Boundedness theorems for some fourth order differential equations.

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Summary. - *In this paper a new approach involving the use of two signum functions together with a suitably chosen Lyapunov function is employed to investigate the boundedness property of solutions of two special cases of* (1.3). *This approach makes for considerable reduction in the conditions imposed on f, g in an earlier paper* [1].

1. - Consider the differential equation

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t) \tag{1.1}$$

in which α_2, α_4 are constants and f, g, p depend on the arguments shown. It was shown in an earlier paper [1], subject to the basic assumptions that $f(z)$, $g'(y)$, $p(t)$ are continuous in z, y, t respectively, that if

(I) $\alpha_2 > 0,\ \alpha_4 > 0$

(II) there are constants $\alpha_1 > 0$, $\alpha_3 > 0$ such that $g(y)/y \geq \alpha_3$ $(y \neq 0)$ and $f(z) \geq \alpha_1$ for all z,

(III) there is a finite constant $\Delta_0 > 0$ such that

$$\{\alpha_1\alpha_2 - g'(y)\}\alpha_3 - \alpha_1\alpha_4 f(z) \geq \Delta_0$$

for all y and z,

(IV) there is a constant $\delta_1 < 2\Delta_0\alpha_4\alpha_1^{-1}\alpha_4\alpha_3^{-2}$ such that

$$g'(y) - g(y)/y \leq \delta_1 \qquad (y \neq 0)$$

(V) there is a constant $\delta_2 < 2\Delta_0\alpha_1^{-1}\alpha_3^{-2}$ such that

$$z^{-1}\int_0^z f(\zeta)d\zeta - f(z) \leq \delta_2 \qquad (z \neq 0),$$

(VI) $\int_0^t |p(\tau)|\,d\tau \leq A < \infty$ $(t \geq 0)$ for some constant A, then for every

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solution $x(t)$ of (1.1) defined by

$$x(0) = x_0, \qquad \dot{x}(0) = y_0, \qquad \ddot{x}(0) = z_0, \qquad \dddot{x}(0) = w_0,$$

there is a finite constant D whose magnitude depends on the initial values x_0, y_0, z_0 and w_0 such that

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq D \tag{1.2}$$

for all $t \geq 0$. The conditions (I), (II), and (III) are suitable generalizations of the ROUTH-HURWITZ conditions

$$\alpha_i > 0 \ (i = 1, 2, 3, 4) \text{ and } (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - \alpha_1^2\alpha_4 > 0$$

for the asymptotic stability (in the large) of the trivial solution of the linear equation

$$x^{(4)} + \alpha_1\dddot{x} + \alpha_2\ddot{x} + \alpha_3\dot{x} + \alpha_4 x = 0.$$

Subsequently TEJUMOLA [2] investigating the more general equation

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.3}$$

in which $p(t, x, y, z, u)$ is bounded for all t, x, y, z and u, succeeded in proving that, under much the same conditions on α_2, α_4, f and g as before, then every solution $x(t)$ of (1.3) *ultimately* satisfies the stronger inequality (1.2) in which the bounding constant D is *independent* of the initial values x_0, y_0, z_0 and w_0.

The main object of the present paper is to draw attention to two special cases of (1.3) which have recently come to our notice (mostly as a result of the work by OGURCOV [3]) for which this boundedness result of the stronger type can be proved subject only to a minimum of «ROUTH-HURWITZ restrictions» and without the use of the conditions (IV), (V).

The first case is the equation

$$x^{(4)} + \alpha_1\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.4}$$

in which α_1, α_2, α_4 are constants, corresponding to $f \equiv \alpha_1$ in (1.3). We shall prove here.

THEOREM 1. - *In the equation* (1.4) *let g, p be continuous in all their arguments and suppose that*

(*i*) $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_4 > 0$, *(ii) there is a constant* $\eta_0 > 0$ *such that*

$$g(y)/y > 0 \qquad (|y| \geq \eta_0),$$

(iii) there is a constant $d_1 > 0$ *such that*

$$\alpha_1\alpha_2\frac{g(y)}{y} - \left\{\frac{g(y)}{y}\right\}^2 - \alpha_1^2\alpha_4 \geq d_1 \qquad (|y| \geq \eta_0), \tag{1.5}$$

(iv) there is a finite constant A_0 *such that*

$$|p(t, x, y, z, u)| \leq A_0 \text{ for all } t, x, y, z \text{ and } u. \tag{1.6}$$

Then there exists a finite constant D whose maginite depends only on α_1, α_2, α_4, η_0, d_1, A_0 and g *such that every solution* $x(t)$ *of* (1.4) *ultimately satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq D. \tag{1.7}$$

Observe here that the existence of $g'(y)$ is not even required. Also no restriction whatever, except that of continuity, has been placed on $g(y)$ in the interval $|y| \leq \eta_0$.

The next special case is the equation

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + \alpha_3\dot{x} + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.8}$$

with α_2, α_3 and α_4 constants, corresponding this time to $g(\dot{x})$ linear in (1.3); and we have here, analogous to Theorem 1,

THEOREM 2. - *In the equation* (1.8) *let f, g be continuous in all their arguments and suppose that*

(i) $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$,

(ii) there is a constant $\zeta_0 > 0$ *such that*

$$f(z) > 0 \qquad (|z| \geq \zeta_0)$$

(iii) there is a constant $d_2 > 0$ *such that*

$$\alpha_2\alpha_3 f(z) - \alpha_3^2 - \alpha_4 f^2(z) \geq d_2 \qquad (|z| \geq \zeta_0),$$

(iv) $p(t, x, y, z, u)$ *satisfies* (1.6).

Then there exists a finite constant D whose magnitude depends only α_2, α_3, α_4, d_2, ζ_0, A_0 *and g such that every solution* $x(t)$ *of* (1.8) *ultimately satisfies* (1.7).

With α_2, α_3 and α_4 constants it is possible to extend Theorem 2 a little further and we shall actually prove here

THEOREM 3. - *Given the equation*

$$x^{(4)} + \psi(\dot{x}, \ddot{x})\dddot{x} + \alpha_2\ddot{x} + \alpha_3\dot{x} + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1.9}$$

in which the function ψ *is such that* $\frac{\partial\psi}{\partial y}(y, z)$ *exist* $\psi(y, z)$, $\frac{\partial\psi}{\partial y}(y, z)$, $p(t, x, y, z, u)$ *are continuous for all* x, y, z, u *and* t, *suppose that*

(*i*) $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$,

(*ii*) *there is a constant* $\zeta_0 > 0$ *such that* $\psi(y, z) > 0$ $(|z| \geq 0)$

(*iii*) *there is a finite constant* F *such that* $\max_{|z|\leq\zeta_0} |\psi(x, z)| \leq F$ *for all* y,

(*iv*) $z\frac{\partial\psi}{\partial y}(y, z) \leq 0$ *for all* y, z,

(*v*) *there is a constant* $d_2 > 0$ *such that*

$$\alpha_2\alpha_3\psi(y, z) - \alpha_3^2 - \alpha_4\psi^2(y, z) \geq d_2 \qquad (|z| \geq \zeta_0), \tag{1.10}$$

(*vi*) $p(t, x, y, z, u)$ *satisfies* (1.6).

Then there exists a finite constant $D > 0$ *whose magnitude depends only on* α_2, α_3, α_4, d_2, ζ_0, A_0 *and* ψ *such that every solution* $x(t)$ *of* (1.9) *satisfies* (1.7).

Note that if ψ is independent of y, then (*iv*) is trivially true, and the existence of F in (*iii*) would follow from the continuity of $\psi(z)$, so that Theorem 2 is indeed a special case of Theorem 3.

2. - Notation for the constants.

We adopt the notation in [2] and the capitals D, D_0, D_1, ... in the text are finite positive constants whose magnitudes are independent of solutions of whatever differential equation is under review: in the context of the equation (1.4), for instance, their magnitudes would depend at most on α_1, α_2, α_4, η_0, d_1, A_0 and g, and in the context of the equation (1.9) on α_2, α_3, α_4, d_2, ζ_0, A_0, and ψ. As usual the D's are not necessarily the same in each place of occurrence unless numbered, but the D's : D_0, D_1, D_2, ... with suffixes attached retain a fixed identity throughout.

3. - A function $V_{3.1}$.

It is convenient in proving Theorem 1 to deal more directly with the differential system

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = u$$

$$\dot{u} = -\alpha_1 u - \alpha_2 z - g(y) - \alpha_4 x + p(t, x, y, z, u) \tag{3.1}$$

which is derived from (1.4) on setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$. We shall prove that there is a continuous function $V = V(x, y, z, u)$ such that

$$V(x, y, z, u) \to +\infty \text{ as } x^2 + y^2 + z^2 + u^2 \to \infty \tag{3.2}$$

and such that the limit

$$\dot{V}^+ \equiv \limsup_{h \to +0} \frac{V(x(t+h), y(t+h), z(t+h), u(t+h)) - V(x(t), y(t), z(t), u(t))}{h} \tag{3.3}$$

exists, corresponding to any solution $(x(t), y(t), z(t), u(t))$ of (3.1), and satisfies

$$\dot{V}^+ \leq -D_0 \text{ if } x^2(t) + y^2(t) + z^2(t) + u^2(t) \geq D, \tag{3.4}$$

for some constants D_0, D_1. As shown in § 4 of [2], the two results (3.2) and (3.4) imply, ultimately that

$$x^2(t) + y^2(t) + z^2(t) + u^2(t) \leq D$$

which is precisely (1.7).

In order to distinguish between the above V and another V, with properties analogous to (3.2) and (3.4), which will arise in the context of Theorem 3 we shall refer to the present V as $V_{3 \cdot 1}$ so as to underline the fact of its association with the system (3.1).

4. - Ogurcov's function V_0.

We were led to the construction of our own $V_{3 \cdot 1}$ by the properties of a certain LYAPUNOV function which we designate here by V_0, which was used by OGURCOV in [3] for investigating the stability of the trivial solution of the equation corresponding to $p \equiv 0$ in (1.4). In the present notation V_0 is given by

$$\begin{aligned} 2V_0 = {} & \alpha_2\alpha_4 x^2 + 2\alpha_1\alpha_4 xy + (\alpha_2^2 - 2\alpha_4)y^2 + \\ & + 4\alpha_4 xz + 2\alpha_1\alpha_2 yz + (\alpha_1^2 + \alpha_2)z^2 + 2\alpha_2 yu + \\ & + 2\alpha_1 zu + 2u^2 + 2\alpha_1 \int_0^y g(\eta)d\eta. \end{aligned} \tag{4.1}$$

with the properties in question contained in the following

Lemma 1. - *Subject to the conditions of Theorem* 1:

(i) there exists a positive definite quadratic form $Q_1(x, y, z, u)$ *and a constant* D_2 *such that*

$$V_0 \geq Q_1(x, y, z, u) - D_2, \tag{4.2}$$

for all x, y, z *and* u

(ii) the derivative $\dot{V}_0 \equiv \dot{V}_0(x(t), y(t), z(t), u(t))$ *corresponding to any solution* $(x(t), y(t), z(t), u(t))$ of (3.1) *satisfies*

$$\dot{V}_0 \leq -D_3(y^2 + u^2) + D_4(|y| + |z| + |u| + 1) \tag{4.3}$$

for some constants D_3 *and* D_4.

Proof. of *(i)*. - V_0 can be rearranged thus:

$$V_0 = V_0^1 + \alpha_1 \int_0^y \left\{ g(\eta) - \frac{\alpha_1\alpha_4}{\alpha_2}\eta \right\} d\eta \tag{4.4}$$

where V_0^1 is the quadratic form given by

$$2V_0^1 = \alpha_2\alpha_4 x^2 + 2\alpha_1\alpha_4 xy + \left(\alpha_2^2 - 2\alpha_4 + \frac{\alpha_1^2\alpha_4}{\alpha_2}\right) y^2 +$$
$$+ 4\alpha_4 xz + 2\alpha_1\alpha_2 yz + (\alpha_1^2 + \alpha_2) z^2 + 2\alpha_2 yu + 2\alpha_1 zu + 2u^2 .$$

For precisely the same reasons as in [3] V_0^1 is positive definite. Also, since $\alpha_1\alpha_2 > 0$ it is clear from (1.5) rewritten in the form

$$\alpha_1\alpha_2 \left\{ \frac{g(y)}{y} - \frac{\alpha_1\alpha_4}{\alpha_2} \right\} \geq d_1 + \left(\frac{g(y)}{y} \right)^2 \qquad (|y| \geq \eta_0)$$

that $g(y)/y > \alpha_1\alpha_4\alpha_2^{-1} > 0$ $(|y| \geq \eta_0)$, and thus, since $g(y)$ is continuous we have evidently that

$$\int_0^y \left\{ g(\eta) - \frac{\alpha_1\alpha_4}{\alpha_2}\eta \right\} d\eta \geq -D \text{ for all } y.$$

The result (4.2) then follows from (4.4).

Proof. of *(ii)*. - Let $(x(t), y(t), z(t), u(t))$ be any solution of (3.1). By a straightforward differentiation from (4.1) we have that

$$\dot{V}_0 = -\chi(y, u) + \{ \alpha_2 y + \alpha_1 z + 2u \} p(t, x, y, z, u), \tag{4.5}$$

where

$$(4.6)\qquad \chi(y,\ u) \equiv \alpha_1 u^2 + 2ug(y) + \alpha_2 y g(y) - \alpha_1\alpha_4 y^2$$
$$= \alpha_1[u + \alpha_1^{-1}g(y)]^2 + \alpha_1^{-1}\{\alpha_1\alpha_2 y g(y) - g^2(y) - \alpha_1^2\alpha_4 y^2\}.$$

We have from (1.5) that

$$(4.7)\qquad \alpha_1\alpha_2 y g(y) - g^2(y) - \alpha_1^2\alpha_4 y^2 \geq d_1 y^2 - D$$

for all y, since, if $|y| \geq \eta_0$ then

$$\chi_1 \equiv \alpha_1\alpha_2 y g(y) - g^2(y) - \alpha_1^2\alpha_4 y^2 - d_1 y^2 + D$$
$$= y^2\left\{\alpha_1\alpha_2\frac{g(y)}{y} - \left(\frac{g(y)}{y}\right)^2 - \alpha_1^2\alpha_4 - d_1\right\} + D$$
$$> 0$$

for arbitrary D, and if $|y| \leq \eta_0$ then any D such that

$$D > (\alpha_1^2\alpha_4 + d_1)\eta_0^2 + \max_{|y|\leq\eta_0} |\alpha_1\alpha_2 y g(y) - g^2(y)|$$

would also secure $\chi_1 > 0$. Thus, by (4.6) and (4.7)

$$(4.8)\qquad \chi \geq \alpha_1\{u + \alpha_1^{-1}g(y)\}^2 + \alpha_1^{-1}d_1 y^2 - D.$$

Next observe from hypothesis (*ii*) and from (1.5), this time taken in the form

$$\frac{g(y)}{y}\left\{\alpha_1\alpha_2 - \frac{g(y)}{y}\right\} \geq d_1 + \alpha_1^2\alpha \qquad (|y| \geq \eta_0)$$

that

$$0 < g(y)/y < \alpha_1\alpha_2 \qquad (|y| \geq \eta_0)$$

so that

$$(4.9)\qquad |g(y)| \leq \alpha_1\alpha_2|y| + D_5 \text{ for all } y,$$

provided that D_5 is sufficiently large.
From (4.9) we have in turn that

$$(4.10)\qquad g^2(y) \leq 2\alpha_1^2\alpha_2^2 y^2 + D$$

for sufficiently large D. Thus, by (4.8) and (4.10),

$$\chi \geq \frac{1}{2}\alpha_1^{-1}d_1y^2 + \frac{1}{4}\alpha_1^{-3}\alpha_2^{-2}d_1g^2(y) + \alpha_1\{u + \alpha_1^{-1}g(y)\}^2 - D. \tag{4.11}$$

Now fix D_6 sufficiently small to ensure that

$$\frac{1}{4}\alpha_1^{-3}\alpha_2^{-2}d_1g^2(y) + \alpha_1\{u + \alpha_1^{-1}g(y)\}^2 \geq D_6[u^2 + g^2(y)].$$

Then, by (4.11), we would have that

$$\chi \geq \frac{1}{2}\alpha_1^{-1}d_1y^2 + D_6u^2 - D$$

from which (4.3) now follows on combining with (4.5) and using (1.6).

5. - Explicit form of $V_{3\cdot1}$.

The function $V_{3\cdot1} = V_{3\cdot1}(x, y, z, u)$ is the composite function:

$$V_{3\cdot1} = V_0 + V_1 + V_2 \tag{5.1}$$

where V_0 is the Ogurcov function (4.1) and $V_1 = V_1(x, u)$ and $V_1 = V_1(y, z)$ are defined by

$$V_1 = \begin{pmatrix} u \operatorname{sgn} x, & \text{if } |x| \geq |u| \\ x \operatorname{sgn} u, & \text{if } |u| \geq |x| \end{pmatrix} \tag{5.2}$$

$$V_2 = \begin{pmatrix} -(2D_4 + \alpha_2)\, y \operatorname{sgn} z, & \text{if } |z| \geq |y| \\ -(2D_4 + \alpha_2)\, z \operatorname{sgn} y, & \text{if } |y| \geq |z| \end{pmatrix}. \tag{5.3}$$

It is clear from their definitions that

$$|V_1| \leq |u|, \qquad |V_2| \leq (2D_4 + \alpha_2)|y|,$$

and hence, by (5.1) and (4.2), that

$$V_{3\cdot1} \geq Q_1(x, y, z, u) - (2D_4 + \alpha_2)|y| - |x| - D_2 \tag{5.4}$$

for all x, y, z and u. Since Q_1 is a positive definite quadratic form, (5.4) implies that

$$V_{3\cdot1} \to +\infty \quad \text{as} \quad x^2 + y^2 + z^2 + u^2 \to \infty$$

so that $V = V_{3\cdot1}$ does satisfy (3.2).

6. - It remains now to verify (3.4) for $V = V_{3\cdot1}$.

Let $(x(t), y(t), z(t), u(t))$ be any solution of (3.1) and let

$$\dot{V}_1^+ \equiv \limsup_{h\to+0} \frac{V_1(x(t+h), u(t+h)) - V_1(x(t), u(t))}{h}$$

with $\dot{V}_2^+$ similarly defined. It is easy to see form (5.2), (5.3) and from (3.1) itself that $\dot{V}_1^+$ and $\dot{V}_2^+$ can be set out in the forms:

$$\dot{V}_1^+ = \begin{pmatrix} -\alpha_1 u + \alpha_2 z + g(y) + \alpha_4 x - p) \operatorname{sgn} x, \text{ if } |x| \geq |u| \\ y \operatorname{sgn} u, \text{ if } |u| \geq |x| \end{pmatrix}$$

$$\dot{V}_2^+ = \begin{pmatrix} -(2D_4 + \alpha_2)|z|, \text{ if } |z| \geq |y| \\ -(2D_4 + \alpha_2) u \operatorname{sgn} y, \text{ if } |y| \geq |z| \end{pmatrix}$$

so that, on using (1.6) and (4.9) as requirred,

$$\dot{V}_1^+ \leq \begin{pmatrix} -\alpha_4 |x| + \alpha_2 |z| + D_7(|y| + |u| + 1), \text{ if } |x| \geq |u| \\ |y|, \text{ if } |u| \geq |x| \end{pmatrix} \tag{6.1}$$

$$\dot{V}_2^+ \leq \begin{pmatrix} -(2D_4 + \alpha_2)|z|, \text{ if } |z| \geq |y| \\ (2D_4 + \alpha_2)|u|, \text{ if } |y| \geq |z| \end{pmatrix}. \tag{6.2}$$

For our estimates of $\dot{V}_{3\cdot1}^+$ we shall use (4.3), (6.1) and (6.2) in conjuuction with the formula

$$\dot{V}_{3\cdot1}^+ = \dot{V}_0 + \dot{V}_1^+ + \dot{V}_2^+ \tag{6.3}$$

from (5.1). If $|z| \geq |y|$, for example, it is clear that $\dot{V}_{3\cdot1}^+$ necessarily satisfies one or other of the following two inequalities

$$\begin{aligned} \dot{V}_{3\cdot1}^+ &\leq -D_3(y^2 + u^2) + D_4(|y| + |z| + |u| + 1) - (2D_4 + \alpha_2)|z| + \\ &\quad + \alpha_2 |z| + D_7(|y| + |u| + 1) - \alpha_4 |x| \\ &= -D_3(y^2 + u^2) + (D_4 + D_7)(|y| + |u| + 1) - \alpha_4 |x| - D_4 |z|, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \dot{V}_{3\cdot1}^+ &\leq -D_3(y^2 + u^2) + D_4(|y| + |z| + |u| + 1) - (2D_4 + \alpha_2)|z| + |y| \\ &= -D_3(y^2 + u^2) + (D_4 + 1)|y| + D_4(|u| + 1) - (D_4 + \alpha_2)|z|, \end{aligned} \tag{6.5}$$

according as $|x| \geq |u|$ or $|x| \leq |u|$, so that, at least,

$$\dot{V}_{3\cdot1}^+ \leq -D_3(y^2 + u^2) + D(|y| + |u| + 1). \tag{6.6}$$

An analogous consideration of the two possibilities that can arise will also show that in the case $|z| \leq |y|$ $\dot{V}^{+}_{3\cdot 1}$ too satisfies (6.6). Thus

$$\dot{V}^{+}_{3\cdot 1} \leq -D_3(y^2 + u^2) + D(|y| + |u| + 1)$$

always, from which it follows that there is a constant D_8 such that

$$\dot{V}^{+}_{3\cdot 1} \leq -1 \text{ if } y^2 + u^2 \geq D_8^2. \tag{6.7}$$

A similar bound can also be established for $\dot{V}^{+}_{3\cdot 1}$ if $y^2 + u^2 \leq D_8^2$ provided that $x^2 + z^2$ is large enough, to be more precise:

$$\dot{V}^{+}_{3\cdot 1} \leq -1 \text{ when } y^2 + u^2 \leq D_8^2 \text{ provided that } x^2 + z^2 \geq D_9^2 \tag{6.8}$$

for some sufficiently large D_9. Indeed let $y^2 + u^2 \leq D_8^2$ and assume, to begin with that $|z| \geq D_8$. Then $|z| \geq |y|$ and so as before $\dot{V}^{+}_{3\cdot 1}$ satisfies one or other of (6.4) or (6.5). Either of these gives at least that

$$\dot{V}^{+}_{3\cdot 1} \leq -D_4|z| + D < -1$$

provided further that $|z|$ is large enough, say $|z| \geq D_{10} (\geq D_8)$; so that we have now shown that

$$\dot{V}^{+}_{3\cdot 1} \leq -1 \text{ if } y^2 + u^2 \leq D_8^2 \text{ and } |z| \geq D_{10}. \tag{6.9}$$

It remains to consider the case

$$y^2 + u^2 \leq D_8^2 \text{ and } |z| \leq D_{10}.$$

Here, we have that

$$\dot{V}_0 \leq D \text{ and } \dot{V}^{+}_2 \leq D$$

by (4.3) and (6.2) respectively. Also, if $|x| \geq D_8$ then, by (6.1),

$$\dot{V}^{+}_{3\cdot 1} \leq -\alpha_4|x| + D,$$

so that, by (6.3),

$$\dot{V}^{+}_1 \leq -\alpha_4|x| + D < -1$$

provided further that $|x|$ is sufficiently large, say, $|x| \geq D_{11} (\geq D_8)$. In other words we also have that

$$\dot{V}^{+}_{3\cdot 1} \leq -1 \text{ if } (y^2 + u^2) \leq D_8^2 \text{ and } |z| \leq D_{10} \text{ but } |x| \geq D_{11}. \tag{6.10}$$

The results (6.9) and (6.10) together show that

$$\dot{V}^{+}_{3\cdot 1} \leq -1 \text{ when } y^2 + u^2 \leq D_8^2 \text{ provided that } x^2 + z^2 \geq D_{10}^2 + D_{11}^2,$$

which is (6.8), with $D_9 = (D_{10}^2 + D_{11}^2)^{1/2}$. In turn (6.7) and (6.8) give that

$$\dot{V}^{+}_{3\cdot 1} \leq -1 \text{ if } x^2 + y^2 + z^2 + u^2 \geq D_8^2 + D_9^2$$

so that the function $V = V_{3\cdot 1}$ also satisfies (3.4).

Theorem 1 now follows as was pointed out.

7. - We turn now to Theorem 3 with the numbering of the *D's* started afresh.

The procedure will follow essentially the pattern for Theorem 1 and we shall skip all inessential details.

8. - **The function** $V_{8\cdot 1}$.

As before we consider the the differential system

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u,\ \dot{u} = -\psi(y,\ z)u - \alpha_2 z - \alpha_3 y - \alpha_4 x + p(t,\ x,\ y,\ z,\ u) \tag{8.1}$$

obtained from (1.9) by setting $y = \dot{x}$, $z = \ddot{x}$ and $u = \dddot{x}$, and to prove Theorem 3 it will be enough to show that there is a function $V = V(x,\ y,\ z,\ u)$ satisfing (3.2) and such that, corresponding to any solution $(x(t),\ y(t),\ z(t),\ u(t))$ of (8.1), (3.3) exists and satisfies (3.4). We shall refer to such a function in what follows as the function $V_{8\cdot 1}$ so as to emphasize the important role which the system (8.1) plays in the characterization of the present V.

9. - **The Ogurcov's functions** W_0.

The particular $V_{8\cdot 1}$ which will be used here was suggested to us by certain properties (stated in Lemma 2 below) of the LYAPUNOV function $W_0 = W_0(x,\ y,\ z,\ u)$ defined by

$$\left\{\begin{aligned} 2W_0 = {} & 2\alpha_4^2 x^2 + 2\alpha_3\alpha_4 xy + (\alpha_2\alpha_4 + \alpha_3^2)y^2 + \\ & + 2\alpha_2\alpha_4 xz + 2\alpha_2\alpha_3 yz + (\alpha_2^2 - 2\alpha_4)z^2 + \\ & + 4\alpha_4 yu + 2\alpha_3 zu + \alpha_2 u^2 + 2\alpha_3 \int_0^z \eta\psi(y,\ \eta)d\eta \end{aligned}\right. \tag{9.1}$$

which was the main tool in OGURCOV's proof of a stability theorem for the equation corresponding to $p \equiv 0$ in (1.9).

LEMMA 2. - *Assume the conditions of Theorem* 3 *hold. Let* $\Phi = \Phi(z)$ *be the differentiable function defined by*

$$\Phi = \begin{cases} \operatorname{sgn} z, & |z| > 2\zeta_0 \\ \sin\left(\dfrac{\pi z}{4\zeta_0}\right), & |z| \leq 2\zeta_0 \end{cases} \tag{9.2}$$

and set

$$U_0 = W_0 - D_2 u \Phi(z) \tag{9.3}$$

where W_0 *is the function* (9.1) *and*

$$D_2 \equiv 8\zeta_0(\alpha_2\alpha_3 F + \alpha_3^2 + F^2 + d_2)/(\pi\alpha_3\sqrt{2}), \tag{9.4}$$

the constants ζ_0, F *and* d_2 *being as given in hypotheses* (*ii*), (*iii*) *and* (*iv*) *of Theorem* 3. *Then*

(*i*) *there exists a positive definite quadratic form* $Q_2(x, y, z, u)$ *and a constant* D_3 *such that*

$$U_0 \geq Q_2(x, y, z, u) - D_3(|u| + 1) \tag{9.5}$$

for all x, y, z *and* u

(*ii*) *the* $\dot{U}_0 = \dot{U}_0(x(t), y(t), z(t), u(t))$ *corresponding to any solution* $(x(t), y(t), z(t), u(t))$ *of* (8.1) *satisfies*

$$\dot{U}_0 \leq -D_4(y^2 + u^2) + D_5(|x| + |y| + |z| + |u| + 1) \tag{9.6}$$

for some constants D_4 *and* D_5.

PROOF OF (*i*). - This part is a consequence of the rearrangement of W_0 in the form:

$$W_0 = W_0^1 + \alpha_3 \int_0^z \left\{ \psi(y, \eta) - \frac{\alpha_3}{\alpha_2} \right\} \eta \, d\eta$$

where W_0^1 is the quadratic form given by:

$$\begin{aligned} 2W_0^1 = {} & 2\alpha_4^2 x^2 + 2\alpha_3\alpha_4 xy + (\alpha_2\alpha_4 + \alpha_3^2) y^2 + \\ & + 2\alpha_2\alpha_4 xz + 2\alpha_2\alpha_3 yz + \left(\alpha_2^2 - 2\alpha_4 - \frac{\alpha_3^2}{\alpha_2}\right) z^2 + \\ & + 4\alpha_4 yu + 2\alpha_3 zu + \alpha_2 u^2. \end{aligned}$$

For, as shown in [3], W_0^1 is positive definite. Next, by (1.10), taken in the form:

$$\alpha_3^2\left(\psi - \frac{\alpha_3}{\alpha_2}\right) \geq \alpha_4\psi^2 + d_2 \qquad |z| \geq \zeta_0$$

$\psi - \dfrac{\alpha_3}{\alpha_2} > 0$ for $|z| \geq \zeta_0$, so that since $|\psi| \leq F < \infty$ for $|z| \leq \zeta_0$,

$$\int_0^z \left\{\psi(y, \eta) - \frac{\alpha_3}{\alpha_2}\right\}\eta d\eta \geq -D$$

for all y, z, and (9.5) now follows on bringing in the fact from the definition (9.2), that $|\Phi(z)| \leq 1$.

PROOF OF (*ii*). - Let $(x(t), y(t), z(t), u(t))$ be any solution of (8.1). Then from (9.1) and (9.3) it can be verified that

$$\dot{U}_0 = -W_1 + \alpha_3 z \int_0^z \eta \frac{\partial\psi}{\partial y}(y, \eta)d\eta + W_2 \tag{9.7}$$

where

$$\begin{aligned} W^1 &= \alpha_3\alpha_4 y^2 + 2\alpha_4\psi(y, z)yu - \alpha_2\psi(y, z)u^2 + \alpha_3 u^2 + D_2\Phi'(z)u^2 \\ W_2 &= (2\alpha_4 y + \alpha_3 z + \alpha_2 u)p + D_2(\psi u + \alpha_2 z + \alpha_3 y + \alpha_4 x - p)\Phi(z). \end{aligned} \tag{9.8}$$

Note that hypothesis (*ii*) of theorem 3 and (1.10), taken this time in the form $\alpha_4\psi\left(\dfrac{\alpha_2\alpha_3}{\alpha_4} - \psi\right) \geq d_2 + \alpha_3^2 \, (|z| \geq \zeta_0)$ yield

$$0 < \psi < \alpha_2\alpha_3\alpha_4^{-1} \qquad (|z| \geq \zeta_0)$$

and, in view of hypothesis (*iii*) of the theorem, this shows that ψ is bounded for all y and z. Thus, since $|\Phi(z)| \leq 1$, it follows on using (1.6) that

$$|W_2| \leq D(|x| + |y| + |z| + |u| + 1). \tag{9.9}$$

Turning to W_1 which we rearrange thus:

$$W_1 = \alpha_3\alpha_4(y + b^{-1}u\psi)^2 + \alpha_3^{-1}[\alpha_2\alpha_3\psi - \alpha_3^2 - \psi^2 + \alpha_3 D_2\Phi'(z)]u^2$$

let $\mu = \mu(y, z)$ denote the terms inside the square brackets here. Since the definition (9.2) implies that $\Phi'(z) \geq 0$ for all z, and that $\Phi'(z) \geq \frac{\pi}{8}\zeta_0^{-1}\sqrt{2}$ when

$|z| \leq \zeta_0$, it is clear at once from (1.10) that

$$\mu(y, z) \geq d_2 \qquad (|z| \geq \zeta_0)$$

and from hypothesis (*ii*) of theorem 3 that, when $|z| \leq \zeta_0$,

$$\mu(y, z) \geq \frac{\pi}{8} \alpha_3 D_2 \zeta_0^{-1} \sqrt{2} - (\alpha_2 \alpha_3 F + \alpha_3^2 + F^2) = d_2,$$

by (9.4). Hence $\mu(y, z) \geq d_2$ always and thus

$$W_1 \geq \alpha_3 \alpha_4 (y + \alpha_3^{-1} u \psi)^2 + \alpha_3^{-1} d_2 u^2 \tag{9.10}$$

In view of the fact, proved earlier, that ψ is bounded, the usual arguments applied to (9.10) will now give that

$$W_1 \geq D(y^2 + u^2) \tag{9.11}$$

for some sufficiently small D.

Hypothesis (*iv*) of Theorem 3 implies that

$$z \int_0^z \eta \frac{\partial \psi}{\partial y}(y, \eta) d\eta \leq 0,$$

and thus (9.6) follows on combining (9.11) and (9.9) with (9.7).

10. - Explicit form of $V_{8 \cdot 1}$.

The function $V_{8 \cdot 1}$ is defined by

$$V_{8 \cdot 1} = U_0 + U_1 + U_2 \tag{10.1}$$

where U_0 is the function (9.3) and U_1, U_2 are given by

$$U_1 = \begin{pmatrix} -2D_5(\alpha_4^{-1}\alpha_2 + 1)\, y \operatorname{sgn} z, & \text{if } |z| \geq |y| \\ -2D_5(\alpha_4^{-1}\alpha_2 + 1)\, z \operatorname{sgn} y, & \text{if } |y| \geq |z| \end{pmatrix} \tag{10.2}$$

$$U_2 = \begin{pmatrix} 2\alpha_4^{-1} D_5\, u \operatorname{sgn} x, & \text{if } |x| \geq |u| \\ 2\alpha_4^{-1} D_5\, x \operatorname{sgn} u, & \text{if } |u| \geq |x| \end{pmatrix}. \tag{10.3}$$

The constant D_5 here is that which appears in (9.6).

It is easy to verify that $V = V_{8\cdot 1}$ satisfies (3.2). For, from (10.2) and (10.3),

$$|U_1| \leq D|y|, \qquad |U_2| \leq |x|$$

and thus, by (9.5),

$$V_{8\cdot 1} \geq Q_2(x, y, z, u) - D(|x| + |y| + |u| + 1)$$

and the right hand side here tends to $+\infty$ as $x^2 + y^2 + z^2 + u^2 \to \infty$ since Q_2 is a positive definite quadratic form in x, y, z and u.

11. - The verification of (3.4).

Let $(x(t), y(t), z(t), u(t))$ be any solution of (8.1), and let $\dot V_{8\cdot 1}^+$, $\dot U_1^+$, $\dot U_2^+$ be defined as before. Then, by (10.1),

$$\dot V_{8\cdot 1}^+ = \dot U_0 + \dot U_1^+ + \dot U_2^+ \tag{11.1}$$

where $\dot U_0$ satisfies (9.6) and $\dot U_1^+$, $\dot U_2^+$ which, in view of (10.2) and (10.3) are given by

$$\dot U_1^+ = \begin{pmatrix} -2D_5(\alpha_2\alpha_4^{-1} + 1)|z| \text{ if } |z| \geq |y| \\ -2D_5(\alpha_2\alpha_4^{-1} + 1)u \operatorname{sgn} y, \text{ if } |y| \geq |z| \end{pmatrix}$$

$$\dot U_2^+ = \begin{pmatrix} -2D_5|x| - 2\alpha_4^{-1}D_5(\psi u + \alpha_2 z + \alpha_3 y - p)\operatorname{sgn} x, \text{ if } |x| \geq |u| \\ 2\alpha_4^{-1}D_5 y \operatorname{sgn} u, \text{ if } |u| \geq |x| \end{pmatrix}$$

satisfy the inequalities:

$$\dot U_1^+ \leq \begin{pmatrix} -2D_5(\alpha_2\alpha_4^{-1} + 1)|z|, \text{ if } |z| \geq |y| \\ 2D_5(\alpha_2\alpha_4^{-1} + 1)|u|, \text{ if } |y| \geq |z| \end{pmatrix} \tag{11.1}$$

$$\dot U_2^+ \leq \begin{pmatrix} -2D_5|x| + 2\alpha_2\alpha_4^{-1}D_5|z| + D_6(|y| + |u| + 1), \text{ if } |x| \geq |u| \\ 2\alpha_4^{-1}D_5|y|, \text{ if } |u| \geq |x| \end{pmatrix} \tag{11.3}$$

In the estimate (11.3) for $\dot U_2^+$ we have used the result (1.6) as well as the fact, pointed out earlier from the hypotheses, that $|\psi(y, z)| \leq D$.

The rest of the verification from now onwards follows exactly as in § 5 with (9.6), (11.1), (11.2) and (11.3) playing the roles of (4.3), (6.3), (6.2) and (6.1) respectively, and we sketch only the outlines.

We have, for instance that, when $|z| \geq |y|$, $\dot V_{8\cdot 1}^+$ necessarily satisfies one or other of the following inequalities:

$$\dot{V}^{+}_{8\cdot 1} \leq - D_4(y^2 + u^2) + D_5(|x| + |y| + |z| + |u| + 1) +$$
$$+ 2D_5\alpha_2\alpha_4^{-1}|z| - 2(\alpha_2\alpha_4^{-1} + 1)D_5|z| - 2D_5|x| +$$
$$+ D_6(|y| + |u| + 1)$$

(11.4) $$\leq - D_4(y^2 + u^2) + D(|y| + |u| + 1) - D_5(|x| + |z|)$$

$$\dot{V}^{+}_{8\cdot 1} \leq - D_4(y^2 + u^2) + D_5(|x| + |y| + |z| + |u| + 1) +$$
$$+ 2\alpha_4^{-1}D_5|y| - 2D_5(\alpha_2\alpha_4^{-1} + 1)|z|$$

(11.5) $$\leq - D_4(y^2 + u^2) + D_5\{(\alpha_4^{-1} + 1)|y| + 2|u| + 1\}$$
$$- D_5(1 + 2\alpha_2\alpha_4^{-1})|z|,$$

according as $|x| \geq |u|$ or $|x| \leq |u|$. The two results (11.4) and (11.5) are the analogue of (6.4) and (6.5) respectively and they show that when $|z| \geq |y|$

$$\dot{V}^{+}_{8\cdot 1} \leq - D_4(y^2 + u^2) + D(|y| + |u| + 1).$$

As in § 5 this last estimate also holds when $|z| \leq |y|$, so that we now have analogous to (6.7) that there is a constant D_7 such that

(11.6) $$\dot{V}^{+}_{8\cdot 1} \leq -1 \quad \text{if} \quad y^2 + u^2 \geq D_7^2.$$

The results (11.4) and (11.5) also show in the same way as before that there is a constant D_8 such that

(11.7) $$\dot{V}^{+}_{8\cdot 1} \leq -1 \quad \text{if} \quad y^2 + u^2 \leq D_7^2 \quad \text{but} \quad |z| \geq D_8.$$

Suppose however that

$$y^2 + u^2 \leq D_7^2 \quad \text{and} \quad |z| \leq D_8.$$

Then, by (9.6) and (11.2), we have respectively

$$\dot{U}_0 \leq D_5|x| + D \quad \text{and} \quad \dot{U}_1^{+} \leq D.$$

Also, by (11.3), provided that $|x| \geq D_7$,

$$\dot{U}_2^{+} \leq - 2D_5|x| + D.$$

Hence, by (11.1),

$$\dot{V}^{+}_{8\cdot1} \leq -D_5 |x| + D$$
$$\leq -1$$

provided further that $|x|$ is sufficiently large, say $|x| \geq D_9 (\geq D_7)$; that is

(11.8) $\dot{V}^{+}_{8\cdot1} \leq -1$ if $y^2 + u^2 \leq D_7^2$ and $|z| \leq D_8$ provided that $|x| \geq D_9$.

The results (11.6), (11.7) and (11.8) show clearly that

$$\dot{V}^{+}_{8\cdot1} \leq -1 \quad \text{if} \quad x^2 + y^2 + z^2 + u^2 \geq D_7^2 + D_8^2 + D_9^2 .$$

Thus the function $V = V_{8\cdot1}$ also satisfies (3.4), and this completes the verification of Theorem 3.

REFERENCES

[1] J. O. C. EZEILO, J. Math. Anal. Appl. Vol. 5, No. 1, 1962, pp. 136-146.
[2] H. O. TEJUMOLA, Ann. Mat. Pura Appl. (IV), Vol. LXXX, 1968, pp. 177-196.
[3] A. I. OGURCOV, Izv. Vyss Ucebn. Zaved Matematika (1958) no 1(2), pp. 124-129.