ for all $x \neq 2/3$. However $\|f - \underline{h}\| = \|g - \underline{h}\| = \|f - \overline{h}\| = \|g - \overline{h}\| = 2 = \theta_1 = \theta$.

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Remark 3. By [1], the L_∞-distance between f and M is given by

$$\theta_f = \sup_{(x,y)\in\triangle} u(x,y)(f(x)-f(y)).$$

Clearly $\theta \geqslant \max(\theta_f, \theta_g)$, because of the assumption following Remark 1.

THEOREM 2. *Let f, g and w be as specified in Section 1. Let θ be as defined above. Then*

(3)
$$\theta = d = \inf_{h\in M} \max(\|f-h\|, \|g-h\|).$$

Hence $\theta \leqslant \max(\|f\|, \|g\|)$.

PROOF: We show first that $\theta = \theta_1 \leqslant \eta = \max(\|f-h\|, \|g-h\|)$ for any arbitrary $h \in M$. So let $(x,y) \in \triangle$. Then

$$w(x)|f(x)-h(x)| \leqslant \|f-h\| \leqslant \eta,$$

and

$$w(y)|g(y)-h(y)| \leqslant \|g-h\| \leqslant \eta.$$

By monotonicity of h, we have $h(y)-h(x) \geqslant 0$, so we obtain

$$\begin{aligned} f(x)-g(y) &\leqslant f(x)-g(y)+h(y)-h(x), \\ &\leqslant (w(x)|f(x)-h(x)|/w(x)) + (w(y)|g(y)-h(y)|/w(y)), \\ &\leqslant \|f-h\|/w(x) + \|g-h\|/w(y), \\ &\leqslant (1/w(x)+1/w(y))\eta = (w(x)+w(y))\eta/w(x)w(y), \end{aligned}$$

or

$$u(x,y)(f(x)-g(y)) \leqslant \eta.$$

Since $(x,y) \in \triangle$ is arbitrary, we conclude that $\theta \leqslant \eta$. Since h was arbitrary, we get $\theta \leqslant d$. Next we show that $\theta = \max(\|f-\underline{h}\|, \|g-\underline{h}\|)$. Let $x \in [a,b]$. By the definition of $\underline{h}$, we have $\underline{h}(x) \geqslant f(x) \vee g(x) - \theta/w(x)$, or equivalently

(4)
$$w(x)(\underline{h}(x)-f(x)) \geqslant -\theta,$$

and

(5)
$$w(x)(\underline{h}(x)-g(x)) \geqslant -\theta.$$

Now, let $\varepsilon > 0$ be given. Then there exists $z \in [a,x]$ such that

$$\underline{h}(x) \leqslant f(z) \vee g(z) - \theta/w(z) + \varepsilon.$$

We have two symmetric cases to consider. It suffices to treat one of them:

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Case 1. $f(z) \geqslant g(z)$,so

$$\underline{h}(x) \leqslant f(z) - \theta/w(z) + \varepsilon. \tag{6}$$

By the definition of θ we have

$$\theta \geqslant (1/w(z) + 1/w(x))^{-1}(f(z) - g(x)), \tag{7}$$

or

$$f(z) - \theta/w(z) \leqslant g(x) + \theta/w(x).$$

Combining (6) and (7) we obtain

$$\underline{h}(x) \leqslant g(x) + \theta/w(x) + \varepsilon.$$

Since ε was arbitrary, we conclude that $\underline{h}(x) \leqslant g(x) + \theta/w(x)$, or

$$w(x)(\underline{h}(x) - g(x)) \leqslant \theta. \tag{8}$$

Thus (5) together with (8) imply that $\|\underline{h} - g\| \leqslant \theta$. It remains to show that $w(x)(\underline{h}(x) - f(x)) \leqslant \theta$. Indeed we have by the definition of θ together with Remark 3 that

$$\theta \geqslant \theta_f \geqslant (1/w(z) + 1/w(x))^{-1}(f(z) - f(x)),$$

or

$$f(z) - \theta/w(z) \leqslant f(x) + \theta/w(x).$$

It follows from (6) that $\underline{h}(x) \leqslant f(x) + \theta/w(x) + \varepsilon$. Since ε was arbitrary, we conclude that $\underline{h}(x) \leqslant f(x) + \theta/w(x)$, or

$$w(x)(\underline{h}(x) - f(x)) \leqslant \theta. \tag{9}$$

Combining (4),(5),(8) and (9) shows that

$$\theta \geqslant \max(\|f - \underline{h}\|, \|g - \underline{h}\|).$$

This establishes the main part of the theorem. The inequality is obtained by putting $h \equiv 0 \in M$. ∎

Remark 4. In light of Theorem 2, we see that in order to exclude the case given by Remark 1(ii) we can not have $\max g \leqslant \max f$ and $\min f \leqslant \min g$ where both of f and g are continuous on $[a, b]$.

THEOREM 3. (Characterisation). *Let* $\underline{h}$, $\overline{h}$, θ *and* d *be as defined earlier. Then* $\underline{h}$, $\overline{h} \in M$, $\underline{h} \leqslant \overline{h}$ and $\theta = d = \max(\|f - \underline{h}\|, \|g - \underline{h}\|) = \max(\|f - \overline{h}\|, \|g - \overline{h}\|)$. *Furthermore, for* $h^* \in M$

$$\theta = d = \max(\|f - h^*\|, \|g - h^*\|) \tag{10}$$

holds if and only if $\underline{h} \leqslant h^* \leqslant \overline{h}$.

PROOF: By Remark 2(iii) we have $\underline{h}$, $\overline{h} \in M$. By Theorem 2 and a similar argument for $\overline{h}$ we obtain

$$\theta = d = \max(\|f - \underline{h}\|, \|g - \underline{h}\|) = \max(\|f - \overline{h}\|, \|g - \overline{h}\|).$$

Suppose now that $h^* \in M$ and $\theta = \max(\|f - h^*\|, \|g - h^*\|) = d$. Let $x \in [a, b]$ be arbitrary but fixed, and let $\varepsilon > 0$ be given. By the definition of $\underline{h}$, there is $z \in [a, x]$ such that $\underline{h}(x) \leqslant f(z) \vee g(z) - \theta/w(z) + \varepsilon$. But

$$\theta \geqslant \max(w(z)(f(z) - h^*(z)), w(z)(g(z) - h^*(z))),$$

which implies that

$$\theta/w(z) \geqslant f(z) - h^*(z), \quad \text{and} \quad \theta/w(z) \geqslant g(z) - h^*(z).$$

Hence,
$$h^*(z) \geqslant f(z) \vee g(z) - \theta/w(z).$$

Thus $\underline{h}(x) \leqslant h^*(z) + \varepsilon \leqslant h^*(x) + \varepsilon$. Since ε was arbitrary we get $\underline{h}(x) \leqslant h^*(x)$. Letting $h^* = \overline{h}$ we end up with $\underline{h} \leqslant \overline{h}$. Similarly we show $h^* \leqslant \overline{h}$.

Next, let $\underline{h} \leqslant h^* \leqslant \overline{h}$. We show that $\max(\|f - h^*\|, \|g - h^*\|) = \theta$. Let $x \in X$.

Then
$$\begin{aligned} \theta &\geqslant \max(w(x)(f(x) - \underline{h}(x)), w(x)(g(x) - \underline{h}(x))), \\ &\geqslant \max(w(x)(f(x) - h^*(x)), (w(x)(g(x) - h^*(x)))). \end{aligned}$$

Also
$$\begin{aligned} \theta &\geqslant \max\left(w(x)(\overline{h}(x) - f(x)), w(x)(\overline{h}(x) - g(x))\right), \\ &\geqslant \max(w(x)(h^*(x) - f(x)), w(x)(h^*(x) - g(x))). \end{aligned}$$

This says that
$$-\theta \leqslant w(x)(f(x) - h^*(x)) \leqslant \theta,$$

and similarly
$$-\theta \leqslant w(x)(g(x) - h^*(x)) \leqslant \theta.$$

Hence
$$\theta \geqslant \max(\|f - h^*\|, \|g - h^*\|).$$

Equality follows from Theorem 2. ∎

LEMMA 4. *Suppose* f , g *and* w *are continuous. Then* $\underline{h}$ *and* $\overline{h}$ *are both continuous.*

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PROOF: By the definition of $\underline{h}$ we may write for $y > x$,

$$\underline{h}(y) = \max\{\underline{h}(x), \max_{z\in[x,y]} (f(z) \vee g(z) - \theta/w(z))\}.$$

Hence

$$\underline{h}(y) - \underline{h}(x) = \max\{0, \max_{z\in[x,y]} (f(z) \vee g(z) - \theta/w(z) - \underline{h}(x))\}.$$

But the fact that $\underline{h}(x) \geqslant f(x) \vee g(x) - \theta/w(x)$ implies that

$$0 \leqslant \underline{h}(y) - \underline{h}(x) \leqslant \max\{0, \max_{z\in[x,y]} ((f(z) \vee g(z) - \theta/w(z)) - (f(x) \vee g(x) - \theta/w(x)))\}.$$

Since f and g are both continuous, we have $f \vee g - \theta/w$ is also continuous. This establishes the continuity of $\underline{h}$. Similarly we obtain the continuity of $\overline{h}$. □

THEOREM 5. *Let f , g and w be continuous with $\theta > 0$. Then*

(11) $$P = \bigcup_{k=1}^{n} [a_k, b_k], \qquad n \geqslant 1,$$

$$a \leqslant a_k \leqslant b_k \leqslant b, \qquad \text{for all } k = 1, \dots, n.$$

For $n \geqslant 2$ $$b_k < a_{k+1}, \qquad k = 1, 2, \dots, n-1,$$

and $$(a_k, b_k) \in T, \qquad \text{for all } k.$$

PROOF: Clearly $m(x,y) : [a,b] \times [a,b] \mapsto R$ is a continuous function. Let

$$\Gamma_i = \{\gamma : \gamma = m(x,y), (x,y) \in T_i\}; \qquad i = 1, 2,$$

Define an equivalence relation $\sim$ on $T_i (i = 1,2)$, by $(x_1, y_1) \sim (x_2, y_2) \iff m(x_1, y_1) = m(x_2, y_2)$, where $(x_1, y_1), (x_2, y_2) \in T_i$. Then the sets

$$T_1^\gamma = \{(x,y) \in T_1 : m(x,y) = \gamma\},$$
$$T_2^\gamma = \{(x,y) \in T_2 : m(x,y) = \gamma\}.$$

are equivalence classes.

For each $\gamma \in \Gamma = \Gamma_1 \cup \Gamma_2$, let

$$T_\gamma = T_1^\gamma \cup T_2^\gamma.$$

Also, let $$a_\gamma = \inf\{x : (x,y) \in T_\gamma\},$$

and $$b_\gamma = \sup\{y : (x,y) \in T_\gamma\}.$$

Clearly $a_\gamma = b_\gamma$ if and only if $T_\gamma = (x,x)$ for a single point $x \in [a,b]$. Suppose $a_\gamma < b_\gamma$. We assert that $m(a_\gamma, b_\gamma) = \gamma$, and so $(a_\gamma, b_\gamma) \in T_\gamma$. Indeed by the definitions

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of inf and sup, there are sequences $(x_n, y_n), (u_n, v_n) \in T, n = 1, 2, \ldots$ such that $x_n \to a_\gamma$ and $v_n \to b_\gamma$. Let us assume without loss of generality that $(x_n, y_n) \in T_1$, so we obtain for all n,

$$(12) \qquad m(x_n, y_n) = (w(x_n) + w(y_n))^{-1}(w(x_n)f(x_n) + w(y_n)g(y_n)) = \gamma,$$

and

$$(13) \qquad \theta = (w(x_n) + w(y_n))^{-1}w(x_n)w(y_n)(f(x_n) - g(y_n)).$$

We now have two cases to consider:

Case 1. There is a subsequence $(u_n, v_n) \in T_1$ such that $v_n \to b_\gamma$, and

$$(14) \qquad m(u_n, v_n) = (w(u_n) + w(v_n))^{-1}(w(u_n)f(u_n) + w(v_n)g(v_n)) = \gamma.$$

and,

$$(15) \qquad \theta = (w(u_n) + w(v_n))^{-1}w(u_n)w(v_n)(f(u_n) - g(v_n)).$$

Hence from (12) and (13) we get

$$\theta/w(x_n) + \theta/w(y_n) = f(x_n) - g(y_n),$$

or

$$f(x_n) - \theta/w(x_n) = g(y_n) + \theta/w(y_n) = \gamma.$$

Similarly (14) and (15) imply that

$$f(u_n) - \theta/w(u_n) = g(v_n) + \theta/w(v_n) = \gamma.$$

Hence,

$$f(x_n) - \theta/w(x_n) = \gamma = g(v_n) + \theta/w(v_n).$$

Letting $n \to \infty$, we conclude by the continuity of f, g and w that

$$(16) \qquad f(a_\gamma) - \theta/w(a_\gamma) = \gamma = g(b_\gamma) + \theta/w(b_\gamma),$$

so that

$$(17) \qquad (w(a_\gamma) + w(b_\gamma))^{-1}w(a_\gamma)w(b_\gamma)(f(a_\gamma) - g(b_\gamma)) = \theta.$$

Thus, $(a_\gamma, b_\gamma) \in T$. Substituting for θ in (17), using the first part of (16), we conclude that $m(a_\gamma, b_\gamma) = \gamma$. This proves the assertion for case 1.

Case 2. There is no sequence $(u_n, v_n) \in T_1^\gamma$ for which $v_n \to b_\gamma$, that is, $v_n \to b_\gamma$ if and only if $(u_n, v_n) \in T_2^\gamma$. In such a case we can argue that $\theta = \theta_f$ which is contradictory to our assumption.Therefore only case 1 is valid.

Next we show that for $(x,y) \in T, x < y$ we have $[x,y] \cap [a_\gamma, b_\gamma] \neq \emptyset$ if and only if $m(x,y) = \gamma$. By the definition of θ , we have

$$f(x) - \theta/w(x) \leqslant g(y) + \theta/w(y). \tag{18}$$

If $[a_\gamma, b_\gamma] \cap [x,y] \neq \emptyset$, then it follows from the definition of a_γ and b_γ that $a_\gamma \leqslant y$ and $b_\gamma \geqslant x$. From (16),(17),(18) and the definition of θ it follows that

$$\begin{aligned} f(x) - \theta/w(x) &\leqslant g(b_\gamma) + \theta/w(b_\gamma) = \gamma \\ &= f(a_\gamma) - \theta/w(a_\gamma) \\ &\leqslant g(y) + \theta/w(y)\,. \end{aligned} \tag{19}$$

Since $(x,y) \in T$, (18) holds with equality, and therefore (19) implies that

$$f(x) - \theta/w(x) = g(y) + \theta/w(y) = \gamma,$$

or alternatively $m(x,y) = \gamma$. The converse follows immediately from the definition of a_γ and b_γ.

By the uniform continuity of f and g we can easily deduce the first part of the theorem, that is, Γ is finite and hence P is a finite union of closed sub-intervals. ∎

THEOREM 6. *Let f , g , w , θ and P be as in the previous Theorem. Then*

$$\underline{h}(x) = \overline{h}(x) \quad \text{if and only if } x \in P,$$

with

$$\underline{h}(x) = \overline{h}(x) = m(a_k, b_k) \quad \text{for all } x \in [a_k, b_k] \text{ and all } k,$$

where

$$m(a_k, b_k) < m(a_{k+1}, b_{k+1}), \quad k = 1, 2, \dots, n-1.$$

Moreover

$$w(x)|f(x) - \underline{h}(x)| = w(y)|g(y) - \overline{h}(y)| = \theta, \quad (x,y) \in T_1,$$

and

$$w(y)|f(y) - \overline{h}(y)| = w(x)|g(x) - \underline{h}(x)| = \theta, \quad (x,y) \in T_2.$$

PROOF: To obtain the first part of the Theorem we start by showing that $\underline{h}(x) = \gamma_k$ for all $x \in [a_k, b_k]$. In (16), let $a_\gamma = a_k$, $b_\gamma = b_k$ and $\gamma = \gamma_k$. Hence we obtain

$$f(a_k) - \theta/w(a_k) = \gamma_k = g(b_k) + \theta/w(b_k), \qquad k = 1, \dots, n.$$

If $a \leqslant z \leqslant b_k$, then by the definition of θ we have

$$f(z) - \theta/w(z) \leqslant g(b_k) + \theta/w(b_k) = \gamma_k.$$

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Since $\theta \geqslant \theta_g$, it follows that

$$g(z) - \theta/w(z) \leqslant g(b_k) + \theta/w(b_k) = \gamma_k,$$

and hence

$$f(z) \vee g(z) - \theta/w(z) \leqslant \gamma_k.$$

From the definition of $\underline{h}$ we conclude that $\underline{h}(x) \leqslant \gamma_k$ for all $x \in [a, b_k]$. But then

$$\underline{h}(a_k) \geqslant f(a_k) - \theta/w(a_k) = \gamma_k.$$

By monotonicity of $\underline{h}$ it follows that for any $x \in [a_k, b_k]$ we have

$$\underline{h}(x) \geqslant \underline{h}(a_k) \geqslant \gamma_k.$$

Hence $\underline{h}(x) = \gamma_k$ for all $x \in [a_k, b_k]$.

Similarly we show that $\overline{h}(x) = \gamma_k$ for all $x \in [a_k, b_k]$.

We now prove the second part of the theorem consisting of the last two equations. Let $(x, y) \in T = T_1 \cup T_2$. Assume without loss of generality that $(x, y) \in T_1$. The other case is similar. Since Γ is finite, we must have $m(x, y) = \gamma_k$ for some $k = 1, 2, \ldots, n$, so it follows that $[x, y] \subseteq [a_k, b_k]$. Since $\underline{h}$ is non-decreasing we have

$$\gamma_k = \underline{h}(a_k) \leqslant \underline{h}(x) \leqslant \underline{h}(y) \leqslant \underline{h}(b_k) = \gamma_k,$$

so that $\underline{h}(x) = \gamma_k$ and

$$\begin{aligned} w(x)|f(x) - \underline{h}(x)| &= w(x)|f(x) - m(x, y)|, \\ &= w(x)|f(x) - (w(x) + w(y))^{-1}(w(x)f(x) + w(y)g(y))|, \\ &= (w(x) + w(y))^{-1} w(x)w(y)|f(x) - g(y)| = \theta. \end{aligned}$$

Similarly,

$$\begin{aligned} w(y)|g(y) - \overline{h}(y)| &= w(y)|g(y) - m(x, y)|, \\ &= w(y)|g(y) - (w(x) + w(y))^{-1}(w(x)f(x) + w(y)g(y))|, \\ &= (w(x) + w(y))^{-1} w(x)w(y)|g(y) - f(x)| = \theta. \end{aligned}$$

□

LEMMA 7. *Suppose that f, g and w are continuous, and that $0 < \theta_f, \theta_g < \theta$.*

Then $\overline{h}(x) > \gamma_k$, *for* $x > b_k$,

and $\underline{h}(x) < \gamma_k$, *for* $x < a_k$.

PROOF: Suppose that for some $x > b_k$ we have $\overline{h}(x) = \gamma_k$. Then by the definition of $\overline{h}(x)$, there exists $y \in [x, b]$ such that

$$\overline{h}(x) = \gamma_k = \min(f(y), g(y)) + \theta/w(y).$$

We have two cases to consider:

Case 1. $f(y) < g(y)$, so that

$$\overline{h}(x) = f(y) + \theta/w(y) = \gamma_k. \tag{20}$$

In such a case we claim that $\theta = u(a_k, b_k)(g(a_k) - f(b_k))$. If not, then we must have $\theta = u(a_k, b_k)(f(a_k) - g(b_k))$. Hence

$$f(a_k) - \theta/w(a_k) = f(a_k) - u(a_k, b_k)(f(a_k) - g(b_k))/w(a_k) = \gamma_k. \tag{21}$$

Combining (20) and (21) results in

$$f(a_k) - \theta/w(a_k) = f(y) + \theta/w(y),$$

or

$$\theta = \frac{w(a_k)w(y)}{(w(a_k) + w(y))}(f(a_k) - f(y)) \leqslant \theta_f.$$

This is a contradiction! Therefore our claim holds and we have

$$\begin{aligned} g(a_k) - \theta/w(a_k) &= g(a_k) - (w(a_k) + w(b_k))^{-1} w(b_k)(g(a_k) - f(b_k)), \\ &= (w(a_k) + w(b_k))^{-1}(w(a_k)g(a_k) + w(b_k)f(b_k)), \\ &= \gamma_k = f(y) + \theta/w(y), \end{aligned}$$

or

$$g(a_k) - f(y) = (1/w(a_k) + 1/w(y))\theta.$$

Hence

$$\theta = u(a_k, y)(g(a_k) - f(y)).$$

which implies that $(a_k, y) \in T$, and by Theorem 5 we conclude that $[a_k, y] \subseteq [a_k, b_k]$ which is a contradiction since $a_k \leqslant b_k < y$. Therefore there is no $x > b_k$, for which $\overline{h}(x) = \gamma_k$,that is, $\overline{h}(x) > \gamma_k$ for $x > b_k$.

Case 2. $g(y) < f(y)$. The same argument applies. This concludes the proof of the first part. The other part is similar. ∎

THEOREM 8. *Let f ,g and w be continuous on $[a, b]$. If $0 < \theta_f$, $\theta_g < \theta$, then*

$$\underline{h}(x) < \overline{h}(x) \text{ for all } x \in [a, a_1] \cup \left(\bigcup_{k=1}^{n-1}\right) \cup (b_n, b].$$

PROOF: Suppose that for some $t \in (b_k, a_{k+1})$, $k = 1, 2, \ldots, n-1$, we have $\underline{h}(t) = \overline{h}(t)$. Then by the definitions of $\underline{h}$ and $\overline{h}$, there exists $u \in [b_k, t]$, $v \in [t, a_{k+1}]$ such that

$$\begin{aligned} \underline{h}(t) &= \max(f(u), g(u)) - \theta/w(u), \\ &= \min(f(v), g(v)) + \theta/w(v) = \overline{h}(t). \end{aligned} \tag{22}$$

Notice that if $u < b_k$, then clearly $\underline{h}(x) = \underline{h}(b_k) = \gamma_k$ for all $x \in (b_k, t]$ which contradicts the definition of b_k. Therefore $u \in [b_k, t]$. Similarly we must have $v \in [t, a_{k+1}]$. In (22) suppose $f(u) > g(u)$. We also have two cases here:

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Case 1. $f(v) > g(v)$, so we obtain

$$\underline{h}(t) = f(u) - \theta/w(u) = g(v) + \theta/w(v) = \overline{h}(t),$$

or

$$\theta = \frac{w(u)w(v)}{(w(u)+w(v))}(f(u) - g(v)), \qquad u < v.$$

This says that $(u,v) \in T$, or there exists some i such that $(u,v) \subseteq [a_i, b_i]$. This is a contradiction, since we have $b_k \leqslant u \leqslant t \leqslant v \leqslant a_{k+1}$.

Case 2. $f(v) < g(v)$, so we obtain

$$\underline{h}(t) = f(u) - \theta/w(u) = f(v) + \theta/w(v) = \overline{h}(t),$$

or

$$\theta = \frac{w(u)w(v)}{(w(u)+w(v))}(f(u) - f(v)) \leqslant \theta_f,$$

contradicting our assumption that $\theta > \theta_f$.

In the cases $t \in [a, a_1]$ or $t \in (b_n, b]$ we follow the same line of argument. Hence, Theorem 8. □

References

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