



BOUNDING THE CONVEX COMBINATION OF ARITHMETIC AND INTEGRAL MEANS IN TERMS OF ONE-PARAMETER HARMONIC AND GEOMETRIC MEANS

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Abstract. In the article, we find the best possible parameters $\lambda_1, \mu_1, \lambda_2$ and μ_2 on the interval $[0, 1/2]$ such that the double inequalities

$$H(a, b; \lambda_1) < \alpha A(a, b) + (1 - \alpha)T(a, b) < H(a, b; \mu_1),$$

$$G(a, b; \lambda_2) < \alpha A(a, b) + (1 - \alpha)T(a, b) < G(a, b; \mu_2)$$

hold for all $\alpha \in [0, 1]$ and $a, b > 0$ with $a \neq b$, where $A(a, b) = (a + b)/2$, $T(a, b) = 2 \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta / \pi$, $H(a, b; \lambda) = 2[\lambda a + (1 - \lambda)b][\lambda b + (1 - \lambda)a] / (a + b)$, $G(a, b; \mu) = \sqrt{[\mu a + (1 - \mu)b][\mu b + (1 - \mu)a]}$ are the arithmetic, integral, one-parameter harmonic and one-parameter geometric means of a and b , respectively.

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1. INTRODUCTION

For $\lambda, \mu \in [0, 1]$, the arithmetic mean $A(a, b)$, harmonic mean $H(a, b)$, geometric mean $G(a, b)$, integral mean $T(a, b)$ [25], one-parameter harmonic mean $H(a, b; \lambda)$ and one-parameter geometric mean $G(a, b; \mu)$ of two distinct positive real numbers a and b are given by

$$A(a, b) = \frac{a + b}{2}, \quad H(a, b) = \frac{2ab}{a + b}, \quad (1.1)$$

$$G(a, b) = \sqrt{ab}, \quad T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta, \quad (1.2)$$

$$H(a, b; \lambda) = H[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a], \quad (1.3)$$

$$G(a, b; \mu) = G[\mu a + (1 - \mu)b, \mu b + (1 - \mu)a], \quad (1.4)$$

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respectively. The integral mean $T(a, b)$ has been the subject of intensive research in recent years due to it has been widely applied in pure and applied mathematics, physics and other natural sciences [2, 4–12, 14–16, 19, 21, 22, 24, 26–30, 36–38, 40].

The identity

$$T(a, b) = \sqrt{ab} I_0 \left(\log \sqrt{b/a} \right) \quad (1.5)$$

and inequalities

$$L(a, b) < T(a, b) < \frac{A(a, b) + G(a, b)}{2} < \frac{2A(a, b) + G(a, b)}{3} < I(a, b) \quad (1.6)$$

for all $a, b > 0$ with $a \neq b$ were established by Qi, Shi, Liu and Yang [18], where

$$I_\nu(t) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \nu + 1)} \left(\frac{t}{2} \right)^{2n + \nu} \quad (1.7)$$

is the modified Bessel function of the first kind [1], $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the classical gamma function [13, 39], and $L(a, b) = (b-a)/(\log b - \log a)$ and $I(a, b) = (b^b/a^a)^{1/(b-a)}/e$ are respectively the logarithmic and inentric means of a and b .

Yang and Chu [31, 32], and Yang, Chu and Song [33] proved that the inequalities

$$\begin{aligned} \lambda_1 \sqrt{L(a, b)A(a, b)} &< T(a, b) < \mu_1 \sqrt{L(a, b)A(a, b)}, \\ L^{\lambda_2}(a, b)A^{1-\lambda_2}(a, b) &< T(a, b) < \mu_2 L(a, b) + (1 - \mu_2)A(a, b), \\ T(a, b) &> L_p(a, b) \\ \lambda_3 \sqrt{L(a, b)I(a, b)} &< T(a, b) < \mu_3 \sqrt{L(a, b)I(a, b)}, \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq \sqrt{2/\pi}$, $\mu_1 \geq 1$, $\lambda_2 \geq 3/4$, $\mu_2 \leq 3/4$, $p \leq 3/2$, $\lambda_3 \leq \sqrt{e/\pi}$ and $\mu_3 \geq 1$, where $L_p(a, b) = [(b^p - a^p)/(p(\log b - \log a))]^{1/p}$ is the p -order generalized logarithmic mean of a and b .

In [20], the authors proved that $p_1 = 0$, $q_1 = 1/4$, $p_2 = 0$ and $q_2 = 1/2 - \sqrt{2}/4$ are the best possible parameters on the interval $[0, 1/2]$ such that the double inequalities

$$H(a, b; p_1) < T(a, b) < H(a, b; q_1), \quad (1.8)$$

$$G(a, b; p_2) < T(a, b) < G(a, b; q_2) \quad (1.9)$$

hold for all $a, b > 0$ with $a \neq b$.

Let $\alpha \in [0, 1]$, $x \in [0, 1/2]$, $a, b > 0$ with $a \neq b$, $f(x) = H(a, b; x)$, $g(x) = G(a, b; x)$. Then we clearly see that both the functions $f(x)$ and $g(x)$ are strictly increasing on $[0, 1/2]$. Inequality (1.6) and the well known inequalities

$$H(a, b) < G(a, b) < L(a, b) < I(a, b) < A(a, b)$$

lead to the conclusion that

$$f(0) = H(a, b) < \alpha A(a, b) + (1 - \alpha)T(a, b) < A(a, b) = f(1/2), \quad (1.10)$$

$$g(0) = G(a, b) < \alpha A(a, b) + (1 - \alpha)T(a, b) < A(a, b) = g(1/2). \quad (1.11)$$

From inequalities (1.10) and (1.11) together with the monotonicity of the functions $f(x)$ and $g(x)$ on the interval $[0, 1/2]$, it is necessary to discover the best possible parameters $\lambda_1, \mu_1, \lambda_2$ and μ_2 on the interval $[0, 1/2]$ such that the double inequalities

$$H(a, b; \lambda_1) < \alpha A(a, b) + (1 - \alpha)T(a, b) < H(a, b; \mu_1),$$

$$G(a, b; \lambda_2) < \alpha A(a, b) + (1 - \alpha)T(a, b) < G(a, b; \mu_2)$$

hold for all $\alpha \in [0, 1]$ and $a, b > 0$ with $a \neq b$.

2. LEMMAS

Lemma 1 (Theorem 2.18 in [3]). *The identity*

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!} \sqrt{\pi}$$

holds for all $n \in \mathbb{N}$.

Lemma 2 ([17]). *Let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be two real sequences with $b_n > 0$ and $\lim_{n \rightarrow \infty} a_n/b_n = s$. Then the power series $\sum_{n=0}^\infty a_n t^n$ is convergent for all $t \in \mathbb{R}$ and*

$$\lim_{t \rightarrow \infty} \frac{\sum_{n=0}^\infty a_n t^n}{\sum_{n=0}^\infty b_n t^n} = s$$

if the power series $\sum_{n=0}^\infty b_n t^n$ is convergent for all $t \in \mathbb{R}$.

Lemma 3 (Lemma 2.2 in [35]). *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}$$

holds for all $x > 0$ and $a \in (0, 1)$.

Lemma 4 ([34]). *Let $A(t) = \sum_{k=0}^\infty a_k t^k$ and $B(t) = \sum_{k=0}^\infty b_k t^k$ be two real power series converging on $(-r, r)$ ($r > 0$) with $b_k > 0$ for all k . If the non-constant sequence $\{a_k/b_k\}_{k=0}^\infty$ is increasing (decreasing) for all k , then the function $t \mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0, r)$.*

Lemma 5 ((3.5) in [23]). *The identity*

$$I_\lambda(t)I_\mu(t) = \sum_{n=0}^\infty \frac{\Gamma(2n + \lambda + \mu + 1)}{n! \Gamma(n + \lambda + \mu + 1) \Gamma(n + \lambda + 1) \Gamma(n + \mu + 1)} \left(\frac{t}{2}\right)^{2n + \lambda + \mu}$$

holds for all $\lambda, \mu > -1$ and $t \in \mathbb{R}$.

Lemma 6. *The identity*

$$\cosh(t)I_0(t) = \sum_{n=0}^\infty \frac{(4n)!}{2^{2n} [(2n)!]^3} t^{2n}$$

holds for all $t \in \mathbb{R}$, where $\cosh(t) = (e^t + e^{-t})/2$ is the hyperbolic cosine functions.

Proof. It follows from (1.7) and Lemmas 1 and 5 that

$$\begin{aligned} I_{-1/2}(t) &= \sqrt{\frac{2}{\pi t}} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \sqrt{\frac{2}{\pi t}} \cosh(t), \\ \cosh(t)I_0(t) &= \sqrt{\frac{\pi t}{2}} I_{-1/2}(t)I_0(t) \\ &= \sqrt{\frac{\pi t}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(2n + \frac{1}{2})}{[n!\Gamma(n + \frac{1}{2})]^2} \left(\frac{t}{2}\right)^{2n-1/2} = \sum_{n=0}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n}. \end{aligned}$$

□

Lemma 7. *The function*

$$f(t) = \frac{\cosh^2(t) - \cosh(t)I_0(t)}{\sinh^2(t)} \quad (2.1)$$

is strictly increasing from $(0, \infty)$ onto $(1/4, 1)$, where $\sinh(t) = (e^t - e^{-t})/2$ is the hyperbolic sine function.

Proof. Let $n \in \mathbb{N}$, and $\{a_n\}$ and $\{b_n\}$ be defined by

$$a_n = \frac{(4n+4)!}{2^{2n+2}[(2n+2)!]^3}, \quad b_n = \frac{2^{2n+2}}{(2n+2)!}, \quad (2.2)$$

respectively.

Then simple computations lead to

$$\frac{a_0}{b_0} = \frac{3}{8}, \quad (2.3)$$

$$\frac{a_n}{b_n} = \frac{(4n+4)!}{2^{4n+4}[(2n+2)!]^2}, \quad (2.4)$$

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{(n+2)(2n+3)(8n+13)(4n+4)!}{2^{4n+5}[(2n+4)!]^2} < 0 \quad (2.5)$$

for all $n \in \mathbb{N}$.

It follows from Lemmas 1, 2, 3 and 6 together with (2.1)-(2.4) that

$$\begin{aligned} f(t) &= 1 - \frac{2[\cosh(t)I_0(t) - 1]}{\cosh(2t) - 1} \\ &= 1 - \frac{2 \sum_{n=1}^{\infty} \frac{(4n)!}{2^{2n}[(2n)!]^3} t^{2n}}{\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n}} = 1 - \frac{2 \sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}}, \end{aligned} \quad (2.6)$$

$$f(0^+) = 1 - \frac{2a_0}{b_0} = \frac{1}{4}, \quad (2.7)$$

$$\frac{1}{\sqrt{\pi(2n+5/2)}} < \frac{a_n}{b_n} = \frac{\Gamma(2n+5/2)}{\sqrt{\pi}\Gamma(2n+3)} < \frac{1}{\sqrt{\pi(2n+2)}},$$

$$f(\infty) = 1 - 2 \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1. \tag{2.8}$$

From Lemma 4, (2.5) and (2.6) we clearly see that the function $f(t)$ is strictly increasing on $(0, \infty)$. Therefore, Lemma 7 follows from (2.7) and (2.8) together with the monotonicity of $f(t)$ on the interval $(0, \infty)$. \square

Lemma 8. *The function*

$$g(t) = \frac{\cosh^2(t) - I_0^2(t)}{\sinh^2(t)} \tag{2.9}$$

is strictly increasing from $(0, \infty)$ onto $(1/2, 1)$.

Proof. Let $n \in \mathbb{N}$, and $\{c_n\}$ and $\{d_n\}$ be defined by

$$c_n = \frac{(2n+2)!}{2^{2n+2}[(n+1)!]^4}, \quad d_n = \frac{2^{2n+2}}{(2n+2)!}, \tag{2.10}$$

respectively. Then simple computations lead to

$$\frac{c_0}{d_0} = \frac{1}{4}, \tag{2.11}$$

$$\frac{c_n}{d_n} = \frac{[(2n+2)!]^2}{2^{4n+4}[(n+1)!]^4}, \tag{2.12}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{(4n+7)(n+2)^2[(2n+2)!]^2}{2^{4n+6}[(n+1)!]^4} < 0 \tag{2.13}$$

for all $n \in \mathbb{N}$.

From Lemmas 1, 2, 3 and 5 together with (2.9)-(2.12) one has

$$I_0^2(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n},$$

$$g(t) = 1 - \frac{2[I_0^2(t) - 1]}{\cosh(2t) - 1}$$

$$= 1 - \frac{2 \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n}}{\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n}} = 1 - \frac{2 \sum_{n=0}^{\infty} c_n t^{2n}}{\sum_{n=0}^{\infty} d_n t^{2n}}, \tag{2.14}$$

$$g(0^+) = 1 - \frac{2c_0}{d_0} = \frac{1}{2}, \tag{2.15}$$

$$\frac{1}{\pi(n+3/2)} < \frac{c_n}{d_n} = \frac{\Gamma^2(n+3/2)}{\pi\Gamma^2(n+2)} < \frac{1}{\pi(n+1)},$$

$$g(\infty) = 1 - 2 \lim_{n \rightarrow \infty} \frac{c_n}{d_n} = 1. \quad (2.16)$$

Therefore, Lemma 8 follows easily from Lemma 4 and (2.13)-(2.16). \square

3. MAIN RESULTS

Theorem 1. Let $\lambda_1, \mu_1, \lambda_2, \mu_2 \in [0, 1/2]$. Then the double inequalities

$$H(a, b; \lambda_1) < \alpha A(a, b) + (1 - \alpha)T(a, b) < H(a, b; \mu_1), \quad (3.1)$$

$$G(a, b; \lambda_2) < \alpha A(a, b) + (1 - \alpha)T(a, b) < G(a, b; \mu_2) \quad (3.2)$$

hold for all $\alpha \in [0, 1]$ and $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/2 - \sqrt{1 - \alpha}/2$, $\mu_1 \geq 1/2 - \sqrt{1 - \alpha}/4$, $\lambda_2 \leq 1/2 - \sqrt{1 - \alpha^2}/2$ and $\mu_2 \geq 1/2 - \sqrt{2(1 - \alpha)}/4$.

Proof. Let $p, q \in [0, 1/2]$. Without loss of generality, we assume that $b > a > 0$ and $t = \log \sqrt{b/a} > 0$ due to $A(a, b)$, $T(a, b)$, $H(a, b; p)$ and $G(a, b; q)$ are symmetric and homogeneous of degree one with respect to a and b . From (1.1)-(1.5) one has

$$A(a, b) = \sqrt{ab} \cosh(t),$$

$$T(a, b) = \sqrt{ab} I_0(t),$$

$$H(a, b; p) = \sqrt{ab} \cosh(t) \left[1 - (1 - 2p)^2 \frac{\sinh^2(t)}{\cosh^2(t)} \right],$$

$$G(a, b; q) = \sqrt{ab} \cosh(t) \sqrt{1 - (1 - 2q)^2 \frac{\sinh^2(t)}{\cosh^2(t)}},$$

$$H(a, b; p) - [\alpha A(a, b) + (1 - \alpha)T(a, b)] = \frac{\sqrt{ab} \sinh^2(t)}{\cosh(t)} [(1 - \alpha)f(t) - (1 - 2p)^2], \quad (3.3)$$

$$\begin{aligned} & G(a, b; q) - [\alpha A(a, b) + (1 - \alpha)T(a, b)] \\ &= \frac{\sqrt{ab} [(1 - \alpha)^2 (\cosh^2(t) - I_0^2(t)) + 2\alpha(1 - \alpha)(\cosh^2(t) - \cosh(t)I_0(t)) - (1 - 2q)^2]}{\cosh(t) \sqrt{1 - (1 - 2q)^2 \frac{\sinh^2(t)}{\cosh^2(t)} + \alpha \cosh(t) + (1 - \alpha)I_0(t)}} \\ &= \frac{\sqrt{ab} \sinh^2(t) [2\alpha(1 - \alpha)f(t) + (1 - \alpha)^2 g(t) - (1 - 2q)^2]}{\cosh(t) \sqrt{1 - (1 - 2q)^2 \frac{\sinh^2(t)}{\cosh^2(t)} + \alpha \cosh(t) + (1 - \alpha)I_0(t)}}, \end{aligned} \quad (3.4)$$

where $f(t)$ and $g(t)$ are defined by Lemmas 7 and 8, respectively.

Therefore, Theorem 1 follows easily from (3.3) and (3.4) together with Lemmas 7 and 8. \square

Remark 1. Let $\alpha = 0$. Then inequalities (3.1) and (3.2) reduce to inequalities (1.8) and (1.9), respectively.

Corollary 1. Let $\lambda_1 = 1/2 - \sqrt{1-\alpha}/2$ and $\mu_1 = 1/2 - \sqrt{1-\alpha}/4$. Then inequality (3.1) leads to

$$\frac{1}{\cosh(t)} < I_0(t) < \frac{3 \cosh(t)}{4} + \frac{1}{4 \cosh(t)}$$

for all $t > 0$.

Corollary 2. Let $\lambda_2 = 1/2 - \sqrt{1-\alpha^2}/2$ and $\mu_2 = 1/2 - \sqrt{2(1-\alpha)}/4$. Then inequality (3.2) leads to

$$\frac{\sqrt{1-\alpha^2 + \alpha^2 \cosh^2(t)} - \alpha \cosh(t)}{1-\alpha} < I_0(t) < \frac{\sqrt{\frac{1-\alpha}{2} + \frac{1+\alpha}{2} \cosh^2(t)} - \alpha \cosh(t)}{1-\alpha}$$

for all $\alpha \in (0, 1)$ and $t > 0$. In particular, if $\alpha = 1/2$, then one has

$$\sqrt{3 + \cosh^2(t)} - \cosh(t) < I_0(t) < \sqrt{1 + 3 \cosh^2(t)} - \cosh(t)$$

for all $t > 0$.

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