Bounded Negativity and Arrangements of Lines

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Abstract

The Bounded Negativity Conjecture predicts that for any smooth complex surface X there exists a lower bound for the selfintersection of reduced divisors on X. This conjecture is open. It is also not known if the existence of such a lower bound is invariant in the birational equivalence class of X. In the present note we introduce certain constants $H(X)$ which measure in effect the variance of the lower bounds in the birational equivalence class of X. We focus on rational surfaces and relate the value of $H(\mathbb{P}^2)$ to certain line arrangements. Our main result is Theorem 3.3 and the main open challenge is Problem 3.11.

Keywords Bounded Negativity, SHGH Conjecture, Arrangements of Lines

Mathematics Subject Classification (2000) MSC 14C20

1 Introduction

In recent years there has been growing interest in constraints on negative curves on algebraic surfaces. The Bounded Negativity Conjecture (BNC for short) is probably the most intriguing open question in this area, see for example [7, Conjecture 1.2.1], [3, Conjecture 1.1].

Conjecture 1.1 (Bounded Negativity Conjecture)**.** *For every smooth projective surface X, there exists an integer $b(X)$ such that $C^2 \geqslant -b(X)$ for every reduced curve $C \subset X$.*

This Conjecture is well known to be false in finite characteristic. In the present paper we work therefore in the setting of complex algebraic varieties, where it remains open.

Conjecture 1.1 is related to a number of interesting questions. The present note is motivated by the following problem.

Problem 1.2 (Birational invariance of the BNC)**.** Let X and Y be birationally equivalent projective surfaces. Does BNC hold for X if and only if it holds for Y? In other words: is the bounded negativity property a birational invariant?

Remark 1.3. Note, that a solution to the above problem is not known even if Y is the blow-up of X in a single point.

Of course, if BNC is true in general, then the above problem has an affirmative solution. However, even in that situation, it is still of interest to know how the bounds $b(X)$ and $b(Y)$ are related in terms of the complexity of a birational map between X and Y.

In the present note we study Problem 1.2 for blow-ups Y_s of $\mathbb{P}^2$ in *arbitrary* sets of s points. A recent preprint by Ciliberto and Roulleau [4] addresses BNC on blow-ups of $\mathbb{P}^2$ in *general* points. These two set-ups are quite different. It is predicted by the Segre-Harbourne-Gimigliano-Hirschowitz (SHGH) Conjecture that (-1)-curves are the only negative curves on blow-ups of $\mathbb{P}^2$ in general points, independent of their number s. On the other hand, if one is allowed to pick any s points, it is elementary to see that one can obtain reduced curves C for which C^2 is arbitrarily negative by allowing s to grow. This raises the question of the boundedness of C^2/s; see Problem 3.11. Our main result, Theorem 3.3, shows that $C^2/s > -4$ for reduced curves C on Y_s which are strict transforms of configurations of lines in $\mathbb{P}^2$. In fact, all examples that we know, of reduced curves C for which $C^2/s \leqslant -2$, are strict transforms of configurations of lines. This has led to our expectation that the most negative values of C^2/s come from configurations of lines, but we have not been able to prove this so far. We also include a short discussion of the only two line arrangements we know of for which $C^2/s \leqslant -3$.

2 Local negativity

Let X be a smooth projective surface for which Conjecture 1.1 is true (such as $X = \mathbb{P}^2$) and let C be a reduced curve on X. Let s be a positive integer and let $P_1, \ldots, P_s$ be mutually distinct smooth points on C. Finally, let $f : Y \to X$ be the blow-up of X at the points $P_1, \ldots, P_s$ with exceptional divisors $E_1, \ldots, E_s$. Then for the proper transform

$$\widetilde{C} = f^*C - \sum_{i=1}^{s} E_i,$$

we have $\widetilde{C}^2 = C^2 - s \geqslant -b(X) - s$, hence $b(Y) \geqslant b(X) + s$. So it is easy to find surfaces birational to a given surface and carrying arbitrarily negative curves. In order to avoid trivial situations of that kind, we introduce the following H–constants, which measure the *local negativity* of curves on surfaces (in analogy to the local positivity measured by Seshadri constants).

Definition 2.1 (H–constants)**.** Let X be a smooth projective surface and let $\mathcal{P} = \{P_1, \ldots, P_s\}$ be a set of mutually distinct $s \geqslant 1$ points in X. Then the H–*constant of X at $\mathcal{P}$* is defined as

$$H(X; \mathcal{P}) := \inf \frac{\left(f^*C - \sum_{i=1}^{s} \operatorname{mult}_{P_i} C \cdot E_i\right)^2}{s}, \tag{1}$$

where $f : Y \to X$ is the blow-up of X at the set $\mathcal{P}$ with exceptional divisors $E_1, \ldots, E_s$ and the infimum is taken over all *reduced* curves $C \subset X$.
Similarly, we define the *s–tuple H–constant of X* as the infimum

$$H(X; s) := \inf_{\mathcal{P}} H(X; \mathcal{P}),$$

where the infimum now is taken over all s–tuples of mutually distinct points in X. Finally, we define the *global H–constant of X* as

$$H(X) := \inf_{s \geqslant 1} H(X; s).$$

We note that H–quotients of the curves $\widetilde{C}$ defined at the beginning of this section are bounded

$$\frac{\widetilde{C}^2}{s} = \frac{C^2}{s} - 1 \geqslant \min(C^2 - 1, -1) \geqslant \min(-b(X) - 1, -1).$$

The relation between the H–constants and the BNC is explained in the following way. Suppose that $H(X)$ is a finite number (typically a negative number). Then for any $s \geqslant 1$ and any reduced curve D on the blow-up of X in s points, we have

$$D^2 \geqslant sH(X).$$

Hence BNC holds on all blow-ups of X at s mutually distinct points. On the other hand, if $H(X) = -\infty$, then BNC might still be true.

It doesn't come as a surprise that H–constants are very hard to compute in general. Therefore as the first step towards understanding them, in the next section, we restrict our attention to blow-ups of the projective plane and to curves C coming from configurations of lines. Restricting to configurations of lines might seem to be a strong restriction, but perhaps in fact it is not; see Problem 3.11.

3   Linear local negativity

In this section we are interested in configurations of lines in the projective plane. By such a configuration we understand a family $\mathcal{L} = \{L_1, \ldots, L_d\}$ of mutually distinct lines L_i. Let $\mathcal{P}(\mathcal{L}) = \{Q_1, \ldots, Q_s\}$ be the set of points in $\mathbb{P}^2$, where at least two of the lines in $\mathcal{L}$ meet; we call these points *singular* points of the configuration. For a point $Q \in \mathbb{P}^2$, denote by $m_Q(\mathcal{L})$ the number of lines in $\mathcal{L}$ passing through that point.

Now we are in a position to introduce the following variant of Definition 2.1.

Definition 3.1. Let $\mathcal{P} = \{P_1, \ldots, P_s\}$ be a set of mutually distinct $s \geqslant 1$ points in the projective $\mathbb{P}^2$. Then the *linear H–constant at $\mathcal{P}$* is defined as

$$H_L(\mathcal{P}) := \inf_{\mathcal{L}} H_L(\mathcal{P}, \mathcal{L}),$$

where for a configuration $\mathcal{L}$ we have

$$H_L(\mathcal{P}, \mathcal{L}) = \frac{d^2 - \sum_{i=1}^{s} m_{P_i}(\mathcal{L})^2}{s}. \tag{2}$$

Similarly as before, we define the *s–tuple linear H–constant* as the infimum

$$H_L(s) := \inf_{\mathcal{P}} H_L(\mathcal{P}),$$

where now the infimum is taken over all s–tuples of mutually distinct points in $\mathbb{P}^2$. Finally, we define the *global linear H–constant of $\mathbb{P}^2$* as

$$H_L := \inf_{s \geqslant 1} H_L(s).$$

Remark 3.2. Note that the quotient on the right in (2) agrees with the quotient under the infimum in (1) for C taken as the union of all lines in $\mathcal{L}$. Note also that it is not required that the points P_i are the singular points of the configuration or even that for each of them there is a configuration line passing through. It turns out however that the quotient (2) is usually minimal when these conditions are fulfilled; see Corollary 3.8.

Our main result is the following bound on the constant H_L. The main point is that the bound holds for an arbitrary number of points s. Moreover the bound is very explicit and close to optimal (see Remark 3.9). This is considerable progress when compared to [2, Section 3.8], where this subject was first taken on.

Theorem 3.3 (Bounded linear negativity on $\mathbb{P}^2$). *With the above notation, we have*

$$H_L \geqslant -4.$$

The main ingredient in the proof of this Theorem is the following inequality due to Hirzebruch combined with some ad hoc arguments. For $k \geqslant 2$, let $t_k(\mathcal{L})$ denote the number of points where exactly k lines from $\mathcal{L}$ meet.

Theorem 3.4 (Hirzebruch inequality). *Let $\mathcal{L}$ be an arrangement of d lines in the complex projective plane $\mathbb{P}^2$. Then*

$$t_2 + \frac{3}{4}t_3 \geqslant d + \sum_{k \geqslant 5}(k-4)t_k, \tag{3}$$

provided $t_d = t_{d-1} = 0$.

Proof. See [9, Section 3 and page 140]. $\qquad\square$

Remark 3.5. Various refinements of the above inequality are known; see for example formula $*(10)$ on page 141 in [1]. However they don't contribute towards improving the lower bound in Theorem 3.3.

Remark 3.6. The proof of the Hirzebruch inequality is based on the logarithmic Miyaoka–Yau–Sakai inequality, which assumes the complex numbers. See [2, Appendix] for a detailed proof and some relevant comments.

Before proving our main result, we observe the next simple fact relating how H-constants $H_L(\mathcal{P}, \mathcal{L})$ change when the set of points $\mathcal{P}$ is modified.

Lemma 3.7. *Let $\mathcal{L}$ be a line configuration and let $\mathcal{P}$ be a set of s points. Let $\mathcal{P}'$ be obtained from $\mathcal{P}$ by adding an additional point $Q \in \mathbb{P}^2$. Then $H_L(\mathcal{P}', \mathcal{L})$ is a weighted average of $H_L(\mathcal{P}, \mathcal{L})$ and $-m_Q(\mathcal{L})^2$. Precisely,*

$$(s+1)H_L(\mathcal{P}', \mathcal{L}) = sH_L(\mathcal{P}, \mathcal{L}) - m_Q(\mathcal{L})^2.$$

Proof of Theorem 3.3. To prove the theorem, we will show that for any configuration of lines $\mathcal{L}$ and points $\mathcal{P}$ we have $H_L(\mathcal{P}, \mathcal{L}) > -4$. First suppose $\mathcal{P} = \mathcal{P}(\mathcal{L})$ is the full set of singularities of $\mathcal{L}$. Say $\mathcal{L}$ has d lines and s singularities, and let t_k for $k \geqslant 2$ denote the number of singularities of multiplicity k. To apply the Hirzebruch inequality, we must first deal with the cases where either t_d or t_{d-1} is nonzero. We assume $d \geqslant 4$ to avoid trivialities.

Case $t_d = 1$. In this case all lines in $\mathcal{L}$ belong to a single pencil. There is a single singularity of multiplicity d, $\mathcal{P}$ is just the singular point, and $H_L(\mathcal{P}, \mathcal{L}) = 0$.

Case $t_{d-1} = 1$. This case is called a *quasi–pencil* by Hirzebruch. We have $t_2 = d - 1$ and all other numbers t_i vanish. In this situation we find

$$H_L(\mathcal{P}, \mathcal{L}) = -2 + \frac{3}{d}.$$

Case $t_d = t_{d-1} = 0$. We now use the Hirzebruch inequality and the following two obvious equalities

$$\text{a) } s = t_2 + \ldots + t_d, \qquad \text{b) } \binom{d}{2} = \sum_{k \geqslant 2}^{d} t_k \binom{k}{2}. \tag{4}$$

to estimate the H_L-constant

$$H_L(\mathcal{P}, \mathcal{L}) = \frac{d^2 - \sum_{k \geqslant 2} k^2 t_k}{s}.$$

By the combinatorial equalities a), b), and the Hirzebruch inequality, we have

$$\sum_{k \geqslant 2} k^2 t_k = 2 \sum_{k \geqslant 2} \binom{k}{2} t_k + \sum_{k \geqslant 2} (k - 4) t_k + 4 \sum_{k \geqslant 2} t_k$$

$$= d^2 - d - 2t_2 - t_3 + \sum_{k \geqslant 5} (k - 4) t_k + 4s$$

$$\leqslant d^2 - 2d - t_2 - \frac{1}{4} t_3 + 4s.$$

We conclude

$$H_L(\mathcal{P}, \mathcal{L}) = \frac{d^2 - \sum_{k \geqslant 2} k^2 t_k}{s} \geqslant -4 + \frac{2d + t_2 + \frac{1}{4} t_3}{s} > -4.$$

Next, fix a configuration $\mathcal{L}$ and let $\mathcal{P}$ be arbitrary. If $H_L(\mathcal{P}, \mathcal{L}) > -1$ we are done, so assume $H_L(\mathcal{P}, \mathcal{L}) \leqslant -1$. Let $\mathcal{P}' = \mathcal{P} \cap \mathcal{P}(\mathcal{L})$ be the subset of points in $\mathcal{P}$ which are singular for $\mathcal{L}$. Then we can use Lemma 3.7 to compute $H_L(\mathcal{P}, \mathcal{L})$ from $H_L(\mathcal{P}', \mathcal{L})$ by repeatedly including new points Q with $-m_Q(\mathcal{L})^2 \geqslant -1$. It follows that $H_L(\mathcal{P}', \mathcal{L}) \leqslant H_L(\mathcal{P}, \mathcal{L})$. Replacing $\mathcal{P}$ by $\mathcal{P}'$ if necessary, we may assume $\mathcal{P} \subset \mathcal{P}(\mathcal{L})$.

We finally use Lemma 3.7 to compute $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L})$ from $H_L(\mathcal{P}, \mathcal{L})$ by repeatedly including new points Q with $-m_Q(\mathcal{L})^2 \leqslant -4$. Since $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) > -4$, we conclude $H_L(\mathcal{P}, \mathcal{L}) \geqslant H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) > -4$, completing the proof. $\qquad\square$

The next corollary follows immediately from the proof of the theorem.

Corollary 3.8. *If $\mathcal{L}$ is any configuration of lines with $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) \leqslant -1$, then*

$$\inf_{\mathcal{P}} H_L(\mathcal{P}, \mathcal{L}) = H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}).$$

Furthermore, if $t_d = t_{d-1} = 0$ then

$$H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) \geqslant -4 + \frac{2d + t_2 + \frac{1}{4} t_3}{s}.$$

Remark 3.9. The theorem shows that H_L is a well-defined real number. A natural question is whether there is a certain line configuration with ratio H_L or if H_L is only a limit of ratios from a sequence of configurations. The least constant $H_L(\mathcal{P}, \mathcal{L})$ known to us so far is $-\frac{225}{67} \approx -3.36$; see section 4.2. There is also an example with $H_L(\mathcal{P}, \mathcal{L}) = -3$ (see section 4.1).

The assumption of reducedness is essential, as we now show.

Example 3.10 (The effect of fattening of the configuration)**.** Let $\mathcal{L}$ be a configuration of lines and $\mathcal{P}$ a configuration of points, and let $k \geqslant 2$ be an integer. Let $k\mathcal{L}$ denote the configuration arising from $\mathcal{L}$ by taking all configuration lines with multiplicity k. Then it is easy to see that

$$H_L(\mathcal{P}, k\mathcal{L}) = k^2 \cdot H_L(\mathcal{P}, \mathcal{L}).$$

The smallest known values of the constants $H(\mathbb{P}^2; \mathcal{P})$ come from line configurations. It is therefore reasonable to ask the following question.

Problem 3.11. Does the lower bound -4 remain valid for $H(\mathbb{P}^2)$? Or, more directly: is $H(\mathbb{P}^2) = H_L$?

As Corollary 3.8 shows, the least H_L–constants are typically achieved by taking the set of points $\mathcal{P}$ as the set of singularities in a configuration $\mathcal{L}$. In this case, if d is the number of lines of a line arrangement $\mathcal{L}$, s is the number of points of intersection of these lines (with $\mathcal{P} = \mathcal{P}(\mathcal{L})$), and the number of lines meeting at the ith point is m_i, then $d(d-1) = \sum_i m_i(m_i - 1)$. Thus $\frac{d^2 - \sum_i m_i^2}{s} = H_L(\mathcal{P}, \mathcal{L}) = \frac{d - \sum_i m_i}{s} = \frac{d}{s} - \overline{m}$, with $\overline{m} = \frac{1}{s}\sum_{i=1}^{s} m_i$, the average of the multiplicities m_i.

If we now define m to be such that $d(d-1) = sm(m-1)$ (and hence $d^2/s > (m-1)^2$ or $\frac{d}{\sqrt{s}} + 1 > m$), then it follows by Lemma 3.13 that $H_L(\mathcal{P}, \mathcal{L}) \geqslant \frac{d}{s} - m$ (and hence $H_L(\mathcal{P}, \mathcal{L}) > d(\frac{1}{s} - \frac{1}{\sqrt{s}}) - 1 > -\frac{d}{\sqrt{s}} - 1$).

Remark 3.12. As an aside we mention that this bound holds for all ground fields in all characteristics. For example, if one takes $\mathcal{L}$ to be all of the lines defined over a finite field of q elements and $\mathcal{P}$ to be all of the points defined over that field, then $s = d = q^2 + q + 1$ and $m = q + 1$, so $H_L(\mathcal{P}, \mathcal{L}) = -q = -(m-1)$. Thus, over an algebraically closed field of finite characteristic, $H_L = -\infty$, so the question of what is the value of H_L, is of interest only in characteristic 0. In fact, note that any line arrangement $\mathcal{L}'$ and choice of points $\mathcal{P}'$ defined over a finite field of q elements has $d' \leqslant q^2 + q + 1$ lines and $s' \leqslant q^2 + q + 1$ points with multiplicity $m_i' \leqslant q + 1$ at each point, so $s'm'(m' - 1) = \sum_i m_i'(m_i' - 1) \leqslant s'q(q+1) \leqslant sq(q+1)$, hence $H_L(\mathcal{P}', \mathcal{L}') \geqslant \frac{d'}{s'} - m' > -m' \geqslant -q - 1$. Therefore, for any s' points defined over a finite field of q elements we have $H_L(s') > -q - 1$.

Lemma 3.13. *Consider any finite set of s positive integers m_i. Let $\overline{m}$ be the average and let m be defined so that $\sum_i m_i(m_i - 1) = sm(m-1)$. Then $m \geqslant \overline{m}$ with equality if and only if all m_i are equal.*

Proof. If we let $c = \sum_i m_i(m_i - 1)/s$, then $m = (1 + \sqrt{1 + 4c})/2$. As is well known and easy to prove, $\sum_i m_i^2/s \geqslant \overline{m}^2$, with equality if and only if all m_i are equal. Thus $c = \sum_i m_i(m_i - 1)/s \geqslant \overline{m}^2 - \overline{m}$, so $1 + 4c \geqslant 4(\overline{m}^2 - \overline{m}) + 1 = (2\overline{m} - 1)^2$, hence $m = (1 + \sqrt{1 + 4c})/2 \geqslant \overline{m}$, with equality if and only if all m_i are equal. $\qquad\square$

In those cases where the m_i are all equal, we of course have $m_i = m = \overline{m}$, but (over the complex numbers) we know of only one such nontrivial configuration with $m > 2$, and hence $m = 3$ by the Hirzebruch inequality (3). This configuration is the dual of the Hesse configuration, for which $d = 9$, $s = 12$ (under duality, the 9 lines are the flex points of a smooth plane cubic, and the 12 points are the lines through pairs of flex points). The series of examples below are constructed in a similar way. We call the resulting configurations s-elliptic.

Theorem 3.14. *Let s be a positive integer. There exists an arrangement of s^2 mutually distinct lines with*

$$t_3(s^2) := \begin{cases} \frac{1}{6}(s^2-1)(s^2-2) & \text{for } s \text{ not divisible by } 3; \\ \frac{1}{6}\left((s^2-1)(s^2-2)+16\right) & \text{for } s \text{ divisible by } 3; \end{cases}$$

triple points. Moreover, there are additional s^2-1 double points in the first case and respectively s^2-9 in the second case (so that for $s=3$ there are no double points at all).

Proof. Let E be an elliptic curve in the complex projective plane given by Weierstrass equation

$$y^2 z = x^3 + axz^2 + bz^3.$$

It is well known that taking the point at infinity $e=(0:1:0)$ as the neutral element for addition on E, the group law is defined geometrically, i.e.

$$P + Q + R = e \quad \text{if and only if} \quad P, Q, R \in E \quad \text{are collinear.} \tag{5}$$

Let E_s denote the subgroup of E consisting of s–torsion points. Then there is an isomorphism $E_s \simeq (\mathbb{Z}/s\mathbb{Z})^2$. Under this isomorphism s–torsion points P, Q, R are collinear if and only if their sum (in $(\mathbb{Z}/s\mathbb{Z})^2$) is equal zero. Moreover if P and Q are two s–torsion points on E, then the line joining these points (or the tangent if $P=Q$) intersects E in another s–torsion point. Now we need to count those lines which pass through exactly 3 mutually distinct s–torsion points. Their number is equal to the number of unordered triples $P, Q, R \in (\mathbb{Z}/s\mathbb{Z})^2$ such that these three points are all distinct and their sum is zero.

There are s^4 ordered triples (P, Q, R) satisfying $P+Q+R=0$ (of course R is determined by P and Q). All triples with at least two points equal are of the form

$$(P, P, -2P), \quad (P, -2P, P) \quad \text{or} \quad (-2P, P, P). \tag{6}$$

Now the counting splits according to divisibility of s by 3. If s is not divisible by 3, then there are no nonzero three–torsion points in E_s, therefore there are exactly $3(s^2-1)+1$ triples as in (6). Thus there are exactly $s^4 - 3(s^2-1) - 1$ ordered triples consisting of three distinct points. For P distinct from zero triples as in (6) give lines passing through only two points. We have $3(s^2-1)$ such triples. We don't consider the tangent line through 0, which corresponds to the triple $(0,0,0)$. Passing to unordered triples we need to divide by 6 the number of triples with 3 distinct elements and by 3 the number of triples with one element repeated. Dualizing the above picture we get the first case of the Theorem.

If s is divisible by 3 then all 9 three–torsion points are in E_s, therefore there are exactly $3(s^2-9)+9$ triples as in (6). Now there are 9 three–torsion points so that the number of pairs with only one element repeated is $3(s^2-9)$. Counting further analogously as above and dualizing we obtain the second case of the Theorem. $\square$

Since in the complex projective plane the dual Hesse configuration is the unique nontrivial configuration of lines with only triple intersection points that we know of, it is a natural problem of independent interest to wonder if this is in fact the only such configuration. The following problem might be viewed as a first step towards understanding configurations having only triple points.

Problem 3.15. Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be a configuration of lines with only triple intersection points. Can $\mathcal{L}$ be equipped with a group structure?

Of course, the question above has an affirmative answer for the dual Hesse configuration. A possible way to introduce a group structure on a configuration $\mathcal{L}$ with only triple intersection points would be as follows. We fix one configuration line, say L_1 and declare it as the neutral element. If a line L intersects L_1 in some point P, then the third line passing through P will be $-L$. This explains the addition for lines L, M in the same pencil as L_1. For the general case, assume that the intersection point $P = L \cap M$ does not belong to L_1. Let N be the third line passing through P. Then we set $L + M = -N$. We were not able to verify if this construction does indeed lead to a group structure on $\mathcal{L}$.

In the next section, we discuss some other interesting configurations coming from unitary reflection groups and compute their linear H–constants.

4   Arrangements of lines with low linear H–constants

As an alternative to asking for configurations with only triple points, one can ask for configurations with no double points. We know of only three kinds of line arrangements (over the complex numbers) for which there are no double points.

For the first, one generalizes the dual of the Hesse configuration. Recall that the original Hesse configuration consists of 12 lines passing through the flexes of a smooth plane cubic. The 9 lines of its dual can be taken to be the linear factors of $(y^3 - z^3)(x^3 - z^3)(x^3 - y^3)$. The generalization consists of the lines $\mathcal{L}_n$ given by the factors of $(y^n - z^n)(x^n - z^n)(x^n - y^n)$ for $n \geqslant 3$. Urzúa calls the resulting configurations *Fermat arrangements* [11, Example II.6]. The corresponding points are the n^2 points of intersection of $x^n - z^n = 0$ and $y^n - z^n = 0$, together with the three coordinate vertices. The three coordinate vertices occur with multiplicity n; the other n^2 points are triple points. For these we have $H_L(\mathcal{P}(\mathcal{L}_n), \mathcal{L}_n) = \frac{3n - 3n - 3n^2}{n^2 + 3} > -3$ with $\lim_{n \to \infty} H_L(\mathcal{P}(\mathcal{L}_n), \mathcal{L}_n) = -3$.

There are only two other arrangements $\mathcal{L}$ with no double points that we know of, one due to Klein [10] with 21 lines (for which $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) = -3$) and another due to Wiman [12] with 45 lines (for which $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) = \frac{-225}{67} \approx -3.36$). These are essentially the only line arrangements we know of with $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) \leqslant -3$; we will see in §4.3 that certain subconfigurations of the Wiman configuration have this property too.

The examples above, i.e., the Fermat, Klein and Wiman arrangements, also are interesting for another reason. Let $I(P) \subset \mathbb{C}[x, y, z]$ be the homogeneous ideal of a point $P \in \mathbb{P}^2$. Then the homogeneous ideal of a finite set of points $\{P_1, \ldots, P_s\} \subset \mathbb{P}^2$ is $I = \bigcap_i I(P_i)$, and the mth symbolic power of I can be defined to be $I^{(m)} = \bigcap_i I(P_i)^m$. It is typically quite rare to have a failure of containment $I^{(3)} \not\subseteq I^2$. But if I is the ideal of the points of intersection of the lines for any of these three cases, we have $I^{(3)} \not\subseteq I^2$ (see [5, 8] for the Fermat arrangements). We suspect that the same holds for s-elliptic configurations, but we are not dwelling on this problem here.

We hope to come back to the ideal theoretic properties related to configurations of lines in a separate paper in the near future.

Now we address the two exotic configurations in more detail.

4.1 The Klein configuration of 21 lines

The Klein configuration is a projective configuration of 21 lines whose intersections consist of precisely 21 quadruple points and 28 triple points, defined over $\mathbb{R}[\sqrt{-7}]$. In this case $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) = \frac{d - \sum_i m_i}{s} = \frac{21 - 168}{49} = -3$. The projective coefficients $(\alpha : \beta : \gamma)$ for each of the lines $\alpha \cdot x + \beta \cdot y + \gamma \cdot z = 0$, taken from [6] and numbered 1 through 21, are as follows:

$$
\begin{array}{lll}
1 : (1 : 0 : 0) & 2 : (0 : 1 : 1) & 3 : (a : 1 : -1) \\
4 : (a : -1 : 1) & 5 : (0 : -1 : 1) & 6 : (1 : a : -1) \\
7 : (a : -1 : -1) & 8 : (-1 : 1 : a) & 9 : (a : 1 : 1) \\
10 : (-1 : a : 1) & 11 : (-1 : -1 : a) & 12 : (1 : 0 : 1) \\
13 : (-1 : a : -1) & 14 : (1 : a : 1) & 15 : (-1 : 0 : 1) \\
16 : (1 : -1 : a) & 17 : (1 : 1 : a) & 18 : (0 : 0 : 1) \\
19 : (-1 : 1 : 0) & 20 : (1 : 1 : 0) & 21 : (0 : 1 : 0)
\end{array}
$$

Here a is a complex root of $x^2 + x + 2$, so the other root is $-(a + 1)$. Now we present projective coordinates of points of intersection of the 21 lines above:

<div align="center">quadruple points</div>

$$
\begin{array}{lll}
1 : (1 : 0 : 0) & 2 : (1 : -a - 1 : -1) & 3 : (1 : a + 1 : 1) \\
4 : (1 : -a - 1 : 1) & 5 : (1 : -1 : 1 + a) & 6 : (0 : 1 : 1) \\
7 : (a + 1 : -1 : 1) & 8 : (0 : 0 : 1) & 9 : (a + 1 : -1 : -1) \\
10 : (a + 1 : 1 : 1) & 11 : (1 : 0 : -1) & 12 : (1 : 1 : a + 1) \\
13 : (1 : -1 : 0) & 14 : (a + 1 : 1 : -1) & 15 : (1 : a + 1 : -1) \\
16 : (1 : 0 : 1) & 17 : (0 : 1 : -1) & 18 : (1 : 1 : 0) \\
19 : (1 : -1 : -1 - a) & 20 : (1 : 1 : -1 - a) & 21 : (0 : 1 : 0)
\end{array}
$$

<div align="center">triple points</div>

$$
\begin{array}{lll}
22 : (0 : 1 : a) & 23 : (0 : 1 : -a) & 24 : (a : -1 : 0) \\
25 : (0 : a : -1) & 26 : (-a + 1 : 1 : -1) & 27 : (1 : -1 : a - 1) \\
28 : (1 : 1 - a : 1) & 29 : (a : 1 : 0) & 30 : (1 : -a : 0) \\
31 : (1 : -1 : 1 - a) & 32 : (1 : 1 : a - 1) & 33 : (1 : 0 : a) \\
34 : (1 : a : 0) & 35 : (1 : -1 : 1) & 36 : (1 : a - 1 : 1) \\
37 : (0 : a : 1) & 38 : (a - 1 : 1 : 1) & 39 : (a : 0 : 1) \\
40 : (1 : 1 : -1) & 41 : (1 : 1 : 1) & 42 : (1 : 0 : -a) \\
43 : (1 : 1 : -a + 1) & 44 : (a - 1 : 1 : -1) & 45 : (1 : -a + 1 : -1) \\
46 : (1 : a - 1 : -1) & 47 : (a - 1 : -1 : -1) & 48 : (1 : -1 : -1) \\
49 : (a : 0 : -1)
\end{array}
$$

The first 21 points are the points where 4 lines meet. The remaining 28 points are where exactly 3 lines meet. Moreover, each line contains 4 of the first 21 points and 4 of the last 28 points. The following array is the incidence matrix for the configuration, so the entry in row i and column j has a 1 if and only if line i contains point j.

```
      5    10   15   20   25   30   35   40   45
      .    .    .    .    .    .    .    .    .
1   000001010000000010001110100000000000010000000000
2   100000100000010010000000001000000001000010001000100000
3   010001000001010000000000011010010000000000000000
```

```
4    0010011000000000001000000000000001000000011001000
5    1000010011000000000000000000000000100100000110
6    0000001000010011000001010000001000000000000000100
7    0000100010000010100000000000000111010000000000000
8    0000000001000010011000001101000000000010000000000
9    0001000001000000100100000000011000000000010010000
10   0100100001000001000000100000100000000000001100000
11   0100001000001000000100000000000000011110000000000
12   0100000000100010000010000000000000000001000011010
13   0001000000100100001001000000100100000100000000000
14   0010000010100000000100110110000000000000000000000
15   0011000000000001000010000001000000110000100000000
16   0010100000001000100000000000000000001000000010101
17   0001000010011000000000001000000000000000000101001
18   1000000000001000010010010000110001000000000000000
19   0000000100010000010100000000001000000011010000000
20   0000100100001000010000000100010001000000000000010
21   1000000100100001000000000000000001000001001000001
```

4.2   The Wiman configuration of 45 lines

The Wiman configuration is a projective configuration of 45 lines whose 201 intersections consist of precisely 36 quintuple points, 45 quadruple points, and 120 triple points. In this case $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) = \frac{d - \sum_i m_i}{s} = \frac{45 - 720}{201} = \frac{-225}{67} \approx -3.36$. This is the most negative example we know. We define A and B in terms of an algebraic element a whose minimal polynomial is $a^4 - a^2 + 4$. Then $A = -(a^3 - 3a - 2)/4$ and $B = (a^3 + a - 2)/4$.

Then the coefficient vectors of the lines are:

$1 : (0 : 1 : 0)$	$2 : (A - 1 : A : 1)$	$3 : (0 : 0 : 1)$
$4 : (-A + 1 : A : -1)$	$5 : (A : 1 : A - 1)$	$6 : (-A : 1 : -A + 1)$
$7 : (A - 1 : -B : -AB - A)$	$8 : (-A + 1 : -B : AB + A)$	$9 : (-A + 1 : A : 1)$
$10 : (A : 1 : -A + 1)$	$11 : (-1 : A - 1 : A)$	$12 : (-AB - 1 : 1 : -B - 1)$
$13 : (-A + 1 : B : -AB - A)$	$14 : (A : -AB + B : -B - 1)$	$15 : (-AB - A : -A + 1 : -B)$
$16 : (A - 1 : A : -1)$	$17 : (-A : 1 : A - 1)$	$18 : (1 : A - 1 : -A)$
$19 : (AB + 1 : 1 : B + 1)$	$20 : (A - 1 : B : AB + A)$	$21 : (-A : -AB + B : B + 1)$
$22 : (AB + A : -A + 1 : B)$	$23 : (B + 1 : -AB - 1 : 1)$	$24 : (1 : A - 1 : A)$
$25 : (1 : -A + 1 : A)$	$26 : (-AB - 1 : 1 : B + 1)$	$27 : (-A - B : AB + A - 1 : 0)$
$28 : (-B : 1 : -AB - A + B)$	$29 : (-AB - 1 : -1 : B + 1)$	$30 : (-B - 1 : A : -AB + B)$
$31 : (-B - 1 : -AB - 1 : -1)$	$32 : (A + B : AB + A - 1 : 0)$	$33 : (B : 1 : AB + A - B)$
$34 : (B + 1 : A : AB - B)$	$35 : (AB + A - 1 : 0 : -A - B)$	$36 : (1 : 0 : 0)$
$37 : (-B - 1 : A : AB - B)$	$38 : (0 : -A - B : AB + A - 1)$	$39 : (-B : 1 : AB + A - B)$
$40 : (B + 1 : -B : -A + B + 1)$	$41 : (1 : -AB : -AB - A + B + 1)$	$42 : (0 : -A - B : -AB - A + 1)$
$43 : (B + 1 : AB + 1 : -1)$	$44 : (A + B : 0 : -AB + B + 1)$	$45 : (B : A - B - 1 : -B - 1)$

The coordinates of the configuration points can be then easily calculated. We refrain however from an explicit but somewhat lengthy list and content ourselves with the remark that the points are equidistributed in this case as well. There are 4 quintuple, 4 quadruple and 8 triple points on each of the configuration lines.

4.3   Subconfigurations of special configurations

We close the paper by noting that the Wiman and Klein configurations also have subconfigurations $\mathcal{L}' \subset \mathcal{L}$ with highly negative constants $H_L(\mathcal{P}(\mathcal{L}'), \mathcal{L}')$ by the next result.

Proposition 4.1. *Let $\mathcal{L}$ be a configuration of d lines and let $\mathcal{P} = \mathcal{P}(\mathcal{L})$ be the set of singularities of $\mathcal{L}$. Suppose each line in $\mathcal{L}$ contains the same number n of points of $\mathcal{P}$. Let $\mathcal{L}' \subset \mathcal{L}$ be a subconfiguration of d' lines. Then*

$$H_L(\mathcal{P}, \mathcal{L}') = H_L(\mathcal{P}, \mathcal{L}) + \frac{(d - d')(n - 1)}{s}.$$

Proof. Say $\mathcal{P} = \{Q_1, \ldots, Q_s\}$ and put $m_i = m_{Q_i}(\mathcal{L})$ and $m_i' = m_{Q_i}(\mathcal{L}')$. Observe that $d'(d' - 1) = \sum_i m_i'(m_i' - 1)$ since $\mathcal{P}$ contains the singularities of $\mathcal{L}'$. Also, since every line in $\mathcal{L}$ contains n points of $\mathcal{P}$, we have $(d - d')n + \sum_i m_i' = \sum_i m_i$. We conclude

$$H_L(\mathcal{P}, \mathcal{L}') = \frac{d' - \sum_i m_i'}{s} = \frac{d' + (d - d')n - \sum_i m_i}{s} = H_L(\mathcal{P}, \mathcal{L}) + \frac{(d - d')(n - 1)}{s},$$

as claimed. $\square$

Remark 4.2. With the hypotheses of the proposition, suppose $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}') \leqslant -1$. Then we have an inequality

$$H_L(\mathcal{P}(\mathcal{L}'), \mathcal{L}') \leqslant H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}') = H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}) + \frac{(d - d')(n - 1)}{s}$$

as in the proof of Theorem 3.3. However, for "large" subcollections $\mathcal{L}' \subset \mathcal{L}$ we often have an equality $\mathcal{P}(\mathcal{L}') = \mathcal{P}(\mathcal{L})$.

Example 4.3. Here are some explicit applications of Proposition 4.1 to the Wiman configuration $\mathcal{L}$. Let $\mathcal{L}' \subset \mathcal{L}$ be a subcollection and let $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$ be its complement.

If $\mathcal{L}''$ is a single line, then $H_L(\mathcal{P}(\mathcal{L}'), \mathcal{L}') = -\frac{220}{67} \approx -3.28$.

Next suppose $\mathcal{L}''$ is a pair of lines. Then $H_L(\mathcal{P}(\mathcal{L}), \mathcal{L}') = -\frac{215}{67} \approx -3.21$ regardless of what pair of lines $\mathcal{L}''$ is. If the lines in $\mathcal{L}''$ meet at a point of multiplicity at least 4 in $\mathcal{L}$, then $\mathcal{P}(\mathcal{L}') = \mathcal{P}(\mathcal{L})$ and $H_L(\mathcal{P}(\mathcal{L}'), \mathcal{L}') = -\frac{215}{67}$. On the other hand, if they meet at a point Q of multiplicty 3 in $\mathcal{L}$ then $\mathcal{L}'$ has only 200 singularities, $\mathcal{P}(\mathcal{L}) = \mathcal{P}(\mathcal{L}') \cup \{Q\}$, and $m_Q(\mathcal{L}') = 1$. We use Lemma 3.7 to compute $H_L(\mathcal{P}(\mathcal{L}'), \mathcal{L}') = -\frac{161}{50} = -3.22$.

Similar computations of the constants $H_L(\mathcal{P}(\mathcal{L}'), \mathcal{L}')$ can be performed as the size of $\mathcal{L}''$ grows, but the combinatorics of $\mathcal{L}$ obviously plays an important role in determining all the constants obtainable in this way.

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