Bounding Methods for Facilities Location Algorithms

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BOUNDING METHODS FOR FACILITIES LOCATION ALGORITHMS

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Single and multi-facility location problems are often solved with iterative computational procedures. Although these procedures have been proven to converge, in practice it is desirable to be able to compute a lower bound on the objective function at each iteration. This enables the user to stop the iterative process when the objective function is within a pre-specified tolerance of the optimum value. In this paper we generalize a new bounding method to include multi-facility problems with ℓ_p distances. A proof is given that for Euclidean distance problems the new bounding procedure is superior to two other known methods. Numerical results are given for the three methods.

When continuous facilities location problems are solved using iterative algorithms such as that given by Weizfeld [1937], some criterion must be utilized to decide when to terminate the solution process. The problem of developing such a criterion was originally addressed in the paper by Love and Yeong [1981] which proposes two methods for computing a lower bound on the total cost function of the facilities location problem. The methods are applicable to both single and multi-facility location problems.

Juel [1978] originally developed the second bound and proved that it must always be at least as good as the first bound. Elzinga and Hearn [1982] have also proved the superiority of the second bound. A discussion of the computational merits of the two bounds are given by Love and Yeong [1981]. A third bound is given by Drezner [1981] for the single-facility case with Euclidean distances.

The purpose of this paper is three-fold. First, we generalize Drezner's bound to apply to single and multi-facility problems with ℓ_p distances. Secondly, we compare the three bounding methods using several randomly generated single-facility test problems. Finally, we prove that for Euclidean distances the third bound is superior to the other two for the single-facility case and, under certain conditions, for the multi-facility case.

1. EXISTING BOUNDING PROCEDURES

The single facility location problem is given by:

$$\text{minimize } W(x) = \sum_{j=1}^{n} w_j d(x, a_j) ,$$

where n is the number of existing fixed locations, $a_j = (a_{j1}, a_{j2})$ is the location of the jth fixed facility and $x = (x_1, x_2)$ is the location of the new facility. The information concerning the cost and/or flow between the new facility and the jth existing facility is incorporated into the non-negative weight w_j. The distance between the new facility and the jth existing facility is $d(x,a_j)$. If x^* represents the optimal facility location, the first two bounds are given by:

$$W(x^*) \geq W(x_k) - \bar{\sigma}(x_k) \|\nabla W(x_k)\| \tag{1}$$

and

$$W(x^*) \geq W(x_k) - \nabla W(x_k)' \cdot x_k + \min_{y \in \Omega} [\nabla W(x_k)' \cdot y] \tag{2}$$

where the prime denotes transpose, Ω is the convex hull of the a_j,

$$\bar{\sigma}(x) = \max_{y \in \Omega} d(x,y)$$

and x_k is the value of x at the k^{th} iteration of the solution procedure. For bounds (1) and (2), the case where $x^* = a_j$ is excluded since typically the first step in a solution algorithm would test each existing facility to determine if it represents the optimal solution. The computational procedure for doing this with Euclidean distances is discussed by Love and Yeong [1981], and a generalization of the procedure for ℓ_p distances is given by Juel and Love [1981]. Convergence properties of the multi-facility ℓ_p distance algorithm are given by Morris [1981].

Drezner [1981] shows that

$$W(x^*) \geq \min_{x_1,x_2} \sum_{j=1}^{n} [w_j/d(a_j,x_k)] \cdot [|x_1 - a_{j1}||a_{j1} - x_1^k| + |x_2 - a_{j2}||a_{j2} - x_2^k|]$$

where $x_k = (x_1{}^k, x_2{}^k)$. At each iteration of a solution process this bound (which we will refer to as the rectangular bound) is evaluated by solving a rectilinear distance problem. The rectangular bound is obtained by solving the following problem:

$$\min_{x_1} \sum_{j=1}^{n} w_j' |x_1 - a_{j1}| + \min_{x_2} \sum_{j=1}^{n} w_j'' |x_2 - a_{j2}|$$

where the "created" weights w_j' and w_j'' are

$$w_j' = [w_j / d(a_j, x_k)] \cdot | a_{j1} - x_1^k |$$

and

$$w_j'' = [w_j / d(a_j, x_k)] \cdot |a_{j2} - x_2^k| \quad \text{for } j = 1, \dots, n.$$

2. THE RECTANGULAR BOUND FOR SINGLE FACILITY ℓ_p DISTANCE PROBLEMS

The single facility rectangular bound can be generalized to ℓ_p distances in the following manner, using the Holder inequality which is given by:

$$\sum_{i=1}^{n} b_i \cdot c_i \le \left(\sum_{i=1}^{n} b_i^p\right)^{1/p} \left(\sum_{i=1}^{n} c_i^q\right)^{1/q}$$

where $\{b_n\}$ and $\{c_n\}$ are real sequences and $1/p + 1/q = 1$. Let $b_i = |x_i - a_{ji}|$ and $c_i = |a_{ji} - x_i^k|$ for $i = 1,2$ and $j = 1,\dots,n$. Then

$$\sum_{i=1}^{2} |x_i - a_{ji}|\,|a_{ji} - x_i^k| \le \{[|x_1 - a_{j1}|^p + |x_2 - a_{j2}|^p]^{1/p}\}\{[|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}\}.$$

This can be written as

$$[|x_1 - a_{j1}|^p + |x_2 - a_{j2}|^p]^{1/p} \ge \frac{|x_1 - a_{j1}|\,|a_{j1} - x_1^k| + |x_2 - a_{j2}|\,|a_{j2} - x_2^k|}{[|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}},$$

or

$$\sum_{j=1}^{n} w_j \ell_p(x, a_j) \ge \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \sum_{j=1}^{n} w_j''|x_2 - a_{j2}|,$$

where

$$w_j' = |a_{j1} - x_1^k| \,/ [|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}$$

and

$$w_j'' = |a_{j2} - x_2^k| \,/ [|a_{j1} - x_1^k|^q + |a_{j2} - x_2^k|^q]^{1/q}.$$

Since

$$W(x) \ge \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \sum_{j=1}^{n} w_j''|x_2 - a_{j2}|,$$

then

$$W(x^*) = \min_{x} W(x) \ge \min_{x_1,x_2}\left[\sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \sum_{j=1}^{n} w_j''|x_2 - a_{j2}|\right],$$

or

$$W(x^*) \ge \min_{x_1} \sum_{j=1}^{n} w_j'|x_1 - a_{j1}| + \min_{x_2} \sum_{j=1}^{n} w_j''|x_2 - a_{j2}|.$$

This result can be used to generate the rectangular bound for the single facility ℓ_p distance model. At each iteration of the solution process, a single facility rectilinear problem

$$W(x_R) = \min_{x_1^R} \sum_{j=1}^{n} w_j'|x_1^R - a_{j1}| + \min_{x_2^R} \sum_{j=1}^{n} w_j''|x_2^R - a_{j2}|,$$

is constructed.

using the fixed facilities a_j, weights w_j and current solution x_k to calculate w_j' and w_j'' for $j = 1,\ldots,n$. The two rectangular distance problems can be solved independently and the optimal solution x_R^* can be used to calculate $W(x_R^*)$, which is the lower bound on $W(x^*)$ at this k^{th} iteration.

While it may appear that adding another optimization problem and solving it has increased the work required to find a lower bound, this procedure has several advantages. The rectilinear problem is separable and each part can be solved rapidly. Also, it is not necessary to find the hull points which are used in both the Love-Yeong (1) and Juel (2) bounds.

In order to test the effectiveness and efficiency of the three bounding methods, several single facility test problems were randomly generated. Comparisons and observations are presented in section 3 for these test runs.

3. BOUND COMPARISONS FOR THE SINGLE FACILITY ℓ_p DISTANCE MODEL

Three programs were written to incorporate the Weiszfeld procedure with each of the lower bound methods. At each iteration of the solution procedure the bound was calculated and tested against the current solution. By entering a percentage error difference, e, a stopping rule calculated as

$$\text{(bound value - objective function value)/(objective function)} \leq e$$

was used to terminate the process. In all sample runs a 1% error difference was entered, i.e. $e = 0.01$, and the initial starting solution used in the Weiszfeld procedure was (0,0). Samples of size $n = 6, 10, 15, 20$ existing facilities were randomly generated. In the first set of runs a unit value was assigned to the w_j weights, and ℓ_p distances were calculated for $p = 2, 1.8, 1.6, 1.4, 1.2$.

For a given value of n and p, a series of test runs was made using each of the three programs. For each bounding method the iterations were terminated using the stopping rule with $e = 0.01$. The number of iterations required, the objective function value, the value of the bound, and the CPU compilation and execution times were recorded in each case. The bound values are displayed in Table 1, where L, J, R refer to the Love-Yeong, Juel and rectangular bounds respectively. The average computation times for various sample runs are in Tables 3 and 4.

From Table 1 it is quite evident that for p values of 2 and 1.8 the rectangular bound provided superior results. However, it is also quite evident that as p decreases in value the rectangular bound may not converge. For example, with $n = 6$ and $p = 1.6$ the rectangular bound did not reach the 1% error difference in 25 iterations. The closest it came was at iteration 9 when the error difference was 1.06%. At successive iterations after the ninth, the percentage error difference increased in value. To further study this phenomena, a second set of test samples were created using weights randomly selected from the range [1,10]. For each n and p combination a series of three runs was made and the data was recorded. Then a new set of weights was generated for the next n and p combination. The results for these test runs are shown in Table 2.

The second series of test runs provided data that supported the earlier observations. The instability of the rectangular bound makes its use impractical except for models with p equal or close to two. However, the test results show that for the Euclidean distance model the rectangular bound was always superior to the Juel bound. Also, the rectangular bound is computationally more efficient than the other two bounds. Average computation and execution times for Euclidean distances are shown in Tables 3 and 4 for a CDC Cyber 170/730.

In the following section we prove that the rectangular bound is superior to the Juel bound for the single facility Euclidean distance model. Thus, practitioners using this type of model need not be concerned about the stability of the rectangular bound as it would provide better results than the other two bounds.

n		6			10			15			20		
P		No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound
2	L	8	25.37	25.13	6	168.2	167.5	7	257.6	255.4	6	390.2	386.4
	J	8	25.38	25.20	5	168.2	167.3	7	257.6	255.7	6	390.2	387.1
	R	4	25.41	25.18	3	168.4	167.2	4	257.8	255.9	3	390.6	388.0
1.8	L	10	25.91	25.72	7	172.9	171.2	9	264.8	262.8	7	400.1	397.1
	J	9	25.91	25.69	7	172.9	171.2	8	264.8	262.1	6	400.1	396.1
	R	5	25.93	25.70	5	172.9	171.8	6	264.8	262.2	4	400.2	397.2
1.6	L	12	26.64	26.42	11	179.2	177.9	10	274.3	271.6	8	413.4	410.7
	J	11	26.65	26.40	11	179.2	177.9	10	274.3	271.7	7	413.4	409.6
	R	*			12	179.2	177.5	*			*		
1.4	L	16	27.71	27.45	16	188.2	186.6	13	287.5	284.9	10	432.1	428.5
	J	15	27.71	27.44	16	188.2	186.6	13	287.5	284.9	9	432.1	428.1
	R	*			*			*			*		
1.2	L	25	29.33	29.02	25	201.6	199.3	20	306.7	303.6	15	460.1	455.7
	J	24	29.33	29.03	25	201.6	199.3	18	306.7	303.7	14	460.1	455.9
	R	*			*			*			*		

*did not converge to within 1% error difference in 25 iterations.

Table 1: Lower Bound Data from Single Facility Samples, $w_j = 1$

	n	6			10			15			20		
P		No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound	No. of Iter	Obj Funct	Bound
	L	9	118.6	117.6	11	839.2	833.0	*			6	2504	2480
2	J	8	118.6	117.7	6	839.4	832.9	8	1574	1571	6	2504	2483
	R	5	118.6	117.7	4	840.0	835.1	5	1580	1574	5	2504	2494
	L	10	121.5	120.6	11	858.0	850.0	*			7	2569	2550
1.8	J	8	121.5	120.4	6	858.2	852.2	10	1623	1619	7	2569	2553
	R	6	121.5	120.4	4	858.7	851.1	5	1634	1619	7	2569	2546
	L	11	125.4	124.3	12	883.1	876.3	*			7	2655	2633
1.6	J	9	125.5	124.3	6	883.3	876.6	12	1688	1680	7	2655	2635
	R	*			*			7	1691	1678	*		
	L	13	130.9	129.6	13	918.2	909.1	*			10	2773	2758
1.4	J	11	130.9	129.7	6	918.4	910.3	16	1780	1772	9	2773	2751
	R	*			*			*			*		
	L	14	138.6	137.2	*			*			25	2946	2915
1.2	J	10	138.6	137.2	10	970.3	964.5	22	1914	1896	14	2946	2923
	R	*			*			*			*		

*did not converge to within 1% error difference in 25 iterations.

Table 2: Lower Bound Data for Singe Facility Samples, $w_j \in [1,10]$

Love-Yeong	Bound Juel	Rectangular
1.961	2.028	1.992

Table 3: Average Compilation Time (secs) for Program and Bound, p = 2

n	6	10	15	20
Love-Yeong	0.545	0.699	0.587	0.524
Juel	0.542	0.487	0.628	0.544
Rectangular	0.358	0.421	0.415	0.414

Table 4: Average Execution Time (secs) for Solution and Bound, p = 2

4. COMPARISON OF RECTANGULAR AND JUEL BOUND FOR EUCLIDEAN DISTANCES

The Juel bound at iteration k is given by

$$J(x_k) = f(x_k) - \nabla f(x_k)' \cdot x_k + \min_{y \in \Omega} [\nabla f(x_k)' \cdot y] .$$

For Euclidean distances,

$$f(x_k) = \sum_{j=1}^{n} w_j d_2(x_k, a_j), \quad d_2(x_k, a_j) = \left[(x_1^k - a_{j1})^2 + (x_2^k - a_{j2})^2 \right]^{1/2}$$

and

$$\nabla f(x_k)' = \left(\sum_{j=1}^{n} w_j (x_1^k - a_{j1}) / d_2(x_k, a_j), \quad \sum_{j=1}^{n} w_j (x_2^k - a_{j2}) / d_2(x_k, a_j) \right) .$$

Substituting in the Juel bound gives

$$J(x_k)= \sum_{j=1}^{n} w_j d_2(x_k,a_j) - \sum_{j=1}^{n} w_j(x_1^k - a_{j1})x_1^k/d_2(x_k,a_j)$$

$$- \sum_{j=1}^{n} w_j(x_2^k - a_{j2})x_2^k/d_2(x_k,a_j) + \min_{(y_1,y_2)\in\Omega}\Bigg[\sum_{j=1}^{n} w_j(x_1^k - a_{j1})y_1/d_2(x_k,a_j)$$

$$+ \sum_{j=1}^{n} w_j(x_2^k - a_{j2})y_2/d_2(x_k,a_j)\Bigg] .$$

The rectangular bound at the $k^{\underline{th}}$ iteration is specified by minimizing

$$R(x_k)= \sum_{j=1}^{n} w_j^{'} | x_1 - a_{j1} | + \sum_{j=1}^{n} w_j^{''} | x_2 - a_{j2} |$$

where

$$w_j^{'} = w_j | a_{j1} - x_1^k | / d_2(x_k, a_j) \text{ and } w_j^{''} = w_j | a_{j2} - x_2^k | / d_2(x_k, a_j) .$$

Let $x_R^* = (x_{R1}^*, x_{R2}^*)$ represent the optimal solution obtained by minimizing $R(x_k)$. Since x_R^* is an element of the rectangular hull specified by the existing facility locations, then $x_{R1}^* \in \{a_{11}, a_{21},\ldots,a_{n1}\}$ and $x_{R2}^* \in \{a_{12},a_{22},\ldots,a_{n2}\}$. Properties of the rectangular hull have been discussed by Juel and Love [1981] and Love and Morris [1975].

It will now be shown that at iteration k the rectangular bound is always as good or better than the Juel bound.

THEOREM 1

For $p = 2$, $R(x_R^*) \geq J(x_k)$.

Proof:

$$R(x_R^*)= \sum_{t=1}^{2} \sum_{j=1}^{n} w_j | a_{jt} - x_t^k | \, | x_{Rt}^* - a_{jt} | / d_2(x_k, a_j) ,$$

$$= \sum_{t=1}^{2} \sum_{j=1}^{n} w_j | a_{jt} - x_t^k | \, | x_{Rt}^* - x_t^k + x_t^k - a_{jt} | / d_2(x_k, a_j)$$

$$= \sum_{t=1}^{2} \sum_{j=1}^{n} w_j |a_{jt} - x_t^k| \, | (x_t^k - a_{jt}) - (x_t^k - x_{Rt}^*) | / d_2(x_k, a_j) .$$

From the triangle inequality, $|x-y| \geq |x| - |y|$, and

$$R(x_R^*) \geq \sum_{t=1}^{2} \sum_{j=1}^{n} w_j | a_{jt} - x_t^k | \, | x_t^k - a_{jt} | / d_2(x_k, a_j)$$

$$- \sum_{t=1}^{2} \sum_{j=1}^{n} w_j | x_t^k - a_{jt} | \, | x_t^k - x_{Rt}^* | / d_2(x_k, a_j)$$

Since $|x|\,|y| \geq x \cdot y$, the right hand side can be expanded giving

$$R(x_R^*) \geq \sum_{j=1}^{n} w_j [(x_1^k - a_{j1})^2 + (x_2^k - a_{j2})^2] / d_2(x_k, a_j)$$

$$- \sum_{t=1}^{2} \sum_{j=1}^{n} w_j (x_t^k - a_{jt}) (x_t^k - x_{Rt}^*) / d_2(x_k, a_j)$$

$$= \sum_{j=1}^{n} w_j d_2(x_k, a_j) - \sum_{t=1}^{2} \sum_{j=1}^{n} w_j (x_t^k - a_{jt}) x_t^k / d_2(x_k, a_j)$$

$$+ \sum_{t=1}^{2} \sum_{j=1}^{n} w_j (x_t^k - a_{jt}) x_{Rt}^* / d_2(x_k, a_j) \, .$$

Thus $R(x_R^*) \geq J(x_k)$ if

$$\sum_{t=1}^{2} \sum_{j=1}^{n} w_j (x_t^k - a_{jt}) x_{Rt}^* / d_2(x_k, a_j) \geq \min_{y \in \Omega} \left[\sum_{t=1}^{2} \sum_{j=1}^{n} w_j (x_t^k - a_{jt}) y_t / d_2(x_k, a_j) \right]$$

The point x_R^* is in the rectangular hull, as defined by Love and Morris [1975]. Juel and Love [1983] have proven that the solution to the rectangular distances problem lies in the convex (Euclidean) hull. Therefore, $R(x_R^*) \geq J(x_k)$, with equality holding only when $x_R^* = y^* \in \Omega$.

This establishes that the rectangular bound can be used without any trepidation about its convergence with Euclidean distances. Considerable time can be saved using this bound, as it required fewer iterations to reach the same level of percentage error difference as the other two bounds. From Tables 2 and 4, when $n = 6$ the rectangular bound provided a 40% (approximate) time saving. Over all the test samples, an average execution time saving of 27% was achieved.

5. EXTENSION TO THE MULTI-FACILITY ℓ_p DISTANCE PROBLEM

The multi-facility ℓ_p distance location problem is to minimize

$$WM_p(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}[|x_{i1} - a_{j1}|^p + |x_{i2} - a_{j2}|^p]^{1/p}$$

$$+ \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}[|x_{i1} - x_{r1}|^p + |x_{i2} - x_{r2}|^p]^{1/p}$$

where

m	is the number of new facilities,
n	is the number of existing facilities,
w_{1ij}	is the nonnegative weight which converts the distance between new facility i and existing facility j into cost,
w_{2ir}	is the nonnegative weight which converts the distance between new facilities i and r into cost ($i \neq r$),
$x_i = (x_{i1}, x_{i2})$	are the location coordinates of new facility i,
$a_j = (a_{j1}, a_{j2})$	are the location coordinates of existing facility j.

Let $x_i^k = (x^k_{i1}, x^k_{i2})$ represent the current solution at the k^{th} iteration of the Weiszfeld procedure for the ith new facility location. Using the Holder inequality and substituting

$$b_i = |x_{i1} - a_{j1}||a_{j1} - x_{i1}^k| \quad \text{and} \quad c_i = |x_{i2} - a_{j2}||a_{j2} - x_{i2}^k|$$

then

$$|x_{i1} - a_{j1}||a_{j1} - x_{i1}^k| + |x_{i2} - a_{j2}||a_{j2} - x_{i2}^k| \leq [|x_{i1} - a_{j1}|^p + |x_{i2} - a_{j2}|^p]^{1/p}$$

$$\cdot [|a_{j1} - x_{i1}^k|^q + |a_{j2} - x_{i2}^k|^q]^{1/q} .$$

This can be rewritten as

$$[|x_{i1}-a_{j1}|^p+|x_{i2}-a_{j2}|^p]^{1/p} \geq \frac{|x_{i1}-a_{j1}||a_{j1}-x_{i1}^k|+|x_{i2}-a_{j2}||a_{j2}-x_{i2}^k|}{[|a_{j1}-x_{i1}^k|^q+|a_{j2}-x_{i2}^k|^q]^{1/q}} .$$

By multiplying both sides of the inequality by the nonnegative weights w_{1ij} and summing, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}[|x_{i1}-a_{j1}|^p+|x_{i2}-a_{j2}|^p]^{1/p}$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{w_{1ij}|x_{i1}-a_{j1}||a_{j1}-x_{i1}^k|}{[|a_{j1}-x_{i1}^k|^q+|a_{j2}-x_{i2}^k|^q]^{1/q}}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{w_{1ij}|x_{i2}-a_{j2}||a_{j2}-x_{i2}^k|}{[|a_{j1}-x_{i1}^k|^q+|a_{j2}-x_{i2}^k|^q]^{1/q}} .$$

Let

$$w'_{1ij}=(w_{1ij}|a_{j1}-x_{i1}^k|)/[|a_{j1}-x_{i1}^k|^q+|a_{j2}-x_{i2}^k|^q]^{1/q}$$

and

$$w''_{1ij}=(w_{1ij}|a_{j2}-x_{i2}^k|)/[|a_{j1}-x_{i1}^k|^q+|a_{j2}-x_{i2}^k|^q]^{1/q} ,$$

then

$$\min_{x_i} \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}\ell_p(x_i', a_j) \geq \min_{x_{i1}} \sum_{i=1}^{m}\sum_{j=1}^{n} w'_{1ij}|x_{i1}-a_{j1}|$$

$$+ \min_{x_{i2}} \sum_{i=1}^{m}\sum_{j=1}^{n} w''_{1ij}|x_{i2}-a_{j2}| .$$

For the terms representing the weighted distances between pairs of new facilities, take any new facility x_i and treat the remaining new facility locations x_r ($r \neq i$) as if they were existing facilities (a_j's). The Holder inequality can be used in the same manner as before to obtain

$$\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,[\,|x_{i1}-x_{r1}|^{p}+|x_{i2}-x_{r2}|^{p}\,]^{1/p} \geq \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w'_{2ir}|x_{i1}-x_{r1}|$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w''_{2ir}|x_{i2}-x_{r2}| \ ,$$

where

$$w'_{2ir}=(w_{2ir}|x_{r1}-x^{k}_{i1}|)/[\,|x_{r1}-x^{k}_{i1}|^{q}+|x_{r2}-x^{k}_{i2}|^{q}]^{1/q}$$

and

$$w''_{2ir}=(w_{2ir}|x_{r2}-x^{k}_{i2}|)/[\,|x_{r1}-x^{k}_{i1}|^{q}+|x_{r2}-x^{k}_{i2}|^{q}]^{1/q}$$

Since the x_r facility locations were treated as fixed, x_{r1} and x_{r2} can be replaced by $x^k{}_{r1}$ and $x^k{}_{r2}$ in the calculation of w'_{2ir} and w''_{2ir}.

Combining these two results gives

$$WM_p(x) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} w'_{1ij}|x_{i1}-a_{j1}|+\sum_{i=1}^{m}\sum_{j=1}^{n} w''_{1ij}|x_{i2}-a_{j2}|$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w'_{2ir}|x_{i1}-x_{r1}|+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w''_{2ir}|x_{i2}-x_{r2}|$$

or

$$WM_p(x) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} w'_{1ij}|x_{i1}-a_{j1}|+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w'_{2ir}|x_{i1}-x_{r1}|$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n} w'_{1ij}|x_{i2}-a_{j2}|+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w''_{2ir}|x_{i2}-x_{r2}| \ .$$

The solution at each iteration of the Weiszfeld procedure for the multi-facility problem is used to construct a multi-facility rectilinear model. The optimal solution to the rectilinear model is used to calculate the rectilinear objective function value, which is the lower bound.

6. COMPARISON OF BOUNDS FOR THE MULTI-FACILITY EUCLIDEAN DISTANCE MODEL

The lower bound for the multi-facility ℓ_p model by Love and Yeong [1981] is given by

$$WM_p(x) \geq WM_p(x_k) - \bar{\bar{\sigma}}(x_k)\,\|\nabla WM_p(x_k)\|,$$

where

$$WM_p(x) = \min WM_p(x),$$

$$\bar{\Omega} = \{s = (s_1, s_2, \ldots, s_m) \mid s_i \in \Omega,\ i = 1, \ldots, m\},$$

$$\bar{\bar{\sigma}}(x) = \max\{d(x,y) \mid y \in \bar{\Omega}\},$$

and $x_k = (x^k{}_{11}, x^k{}_{12}, \ldots, x^k{}_{m1}, x^k{}_{m2})$ is a point generated by any procedure at the $k^{\underline{th}}$ iteration.

For the same model, the lower bound by Juel [1978] is

$$WM_p(x) \geq WM_p(x_k) - \nabla WM_p(x_k)' \cdot x_k + \min_{y \in \bar{\Omega}} \{\nabla WM_p(x_k)' \cdot y\}.$$

For $p = 2$, the gradient $\nabla WM_2(x)$ has components $\partial WM_2(x)/\partial x_{it}$, where

$$\frac{\partial WM_2(x)}{\partial x_{it}} = \sum_{j=1}^{n} w_{1ij}(x_{it} - a_{jt})/d_2(x_i, a_j) + \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir}(x_{it} - x_{rt})/d_2(x_i, x_r),$$

$$w_{2ir} = \begin{cases} w_{2ri} & r \neq i,\ r = 1, 2, \ldots, m \\ 0 & r = i \end{cases}$$

for $t = 1,2$, and $i = 1,\ldots,m$.

By substituting for the gradient, the Juel bound for $p = 2$ can be expressed as

$$J(x_k) = WM_2(x_k) - \sum_{t=1}^{2} \left[\sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij}(x^k_{it} - a_{jt})x^k_{it}/d_2(x^k_i, a_j) \right.$$

$$\left. + \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir}(x^k_{it} - x^k_{rt})x^k_{it}/d_2(x^k_i, x^k_r) \right] + \min_{y \in \bar{\Omega}} \{\nabla WM_2(x_k)' \cdot y\}.$$

Since the Juel bound is as good or better than the Love-Yeong bound at each iteration, only the Juel bound need be compared to the multi-facility rectangular bound. It will now be shown that under certain conditions the multi-facility rectangular bound will be better than the Juel bound. Before proceeding with this proof, the following Lemma is required.

LEMMA 1

$$\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_{i1} - x_{r1})x_{i1} - \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir}(x_{i1} - x_{r1})x_{r1}$$

$$= \sum_{i=1}^{m-1} \left[\sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir}(x_{i1} - x_{r1}) \right] x_{i1}$$

Proof:

By expanding the left hand side and grouping like terms and then using the fact that

$w_{2ir} = w_{2ri}$, for $i, r = 1,\ldots,m$ and $i \neq r$, and that $w_{2ii} = 0$, for $i = 1,\ldots m$,

it is easy to see that the resulting expression is equal to the expanded right hand side.

Let x_R^* represent the optimal solution at the kth iteration obtained by solving the multi-facility rectilinear model. Since the optimal solution for each facility location is an element of the single facility rectangular hull then $x_R^* = (x_{11}^*, x_{12}^*, \ldots, x_{m1}^*, x_{m2}^*)$, where

$x_{i1}^* \in \{a_{11}, a_{21}, \ldots, a_{n1}\}$ and $x_{i2}^* \in \{a_{21}, a_{22}, \ldots, a_{2n}\}$, $i = 1,\ldots,m$.

Let

$$R(x_R^*) = \min_x R(x)$$

where

$$R(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} w'_{1ij}|x_{i1} - a_{j1}| + \sum_{i=1}^{m} \sum_{j=1}^{n} w''_{1ij}|x_{i2} - a_{j2}|$$

$$+ \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w'_{2ir}|x_{i1} - x_{r1}| + \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w''_{2ir}|x_{i2} - x_{r2}| .$$

THEOREM 2

For $p=2$,

$$R(x^*_R) \geq J(x_k) \text{ if } \sum_{t=1}^{2}\left[\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}(x^k_{it}-x^k_{rt})(x^k_{it}-x^*_{it})/d_2(x^k_i,x^k_r)\right.$$

$$\left. - \sum_{i=1}^{m-1}\sum_{\substack{r=1\\ r\neq i}}^{m} w_{2ir}(x^k_{it}-x^k_{rt})x^*_{it}/d_2(x^k_i,x^k_r)\right] \geq 0 \ .$$

Proof:

By substituting for w'_{1ij}, w''_{1ij}, w'_{2ir} and w''_{2ir},

$$R(x^*_R) = \sum_{t=1}^{2}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|a_{jt}-x^k_{it}|\,|x^*_{it}-a_{jt}|\,/\,d_2(x^k_i,a_j)\right.$$

$$\left. + \sum_{i=1}^{m-1}\sum_{r=i+1}^{n} w_{2ir}(x_{it}-x_{rt})(x^*_{it}-x^k_{it})/d_2(x^k_i,x^k_r)\right]$$

$$= \sum_{t=1}^{2}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|x^k_{it}-a_{jt}|\,|x^*_{it}-x^k_{it}+x^k_{it}-a_{jt}|\,/|d_2(x^k_i,a_j)\right.$$

$$\left. + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}|x^k_{it}-x^k_{rt}|\,|x^k_{it}-x^k_{rt}+x^k_{rt}-x^*_{it}|\,/\,d_2(x^k_i,x^k_r)\right]$$

$$= \sum_{t=1}^{2}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|x^k_{it}-a_{jt}|\,|(x^k_{it}-a_{jt})-(x^k_{it}-x^*_{it})|\,/\,d_2(x^k_i,a_j)\right.$$

$$\left. + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}|x^k_{it}-x^k_{rt}|\,|(x^k_{it}-x^k_{rt})-(x^*_{it}-x^k_{rt})|\,/\,d_2(x^k_i,x^k_r)\right] .$$

As before, using the inequality $|x-y| \geq |x| - |y|$,

$$R(x_R^*) \geq \sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|x_{it}^k - a_{jt}|^2 / d_2(x_i^k, a_j)$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}|x_{it}^k - a_{jt}|\,|(x_{it}^k - a_{jt}|\,|x_{it}^k - x_{it}^*| / d_2(x_i^k, a_j)$$

$$+ \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}|x_{it}^k - x_{rt}^k|^2 / d_2(x_i^k, x_r^k)$$

$$- \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}|x_{it}^k - x_{rt}^k|\,|x_{it}^* - x_{rt}^k| / d_2(x_i^k, x_r^k)\Big] .$$

Again, using $|x|\,|y| \geq x\cdot y$,

$$R(x_R^*) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij} d(x_i^k, a_j) + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\, d_2(x_i^k, x_r^k)$$

$$- \sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^k - a_{jt})(x_{it}^k - x_{it}^*)| / d_2(x_i^k, a_j)$$

$$+ \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^k - x_{rt}^k)\cdot(x_{it}^* - x_{rt}^k) / d_2(x_i^k, x_r^k)\Big] ,$$

or

$$R(x_R^*) \geq WM_2(x_k) - \sum_{t=1}^{2}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^k - a_{jt})x_{it}^k / d_2(x_i^k, a_j)$$

$$- \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^k - x_{rt}^k)x_{rt}^k / d_2(x_i^k, x_r^k)$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}\,(x_{it}^k - a_{jt})x_{it}^* / d_2(x_i^k, a_j)$$

$$+ \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^k - x_{rt}^k)x_{it}^* / d_2(x_i^k, x_r^k)\Big] .$$

Lemma 1 can now be used to rewrite the expression involving the quantity

$$-\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^{k}-x_{rt}^{k})x_{rt}^{k}.$$

$$R(x_R^{*}) \geq WM_2(x_k) - \sum_{t=1}^{2}\Bigg[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^{k}-a_{jt})x_{it}^{k}/d_2(x_i^{k},a_j)$$

$$+\sum_{i=1}^{m}\sum_{\substack{r=1\\ r\neq i}}^{m} w_{2ir}\,(x_{it}^{k}-x_{rt}^{k})x_{it}^{k}/d_2(x_i^{k},x_r^{k})$$

$$-\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^{k}-x_{rt}^{k})\,x_{it}^{k}/d_2(x_i^{k},x_r^{k})$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^{k}-a_{jt})x_{it}^{*}/d_2(x_i^{k},a_j)$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^{k}-x_{rt}^{k})\,x_{it}^{*}/d_2(x_i^{k},x_r^{k})$$

$$= WM_2(x_k) - \nabla WM_2(x_k)^{T}\cdot x_k + \sum_{t=1}^{2}\Bigg[\sum_{i=1}^{m}\sum_{j=1}^{n} w_{1ij}(x_{it}^{k}-a_{jt})x_{it}^{*}/d_2(x_i^{k},a_j)$$

$$+\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}-x_{rt})\,x_{it}^{k}/d_2(x_i^{k},x_r^{k})$$

$$-\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} w_{2ir}\,(x_{it}^{k}-x_{rt}^{k})\,x_{it}^{*}/d_2(x_i^{k},x_r^{k})\Bigg].$$

By adding and subtracting

$$\sum_{i=1}^{m}\sum_{\substack{r=1\\ r\neq i}}^{m} w_{2ir}\,(x_{it}^{k}-x_{rt}^{k})x_{it}^{*}/d_2(x_i^{k},x_r^{k}),$$

$\nabla WM_2(x_k)' \cdot x_R^*$ can be created, resulting in

$$R(x_k^*) \geq WM_2(x_k) - \nabla WM_2(x_k)' \cdot x_k + \nabla WM_2(x_k)' \cdot x_R^*$$

$$+ \sum_{t=1}^{2} \Bigg[\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) x_{it}^k / d_2(x_i^k, x_r^k)$$

$$- \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) x_{it}^* / d_2(x_i^k, x_r^k)$$

$$- \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) x_{it}^* / d_2(x_i^k, x_r^k) \Bigg] .$$

$$= WM_2(x_k) - \nabla WM_2(x_k)' \cdot x_k + \nabla WM_2(x_k)' \cdot x_R^*$$

$$+ \sum_{t=1}^{2} \Bigg[\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) (x_{it}^k - x_{it}^*) / d_2(x_i^k, x_r^k)$$

$$- \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) x_{it}^* / d_2(x_i^k, x_r^k) \Bigg] .$$

This expression can be compared to the Juel bound,

$$J(x_k) = WM_2(x_k) - \nabla WM_2(x_k)' \cdot x_k + \min_{y \in \bar{\Omega}} \{\nabla WM_2(x_k)' \cdot y\} .$$

Since x_R^* is an element the Euclidean hull $\bar{\Omega}$, then for any value of p

$$\min_{y \in \bar{\Omega}} \{\nabla WM_p(x_k)' \cdot y\} \leq \nabla WM_p(x_k)' \cdot x_R^* .$$

Thus,

$$R(x_R^*) \geq J(x_k) \text{ if } \sum_{t=1}^{2} \Bigg[\sum_{i=1}^{m-1} \sum_{r=i+1}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) (x_{it}^k - x_{it}^*) / d(x_i^k, x_r^k)$$

$$- \sum_{i=1}^{m} \sum_{\substack{r=1 \\ r \neq i}}^{m} w_{2ir} (x_{it}^k - x_{rt}^k) x_{it}^* / d(x_i^k, x_r^k) \Bigg| \geq 0 .$$

The superiority of the rectangular bound for the multi-facility Euclidean distance model is not as decisive as in the single facility case. If the sum of the residual terms involving the w_{2ir} weights is positive, then the rectangular bound is always better than the Juel bound. If this sum is negative, it is possible that the rectangular bound may not be superior to the Juel bound. However, if the difference between

$$\min_{y \in \bar{\Omega}} \{\nabla WM_p(x_k)' \cdot y\} \quad \text{and} \quad \nabla WM_p(x_k)' \cdot x_R^*$$

compensates for this negative sum, the rectangular bound would be as good or better than the Juel bound.

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