Research Article

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Boundedness of vector-valued sublinear operators on weighted Herz-Morrey spaces with variable exponents

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Abstract: If vector-valued sublinear operators satisfy the size condition and the vector-valued inequality on weighted Lebesgue spaces with variable exponent, then we obtain their boundedness on weighted Herz-Morrey spaces with variable exponents.

Keywords: subilinear operator, vector-valued inequality, Muckenhoupt weight, variable exponent, Herz-Morrey space

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1 Introduction

Since the fundamental paper [1] by Kováčik and Rákosník appeared in 1991, the Lebesgue spaces with variable exponent have been extensively studied by many authors; see [2–4]. Motivated by applications to fluid dynamics, image restoration and partial differential equations with non-standard growth conditions, many variable spaces were introduced, such as Besov and Triebel-Lizorkin spaces with variable exponents [5–12], Besov-type and Triebel-Lizorkin-type spaces with variable exponents [13–21], Hardy spaces with variable exponent [22], Bessel potential spaces with a variable exponent [23,24] and Morrey spaces with variable exponents [25]. The list is not exhausted.

Herz spaces were introduced in [26]. After that the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [27], in the summability of Fourier transforms [28] and in regularity theory for elliptic equations in divergence form [29]. For more details of the theory and applications of Herz spaces, we refer the reader to the monograph [30]. Herz spaces with variable exponents were studied in [31–34]. As a generalization, Herz-Morrey spaces with variable exponents were introduced in [35]. Indeed, Izuki [35] obtained the boundedness of vector-valued sublinear operators satisfying a size condition on Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. Furthermore, Dong and Xu of the paper generalized Izuki's result for the $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ in [36]. Wang and Shu [37] obtained the boundedness of some sublinear operators on weighted variable Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n, w)$.

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Motivated by the mentioned work, in this paper, we will prove the boundedness of vector-valued sublinear operators on weighted Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)$. As a result, we obtain the boundedness of vector-valued Hardy-Littlewood maximal operator on weighted Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)$. It is well known that the boundedness of vector-valued Hardy-Littlewood maximal operator on non-weighted and weighted Lebesgue spaces play a key role in the theory of function spaces. The paper is organized as follows. In Section 2, we collect some notations and state the main result. The proof of the main result is given in Section 3.

2 Notations and main result

In this section, we first recall some definitions and notations, then we state our result. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ taking values in $[1,\infty)$, then the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable: } \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0 \right\}.$$

The Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1 \right\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for all compact subsets $K \subset \mathbb{R}^n\}$, where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, we denote $p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$, $p_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all measurable function $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable function $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as above for $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. $p'(\cdot)$ is the conjugate exponent of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, which means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a nonnegative measurable function on $\mathbb{R}^n$. Then the weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable functions f such that $fw \in L^{p(\cdot)}$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw\|_{L^{p(\cdot)}}.$$

Definition 1. Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.

(i) The function $\alpha(\cdot)$ is locally log-Hölder continuous if there exists a constant C_1 such that

$$|\alpha(x) - \alpha(y)| \le \frac{C_1}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n, \ |x-y| < \frac{1}{2}.$$

(ii) The function $\alpha(\cdot)$ is log-Hölder continuous at the origin if there exists a constant C_2 such that

$$|\alpha(x) - \alpha(0)| \le \frac{C_2}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}^{\log}_0(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at the origin.

(iii) The function $\alpha(\cdot)$ is log-Hölder continuous at infinity if there exists $\alpha_\infty \in \mathbb{R}$ and a constant C_3 such that

$$|\alpha(x) - \alpha_\infty| \le \frac{C_3}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.

(iv) The function $\alpha(\cdot)$ is global log-Hölder continuous if $\alpha(\cdot)$ are both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions.

Definition 2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a positive measurable function w is said to be in $A_{p(\cdot)}$, if exists a positive constant C for all balls B in $\mathbb{R}^n$ such that

$$\frac{1}{|B|}\|w\chi_B\|_{L^{p(\cdot)}}\|w^{-1}\chi_B\|_{L^{p'(\cdot)}} \le C.$$

Remark 1. The variable Muckenhoupt $A_{p(\cdot)}$ was introduced by Cruz-Uribe et al. in [38]. For more details, see [38–42]. It is easy to see that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $w^{-1} \in A_{p'(\cdot)}$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls Q containing x in $\mathbb{R}^n$.

In general, the Hardy-Littlewood maximal operator is not bounded on weighted variable Lebesgue spaces. But one has the following lemma [38, Theorem 1.5, p. 746].

Lemma 1. *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there is a positive constant C such that for each $f \in L^{p(\cdot)}(w)$,*

$$\|(Mf)w\|_{L^{p(\cdot)}} \le C\|fw\|_{L^{p(\cdot)}}.$$

To give the definitions of the weighted Herz-Morrey space with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$ we define $B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $D_k := B_k \backslash B_{k-1}$, $\chi_k := \chi_{D_k}$, $\widetilde{\chi}_m = \chi_m$, $m \ge 1$, $\widetilde{\chi}_0 = \chi_{B_0}$. We also need the notation of the variable mixed sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$, which is first defined by Almeida and Hästö in [5]. Let w be a nonnegative measurable function. Given a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, define the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{f_j\}_j) := \sum_{j\in\mathbb{Z}} \inf\left\{\lambda_j : \int_{\mathbb{R}^n}\left(\frac{|f_j(x)w(x)|}{\lambda_j^{\frac{1}{q(x)}}}\right)^{p(x)} \mathrm{d}x \le 1\right\},$$

where $\lambda^{1/\infty} = 1$. If $q^+ < \infty$ or $q(\cdot) \le p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{f_j\}_j) = \sum_{j\in\mathbb{Z}} \||f_j w|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

The norm is

$$\|\{f_j\}_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} := \inf\Big\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{f_j/\mu\}_j) \le 1\Big\}.$$

Definition 3. Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $\lambda \in [0, \infty)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous weighted Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)$ and non-homogeneous weighted Herz-Morrey space $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)$ are defined, respectively, by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w) := \left\{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}, w) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} < \infty\right\}$$

and

$$MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w) := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n, w) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \|(2^{\alpha(\cdot)k} f\chi_k)_{k\le L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

and

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} := \sup_{L\in\mathbb{N}_0} 2^{-L\lambda} \|(2^{\alpha(\cdot)k} f\tilde{\chi}_k)_{k=0}^{L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.$$

For any quantities A and B, if there exists a constant $C > 0$ such that $A \le CB$, we write $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$. The following Proposition 1 is from [43, Proposition 1, pp. 5–6].

Proposition 1. *Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, w be a weight, $\lambda \in [0,\infty)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.*

(i) If $\alpha(\cdot), q(\cdot) \in \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then for any $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}, w)$,

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} \approx \max\left\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \|(2^{k\alpha(0)} f\chi_k)_{k\le L}\|_{\ell^{q_0}(L^{p(\cdot)}(w))}, \sup_{L>0, L\in\mathbb{Z}} \left[2^{-L\lambda} \|(2^{k\alpha(0)} f\chi_k)_{k<0}\|_{\ell^{q_0}(L^{p(\cdot)}(w))} + 2^{-L\lambda} \|(2^{k\alpha_\infty} f\chi_k)_{k=0}^{L}\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))} \right] \right\},$$

where and hereafter, $q_0 := q(0)$.

(ii) If $\alpha(\cdot), q(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then

$$MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w) = MK^{\alpha_\infty,\lambda}_{q_\infty,p(\cdot)}(w).$$

Lemma 2 has been proved by Izuki and Noi [44, pp. 9–10].

Lemma 2. *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there exist constants $\delta_1, \delta_2 \in (0,1)$ and $C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_B\|_{L^{p(\cdot)}(w)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \tag{1}$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(w^{-1})}}{\|\chi_B\|_{L^{p'(\cdot)}(w^{-1})}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \tag{2}$$

Our main result is as follows.

Theorem 1. *Let $r \in (1,\infty)$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot), q(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$, $\lambda - n\delta_1 < \alpha(0)$, $\alpha_\infty < n\delta_2$, where $\delta_1, \delta_2 \in (0,1)$ are the constants in* Lemma 2 *for the exponent $p(\cdot)$ and the weight w. Suppose that T is a sublinear operator satisfies the size condition,*

$$|Tf(x)| \le C \int_{\mathbb{R}^n} |x-y|^{-n} |f(y)| \mathrm{d}y \tag{3}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a.e. $x \notin$ supp f. If the sublinear operator T satisfies vector-valued inequality on $L^{p(\cdot)}(w)$,

$$\left\| \left(\sum_{j=1}^{\infty} |Tf_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)} \tag{4}$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ *of locally integrable functions on* $\mathbb{R}^n$, *then*

$$\left\| \left(\sum_{j=1}^{\infty} |Tf_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)},$$

where C *is independent of* $\{f_j\}_{j=1}^{\infty}$.

The following Lemma 3 is Corollary 3.2 in [42, p. 11].

Lemma 3. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* w *be a weight. If the maximal operator* M *is bounded on* $L^{p(\cdot)}(w)$ *and* $L^{p'(\cdot)}(w^{-1})$ *and* $r \in (1, \infty)$, *then there is a positive constant* C *such that*

$$\left\| \left(\sum_{j=1}^{\infty} (Mf_j)^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)}.$$

From Theorem 1 and Lemma 3, we obtain the following corollary.

Corollary 1. *Let* $r \in (1, \infty)$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot)$, $q(\cdot) \in L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_{\infty}^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$, $\lambda - n\delta_1 < \alpha(0)$, $\alpha_{\infty} < n\delta_2$, *where* δ_1, $\delta_2 \in (0, 1)$ *are the constants in* Lemma 2 *for the exponent* $p(\cdot)$ *and the weight* w, *then*

$$\left\| \left(\sum_{j=1}^{\infty} |Mf_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)},$$

where C *is independent of* $\{f_j\}_{j=1}^{\infty}$ *of locally integrable functions on* $\mathbb{R}^n$.

3 Proof of Theorem 1

To prove Theorem 1, we need the following lemma, which is well known. For example, see [45, Proposition 1.2, p. 6].

Lemma 4. *Let* $0 < p \le \infty$, $\varepsilon > 0$. *Then there is a positive constant* C *such that*

$$\left(\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\varepsilon} a_k \right)^p \right)^{1/p} \le C \left(\sum_{j=-\infty}^{\infty} a_j^p \right)^{1/p} \tag{5}$$

for non-negative sequences $\{a_j\}_{j=-\infty}^{\infty}$. *Here, when* $p = \infty$, *it is understood that* (5) *stands for*

$$\sup_{j \in \mathbb{Z}} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\varepsilon} a_k \right) \le C \sup_{j \in \mathbb{Z}} a_j.$$

Proof of Theorem 1. Since the set of all bounded compact supported functions is dense in weighted variable Lebesgue spaces (see [42, Lemma 3.1, p. 10]), we only consider bounded compact supported functions. Let $\{f_j\}$ be a sequence of bounded compact supported functions, we decompose

$$f_j(x) = \sum_{l=-\infty}^{\infty} f_j^l \chi_l =: \sum_{l=-\infty}^{\infty} f_j^l, \quad j \in \mathbb{N}.$$

By Proposition 1, we have

$$\left\|\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} \approx \max\left\{\sup_{L\le 0,\,L\in\mathbb{Z}} 2^{-L\lambda}\left\|\left(2^{k\alpha(0)}\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right)_{k\le L}\right\|_{l^{q_0}(L^{p(\cdot)}(w))},\right.$$
$$\sup_{L>0,L\in\mathbb{Z}}\left[2^{-L\lambda}\left\|\left(2^{k\alpha(0)}\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right)_{k<0}\right\|_{l^{q_0}(L^{p(\cdot)}(w))}\right.$$
$$\left.\left.+\,2^{-L\lambda}\left\|\left(2^{k\alpha_\infty}\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right)_{k=0}^{L}\right\|_{l^{q_\infty}(L^{p(\cdot)}(w))}\right]\right\}$$
$$\coloneqq \max\{E,H\},$$

where

$$E \coloneqq \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\left\|\left(2^{k\alpha(0)}\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right)_{k\le L}\right\|_{l^{q_0}(L^{p(\cdot)}(w))},$$
$$H \coloneqq \sup_{L>0,L\in\mathbb{Z}}\{F+G\},$$
$$F \coloneqq 2^{-L\lambda}\left\|\left(2^{k\alpha(0)}\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right)_{k<0}\right\|_{l^{q_0}(L^{p(\cdot)}(w))},\quad L>0,$$
$$G \coloneqq 2^{-L\lambda}\left\|\left(2^{k\alpha_\infty}\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right)_{k=0}^{L}\right\|_{l^{q_\infty}(L^{p(\cdot)}(w))},\quad L>0.$$

Since to estimate F is essentially similar to estimate E, so we suffice to show that

$$E,G \lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.$$

To do so, we have

$$E \le C\sum_{i=i}^{3}E_i,\quad G\le C\sum_{i=i}^{3}G_i,$$

where

$$E_1 \coloneqq \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=-\infty}^{k-2}Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$
$$E_2 \coloneqq \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=k-1}^{k+1}Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_3 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=k+2}^{\infty} Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$G_1 := 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=-\infty}^{k-2} Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_2 := 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=k-1}^{k+1} Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_3 := 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=k+2}^{\infty} Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}}.$$

We shall use the following estimates. If $l < k-1$, then by Lemma 2 and Definition 2, we have

$$\begin{aligned}\left\|2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^l(y)|^r\right)^{\frac{1}{r}}\mathrm{d}y\chi_k\right\|_{L^{p(\cdot)}(w)} &\le C2^{-kn}\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}w\chi_l\right\|_{L^{p(\cdot)}}\|\chi_l w^{-1}\|_{L^{p'(\cdot)}}\\ &\le C2^{-kn}|B_k|\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}(w^{-1})}\|\chi_{B_l}\|_{L^{p'(\cdot)}(w^{-1})}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}\\ &\le C2^{(l-k)n\delta_2}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}.\end{aligned}\tag{6}$$

If $l \ge k+1$, then

$$\begin{aligned}\left\|2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^l(y)|^r\right)^{\frac{1}{r}}\mathrm{d}y\chi_k\right\|_{L^{p(\cdot)}(w)} &\le C2^{-kn}\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}w\chi_l\right\|_{L^{p(\cdot)}}\|\chi_l w^{-1}\|_{L^{p'(\cdot)}}\\ &\le C2^{-kn}\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}\|\chi_{B_l}\|^{-1}_{L^{p(\cdot)}(w)}\|\chi_{B_l}\|_{L^{p'(\cdot)}(w^{-1})}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}\\ &\le C2^{(l-k)n(1-\delta_1)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}.\end{aligned}\tag{7}$$

To estimate E_1, since $l \le k-2$, we deduce that

$$|x-y| \ge |x| - |y| > 2^{k-1} - 2^l \ge 2^{k-2}, \quad x\in D_k,\ y\in D_l.$$

Thus, by (3) for $\forall x \in D_k$, we have

$$|Tf_j^l| \lesssim 2^{-kn}\int_{\mathbb{R}^n}|f_j^l(y)|\mathrm{d}y.$$

Therefore, by the Minkowski inequality, we obtain

$$\begin{aligned}\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=-\infty}^{k-2}Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)} &\lesssim \left\|\left(\sum_{j=1}^{\infty}\left(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_j^l(y)|\mathrm{d}y\right)^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^l(y)|^r\right)^{\frac{1}{r}}\mathrm{d}y\chi_k\right\|_{L^{p(\cdot)}(w)}.\end{aligned} \tag{8}$$

By (6) and Lemma 4, we obtain

$$\begin{aligned}&\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^l(y)|^r\right)^{\frac{1}{r}}\mathrm{d}y\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}} \\ &\lesssim \left\{\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left(\sum_{l=-\infty}^{k-2}2^{(l-k)n\delta_2}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}(w)}\right)^{q(0)}\right\}^{\frac{1}{q(0)}} \\ &= \left\{\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-2}2^{l\alpha(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}2^{(l-k)(n\delta_2-\alpha(0))}\right)^{q(0)}\right\}^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{l=-\infty}^{L-2}2^{l\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},\end{aligned}$$

where $2^{-|k-l|(n\delta_2-\alpha(0))} = 2^{-|k-l|\varepsilon}$ for $\varepsilon = n\delta_2 - \alpha(0) > 0$. Hence,

$$E_1 \lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.$$

To estimate E_2. For $k-1 \le l \le k+1$, $\forall x \in D_k$, since T satisfies (4), then by the Minkowski inequality, we obtain

$$\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=k-1}^{k+1}Tf_j^l\right|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)} \lesssim \left\|\sum_{l=k-1}^{k+1}\left(\sum_{j=1}^{\infty}|Tf_j^l|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)} \lesssim \sum_{l=k-1}^{k+1}\left\|\left(\sum_{j=1}^{\infty}|Tf_j^l|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}(w)} \lesssim \sum_{l=k-1}^{k+1}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}. \tag{9}$$

Thus, we have

$$\begin{aligned}E_2 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{l=k-1}^{k+1}\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{l=-\infty}^{L+1}2^{l\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}} \lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.\end{aligned}$$

To estimate E_3, since $l \geq k+2$, we have

$$|x-y| \geq |y| - |x| > 2^{l-2}, \quad x \in D_k, \ y \in D_l.$$

For $\forall x \in D_k$, since the sublinear operator T satisfies (3), we have

$$|Tf_j^l| \lesssim 2^{-ln} \int_{\mathbb{R}^n} |f_j^l(y)| \mathrm{d}y.$$

Therefore, by the Minkowski inequality, we have

$$\begin{aligned} \left\| \left(\sum_{j=1}^{\infty} \left| \sum_{l=k+2}^{\infty} Tf_j^l \right|^r \right)^{\frac{1}{r}} \chi_k \right\|_{L^{p(\cdot)}(w)} &\lesssim \left\| \left(\sum_{j=1}^{\infty} \left(\sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_j^l(y)| \mathrm{d}y \right)^r \right)^{\frac{1}{r}} \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j^l(y)|^r \right)^{\frac{1}{r}} \mathrm{d}y \chi_k \right\|_{L^{p(\cdot)}(w)}. \end{aligned} \tag{10}$$

By (7), we obtain

$$\begin{aligned} &\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j^l(y)|^r \right)^{\frac{1}{r}} \mathrm{d}y \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left(\sum_{l=k+2}^{\infty} 2^{(k-l)n\delta_1} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(w)} \right)^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{k=-\infty}^{L} \left(\sum_{l=k+2}^{L} 2^{l\alpha(0)} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)} 2^{(k-l)(n\delta_1+\alpha(0))} \right)^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\quad + \left(\sum_{k=-\infty}^{L} \left(2^{k\alpha(0)} \sum_{l=L+1}^{0} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)} 2^{(k-l)n\delta_1} \right)^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\quad + \left(\sum_{k=-\infty}^{L} \left(2^{k\alpha(0)} \sum_{l=1}^{\infty} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)} 2^{(k-l)n\delta_1} \right)^{q(0)} \right)^{\frac{1}{q(0)}} \\ &:= I_{3,1} + I_{3,2} + I_{3,3}. \end{aligned}$$

Therefore,

$$\begin{aligned} E_3 &\lesssim \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} I_{3,1} + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} I_{3,2} + I \sup_{L \leq 0, L \in \mathbb{Z}} 2_{3,3}^{-L\lambda} \\ &:= E_{3,1} + E_{3,2} + E_{3,3}. \end{aligned}$$

We consider $E_{3,1}$. By Lemma 4, we have

$$E_{3,1} \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}\left(\sum_{l=k+2}^{L} 2^{l\alpha(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)} 2^{(k-l)(n\delta_1+\alpha(0))}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{l=-\infty}^{L+2} 2^{l\alpha(0)q(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)},$$

where $2^{-|k-l|(n\delta_1+\alpha(0))} = 2^{-|k-l|\eta}$ for $\eta = n\delta_1 + \alpha(0) > 0$.

We consider $E_{3,2}$. Since $n\delta_1 + \alpha(0) - \lambda > 0$, we obtain

$$E_{3,2} \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_1+\alpha(0))}\sum_{l=L+1}^{0} 2^{l\alpha(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)} 2^{-l(n\delta_1+\alpha(0))}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \sup_{L\le 0, L\in\mathbb{Z}}\sup_{l\le 0} 2^{-l\lambda}2^{l\alpha(0)}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_1+\alpha(0))}\sum_{l=L+1}^{0} 2^{-l(n\delta_1+\alpha(0)-\lambda)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{L(-n\delta_1-\alpha(0))}\left(\sum_{k=-\infty}^{L} 2^{k(n\delta_1+\alpha(0))q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)}.$$

We consider $E_{3,3}$. Since $n\delta_1 + \alpha(0) - \lambda > 0$, we obtain

$$E_{3,3} \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_1+\alpha(0))}\sum_{l=1}^{\infty} 2^{l\alpha_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)} 2^{-l(n\delta_1+\alpha_\infty)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \sup_{L\le 0, L\in\mathbb{Z}}\sup_{l\ge 1} 2^{-l\lambda}2^{l\alpha_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}\left(2^{k(n\delta_1+\alpha(0))}\sum_{l=1}^{\infty} 2^{-l(n\delta_1+\alpha_\infty-\lambda)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k(n\delta_1+\alpha(0))q(0)}\right)^{\frac{1}{q(0)}}$$

$$\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{L(-\lambda+n\delta_1+\alpha(0))}$$

$$\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)}.$$

To go on, we need further preparation.

If $l<0$, by Proposition 1, we have

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)} &= 2^{-l\alpha(0)}\left(2^{l\alpha(0)q(0)}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}}\\ &\lesssim 2^{-l\alpha(0)}\left(\sum_{t=-\infty}^{l}2^{t\alpha(0)q(0)}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_t\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}}\\ &\lesssim 2^{l(\lambda-\alpha(0))}\left(2^{-l\lambda}\left(\sum_{t=-\infty}^{l}\Big\|2^{t\alpha(0)}\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_t\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\right)\right)^{\frac{1}{q(0)}}\\ &\lesssim 2^{l(\lambda-\alpha(0))}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.\end{aligned}\tag{11}$$

Next, we estimate G. To estimate G_1, by (8) and (6), we have

$$\begin{aligned}G_1 &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left(\sum_{l=-\infty}^{k-2}2^{(l-k)n\delta_2}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}\right)^{q_\infty}\right\}^{\frac{1}{q_\infty}}\\ &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}2^{(l-k)n\delta_2}+\sum_{l=0}^{k}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}2^{(l-k)n\delta_2}\right)^{q_\infty}\right\}^{\frac{1}{q_\infty}}\\ &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}2^{(l-k)n\delta_2}\right)^{q_\infty}\right\}^{\frac{1}{q_\infty}}\\ &\quad+2^{-L\lambda}\left\{\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left(\sum_{l=0}^{k}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}2^{(l-k)n\delta_2}\right)^{q_\infty}\right\}^{\frac{1}{q_\infty}}\\ &=: G_{1,1}+G_{1,2}.\end{aligned}$$

If $q_\infty\ge 1$, since $n\delta_2-\alpha_\infty>0$ and $n\delta_2-\alpha(0)>0$, then by the Minkowski inequality and (11), we obtain

$$\begin{aligned}G_{1,1} &= 2^{-L\lambda}\left\{\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}2^{(l-k)n\delta_2}\right)^{q_\infty}\right\}^{\frac{1}{q_\infty}}\\ &\lesssim 2^{-L\lambda}\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}\left\{\sum_{k=0}^{L}(2^{k\alpha_\infty}2^{(l-k)n\delta_2})^{q_\infty}\right\}^{\frac{1}{q_\infty}}\\ &\lesssim 2^{-L\lambda}\sum_{l=-\infty}^{-1}2^{ln\delta_2}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{\frac{1}{r}}\chi_l\Big\|_{L^{p(\cdot)}(w)}\left\{\sum_{k=0}^{L}2^{-k(n\delta_2-\alpha_\infty)q_\infty}\right\}^{\frac{1}{q_\infty}}\end{aligned}$$

$$\lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} 2^{-L\lambda} \sum_{l=-\infty}^{-1} 2^{l(n\delta_2+\lambda-\alpha(0))}$$

$$\lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.$$

If $q_\infty < 1$, since $n\delta_2 - \alpha_\infty > 0$ and $n\delta_2 - \alpha(0) > 0$, then by (11), we have

$$\begin{aligned}
G_{1,1} &\lesssim 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \sum_{l=-\infty}^{-1} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)}^{q_\infty} 2^{(l-k)n\delta_2 q_\infty} \right)^{\frac{1}{q_\infty}} \\
&= 2^{-L\lambda} \left(\sum_{l=-\infty}^{-1} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)}^{q_\infty} 2^{ln\delta_2 q_\infty} \sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} 2^{-kn\delta_2 q_\infty} \right)^{\frac{1}{q_\infty}} \\
&= 2^{-L\lambda} \left(\sum_{l=-\infty}^{-1} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)}^{q_\infty} 2^{ln\delta_2 q_\infty} \sum_{k=0}^{L} 2^{-k(n\delta_2-\alpha_\infty) q_\infty} \right)^{\frac{1}{q_\infty}} \\
&\lesssim 2^{-L\lambda} \left(\sum_{l=-\infty}^{-1} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)}^{q_\infty} 2^{ln\delta_2 q_\infty} \right)^{\frac{1}{q_\infty}} \\
&\lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)} 2^{-L\lambda} \left(\sum_{l=-\infty}^{-1} 2^{l(n\delta_2+\lambda-\alpha(0)) q_\infty} \right)^{\frac{1}{q_\infty}} \\
&\lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.
\end{aligned}$$

We consider $G_{1,2}$. Since $n\delta_2 - \alpha_\infty > 0$, by Lemma 4, we have

$$\begin{aligned}
G_{1,2} &= 2^{-L\lambda} \left\{ \sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \left(\sum_{l=0}^{k} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)} 2^{(l-k)n\delta_2} \right)^{q_\infty} \right\}^{\frac{1}{q_\infty}} \\
&= 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \left(\sum_{l=0}^{k} 2^{l\alpha_\infty} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)} 2^{(l-k)(n\delta_2-\alpha_\infty)} \right)^{q_\infty} \right\}^{\frac{1}{q_\infty}} \\
&\lesssim 2^{-L\lambda} \left(\sum_{l=0}^{k} 2^{l\alpha_\infty q_\infty} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_l \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\
&\lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)},
\end{aligned}$$

where $2^{-|k-l|(n\delta_2-\alpha_\infty)} \lesssim 2^{-|k-l|\eta}$ for $\eta = n\delta_2 - \alpha_\infty$.

To estimate G_2, by (9), we have

$$\begin{aligned} G_2 &\lesssim 2^{-L\lambda}\left(\sum_{l=k-1}^{k+1}\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &\lesssim 2^{-L\lambda}\left(\sum_{l=-1}^{L+1}2^{l\alpha_\infty q_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)}. \end{aligned}$$

To estimate G_3, by (10), we have

$$G_3 \le 2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^l(y)|^r\right)^{\frac{1}{r}}\mathrm{d}y\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}}.$$

By (7), we obtain

$$\begin{aligned} G_3 &\lesssim 2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left(\sum_{l=k+2}^{\infty}2^{(k-l)n\delta_1}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &\lesssim 2^{-L\lambda}\left(\sum_{k=0}^{L}\left(\sum_{l=k+2}^{L+2}2^{l\alpha_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}2^{(k-l)(n\delta_1+\alpha_\infty)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &\quad + 2^{-L\lambda}\left(\sum_{k=0}^{L}\left(2^{k\alpha_\infty}\sum_{l=L+3}^{\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}2^{(k-l)n\delta_1}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &:= G_{3,1} + G_{3,2}. \end{aligned}$$

Now we estimate $G_{3,1}$. By Lemma 4, we obtain

$$\begin{aligned} G_{3,1} &\lesssim 2^{-L\lambda}\left(\sum_{k=0}^{L}\left(\sum_{l=k+2}^{L+2}2^{l\alpha_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}2^{(k-l)(n\delta_1+\alpha_\infty)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &\lesssim 2^{-L\lambda}\left(\sum_{l=0}^{L+2}2^{l\alpha_\infty q_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}} \\ &\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q(\cdot),p(\cdot)}^{\alpha(\cdot),\lambda}(w)}, \end{aligned}$$

where $2^{-|k-l|(n\delta_1+\alpha_\infty)} = 2^{-|k-l|\zeta}$ for $\zeta = n\delta_1 + \alpha_\infty > 0$.

Then we estimate $G_{3,2}$. Since $n\delta_1 + \alpha_\infty - \lambda > 0$,

$$
\begin{aligned}
G_{3,2} &\lesssim 2^{-L\lambda}\left(\sum_{k=0}^{L}\left(2^{k(n\delta_1+\alpha_\infty)}\sum_{l=L+3}^{\infty}2^{l\alpha_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}2^{-l(n\delta_1+\alpha_\infty)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\
&\lesssim \sup_{l\geq 1}2^{-l\lambda}2^{l\alpha_\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_l\right\|_{L^{p(\cdot)}(w)}2^{-L\lambda}\left(\sum_{k=0}^{L}\left(2^{k(n\delta_1+\alpha_\infty)}\sum_{l=L+3}^{\infty}2^{-l(n\delta_1+\alpha_\infty-\lambda)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\
&\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k(n\delta_1+\alpha_\infty)q_\infty}2^{-L(n\delta_1+\alpha_\infty-\lambda)}\right)^{\frac{1}{q_\infty}}\\
&\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}2^{-L\lambda}2^{(n\delta_1+\alpha_\infty)L}2^{-L(n\delta_1+\alpha_\infty-\lambda)}\\
&\lesssim \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(w)}.
\end{aligned}
$$

This completes the proof. □

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