

## BOUNDEDNESS OF OPERATORS ON HARDY SPACES VIA ATOMIC DECOMPOSITIONS

MARCIN BOWNIK

(Communicated by Andreas Seeger)

ABSTRACT. An example of a linear functional defined on a dense subspace of the Hardy space  $H^1(\mathbb{R}^n)$  is constructed. It is shown that despite the fact that this functional is uniformly bounded on all atoms, it does not extend to a bounded functional on the whole  $H^1$ . Therefore, this shows that in general it is not enough to verify that an operator or a functional is bounded on atoms to conclude that it extends boundedly to the whole space. The construction is based on the fact due to Y. Meyer which states that quasi-norms corresponding to finite and infinite atomic decompositions in  $H^p$ ,  $0 < p \leq 1$ , are not equivalent.

### 1. INTRODUCTION

The intended purpose of this work is not only of research, but also of pedagogical nature, since it is based on an already published, but quite possibly not well-known, example of Y. Meyer.

In this note we give a rather surprising example of a linear functional defined on a dense subspace of  $H^1$ , which maps all atoms into bounded scalars, but yet it cannot be extended to a bounded functional on the whole space  $H^1$ . As a consequence of this example, it follows that in general it does not suffice to check that an operator from a Hardy space  $H^p$ ,  $0 < p \leq 1$ , into some other quasi-Banach space  $X$  maps atoms into bounded elements of  $X$  to verify that this operator extends to a bounded operator on  $H^p$ . An untrained reader might inadvertently draw such a conclusion by reading literature on atomic decompositions of Hardy spaces. Here we list a few references, which could potentially lead someone into this not fully justified belief [3, Proof of Lemma II.2], [5, Corollary 6.3], [8, Lemma 5.1], [10, Chapter 6.7.c], [14, Chapter 6.3], [16, Chapter III.3.3], [17, Chapter 1], and [18, Proposition 6.13].

Despite this, it is important to emphasize that to verify boundedness for many important classes of operators defined on  $H^p$  spaces, it is indeed sufficient to check that atoms are mapped into bounded elements of  $X$ . Probably the best known example of a class with this property are Calderón-Zygmund operators. The complete proof of this fact (based on atomic decomposition of  $H^p$  spaces) can be found, for

---

Received by the editors July 8, 2004.

2000 *Mathematics Subject Classification*. Primary 42B30.

The author was partially supported by NSF grant DMS-0441817.

©2005 American Mathematical Society  
Reverts to public domain 28 years from publication

example, in [9, Chapter III.7] or [10, Theorem 6.7.1] for convolution-type Calderón-Zygmund operators and in [1, Chapter 1.9] or [15, Lemma 1 in Chapter 7.3] for non-convolution operators.

A rudimentary set of facts about real-variable theory of Hardy spaces can be found in [4, 7, 8, 9, 10, 16]. Here, we limit ourselves to the basic definition of real-variable  $H^p$  spaces due to Fefferman and Stein [7].

**Definition 1.** We say that a tempered distribution  $f \in \mathcal{S}'(\mathbb{R}^n)$  belongs to the Hardy space  $H^p(\mathbb{R}^n)$ ,  $0 < p < \infty$ , if its radial maximal function  $M_\varphi^0 f$  (or equivalently non-tangential maximal function  $M_\varphi f$ ) is in  $L^p$ . Here,  $\varphi$  is any test function in the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$  with  $\int \varphi \neq 0$ , and

$$(1) \quad M_\varphi^0 f(x) = \sup_{t>0} |f * \varphi_t(x)|,$$

$$(2) \quad M_\varphi f(x) = \sup_{t>0} \sup_{|y-x|<t} |f * \varphi_t(y)|,$$

where  $\varphi_t(x) = t^{-n}\varphi(x/t)$ .

A fundamental result of Fefferman and Stein asserts that this definition does not depend on the choice of  $\varphi \in \mathcal{S}$  (as long as  $\int \varphi \neq 0$ ) and  $H^p(\mathbb{R}^n)$  with the quasi-norm  $\|f\|_{H^p} = \|M_\varphi^0 f\|_{L^p}$  (or  $\|f\|_{H^p} = \|M_\varphi f\|_{L^p}$ ) is a quasi-Banach space. Moreover,  $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$  for  $p > 1$ . For proofs of these facts we refer to [16, Chapter III.1].

## 2. EXAMPLE OF MEYER

In this section we present an example of an atom in  $H^p$  whose norm is not achieved by its finite atomic decomposition. The first example of this kind for  $H^1$  was exhibited by Y. Meyer [13]; see also [9, Section III.8.3]. Here, we merely adapt this example to a more general  $H^p(\mathbb{R}^n)$ ,  $0 < p \leq 1$ , case.

We start by recalling a definition of an atom for  $H^p$  spaces. For the sake of simplicity, we will use only  $L^\infty$  normalization for our atoms and we will limit ourselves to the classical isotropic Hardy spaces  $H^p(\mathbb{R}^n)$  given by Definition 1.

**Definition 2.** We say that a function  $a$  is a  $p$ -atom, where  $0 < p \leq 1$ , if

$$(3) \quad \text{supp } a \subset B(x_0, r) \quad \text{for some } x_0 \in \mathbb{R}^n, r > 0,$$

$$(4) \quad \|a\|_\infty \leq |B(x_0, r)|^{-1/p},$$

$$(5) \quad \int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \quad \text{for all } |\alpha| \leq \lfloor (1/p - 1)n \rfloor.$$

Here,  $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ .

Let  $\Theta^k(\mathbb{R}^n)$  be the space of all (finite) linear combinations of  $p$ -atoms, that is,

$$\Theta^k(\mathbb{R}^n) = \{f \in L^\infty(\mathbb{R}^n) : \text{supp } f \text{ is bounded and } \int_{\mathbb{R}^n} f(x)x^\alpha dx = 0 \text{ for } |\alpha| \leq k\}.$$

It is well known that  $\Theta^k(\mathbb{R}^n)$  is a dense subspace of  $H^p(\mathbb{R}^n)$ ,  $0 < p \leq 1$ , for sufficiently large  $k$ , that is, for  $k \geq \lfloor (1/p - 1)n \rfloor$ . In fact, an even smaller space  $\Theta^k(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$  is also a dense subspace of  $H^p(\mathbb{R}^n)$  for the same range of  $k$ 's.

On the space  $\Theta^k(\mathbb{R}^n)$  we consider two quasi-norms corresponding to finite and infinite atomic decompositions:

$$(6) \quad \|f\|_{H^p, \infty} = \inf \left\{ \left( \sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p} : f = \sum_{i=1}^{\infty} \lambda_i a_i, \quad a_i \text{ is a } p\text{-atom for } i \in \mathbb{N} \right\}.$$

$$(7) \quad \|f\|_{H^p, < \infty} = \inf \left\{ \left( \sum_{i=1}^N |\lambda_i|^p \right)^{1/p} : f = \sum_{i=1}^N \lambda_i a_i, \right. \\ \left. a_i \text{ is a } p\text{-atom for } 1 \leq i \leq N, \text{ and } N \in \mathbb{N} \right\}.$$

It should be emphasized that the equality  $f = \sum_{i=1}^{\infty} \lambda_i a_i$  in (6) is understood in the sense of tempered distributions  $\mathcal{S}'(\mathbb{R}^n)$ . This follows from a standard  $H^p$  theory fact stating that for any choice of coefficients  $(\lambda_i)_{i=1}^{\infty} \in \ell^p(\mathbb{N})$  and  $p$ -atoms  $a_i$ 's, the series  $\sum_{i=1}^{\infty} \lambda_i a_i$  converges in  $\|\cdot\|_{H^p}$  quasi-norm, and hence in  $\mathcal{S}'$ .

The atomic decomposition theorem of Coifman [2] for  $H^p$  spaces states that the converse is also true, i.e., every element  $f \in H^p(\mathbb{R}^n)$  can be decomposed as  $f = \sum_{i=1}^{\infty} \lambda_i a_i$  for some choice of  $\lambda_i$ 's and  $p$ -atoms  $a_i$ 's. Moreover,

$$\|f\|_{H^p} \asymp \|f\|_{H^p, \infty} \quad \text{for all } f \in H^p,$$

and hence for all  $f \in \Theta^k(\mathbb{R}^n)$ , where  $k \geq \lfloor (1/p - 1)n \rfloor$ .

A less-known result due to Y. Meyer states that the above is not true when the quasi-norm  $\|\cdot\|_{H^p, \infty}$  is replaced by  $\|\cdot\|_{H^p, < \infty}$ . Hence, the quasi-norms  $\|\cdot\|_{H^p, \infty}$  and  $\|\cdot\|_{H^p, < \infty}$  are **not** equivalent on  $\Theta^k(\mathbb{R}^n)$ .

**Theorem 1.** *Suppose  $0 < p \leq 1$  and  $k \geq \lfloor (1/p - 1)n \rfloor$ . Then for arbitrarily small  $\varepsilon > 0$ , there exists  $f \in \Theta^k(\mathbb{R}^n)$  such that*

$$(8) \quad \|f\|_{H^p, \infty} < \varepsilon \quad \text{and} \quad \|f\|_{H^p, < \infty} = 1.$$

*Proof.* Let  $a$  be a  $p$ -atom supported on the unit ball  $B(0, 1)$  with

$$(9) \quad \int_{\mathbb{R}^n} a(x) x^\alpha dx = 0 \quad \text{for all } |\alpha| \leq k,$$

and such that

$$(10) \quad |a(x)| \geq c|B(0, 1)|^{-1/p} > 0 \quad \text{for a.e. } x \in B(0, 1).$$

To show that an atom  $a$  satisfying (10) exists, let  $K = \sum_{i=0}^k \binom{n-1+i}{i}$  be the cardinality of the collection of all multi-indices  $\alpha$  with  $|\alpha| \leq k$ . Then, we claim that it suffices to construct a finite partition  $\{E_i\}_{i=1}^m$  of  $B(0, 1)$  such that the vectors

$$(11) \quad v_i = \left( \int_{E_i} x^\alpha dx \right)_{|\alpha| \leq k} \in \mathbb{R}^K, \quad i = 1, \dots, m,$$

span the whole space  $\mathbb{R}^K$  even if one of them is removed, i.e.,

$$(12) \quad \forall 1 \leq i_0 \leq m \quad \text{span}\{v_i : 1 \leq i \leq m, i \neq i_0\} = \mathbb{R}^K.$$

Indeed, (12) implies that there exist non-zero coefficients  $c_1, \dots, c_m$  such that  $\sum_{i=1}^m c_i v_i = 0$ . Moreover, by scaling we may also assume that  $\sup_{1 \leq i \leq m} |c_i| \leq |B(0, 1)|^{-1/p}$ . Then, one can immediately verify that

$$a(x) = \sum_{i=1}^m c_i \mathbf{1}_{E_i}(x)$$

is a required atom satisfying (10) with  $c = |B(0, 1)|^{1/p} \inf_{1 \leq i \leq m} |c_i|$ . Finally, we remark that a partition  $\{E_i\}_{i=1}^m$  satisfying (11) and (12) can easily be found by an inductive partitioning of the ball.

Next, we choose a collection of pairwise disjoint balls  $\{B_i\}_{i \in \mathbb{N}}$  such that  $B_i \subset B(0, 1)$  for all  $i \in \mathbb{N}$ ,

$$U := \bigcup_{i \in \mathbb{N}} B_i \text{ is dense in } B(0, 1) \quad \text{and} \quad |U| = \sum_{i \in \mathbb{N}} |B_i| < c\varepsilon^p.$$

For each  $i \in \mathbb{N}$ , let  $a_i$  be a dilated and translated copy of the atom  $a$  with support adjusted to the ball  $B_i$ . That is, if  $B_i = B(x_0, r)$ , then  $a_i(x) = r^{-n/p} a((x - x_0)/r)$ . As a consequence of (10) each  $a_i$  is an atom supported on  $B_i$  and satisfying

$$(13) \quad |a_i(x)| \geq c|B_i|^{-1/p} \quad \text{for a.e. } x \in B_i.$$

Let

$$(14) \quad f(x) = c^{-1/p} \sum_{i \in \mathbb{N}} |B_i|^{1/p} a_i(x).$$

Then, it is obvious that  $\|f\|_{H^p, \infty}^p \leq \sum_{i \in \mathbb{N}} |B_i|/c < \varepsilon^p$ . On the other hand, we claim that  $\|f\|_{H^p, \infty}$  must remain large.

Indeed, suppose that  $f$  has a finite atomic decomposition  $f = \sum_{i=1}^N \lambda_i b_i$ , where each  $b_i$  is supported on a ball  $\tilde{B}_i$ . Let  $g$  be a majorant of  $f$  given by

$$g = \left( \sum_{i=1}^N |\lambda_i|^p |\tilde{B}_i|^{-1} \mathbf{1}_{\tilde{B}_i} \right)^{1/p}.$$

By (13) and (14)

$$(15) \quad \mathbf{1}_U(x) \leq |f(x)| \leq \sum_{i=1}^N |\lambda_i| |b_i(x)| \leq \left( \sum_{i=1}^N |\lambda_i|^p |b_i(x)|^p \right)^{1/p} \\ \leq \left( \sum_{i=1}^N |\lambda_i|^p |\tilde{B}_i|^{-1} \mathbf{1}_{\tilde{B}_i}(x) \right)^{1/p} = g(x).$$

Since  $g$  is continuous everywhere almost everywhere (possibly with the exception of the union of boundaries of a finite collection of balls  $\bigcup_{i=1}^N \partial(\tilde{B}_i)$ ) and  $U$  is dense in  $B(0, 1)$ , hence  $g(x) \geq 1$  for a.e.  $x \in B(0, 1)$ . Therefore,

$$|B(0, 1)| \leq \int_{B(0, 1)} g(x)^p dx = \sum_{i=1}^N |\lambda_i|^p.$$

Consequently,  $\|f\|_{H^p, < \infty} \geq |B(0, 1)|^{1/p}$ . It is also immediate from (14) that  $\|f\|_{H^p, < \infty} \leq c^{-1/p} |B(0, 1)|^{1/p}$ . Since  $\varepsilon > 0$  was arbitrary, by a simple rescaling we find  $f$  satisfying (8), which completes the proof of Theorem 1.  $\square$

It is perhaps worthwhile to recall the original example of Meyer, which through its simplicity better illustrates the idea of the above proof; see also [9, Chapter III.8].

**Example 1.** For arbitrarily small  $\varepsilon > 0$ , we will construct a function  $f \in \Theta^0(\mathbb{R})$  such that

$$\|f\|_{H^1, \infty} < \varepsilon \quad \text{and} \quad \|f\|_{H^1, < \infty} = 1.$$

Let  $\{B_i\}_{i \in \mathbb{N}}$  be a collection of pairwise disjoint intervals  $\subset [0, 1]$  such that  $U := \bigcup_{i \in \mathbb{N}} B_i$  is dense in  $[0, 1]$  and  $|U| < \varepsilon$ . Let  $a_i$  be a 1-atom supported on  $B_i$ , which equals  $1/|B_i|$  on the left half of  $B_i$  and  $-1/|B_i|$  on the other half. Let  $f(x) = \sum_{i \in \mathbb{N}} |B_i| a_i(x)$ . It is clear that  $\|f\|_{H^1, \infty} \leq \sum_{i \in \mathbb{N}} |B_i| < \varepsilon$  and  $|f(x)| = 1$  for a.e.  $x \in U$ .

To see that  $\|f\|_{H^1, < \infty} = 1$ , consider a finite atomic decomposition  $f = \sum_{i=1}^N \lambda_i b_i$ , where each  $b_i$  is supported on the interval  $\tilde{B}_i$ . Then

$$\mathbf{1}_U(x) = |f(x)| \leq \sum_{i=1}^N |\lambda_i| |a_i(x)| \leq \sum_{i=1}^N |\lambda_i| |\tilde{B}_i|^{-1} \mathbf{1}_{\tilde{B}_i}(x) =: g(x).$$

Since  $g$  is discontinuous only on a finite number of points and  $U \subset [0, 1]$  is dense, hence  $g(x) \geq 1$  for a.e.  $x \in [0, 1]$ . Integrating  $g(x)$  over  $[0, 1]$  yields  $\|f\|_{H^1, < \infty} \geq 1$ . Since  $f$  is itself a 1-atom supported on  $[0, 1]$ , hence  $\|f\|_{H^1, < \infty} = 1$ .

### 3. UNBOUNDED LINEAR FUNCTIONALS ON $H^1$

The goal of this section is to show the existence of a linear functional on a dense subspace of  $H^1$ , which does not extend to a bounded functional on the whole  $H^1$  despite the fact that it maps all 1-atoms into scalars with universally bounded absolute values. The existence of such a functional will follow from Meyer's example and an application of the Hahn-Banach Theorem.

**Theorem 2.** *There exists a linear functional  $l$  on  $\Theta^0(\mathbb{R}^n)$  such that*

$$(16) \quad |l(f)| \leq \|f\|_{H^1, < \infty} \quad \text{for all } f \in \Theta^0(\mathbb{R}^n),$$

*which does not extend to a bounded functional on  $H^1(\mathbb{R}^n)$ , i.e.,*

$$(17) \quad \sup_{f \in \Theta^0(\mathbb{R}^n)} |l(f)| / \|f\|_{H^1, \infty} = \infty.$$

*In particular,  $l$  is uniformly bounded on all atoms in  $H^1(\mathbb{R}^n)$ . That is,  $|l(a)| \leq 1$  for every 1-atom  $a$ .*

*Proof.* Suppose  $\{x_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$  is any sequence such that  $B(x_i, 1) \cap B(x_j, 1) = \emptyset$  for every  $i \neq j$ . For each  $i \in \mathbb{N}$ , let  $a_i(x)$  be a function in  $\Theta^0(\mathbb{R}^n)$  supported on the ball  $B(x_i, 1)$  and satisfying

$$(18) \quad \|a_i\|_{H^1, \infty} < 1/i \quad \text{and} \quad \|a_i\|_{H^1, < \infty} = 1.$$

In addition, from the proof of Theorem 1 we can also assume that

$$(19) \quad |a_i(x)| \geq c/|B(0, 1)| > 0 \quad \text{for } x \in U_i, \text{ where } U_i \subset B(x_i, 1) \text{ is dense.}$$

Here,  $c$  is a constant independent of  $i \in \mathbb{N}$ . In fact, we can choose  $a_i$ 's such that  $c = 1$  is the largest possible by taking atoms taking only two non-zero and opposite values as in Example 1.

Let  $V = \text{span}\{a_i(x) : i \in \mathbb{N}\} \subset \Theta^0(\mathbb{R}^n)$  be the space of all finite linear combinations of the above functions. We claim that

$$(20) \quad c \sum_{i \in \mathbb{N}} |c_i| \leq \|f\|_{H^1, < \infty} \leq \sum_{i \in \mathbb{N}} |c_i| \quad \text{for all } f(x) = \sum_{i \in \mathbb{N}} c_i a_i(x) \in V.$$

Therefore,  $V$  is isomorphic to the subspace of  $\ell^1(\mathbb{N})$  consisting of sequences with finite support.

To show (20), we proceed as in the proof of Theorem 1. Suppose that  $f(x) = \sum_{i \in \mathbb{N}} c_i a_i(x) \in V$  has a finite atomic decomposition  $f(x) = \sum_{j=1}^N \lambda_j b_j(x)$ , where each  $b_j$  is supported on a ball  $B_j$ . By (19)

$$\frac{c}{|B(0,1)|} \sum_{i \in \mathbb{N}} |c_i| \mathbf{1}_{U_i}(x) \leq |f(x)| \leq \sum_{j=1}^N |\lambda_j| |b_j(x)| \leq \sum_{j=1}^N |\lambda_j| |B_j|^{-1} \mathbf{1}_{B_j}(x) =: g(x).$$

Since  $g$  is continuous everywhere almost everywhere (possibly with the exception of the union of boundaries of a finite collection of balls  $\bigcup_{j=1}^N \partial(B_j)$ ) and each  $U_i$  is dense in  $B(x_i, 1)$ , hence

$$g(x) \geq \frac{c}{|B(0,1)|} \sum_{i \in \mathbb{N}} |c_i| \mathbf{1}_{B(x_i,1)}(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Therefore,

$$c \sum_{i \in \mathbb{N}} |c_i| \leq \int_{\mathbb{R}^n} g(x) dx = \sum_{j=1}^N |\lambda_j|.$$

This shows the lower bound in (20). The upper bound in (20) is trivial by the triangle inequality and (18).

Define a linear functional  $l$  initially on  $V$  by

$$l(f) = \sum_{i \in \mathbb{N}} c_i \quad \text{for } f(x) = \sum_{i \in \mathbb{N}} c_i a_i(x) \in V.$$

By (20),  $l$  is a bounded functional on a subspace of  $V$  of a normed space  $\Theta^0(\mathbb{R}^n)$  equipped with the norm  $\|\cdot\|_{H^1, < \infty}$ . Moreover, the norm of  $l$  is at most 1. Therefore, by the Hahn-Banach Theorem,  $l$  extends to a bounded functional on the whole space  $\Theta^0(\mathbb{R}^n)$  such that (16) holds. Since,  $l(a_i)/\|a_i\|_{H^1, \infty} \geq i$  and  $i \in \mathbb{N}$  is arbitrary, we also have (17), which completes the proof of Theorem 2.  $\square$

*Remark 1.* We remark that the proof of Theorem 2 can be easily modified to show the existence of a linear functional  $l$  defined on some subspace of  $V \subset \Theta^k(\mathbb{R}^n)$ , where  $k \geq \lfloor (1/p - 1)n \rfloor$ ,  $0 < p \leq 1$ , which is bounded on  $V$  equipped with the quasi-norm  $\|\cdot\|_{H^p, < \infty}$ , but is unbounded as a functional on  $V$  with the quasi-norm  $\|\cdot\|_{H^p, \infty}$ . However, since the Hahn-Banach Theorem is not valid on general quasi-normed spaces, there is no guarantee that this functional can be boundedly extended to the whole  $\Theta^k(\mathbb{R}^n)$ .

In fact, Duren, Romberg, and Shields [6] characterized the duals of classical Hardy spaces on the unit complex disc  $H^p(\mathbb{D})$  for  $0 < p < 1$  and used it to show that the Hahn-Banach Theorem fails for these spaces. Furthermore, Kalton [11, 12] showed that a quasi-Banach space  $X$  has the Hahn-Banach Extension property (continuous linear functionals on a closed subspace extend to the whole space) if and only if it is a Banach space.

Finally, we discuss how Theorem 2 relates to the problem of showing boundedness of operators on Hardy spaces via atomic decompositions. A typical argument invoked for that purpose is as follows.

Suppose  $T$  is a linear operator defined on some dense subspace  $D$  of  $H^p(\mathbb{R}^n)$ ,  $0 < p \leq 1$ , into some quasi-Banach space  $X$ , with the property that  $\|T(a)\|_X \leq C < \infty$  for all  $p$ -atoms  $a$  and some universal constant  $C$ . Here, we implicitly require that  $\Theta^k(\mathbb{R}^n) \subset D$ , where  $k = \lfloor (1/p - 1)n \rfloor$  and  $\|\cdot\|_X$  satisfies  $p$ -triangle inequality

$\|f + g\|_X^p \leq \|f\|_X^p + \|g\|_X^p$ . To show that  $T$  extends to a bounded operator from  $H^p$  to  $X$ , consider an arbitrary element  $f \in \Theta^k(\mathbb{R}^n)$ . By the atomic decomposition theorem for  $H^p$  spaces, we can represent  $f = \sum_{i \in \mathbb{N}} \lambda_i a_i$ , where the  $a_i$ 's are  $p$ -atoms and  $\sum_{i \in \mathbb{N}} |\lambda_i|^p \leq C_0 \|f\|_{H^p}$  for some universal constant  $C_0$ . Since

$$(21) \quad Tf = T\left(\sum_{i \in \mathbb{N}} \lambda_i a_i\right) = \sum_{i \in \mathbb{N}} \lambda_i T(a_i),$$

hence

$$(22) \quad \|Tf\|_X^p \leq \sum_{i \in \mathbb{N}} \|\lambda_i T(a_i)\|_X^p \leq C \sum_{i \in \mathbb{N}} |\lambda_i|^p \leq CC_0 \|f\|_{H^p}.$$

Since  $f$  was arbitrary, (22) shows that  $T$  extends to a bounded operator  $T : H^p \rightarrow X$ .

The main problem with this argument is that in general there is no guarantee that (21) is valid due the fact that the sum in (21) is infinite. Theorem 2 shows that this is not only a theoretical possibility, but (21) may indeed fail in certain situations (at least when  $p = 1$ ).

The above argument also has a variant, where infinite atomic decomposition is replaced with a finite one  $f = \sum_{i=1}^N \lambda_i a_i$ , where the  $a_i$ 's are  $p$ -atoms and  $\sum_{i=1}^N |\lambda_i|^p \leq C_0 \|f\|_{H^p}$  for some constant  $C_0$ . This time the problem lies with the fact that  $C_0$  cannot be chosen universally for all  $f \in \Theta^k(\mathbb{R}^n)$  as it is evidenced by Theorem 1.

Therefore, in light of Theorems 1 and 2 we must undoubtedly admit that in general it is not enough to verify that an operator or a functional is merely bounded on  $p$ -atoms to conclude that it extends boundedly to the whole space  $H^p$ ,  $0 < p \leq 1$ . It is also necessary to verify an identity such as (21), asserting that  $T$  behaves well with respect to infinite atomic decompositions. This in turn is not always a trivial task, e.g. in the case of Calderón-Zygmund operators it requires use of certain approximation arguments. For further details, we refer to [1, 9, 15].

## REFERENCES

1. M. Bownik, Anisotropic Hardy spaces and wavelets, *Mem. Amer. Math. Soc.* **164** (2003), no. 781, vi+122 pp. MR1982689 (2004e:42023)
2. R.R. Coifman, A real variable characterization of  $H^p$ , *Studia Math.* **51** (1974), 269–274. MR0358318 (50:10784)
3. R.R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures Appl.* **72** (1993), 247–286. MR1225511 (95d:46033)
4. R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645. MR0447954 (56:6264)
5. J. Duoandikoetxea, *Fourier Analysis*, American Math. Soc., Providence (2001). MR1800316 (2001k:42001)
6. P.L. Duren, B.W. Romberg, and A.L. Shields, Linear functionals on  $H^p$  spaces with  $0 < p < 1$ , *J. Reine Angew. Math.* **238** (1969), 32–60. MR0259579 (41:4217)
7. C. Fefferman, E.M. Stein,  $H^p$  spaces of several variables, *Acta Math.* **129** (1972), 137–193. MR0447953 (56:6263)
8. G.B. Folland, E.M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, N.J., 1982. MR0657581 (84h:43027)
9. J. García-Cuerva, J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland (1985). MR0807149 (87d:42023)
10. L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson (2004).
11. N.J. Kalton, Basic sequences in  $F$ -spaces and their applications, *Proc. Edinburgh Math. Soc.* **19** (1974/75), 151–167. MR0415259 (54:3350)

12. N.J. Kalton, Quasi-Banach Spaces in “Handbook of geometry of Banach spaces, Vol. II”, 1099–1130, North-Holland, Amsterdam (2003). MR1999192 (2005a:46004)
13. Y. Meyer, M. Taibleson, and G. Weiss, Some functional analytic properties of the spaces  $B_q$  generated by blocks, *Indiana Univ. Math. J.* **34** (1985), 493–515. MR0794574 (87c:46036)
14. Y. Meyer, Wavelets and operators, Cambridge University Press, Cambridge (1992). MR1228209 (94f:42001)
15. Y. Meyer, R. Coifman, Wavelets. Calderón-Zygmund and multilinear operators, Cambridge University Press, Cambridge (1997). MR1456993 (98e:42001)
16. E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press (1993). MR1232192 (95c:42002)
17. M.H. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces, *Astérisque* **77** (1980), 67–149. MR0604370 (83g:42012)
18. P. Wojtaszczyk, A mathematical introduction to wavelets, Cambridge University Press, Cambridge (1997). MR1436437 (98j:42025)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403–1222  
*E-mail address:* [mownik@uoregon.edu](mailto:mownik@uoregon.edu)