BOUNDS FOR DISTRIBUTIONS WITH MONOTONE HAZARD RATE, I

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1. Introduction. If F is a probability distribution such that $F(0-) = 0$ and $\int_0^\infty x^r \, dF(x) = \mu_r < \infty$, and if $r, t > 0$, then according to Markov's inequality,

$$(1.1) \qquad \begin{aligned} 0 \leqq 1 - F(t) &\leqq \mu^r/t^r, & t \geqq \mu_r \\ &\leqq 1, & t \leqq \mu_r . \end{aligned}$$

This inequality is known to be sharp; indeed, for each positive r and t there exist distributions satisfying the conditions of (1.1) and attaining equality.

A number of improvements of (1.1) have been obtained under additional assumptions about the distribution F. Perhaps the most notable of these is the result of Gauss (1821) which applies in case $1 - F(x)$ is convex in $x \geqq 0$, and predates any version of (1.1). Hypotheses similar to that of Gauss have been used by a number of authors to obtain improvements; much of this work has been summarized by Fréchet (1950). Improvements of the classical bounds were studied by Mallows (1956) under restrictions on the number of sign changes of some derivative of the distribution, and also with restrictions on the size of the derivative. This work extends the result of Gauss as well as that of Markov (1898) which utilized bounds on the density. Recently, Mallows (1963) has utilized the methods and results of Krein (1951) to extend his earlier work, and has obtained inequalities on distributions having n specified moments and whose first s derivatives satisfy certain boundedness and sign change conditions.

In this paper, we obtain sharp upper and lower bounds for $1 - F(t)$ under a variety of conditions, particularly that the hazard rate is monotone. These conditions are of interest for two reasons: First, they are sufficient to yield quite striking improvements of (1.1), and second, they are natural to many situations in life testing, reliability, actuarial science, and other areas of statistical interest.

A distribution F is said to have increasing (decreasing) hazard rate, denoted by IHR(DHR), if $\log[1 - F(x)]$ is concave where finite (convex on $[0, \infty)$). If F has a density f, then the ratio $q(x) = f(x)/[1 - F(x)]$ is defined for $F(x) < 1$, and is called the *hazard rate*. It is easily seen that $\log[1 - F(x)]$ is concave (convex) in $x \geqq 0$ if and only if there exists a version of the density f for which $q(x)$ is increasing (decreasing) in $x \geqq 0$.

The practical interest of the hazard rate derives from its probabilistic interpretation: If F is a life distribution, then $q(x)\,dx$ may be regarded as the conditional probability of death in $(x, x + dx)$ given survival to age x.

The property of monotone hazard rate is connected with the theory of total

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Received 20 June 1963; revised 30 April 1964.

[1] Work done under the auspices of Boeing Scientific Research Laboratories.

positivity in the following way. A distribution F is IHR if and only if $1 - F(x - y)$ is totally positive of order 2 in real x and y (see Schoenberg (1951) for a definition of terms). A distribution F is DHR if and only if $1 - F(x + y)$ is totally positive of order 2 in $x + y \geqq 0$. Properties of distributions with monotone hazard rate have been investigated by Barlow, Marshall and Proschan (1963).

We pay particular attention to the question of sharpness of the inequalities given, and to the conditions for equality. Examples attaining equality serve not only to prove sharpness, but also indicate what stronger assumptions may yield a further improvement of the inequality. For if a property is enjoyed by a distribution attaining equality, then the assumption of that property cannot result in further improvement. Where uniqueness of a distribution attaining equality can be shown, then of course strict inequality holds in all other cases.

The statement of (1.1) for $r > 0$ is in reality no more general than its statement for $r = 1$. This is because of the fact that for $r = 1$, (1.1) may be written in the form $P\{X \geqq t\} \leqq \mu/t$ where $\mu = E(X)$. With $X = Y^r$, one then obtains (1.1) for arbitrary $r > 0$. The results of this paper cannot be so simply extended, because the property of monotone hazard rate need not be preserved under a transformation of the form $X = Y^r$.

Throughout this paper we assume unless otherwise stated that distributions are left continuous.

2. Methods of proof. If X is a random variable satisfying $P\{X \varepsilon I\} = 1$ and certain moments of X are known, there is a standard method for obtaining a sharp upper bound for the probability that X lies in some specified set $\mathfrak{J} \subset I$. If $\mathfrak{A}$ is the class of polynomials $h(x) = \sum a_j x^j$ where (i) $a_j = 0$ unless the jth moment of X is known, and (ii) $h(x)$ dominates the characteristic (indicator) function of $\mathfrak{J}$ on I, then

$$P\{X \varepsilon \mathfrak{J}\} \leqq \inf_{\mathfrak{A}} Eh(X) \tag{2.1}$$

(see Marshall and Olkin (1961) for a more general discussion). The usual proof of Markov's inequality (1.1) is of this form where the minimizing polynomial is x^r/t^r.

This proof of Markov's inequality does not seem adaptable to the case in which other kinds of information are available about the distribution F. We consider an alternate proof based upon the following lemma: *If ζ is an increasing function on* $[0, \infty)$ *and* G_1, G_2 *are probability distributions satisfying* $G_1(x) \leqq G_2(x)$ *for all* x, *then*

$$\int_{0-}^{\infty} \zeta(x)\, dG_1(x) \geqq \int_{0-}^{\infty} \zeta(x)\, dG_2(x). \tag{2.2}$$

To apply this, observe that

$$\begin{aligned} 1 - F(x) \geqq 1 - G(x) &= 1, & x < 0 \\ &= 1 - F(t), & 0 \leqq x \leqq t \\ &= 0, & x > t. \end{aligned} \tag{2.3}$$

Then since x^r/t^r is increasing in x,

$$\frac{\mu_r}{t^r} = \int_{0-}^{\infty} \frac{x^r}{t^r}\, dF(x) \geqq \int_{0-}^{\infty} \frac{x^r}{t^r}\, dG(x) = 1 - F(t).$$

Some kinds of information about F readily yield a sharpening of (2.3) with consequent improvement of (1.1), and we illustrate with two simple examples.

Example 2.1. If $F(x)$ is convex in $(0, t)$, then

$$\begin{aligned} 1 - F(x) &\geqq 1 - xF(t)/t, & x &\leqq t \\ &\geqq 0, & x &> t. \end{aligned}$$

Using this, one obtains

$$(2.4) \qquad 1 - F(t) \leqq \mu_r/t^r - (1/r)(1 - \mu_r/t^r),$$

an improvement of (1.1) due to Narumi (1923).

Example 2.2. If $1 - F(x)$ is convex, $x \geqq 0$ (e.g., if F is the distribution of a random variable $X = |Y|$ where Y has a density with unique mode at 0), then $1 - F(x)$ has a supporting line at $t > 0$, so that there exists $\alpha \leqq 1$ such that

$$(2.5) \quad \begin{aligned} 1 - F(x) &\geqq 1, & x &< 0 \\ &\geqq \alpha + [1 - F(x) - \alpha]x/t, & 0 \leqq x &< \alpha t/[\alpha - 1 + F(t)] \\ &\geqq 0, & x &\geqq \alpha t/[\alpha - 1 + F(t)]. \end{aligned}$$

Thus for some $\alpha \leqq 1$, $\mu_r \geqq (\alpha t)^{r+1}/(r + 1)t(\alpha - 1 + F(t))^r$, or

$$(2.6) \qquad 1 - F(t) \leqq \alpha - \alpha^{1+1/r}t/(r + 1)^{1/r}\mu_r^{1/r} \equiv \varphi(\alpha).$$

Though we have no way of obtaining α to satisfy (2.5), we do obtain a valid bound by maximizing $\varphi(\alpha)$ for $\alpha \leqq 1$. This maximum occurs at $\alpha = r\mu_r^{1/r}/t(r + 1)^{1-1/r}$ if $t \geqq r\mu_r^{1/r}/(r + 1)^{1-1/r} = t_0$, and at $\alpha = 1$ if $t \leqq t_0$. Thus

$$(2.7) \quad \begin{aligned} 1 - F(t) &\leqq 1 - t/(r + 1)^{1/r}\mu_r^{1/r}, & t &\leqq t_0 \\ &\leqq [\mu_r/t^r][r/(r + 1)]^r & t &\geqq t_0 . \end{aligned}$$

This result was partially obtained by Camp (1922) and Meidell (1922) and has been given by Fréchet (1950). For $r = 2$, it is essentially equivalent to Gauss' result of 1821, and the method of the above proof is due to Gauss.

The method of Example 2.2 has the disadvantage of providing no inequality unless the problem of maximizing φ can be solved; this is in contrast to the method utilizing (2.1), where a valid bound is provided by any h satisfying (i) and (ii).

We mention two other useful methods. Inequality (3.8) can be obtained by an application of Jensen's inequality, as can (2.7) in case $t \leqq t_0$. Finally, we give another proof of (1.1) which, suitably modified, yields a simple proof of Theorem 3.9. The distribution G defined by

$$1 - G(x) = 1, \qquad x < 0$$

$$= \mu_r/t^r, \qquad 0 \leqq x < t$$
$$= 0, \qquad x \geqq t$$

has rth moment $\mu_r = \int_0^\infty x^r \, dF(x)$. Hence F and G must cross at least once; such a crossing can occur only in the interval $(0, t)$, and thus $1 - F(t) \leqq 1 - G(t) = \mu_r/t^r$. The ideas of this proof are also useful in the following companion paper (Barlow and Marshall (1964)), where more than one moment is known.

3. Bounds for $1 - F$ when F has monotone hazard rate. We introduce this section with some general lemmas that are later applied to obtain more specific results.

Let $t > 0$,

$$1 - G_{z;w}(x) = 1, \qquad x \leqq z,$$
$$= w^{(x-z)/(t-z)}, \qquad x \geqq z,$$

and let $1 - G_z(x) = 1 - G_{z;1-F(t)}(x)$.

LEMMA 3.1. *Let F be* IHR, $F(0) = 0$. *Let ζ be a function strictly increasing on* $[0, \infty)$ *such that* $\int_0^\infty \zeta(x) \, dF(x) = \nu$ *exists finitely. Then*

$$\psi(w) = \sup_{0 \leqq z \leqq t} \int_0^\infty \zeta(x) \, dG_{z;w}(x)$$

is strictly increasing, and if $\psi(1 - F(t)) < \infty$,

$$(3.1) \qquad 1 - F(t) \geqq \psi^{-1}(\nu), \qquad \zeta(t) \leqq \nu$$
$$\geqq 0, \qquad \zeta(t) > \nu,$$

where $\psi^{-1}(\nu) = \sup \{w : \psi(w) \leqq \nu\}$.

PROOF. Note that $G_{z;w}(x)$ is decreasing in w for fixed x and z. Since ζ is strictly increasing, this means $\int_0^\infty \zeta(x) \, dG_{z;w}(x)$ is strictly increasing in w, so $\psi(w)$ is strictly increasing. Since $\psi(0) = \zeta(t)$, $\psi^{-1}(\nu)$ is defined when $\zeta(t) \leqq \nu$.

Since $\log[1 - F(x)]$ is concave, there exists z_0, $0 \leqq z_0 \leqq t$ such that $F(x) \geqq G_{z_0}(x)$ for all x. Since ζ is increasing,

$$(3.2) \qquad \nu = \int_0^\infty \zeta(x) \, dF(x) \leqq \int_0^\infty \zeta(x) \, dG_{z_0}(x)$$
$$\leqq \sup_{0 \leqq z \leqq t} \int_0^\infty \zeta(x) \, dG_z(x) = \psi(1 - F(t))$$

and (3.1) follows.||

Note that no use was made of the condition $F(0) = 0$ other than to confine z_0 to $[0, t]$ rather than $(-\infty, t]$.

LEMMA 3.1$'$. *If $\zeta(t) \leqq \nu$ and ψ is continuous at ν, equality is attained in* (3.1) *uniquely by the distribution $G_{z^*;\psi^{-1}(\nu)}(x)$, where z^* is defined by*

$$\int_0^\infty \zeta(x) \, dG_{z^*;\psi^{-1}(\nu)}(x) = \sup_{0 \leqq z \leqq t} \int_0^\infty \zeta(x) \, dG_{z;\psi^{-1}(\nu)}(x).$$

If $\zeta(t) > \nu$ and $\zeta(s) = \nu$ has a solution, then, e.g., the distribution degenerate at s achieves equality.

PROOF. If $\zeta(t) \leqq \nu$, then $\psi^{-1}(\nu)$ exists. Since ψ is continuous at ν,

$$\nu = \psi(\psi^{-1}(\nu)) = \sup_{0 \leqq z \leqq t} \int_0^\infty \zeta(x)\, dG_{z;\psi^{-1}(\nu)}(x)$$

$$= \int_0^\infty \zeta(x)\, dG_{z^*;\psi^{-1}(\nu)}(x) = \psi[1 - G_{z^*;\psi^{-1}(\nu)}(t)],$$

so that the hypotheses of Lemma 3.1 are satisfied when $F = G_{z^*;\psi^{-1}(\nu)}$ and equality is attained. Uniqueness follows from the fact that equality must hold in (3.2) if it holds in (3.1).||

LEMMA 3.2. *If the conditions of Lemma 3.1 are satisfied and if, in addition, ζ is convex, then $\psi(1 - F(t)) = \int_0^\infty \zeta(x)\, dG_0(x)$ whenever $\zeta(t) \leqq \nu$.*

PROOF.

$$\int_0^\infty \zeta(x)\, dG_z(x) = -\frac{L}{t - z} \int_z^\infty \zeta(x) \exp\left(\frac{x - z}{t - z} L\right) dx$$

$$= -L \int_0^\infty \zeta(z(1 - y) + ty) e^{yL}\, dy \equiv \varphi(z),$$

where $L = \log[1 - F(t)]$. Since ζ is convex, φ is also convex, and $\sup_{0 \leqq z \leqq t} \varphi(z) = \varphi(0)$ or $\varphi(t)$. If $\zeta(t) < \nu$, then $\sup_{0 \leqq z \leqq t} \varphi(z) = \varphi(t)$ implies by (3.2) that $\psi(1 - F(t)) = \varphi(t) = \zeta(t) \geqq \nu$, a contradiction, so that $\psi(1 - F(t)) = \sup_{0 \leqq z \leqq t} \varphi(z) = \varphi(0) = \int_0^\infty \zeta(x)\, dG_0(x)$. If $\zeta(t) = \nu$, the result follows by limiting arguments.||

From (3.2) and Lemma 3.2, it follows that if ζ is strictly increasing and convex on $[0, \infty)$, and if $\zeta(t) \leqq \nu$, then

$$\nu = \int_0^\infty \zeta(x)\, dF(x) \leqq \int_0^\infty \zeta(x) \omega e^{-\omega x}\, dx$$

where $\omega = -t^{-1} \log(1 - F(t))$. This inequality is to be compared with the inequality

$$\int_0^\infty \zeta(x)\, dF(x) \leqq \int_0^\infty \zeta(x) \cdot \frac{1}{\mu_1} e^{-x/\mu_1}\, dx \tag{3.3}$$

where $\mu_1 = \int_0^\infty x\, dF(x)$ and ζ need only be convex. Inequality (3.3) follows from an integration by parts and the fact (Karlin, Proschan and Barlow (1961)) that $1 - F(x)$ crosses e^{-x/μ_1} exactly once, the crossing being from above. Inequality (3.3) is due to Karlin and Novikoff (1963).

LEMMA 3.3. *Let F be* IHR, $F(0) = 0$. *Let ζ be a function strictly decreasing on $[0, \infty)$ such that $\int_0^\infty \zeta(x)\, dF(x) = \nu$ exists finitely. Then $\nu \geqq \inf_{0 \leqq z \leqq t} \int_0^\infty \zeta(x)\, dG_z(x)$,*

$$\psi(w) = \inf_{0 \leqq z \leqq t} \int_0^\infty \zeta(x)\, dG_{z;w}(x)$$

is strictly decreasing, and

$$(3.4) \qquad \begin{aligned} 1 - F(t) &\geqq \psi^{-1}(\nu), & \zeta(t) &\geqq \nu \\ &\geqq 0, & \zeta(t) &< \nu, \end{aligned}$$

where $\psi^{-1}(\nu) = \inf \{w : \psi(w) \leqq \nu\}$.

The proof of Lemma 3.3 is essentially the same as the proof of Lemma 3.1, and will be omitted. The obvious analogs of Lemma 3.1′ and of Lemma 3.2 with concavity replacing convexity are also omitted. Let

$$\begin{aligned} 1 - H_a(x) &= e^{-ax}, & x &\leqq t \\ &= 0, & x &> t. \end{aligned}$$

LEMMA 3.4. *Let F be* IHR, $F(0) = 0$, *and let ζ be a function strictly increasing on* $[0, \infty)$ *such that* $\int_0^\infty \zeta(x)\, dF(x) = \nu$ *exists finitely. Then* $\nu = \int_0^\infty \zeta(x)\, dH_a(x)$ *has a solution a_0 if and only if* $\nu \leqq \zeta(t)$; *in this case, a_0 is unique, and*

$$(3.5) \qquad \begin{aligned} 1 - F(t) &\leqq 1 & \nu &\geqq \zeta(t) \\ &\leqq e^{-a_0 t} & \nu &< \zeta(t). \end{aligned}$$

PROOF. There is at most one crossing of $1 - H_a(x)$ by $1 - F(x)$ in $(0, t)$, and if such a crossing exists, it is from above (Karlin, Proschan, Barlow (1961)). If a_0 exists, $\nu = \int_0^\infty \zeta(x)\, dF(x) = \int_0^\infty \zeta(x)\, dH_{a_0}(x)$ and ζ strictly increasing implies F and H_{a_0} are not stochastically ordered. Thus $1 - F(t) \leqq e^{-a_0 t}$.

If a_0 exists, then $\nu = \int_0^\infty \zeta(x)\, dH_{a_0}(x) \leqq \int_0^\infty \zeta(x)\, dH_0(x) = \zeta(t)$; if $\zeta(t) \geqq \nu$, then $\zeta(t) = \int_0^\infty \zeta(x)\, dH_0(x) \geqq \nu \geqq \zeta(0) = \lim_{a \to \infty} \int_0^\infty \zeta(x)\, dH_a(x)$ together with continuity of $\int_0^\infty \zeta(x)\, dH_a(x)$ implies a_0 exists. Uniqueness of a_0 follows from the stochastic ordering of the H_a and monotonicity of ζ.||

REMARK. Examination of the above proof shows that (3.5) still holds if the hypothesis that F is IHR is replaced by the weaker condition that $x^{-1} \log[1 - F(x)]$ is decreasing in $x \leqq t$.

LEMMA 3.4′. *If a_0 exists, then equality in* (3.5) *is uniquely attained by* H_{a_0}. *If a_0 does not exist and ζ is continuous, then* $\zeta(s) = \nu$ *has a solution* $s_0 > t$ *and the distribution degenerate at s_0 attains equality.*

PROOF. We need only prove uniqueness when a_0 exists. Since $\log[1 - F(x)]$ is concave, $1 - F(t) = 1 - H_{a_0}(t)$ implies $1 - F(x) \geqq 1 - H_{a_0}(x)$ for all x in $[0, t]$, and hence for all x. This together with $\nu = \int_0^\infty \zeta(x)\, dH_{a_0}(x) = \int_0^\infty \zeta(x)\, dF(x)$ implies $1 - F(x) = 1 - H_{a_0}(x)$ for all x.||

REMARK. In case $\nu \leqq \zeta(t)$, and F is *right* continuous equality cannot be attained in (3.5), but the bound can be approximated by a distribution of the form

$$\begin{aligned} 1 - H(x) &= e^{-a_\epsilon x}, & x &< t + \epsilon \\ &= 0, & x &\geqq t + \epsilon, \end{aligned}$$

where a_ϵ is determined by $\int_0^\infty \zeta(x)\, dH(x) = \nu$.

The analog of Lemma 3.4 for decreasing ζ is straightforward, and is omitted. Let $t > 0$,

$$1 - K_{\alpha;w}(x) = \alpha(w/\alpha)^{x/t}, \qquad 0 \leqq w < \alpha \leqq 1,\ x \geqq 0$$
$$= 1, \qquad x < 0,$$

and let

$$1 - K_\alpha(x) = 1 - K_{\alpha,1-F(t)}(x).$$

LEMMA 3.5. *Let F be* DHR, $F(0-) = 0$. *Let ζ be a function strictly increasing on* $[0, \infty)$ *such that* $\int_{0-}^{\infty} \zeta(x)\, dF(x) = \nu$ *exists finitely. Then*

$$\psi(w) = \inf_{1 \geqq \alpha > w} \int_{0-}^{\infty} \zeta(x)\, dK_{\alpha;w}(x)$$

is strictly increasing, and

$$1 - F(t) \leqq \psi^{-1}(\nu), \tag{3.6}$$

where $\psi^{-1}(\nu) = \inf\{w : \psi(w) \geqq \nu\} < 1$.

PROOF. Since $\log[1 - F(x)]$ is convex and $t > 0$, there exists $\alpha_0 < 1$ such that $1 - K_{\alpha_0}(x) \leqq 1 - F(x)$ for all x. Since ζ is increasing,

$$\nu = \int_{0-}^{\infty} \zeta(x)\, dF(x) \geqq \int_{0-}^{\infty} \zeta(x)\, dK_{\alpha_0}(x)$$
$$\geqq \inf_{1 \geqq \alpha > 1-F(t)} \int_{0-}^{\infty} \zeta(x)\, dK_\alpha(x) = \psi(1 - F(t)).$$

As in the proof of Lemma 3.1, $\psi(w)$ is strictly increasing in w, so that (3.6) follows if $\psi^{-1}(w)$ is defined. But $\{w : \psi(w) \geqq \nu\}$ is not empty, since

$$\lim_{w \uparrow 1} \psi(w) = \lim_{a \downarrow 0} \int_0^{\infty} \zeta(x) a e^{-ax}\, dx$$
$$= \lim_{M \to \infty} \lim_{a \downarrow 0} \int_M^{\infty} \zeta(x) a e^{-ax}\, dx \geqq \lim_{M \to \infty} \zeta(M) > \nu.$$

Since $\lim_{w \to 1} \psi(w) > \nu$, there exists $w < 1$ satisfying $\psi(w) > \nu$. This implies $\psi^{-1}(\nu) < 1$.||

LEMMA 3.5′. *Equality is attained in* (3.6) *uniquely by the distribution* $K_{\alpha^*,\psi^{-1}(\nu)}(x)$, *where* α^* *is defined by*

$$\int_0^{\infty} \zeta(x)\, dK_{\alpha^*,\psi^{-1}(\nu)}(x) = \inf_{\alpha \geqq \psi^{-1}(\nu)} \int_0^{\infty} \zeta(x)\, dK_{\alpha,\psi^{-1}(\nu)}(x).$$

The proof of this is similar to the proof of Lemma 3.1′. We omit the analog of Lemma 3.5 for decreasing ζ; its statement is obtained by substituting the words "decreasing" for "increasing" and "supremum" for "infimum" in the statement of Lemma 3.5. The direction of inequality (3.6) is then unchanged.

LEMMA 3.6. *Let F be DHR, $F(0) = 0$, and let ζ be a strictly decreasing positive function on $[0, \infty)$ such that $\int_{0-}^{\infty} \zeta(x)\, dF(x) = \nu$ exists finitely. Then*

$$\psi(w) = \int_0^t \frac{-\log w}{t} w^{x/t} \zeta(x)\, dx$$

is continuous and strictly increasing in $w \in [0, 1]$, and

$$1 - F(t) \geqq \psi^{-1}(\nu) > 0. \tag{3.7}$$

PROOF. Since ζ is positive and $\log[1 - F(x)]$ is convex,

$$\nu = \int_0^{\infty} \zeta(x)\, dF(x) \geqq \int_0^t \zeta(x)\, dF(x)$$

$$\geqq \int_0^t \zeta(x)\, d(1 - [1 - F(t)]^{x/t}) = \psi(1 - F(t)).$$

One concludes that ψ is continuous and strictly decreasing, that $\lim_{w \to 0} \psi(w) = \zeta(0) > \nu$ and that $\lim_{w \to 1} \psi(w) = 0$ by considering the integrand in the definition of ψ. Thus $\psi^{-1}(\nu)$ exists. Since $\nu > 0$, it follows that $\psi^{-1}(\nu) > 0$.||

LEMMA 3.6′. *Equality is attained in (3.7) uniquely by the (improper) distribution*

$$1 - G(x) = [\psi^{-1}(\nu)]^{x/t}, \qquad 0 \leqq x \leqq t$$
$$= \psi^{-1}(\nu), \qquad x \geqq t.$$

If $\lim_{m \to \infty} \int_m^{\infty} \zeta(x) a e^{-ax}\, dx = 0$ uniformly in a, $0 < a < \delta$ for some $\delta > 0$, then for sufficiently small $\epsilon > 0$, there exists a proper distribution satisfying the conditions of Lemma 3.6 with the value $1 - \psi^{-1}(\nu) - \epsilon$ at t, so that no sharpening of (3.7) is possible.

Before proving this result, we note that $\lim_{x \to \infty} \zeta(x) = 0$ implies

$$\lim_{m \to \infty} \int_m^{\infty} \zeta(x) a e^{-ax}\, dx \leqq \lim_{m \to \infty} \zeta(m) = 0,$$

so that the limit is uniform in a.

PROOF. Choose ϵ so small that $\int_{0-}^{\infty} \zeta(x)\, d\{1 - [\psi^{-1}(\nu) + \epsilon]^{x/t}\} > \nu$, possible since $\lim_{\epsilon \to 0} \int_{0-}^{t} \zeta(x)\, d\{1 - [\psi^{-1}(\nu) + \epsilon]^{x/t}\} = \int_{0-}^{t} \zeta(x)\, d\{1 - [\psi^{-1}(\nu)]^{x/t}\} = \nu$, and since $\zeta(x) > 0$ for all x. Choose a_0 to satisfy $\int_0^{\infty} \zeta(x)\, dG_a(x) = \nu$, where

$$1 - G_a(x) = [\psi^{-1}(\nu) + \epsilon]^{x/t}, \qquad 0 \leqq x \leqq t$$
$$= [\psi^{-1}(\nu) + \epsilon] e^{-a(x-t)}, \qquad x \geqq t.$$

In order to show that a_0 exists, note first that by choice of ϵ, $\int_{0-}^{\infty} \zeta(x)\, dG_a(x) > \nu$ when $a = -t^{-1} \log[\psi^{-1}(\nu) + \epsilon]$. Then since $\zeta(x)$ is uniformly integrable with respect to G_a, $a < \delta$, $\lim_{a \to 0} \int_{0-}^{\infty} \zeta(x)\, dG_a(x) = \int_{0-}^{\infty} \zeta(x)\, dG_0(x) < \nu$ [Loève (1960), p. 183]. By continuity of $\int_{0-}^{\infty} \zeta(x)\, dG_a(x)$, a_0 exists. Since $a_0 < -t^{-1} \log[\psi^{-1}(\nu) + \epsilon]$, G_{a_0} is DHR.||

We do not give an analog to Lemma 3.6 for $\zeta(t)$ increasing; instead we prove

LEMMA 3.7. *Let F be* DHR, $F(0) = 0$. *If $\zeta(x)$ is an increasing function on* $[0, \infty)$ *such that* $\lim_{x\to\infty} \zeta(x) = \infty$ *and such that* $\int_{0-}^{\infty} \zeta(x)\, dF(x) = \nu < \infty$, *then the inequality* $1 - F(t) \geqq 0$ *is sharp for all* $t > 0$. *That is, no non-trivial lower bound can be given.*

PROOF. Since $\nu < \infty$, $\int_0^\infty \zeta(x) be^{-bx}\, dx < \infty$ for all $b > b_0 = \lim_{x\to\infty} F'(x)/[1 - F(x)]$, where $F'(x) = dF(x)/dx$. Let

$$a = [\nu - \zeta(0)] \Big/ \left[\int_0^\infty \zeta(x) be^{-bx}\, dx - \zeta(0)\right].$$

Then $\lim_{b\downarrow b_0} a = [\nu - \zeta(0)]/[\lim_{x\to\infty} \zeta(x) - \zeta(0)] = 0$, so that for $b - b_0$ sufficiently small,

$$\begin{aligned} 1 - G_b(x) &= 1, & x &< 0 \\ &= ae^{-bx}, & x &\geqq 0 \end{aligned}$$

is a distribution function satisfying the conditions of the lemma. But $\lim_{b\to b_0} 1 - G_b(t) = 0$.||

Lemma 3.7 is still true even when a density is required to exist, as can be seen by considering distributions of the form

$$\begin{aligned} 1 - G(x) &= e^{-\alpha x}, & 0 \leqq x &< t \\ &= e^{-\beta x - (\alpha - \beta)t}, & x &\geqq t, \end{aligned}$$

where $\int_0^\infty \zeta(x) \alpha e^{-\alpha x}\, dx \leqq \nu$ and β is determined by $\int_0^\infty \zeta(x)\, dG(x) = \nu$.

3.1. *Bounds for* $1 - F$, rth *moment given.*

THEOREM 3.8. *If F is* IHR, $F(0) = 0$, $r \geqq 1$ *and* $\int_0^\infty x^r\, dF(x) = \mu_r$, *then*

$$(3.8) \qquad \begin{aligned} 1 - F(t) &\geqq \exp[t/\lambda_r^{1/r}], & t &\leqq \mu_r^{1/r} \\ &\geqq 0, & t &> \mu_r^{1/r}, \end{aligned}$$

where $\lambda_r = \mu_r/\Gamma(r + 1)$. *This inequality is sharp.*

PROOF. This theorem is an immediate application of Lemmas 3.1, 3.1′ and 3.2, where $\zeta(x) = x^r$.||

In case $r = 1$ and F is continuous, (3.8) has an elegant direct proof. Since $\log[1 - F(x)]$ is concave, it follows from Jensen's inequality that

$$\log[1 - F(\lambda_1)] \geqq \int_0^\infty \log[1 - F(x)]\, dF(x) = \int_0^1 \log(1 - u)\, du = -1,$$

hence $1 - F(\lambda_1) \geqq e^{-1}$. Since $[1 - F(t)]^{1/t} \geqq [1 - F(\lambda_1)]^{1/\lambda_1}$ for $t \leqq \lambda_1$ (see Barlow, Proschan and Marshall, 1963), we have $1 - F(t) \geqq e^{-t/\lambda_1}$.||

The above proof can be easily modified with limiting arguments to include the case that F is not continuous. S. Karlin has pointed out that this proof can also be generalized to include the cases $r > 1$.

THEOREM 3.9. *Let F be* IHR, $F(0) = 0$, $r > 0$, *and* $\int_0^\infty x^r\, dF(x) = \mu_r$. *Then*

$$\mu_r = rt^r \int_0^1 x^{r-1} w^x \, dx \tag{3.9}$$

has a solution w_0 *if and only if* $t \geqq \mu_r^{1/r}$. *In this case,* w_0 *is unique, and*

$$\begin{aligned} 1 - F(t) &\leqq 1, & t &< \mu_r^{1/r} \\ &\leqq w_0, & t &\geqq \mu_r^{1/r}. \end{aligned} \tag{3.10}$$

This inequality is sharp.

PROOF. This theorem is a special case of Lemmas 3.4 and 3.4′.||

Again we give a simple, direct proof different from that given for Lemma 3.4. Since F is IHR, $[1 - F(x)]^{1/x}$ is decreasing, and

$$\begin{aligned} \mu_r = \int_0^\infty rx^{r-1}[1 - F(x)] \, dx &\geqq \int_0^t rx^{r-1}[1 - F(t)]^{x/t} \, dx \\ &= rt^{r-1} \int_0^1 y^{r-1}[1 - F(t)]^y \, dy \equiv \varphi(1 - F(t)). \end{aligned} \tag{3.11}$$

Differentiation easily yields the result that $\varphi(w)$ is strictly increasing in [0, 1]; furthermore, $\varphi(0) = 0$, $\varphi(1) = t^r$. Since $t^r \geqq \mu_r$, there exists a unique w_0 such that $\varphi(w_0) = \mu_r$. Monotonicity of φ together with $\varphi(1 - F(t)) \leqq \mu_r$ implies that $w_0 \geqq 1 - F(t)$.||

Of course, bounds for distribution functions also yield bounds for percentiles. Specifically, for $0 < p < 1$, let ξ_p be a solution of $F(\xi_p) \leqq p \leqq F(\xi_p+)$. If $L(t) \leqq F(t) \leqq U(t)$, these inequalities together imply $L(\xi_p) \leqq p \leqq U(\xi_p+)$, and if we define $L^{-1}(x) = \sup\{y : L(y) \leqq x\}$, $U^{-1}(x) = \inf\{y : U(y) \geqq x\}$, then $U^{-1}(p) \leqq \xi_p \leqq L^{-1}(p)$. Bounds for ξ_p obtainable in this way from (3.8) and (3.9) are given in

COROLLARY 3.10. *If* F *is* IHR, $F(0) = 0$ *and* $\int_0^\infty x^r \, dF(x) = \mu_r$, *then*

$$\begin{aligned} \mu_r^{1/r}\left[\int_0^1 ry^{r-1}(1-p)^y \, dy\right]^{-1/r} \geqq \xi_p &\geqq -\lambda_r^{1/r} \log(1-p), \\ & \qquad p \leqq 1 - \exp\{-[\Gamma(r+1)]^{1/r}\} \\ &\geqq \mu_r^{1/r}, \\ & \qquad p > 1 - \exp\{-[\Gamma(r+1)]^{1/r}\} \end{aligned} \tag{3.12}$$

where $\lambda_r = \mu_r / \Gamma(r+1)$.

PROOF. The lower bound for ξ_p follows directly from (3.8) and the definition of U^{-1}. The upper bound follows from (3.11) with $t = \xi_p$, $1 - F(t) = p$.||

Note that distributions which attain equality in (3.8) and (3.10) also attain equality in Corollary 3.10.

The case $p = \frac{1}{2}$, $r = 1$ is of special interest, and yields $\mu_1 \log 2 \leqq M \leqq 2\mu_1 \log 2$ where M is the median.

THEOREM 3.11. *If* F *is* DHR, $F(0) = 0$, $r > 0$ *and* $\int_0^\infty x^r \, dF(x) = \mu_r < \infty$, *then*

$$(3.13) \qquad \begin{aligned} 1 - F(t) &\leqq \exp(-t/\lambda_r^{1/r}), & t &\leqq r\lambda_r^{1/r} \\ &\leqq [r^r e^{-r}/\Gamma(r+1)][\mu_r/t^r] = r^r t^{-r} e^{-r}\lambda_r\,, & t &\geqq r\lambda_r^{1/r}. \end{aligned}$$

This inequality is sharp.

PROOF. We obtain the bound from Lemma 3.5, with $\zeta(x) = x^r$, and

$$\int_{0-}^{\infty} \zeta(x)\, dK_{\alpha;w}(x) = \alpha t^r \Gamma(r+1)(\log \alpha/w)^{-r},$$

so that

$$\begin{aligned} \psi(w) &= w e^r t^r r^{-r} \Gamma(r+1), & w &< e^{-r} \\ &= t^r \Gamma(r+1)(-\log w)^{-r}, & w &\geqq e^{-r}. \end{aligned}$$

Computing $\psi^{-1}(\mu_r)$, we obtain (3.13) from (3.6). Sharpness follows from Lemma 3.5′.||

THEOREM 3.12. *Under the hypotheses of Theorem* 3.11, *no non-trivial lower bound for* $1 - F(t)$ *can be given.*

PROOF. This follows immediately from Lemma 3.7.||

3.2 *Bounds for* $1 - F$, *Laplace transform given at a point.* In this section, we compute explicitly the bounds of the various foregoing lemmas for the case that $\zeta(x) = e^{-sx}$ and $\nu = \int_{0-}^{\infty} e^{-sx}\, dF(x) \doteq f^*(s)$. Bounds of this kind do not seem to be generally known even without the assumption of a monotone hazard rate, although they are easily obtainable by standard methods.

We remark that inequalities given the first moment are obtained from those given $f^*(s)$ by letting $s \to 0$, $F(0) = 0$.

Before giving the improved bounds for distributions with monotone hazard rate, we prove the following

THEOREM 3.13. *If* $s > 0$, $\int_{0-}^{\infty} e^{-sx}\, dF(x) = f^*(s)$ *and* $s_0 = -s^{-1} \log f^*(s)$, *then*

$$(3.14) \qquad \begin{aligned} 1 - F(t) &\leqq [1 - f^*(s)]/[1 - e^{-st}], & t &\geqq s_0 \\ &\leqq 1, & t &< s_0\,; \end{aligned}$$

$$(3.15) \qquad \begin{aligned} 1 - F(t+) &\geqq 1 - [f^*(s)/e^{-st}], & t &\leqq s_0 \\ &\geqq 0, & t &> s_0\,. \end{aligned}$$

PROOF.

$$\begin{aligned} 1 - f^*(s) = \int_{0-}^{\infty} (1 - e^{-sx})\, dF(x) &\geqq \int_{t-}^{\infty} (1 - e^{-sx})\, dF(x) \\ &\geqq (1 - e^{-st}) \int_{t-}^{\infty} dF(x) = (1 - e^{-st})[1 - F(t)] \end{aligned}$$

which gives (3.14).

$$f^*(s) = \int_0^{\infty} e^{-sx}\, dF(x) \geqq \int_0^{t+} e^{-sx}\, dF(x) \geqq e^{-st} \int_0^{t+} dF(x) = e^{-st} F(t)$$

which gives (3.15).||

THEOREM 3.13′. *Inequalities* (3.14) *and* (3.15) *are sharp.*

PROOF. For fixed s and t, we consider the following examples;

for (3.14), $t \geqq s_0$, place probability p at t, $1 - p$ at 0 where $p = [1 - f^*(s)]/[1 - e^{-st}]$;

for (3.14), $t \leqq s_0$, place probability 1 at s_0;

for (3.15), $t \geqq s_0$, place probability 1 at s_0;

for (3.15), $t \leqq s_0$, place probability p_m at t, $1 - p_m$ at m where $p_m = (f^*(s) - e^{-sm})/(e^{-st} - e^{-sm})$. Then $\lim_{m\to\infty} p_m = f^*(s)/e^{-st}$.

In each case the distributions satisfy the hypothesis of Theorem 3.13. Equality is attained except in the last case, where the bound is approached asymptotically.

If F is IHR, it follows from (3.3) that $f^*(s) = \int_0^\infty e^{-sx}\,dF(x) \leqq (1 + \mu_1 s)^{-1}$ so that $f^*(s) < \infty$ for all $s > -\mu_1^{-1}$. Thus the following theorem has meaning for at least some values of $s < 0$.

THEOREM 3.14. *Let F be* IHR, $F(0) = 0$, *let* $s \neq 0$, *and let* $f^*(s) = \int_0^\infty e^{-sx}\,dF(x) < \infty$. *Then*

$$
(3.16)\quad
\begin{aligned}
1 - F(t) &\geqq \exp[-stf^*(s)/(1 - f^*(s))], && 0 < t \leqq (1 - f^*(s))/s\\
&\geqq \exp(L_0), && s^{-1}[1 - f^*(s)] < t \leqq -s^{-1}\log f^*(s)\\
& && \textit{and } s > 0\\
&\geqq 0, && t > -s^{-1}\log f^*(s) \textit{ and } s > 0;\text{ or}\\
& && t > s^{-1}[1 - f^*(s)] \textit{ and } s < 0,
\end{aligned}
$$

where L_0 is the unique solution satisfying $-1 \leqq L < 0$ *of*

$$f^*(s) = -L\exp(-st + 1 + L).$$

The inequality is sharp.

PROOF. Suppose first that $s > 0$. We note for later reference that

$$s^{-1}[1 - f^*(s)] \leqq -s^{-1}\log f^*(s) \leqq \mu_1;$$

the first inequality is the well-known inequality $\log x \leqq x - 1$. The second inequality is equivalent to $\int_0^\infty e^{-sx}\,dF(x) \geqq e^{-s\mu_1}$, which follows from the convexity of e^{-sx} by Jensen's inequality.

Now let $\zeta(x) = e^{-sx}$, so that the conditions of Lemma 3.3 are satisfied, and

$$f^*(s) \geqq \inf_{0\leqq z\leqq t}\int_0^\infty e^{-sx}\,dG_z(x) = \inf_{0\leqq z\leqq t}\varphi(z)$$

where $\varphi(z) = -Le^{-sz}/[s(t - z) - L]$ and $L = \log[1 - F(t)]$. Since e^{-sx} is convex, φ is also convex (see proof of Lemma 3.2). Setting $(d/dz)\varphi(z)\,|_{z=\bar z} = 0$, we see that $\bar z = t - (1 + L)/s$. Note that $\bar z \leqq t$ whenever $t \leqq -s^{-1}\log f^*(s)$, since in this case $t \leqq -s^{-1}\log f^*(s) \leqq \mu_1$ implies $1 + L \geqq 0$ by (3.8) with $r = 1$.

In case $s^{-1}[1 - f^*(s)] \geqq t$, we claim $\bar z \leqq 0$. Suppose the contrary, $0 < \bar z \leqq t$. Then

$$(3.17)\quad f^*(s) \geqq \inf_{0\leqq z\leqq t}\varphi(z) = \varphi(z_0) = -L\exp(-st + 1 + L),$$

or $f^*(s)e^{st-1} \geqq -Le^L$, and since $z_0 > 0$, $-L > 1 - st$. But $1 + L \geqq 0$ so that $1 - st < -L \leqq 1$. Hence $f^*(s)e^{st-1} \geqq -Le^L > (1 - st)e^{st-1}$, or $f^*(s) > 1 - st$, contradicting $t \leqq s^{-1}[1 - f^*(s)]$. Thus $\tilde{z} \leqq 0$ and we conclude $\inf_{0 \leqq z \leqq t} \varphi(z) = \varphi(0)$. But $f^*(s) \geqq \varphi(0)$ yields (3.16) for $t \leqq s^{-1}[1 - f^*(s)]$ and $s > 0$.

Next, suppose that $s^{-1}[1 - f^*(s)] < t \leqq -s^{-1} \log f^*(s)$. The function $xe^{-x-st+1}$ is monotone increasing (decreasing) in $x \in [0, 1]$ (in $[1, \infty)$), and attains the maximum e^{-st} at $x = 1$. Since $t \leqq -s^{-1} \log f^*(s)$, there exist solutions $0 < c_0 < 1$, $c_1 > 1$ of $f^*(s) = xe^{-x+1-st}$, and setting $c = -L$ we obtain from $f^*(s) \geqq ce^{-c+1-st}$ (i.e., (3.17)) that $c \leqq c_0$ or $c \geqq c_1$. But $1 + L > 0$ implies $c < 1$, so that $c \leqq c_0$. This yields (3.16) in case $s^{-1}[1 - f^*(s)] < t \leqq -s^{-1} \log f^*(s)$ and $s > 0$.

If $s < 0$, then let $\zeta(x) = e^{-sx} - 1$, and the inequality follows from Lemmas 3.1 and 3.2.

Sharpness of (3.16) follows from Lemma 3.1′, and its analog giving sharpness of Lemma 3.3.||

THEOREM 3.15. *If F is* IHR, $F(0) = 0$, $s \neq 0$ *and* $\int_0^\infty e^{-sx}\, dF(x) = f^*(s) < \infty$, *then*

$$(3.18) \qquad \begin{aligned} 1 - F(t) &\leqq 1, & t &\leqq -s^{-1} \log f^*(s) \\ &\leqq e^{-a_0 t}, & t &> -s^{-1} \log f^*(s) \end{aligned}$$

where a_0 is the unique solution of $f^*(s) = [s/(s + a)]e^{-(s+a)t} + a/(s + a)$. *The inequality is sharp.*

PROOF. In case $s < 0$, the inequality follows from Lemma 3.4, and for $s > 0$, it follows from the analog of Lemma 3.4 for decreasing ζ. Sharpness follows from Lemma 3.4′.||

THEOREM 3.16. *If F is* DHR, $F(0) = 0$, $s \neq 0$ *and* $\int_0^\infty e^{-sx}\, dF(x) = f^*(s) < \infty$, *then*

$$(3.19) \qquad \begin{aligned} 1 - F(t) &\leqq \exp\{-stf^*(s)/[1 - f^*(s)]\}, & t &\leqq [1 - f^*(s)]/s \\ &\leqq e^{st-1}[1 - f^*(s)]/st, & [1 - f^*(s)]/s &\leqq t \text{ and } st < 1 \\ &\leqq 1 - f^*(s), & st &\geqq 1. \end{aligned}$$

The inequality is sharp.

PROOF. We compute $\int_{0-}^\infty e^{-sx}\, dK_\alpha(x) = \alpha\eta/(st + \eta) + 1 - \alpha$ where $\eta = \log \alpha - \log[1 - F(t)]$. The inequality then follows from Lemma 3.5 and its analog for decreasing ζ. Sharpness follows from Lemma 3.5′. In case $t \leqq [1 - f^*(s)]/s$, the inequality is more easily obtained as follows. From the proof of Lemma 3.5 and its analog, $sf^*(s) \leqq s \int_0^\infty e^{-sx}\, dK_\alpha(x)$ for some α, $1 - F(t) < \alpha \leqq 1$; solving this for $1 - F(t)$ yields $1 - F(t) \leqq \sup_{1-F(t)<\alpha\leqq 1} \alpha \exp\{[st(f^* - 1) + \alpha st]/[f^* - 1]\}$. From this one easily obtains (3.19) for $st < 1$.||

If $st \geqq 1$, the distribution achieving equality in (3.19) is improper, but can be approximated by proper distribution functions.

THEOREM 3.17. *If F is* DHR, $F(0) = 0$, $s > 0$ *and* $\int_0^\infty e^{-sx}\, dF(x) = f^*(s) < \infty$, *then*

$$1 - F(t) \geqq e^{L_0} \tag{3.20}$$

where L_0 *is the unique solution of* $f^*(s) = L(1 - e^{L-st})/(L - st)$. *The inequality is sharp.*

PROOF. This is a direct consequence of Lemmas 3.6 and 3.6′.||

Note that by Lemma 3.7, a non-trivial lower bound cannot be given under the conditions of Theorem 3.17 if $s < 0$.

4. Bounds for $1 - F$ **utilizing bounds on the hazard rate.** In this section, we obtain bounds for $1 - F(t)$ utilizing a new condition on the hazard rate $q(x)$, i.e. on the slope of $-\log[1 - F(x)]$. The condition, assumed both with and without the IHR property, is that $q(x)$ is bounded. In the IHR case, the inequalities of Section 3 are applicable, but with the additional condition they are no longer sharp.

THEOREM 4.1. *If* $F(0) = 0$, $q(x) \geqq \alpha$ *for all* $x \geqq 0$, *and* $\int_0^\infty xf(x)\,dx = \mu$, *then*

$$(4.1)\qquad \begin{aligned} 1 - F(t) &\leqq e^{-\alpha t}, & t \leqq -(1/\alpha)\log(1 - \alpha\mu) = t_0 \\ &\leqq \alpha\mu e^{-\alpha t}/(1 - e^{-\alpha t}), & t \geqq t_0\,; \end{aligned}$$

$$(4.2)\qquad \begin{aligned} 1 - F(t) &\geqq \alpha\mu - 1 + e^{-\alpha t}, & t \leqq t_0 \\ &\geqq 0, & t \geqq t_0\,. \end{aligned}$$

We remark that $q(x) \geqq \alpha$ implies $\alpha\mu \leqq 1$ so that t_0 is defined. More generally, by integrating both sides of $x^r f(x) \geqq \alpha x^r[1 - F(x)]$ it follows that

$$\mu_r \geqq \alpha\mu_{r+1}/(r + 1), \qquad r > -1. \tag{4.3}$$

It will be seen from the proof that the bound $1 - F(t) \leqq e^{-\alpha t}$ is valid for all t; this is a sharp bound for all t in case μ is unknown.

PROOF OF (4.1). $q(x) \geqq \alpha$ implies $1 - F(t) = \exp(-\int_0^t q(x)\,dx) \leqq e^{-\alpha t}$, which is the upper bound for $t \leqq t_0$. To obtain the upper bound for $t \geqq t_0$, note first that $q(w) \geqq \alpha$ implies for $t > x$,

$$[1 - F(t)]/[1 - F(x)] = \exp\left(-\int_x^t q(w)\,dw\right) \leqq e^{-\alpha(t-x)}.$$

Thus

$$\begin{aligned} \mu = \int_0^t xf(x)\,dx + \int_t^\infty xf(x)\,dx &\geqq \int_0^t x\alpha[1 - F(x)]\,dx + t[1 - F(t)] \\ &\geqq \int_0^t x\alpha[1 - F(t)]e^{\alpha(t-x)}\,dx + t[1 - F(t)] = [1 - F(t)][1 - e^{-\alpha t}]/\alpha e^{-\alpha t}.\| \end{aligned}$$

PROOF OF (4.2).

$$\begin{aligned} \mu = \int_0^\infty [1 - F(x)]\,dx &= \int_0^t [1 - F(x)]\,dx + \int_t^\infty [1 - F(x)]\,dx \\ &\leqq \int_0^t e^{-\alpha x}\,dx + \int_t^\infty \frac{f(x)}{\alpha}\,dx = \frac{1 - e^{-\alpha t}}{\alpha} + \frac{1 - F(t)}{\alpha}.\| \end{aligned}$$

THEOREM 4.1$'$. *Equality is attained in* (4.1) *uniquely by the distribution*

$$\begin{aligned} 1 - G(x) &= e^{-\alpha x}, && 0 \leqq x \leqq t_0, \\ &= 0, && x > t_0, \end{aligned} \quad t \leqq t_0; \tag{4.4}$$

$$\begin{aligned} 1 - G(x) &= \alpha\mu e^{-\alpha x}/(1 - e^{-\alpha t}), && x \leqq t, \\ &= 0, && x > t, \end{aligned} \quad t \geqq t_0.$$

Equality is attained in (4.2) *uniquely when* $t < t_0$, *by the distribution given by* (4.4) *for* $t \geqq t_0$, *and by*

$$\begin{aligned} 1 - G(x) &= e^{-\alpha x}, && x < t \\ &= (\alpha\mu - 1 + e^{-\alpha t})e^{-\alpha(x-t)}, && x \geqq t, \end{aligned} \quad t \leqq t_0.$$

The remark following Lemma 3.4$'$ is appropriate to the above left continuous distributions. The above distributions do not have densities at all points, but can be approximated by distributions satisfying the hypotheses of Theorem 4.1 and having densities. Inequalities (4.1) and (4.2) hold when no density exists, providing $\lim_{\Delta \to 0} [F(x + \Delta) - F(x)]/\Delta[1 - F(x)] \geqq \alpha$.

Note that if $t \leqq t_0$, under the hypotheses of Theorem 4.1,

$$f(t) \geqq \alpha[1 - F(t)] \geqq \alpha(\alpha\mu - 1 + e^{-\alpha t}). \tag{4.5}$$

THEOREM 4.2. *If* $F(0) = 0$, $q(x) \geqq \alpha$, $q(x)$ *is increasing and* $\int_0^\infty xf(x)\,dx = \mu$, *then*

$$\begin{aligned} 1 - F(t) &\leqq e^{-\alpha t}, && t \leqq -(1/\alpha)\log(1 - \alpha\mu) = t_0, \\ &\leqq e^{-yt}, && t \geqq t_0, \end{aligned} \tag{4.6}$$

where y *is determined by* $(1 - e^{-yt})/y = \mu$;

$$\begin{aligned} 1 - F(t) &\geqq e^{-t/\mu}, && t \leqq \mu, \\ &\geqq e^{-(\alpha z_0 + 1)}, && \mu < t < t_0, \\ &\geqq 0, && \mu \geqq t_0, \end{aligned} \tag{4.7}$$

where z_0 *is determined by* $1 - \alpha\mu = [1 - \alpha(t - z_0)]e$.

PROOF OF (4.6). For $t \leqq t_0$, (4.6) follows from (4.1); for $t \geqq t_0$, (4.6) follows from (3.10) with $r = 1$.||

PROOF OF (4.7). For $t < \mu$, (4.7) follows from (3.8) with $r = 1$. To obtain the bound for $\mu < t < t_0$, note first that $q(x) \geqq \alpha$ implies $\log[1 - F(x)] \leqq -\alpha x$. Since $\log[1 - F(x)]$ is concave, there exists z, $0 \leqq z \leqq t$, such that

$$\begin{aligned} \log[1 - F(x)] &\leqq -\alpha x, && 0 \leqq x \leqq z \\ &\leqq \{[\alpha z - (\alpha t + A)]/(t - z)\}(x - z) - \alpha z, && x \geqq z, \end{aligned}$$

where $\log[1 - F(t)] = -(\alpha t + A)$.

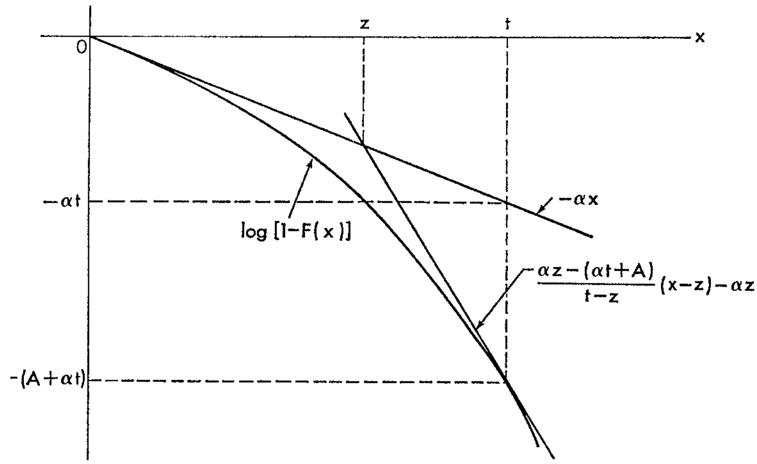

FIG. 4.1

Thus for some z, $0 \leqq z \leqq t$,

$$\mu = \int_0^\infty [1 - F(x)]\, dx \leqq \int_0^z \exp(-\alpha x)\, dx + \int_z^\infty \exp\left[\frac{\alpha z - (\alpha t + A)}{t - z}(x - z) - \alpha z\right] dx$$

$$= \frac{1 - e^{-\alpha z}}{\alpha} - \frac{t - z}{\alpha z - (\alpha t + A)} e^{-\alpha z},$$

or

$$[\mu - \alpha^{-1}(1 - e^{-\alpha z})]/(t - z)e^{-\alpha z} \leqq 1/[A + \alpha(t - z)].$$

Since $z \leqq t < t_0$, it follows that $\mu - \alpha^{-1}(1 - e^{-\alpha z}) > 0$ and

$$A \leqq (t - z)e^{-\alpha z}/[\mu - \alpha^{-1}(1 - e^{-\alpha z})] - \alpha(t - z) \equiv \varphi(z).$$

We compute that $\varphi'(z) = 0$ if and only if $1 - \alpha\mu = [1 - \alpha(t - z)]e^{-\alpha z} \equiv \psi(z)$. $\psi(z)$ is increasing in z; $\psi(0) = 1 - \alpha t \leqq 1 - \alpha\mu$ since $t \geqq \mu$; $\psi(t) = e^{-\alpha t} > 1 - \alpha\mu$ since $t < t_0$. Thus for some z_0, $0 \leqq z_0 \leqq t$, $\varphi'(z_0) = 0$. Since $\varphi(z_0) \geqq \varphi(z)$, $0 \leqq z \leqq t$, $A \leqq \varphi(z_0)$, or

$$A + \alpha t \leqq \alpha z_0 + \frac{(t - z_0)e^{-\alpha z_0}}{\mu - \alpha^{-1}(1 - e^{-\alpha z_0})} = \alpha z_0 + 1. \|$$

THEOREM 4.2′. *Equality is attained in* (4.6) *uniquely by the distribution given in* (4.4) *for* $t \leqq t_0$, *and by*

$$1 - G(x) = e^{-xy}, \quad x \leqq t, \qquad t > t_0.$$
$$= 0, \quad x > t,$$

Equality is attained in (4.7), *uniquely when* $t < t_0$, *by the distribution given in* (4.4) *for* $t > t_0$, *and by*

$$1 - G(x) = e^{-x/\mu}, \qquad x \geqq 0, t \leqq \mu;$$

$$\left.\begin{aligned} 1 - G(x) &= e^{-\alpha x}, && 0 \leqq x \leqq z_0, \\ &= \exp\left(-\frac{x - z_0}{t - z_0} - \alpha z_0\right), && x > z_0, \end{aligned}\right\} \quad \mu < t < t_0.$$

PROOF. For equality in (4.6) and $t > t_0$, G has hazard rate

$$\begin{aligned} q_G(x) &= y, && x \leqq t \\ &= \infty, && x > t. \end{aligned}$$

To see that $y \geqq \alpha$, let $\theta(y, t) = (1 - e^{-yt})/y$. Then $\partial\theta/\partial y \leqq 0$, $\partial\theta/\partial t \geqq 0$, and $t = t_0$ implies $y = \alpha$. Therefore if $\theta(y, t) = \mu$ and $t \geqq t_0$, $y \geqq \alpha$.

For equality in (4.7) and $\mu < t < t_0$, G has hazard rate

$$\begin{aligned} q_G(x) &= \alpha, && x \leqq z_0, \\ &= (t - z_0)^{-1}, && x > z_0. \end{aligned}$$

Since $\alpha\mu < 1$, $(1 - \alpha\mu)e^{\alpha z_0} = 1 - \alpha(t - z_0) > 0$, or $(t - z_0)^{-1} \geqq \alpha$.||

THEOREM 4.3. *If* $F(0) = 0$, $q(x) \leqq \beta < \infty$ *for all* $x \geqq 0$, *and* $\int_0^\infty xf(x)\,dx = \mu$, *then*

$$(4.8) \qquad \begin{aligned} 1 - F(t) &\leqq e^{-\beta z_0}, && t > \mu - \beta^{-1}, \\ &\leqq 1, && t \leqq \mu - \beta^{-1}, \end{aligned}$$

where z_0 *is the unique solution of* $(t - z)e^{-\beta z} = \mu - \beta^{-1}$ *satisfying* $0 \leqq z_0 \leqq t$;

$$(4.9) \qquad 1 - F(t) \geqq e^{-\beta t}.$$

PROOF OF (4.9). $q(x) \leqq \beta$ implies immediately that $1 - F(t) = \exp(-\int_0^t q(x)\,dx) \geqq e^{-\beta t}$.||

PROOF OF (4.8). From (4.9), it follows that if $z > 0$,

$$\int_0^z [1 - F(x)]\,dx \geqq \int_0^z e^{-\beta x}\,dx = (1 - e^{\beta z})/\beta.$$

Since $q(x) \leqq \beta$, $1 - F(x) \geqq \beta^{-1}f(x)$, and $\int_t^\infty [1 - F(x)]\,dx \geqq \beta^{-1}[1 - F(t)]$. Thus for $0 < z < t$,

$$\begin{aligned} \mu &= \int_0^z [1 - F(x)]\,dx + \int_z^t [1 - F(x)]\,dx + \int_t^\infty [1 - F(x)]\,dx \\ &\geqq \beta^{-1}(1 - e^{-\beta z}) + (t - z)\,[1 - F(t)] + \beta^{-1}\,[1 - F(t)], \end{aligned}$$

or

$$1 - F(t) \leqq [\beta\mu - 1 + e^{-\beta z}]/[\beta(t - z) + 1] \equiv \varphi(z).$$

Setting $\varphi'(z) = 0$ in order to minimize the bound, we obtain $\mu - \beta^{-1} =$

$(t - z)e^{-\beta z} \equiv \psi(z)$. From the facts that $\psi(z)$ is decreasing in $0 \leqq z \leqq t$, $\psi(t) = 0$, $\psi(0) = t$, it follows that for $t \geqq \mu - \beta^{-1}$ the equation $\psi(z) = \mu - \beta^{-1}$ has a unique solution z_0 satisfying $0 \leqq z_0 \leqq t$. To complete the proof, note that since $(t - z_0)e^{-\beta z_0} = \mu - \beta^{-1}$, $\varphi(z_0) = (\beta\mu - 1 + e^{-\beta z_0})/[(\beta\mu - 1)e^{\beta z_0} + 1] = e^{-\beta z_0}$.||

THEOREM 4.3′. *Equality is attained in* (4.8), *uniquely when* $t > \mu - \beta^{-1}$, *by the distribution*

$$(4.10)\qquad \begin{aligned} 1 - G(x) &= e^{-\beta x}, && 0 \leqq x \leqq z_0, && \\ &= e^{-\beta z_0}, && z_0 < x < t, && t > \mu - \beta^{-1}, \\ &= e^{-\beta(x - t + z_0)}, && x \geqq t, && \\ 1 - G(x) &= 1, && x \leqq t && t \leqq \mu - \beta^{-1}, \\ &= e^{-a(x-t)}, && x > t, && \end{aligned}$$

where $a^{-1} = \mu - t > \beta^{-1}$. *Equality is attained in* (4.9) *by the distribution*

$$\begin{aligned} 1 - G(x) &= e^{-\beta x}, && 0 < x \leqq t \\ &= e^{-t(\beta - a) - ax}, && x > t, \end{aligned}$$

where $a = e^{-\beta t}/[\mu - \beta^{-1}(1 - e^{-\beta t})]$.

We omit a proof of this theorem.

If $t > \mu - \beta^{-1}$, then (4.8) yields

$$(4.11)\qquad f(t) \leqq \beta[1 - F(t)] \leqq \beta e^{-\beta z_0}.$$

In place of (4.8), it is possible to give the non-sharp, but explicit upper bound $1 - F(t) \leqq \mu/(t + \beta^{-1})$. To obtain this, note that

$$\mu = \int_0^t [1 - F(x)]\,dx + \int_t^\infty [1 - F(x)]\,dx \geqq t[1 - F(t)] + \int_t^\infty \beta^{-1} f(x)\,dx = (t + \beta^{-1})[1 - F(t)].$$

The hypotheses of Theorem 4.3 yields the moment inequality $\mu_r \leqq \beta\mu_{r+1}/(r + 1)$, $r > -1$ which is to be compared with (4.3).

THEOREM 4.4. *If* $F(0) = 0$, $q(x) \leqq \beta$, q *is increasing and* $\int_0^\infty xf(x)\,dx = \mu$, *then*

$$(4.12)\qquad \begin{aligned} 1 - F(t) &\leqq 1, && t \leqq \mu - \beta^{-1}, \\ &\leqq w_0, && t > \mu - \beta^{-1}. \end{aligned}$$

where w_0 *is the unique solution of* $\mu = -[t(1 - w)/\log w] + w/\beta$;

$$(4.13)\qquad \begin{aligned} 1 - F(t) &\geqq e^{-t/\mu}, && t \leqq \mu, \\ &\geqq e^{-\beta(t-\mu)-1}, && t > \mu. \end{aligned}$$

PROOF OF (4.12). If $L = \log[1 - F(t)]$ then since $\log[1 - F(x)]$ is concave, $1 - F(x) \geqq e^{Lx/t}$, $x \leqq t$. Since $q(x) \leqq \beta$, $1 - F(x) \geqq f(x)/\beta$, so that

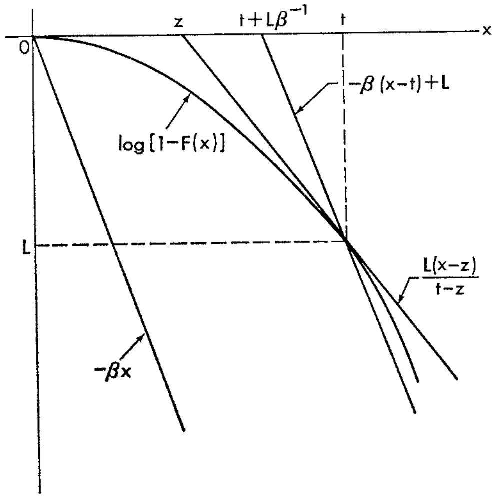

FIG. 4.2

$$\mu = \int_0^\infty [1 - F(x)]\, dx \geqq \int_0^t \exp\,(Lx/t)\, dx + \int_t^\infty \beta^{-1} f(x)\, dx = L^{-1}\, t(e^L - 1)$$

$$+ \beta^{-1}[1 - F(t)] = -\frac{tF(t)}{\log[1 - F(t)]} + \frac{1 - F(t)}{\beta} \equiv \varphi(1 - F(t)).$$

Since $\lim_{w\to 0} \varphi(w) = 0$, $\lim_{w\to 1} \varphi(w) = \beta^{-1} + t$ and $\varphi(w)$ is increasing in w, there exists a unique w_0 satisfying $\varphi(w_0) = \mu$ whenever $t > \mu - \beta^{-1}$. Furthermore, $1 - F(t) \leqq w_0$.||

PROOF OF (4.13). Again let $L = \log[1 - F(t)]$. Since $q(x)$ is increasing, there exists z such that $\log[1 - F(x)] \leqq L(x - z)/(t - z)$, $x \geqq z$. Since $q(x) \leqq \beta$, and $F(0) = 0$, it follows that $0 \leqq z \leqq t + L\beta^{-1}$. Thus for some z, $0 \leqq z < t + L\beta^{-1}$,

$$\begin{aligned} 1 - F(x) &\leqq 1, & x &< z \\ &\leqq \exp(L(x - z)/(t - z)), & x &\geqq z, \end{aligned}$$

and

$$\mu = \int_0^\infty [1 - F(x)]\, dx \leqq z + \int_z^\infty \exp\left(L\,\frac{x - z}{t - z}\right) dz = z - (t - z)\, L^{-1}.$$

Since $t > z$, $\psi(z) = (\mu - z)/(t - z) \leqq -L^{-1}$. $\psi'(z) = (\mu - t)/(t - z)^2$, so that if $t \geqq \mu$, $\psi(z)$ is decreasing and $\min_{0 \leqq z \leqq t + L\beta^{-1}} \psi(z) = \psi(t + L\beta^{-1}) \leqq -L^{-1}$, or $L \geqq \beta(\mu - t) - 1$. In case $t \leqq \mu$ the bound follows from (3.8) with $r = 1$.||

THEOREM 4.4$'$. *Equality is attained in* (4.12), *uniquely for* $t > \mu - \beta^{-1}$, *by the distribution given in* (4.10) *for* $t \leqq \mu - \beta^{-1}$, *and by*

$$\begin{aligned} 1 - G(x) &= w_0^{x/t}, & 0 &\leqq x \leqq t \\ &= w_0 e^{-\beta(x-t)}, & x &> t, \end{aligned} \qquad t > \mu - \beta^{-1}.$$

Equality is attained in (4.13) uniquely by the distribution

$$1 - G(x) = e^{-x/\mu}, \qquad t \leqq \mu;$$

$$\begin{aligned} 1 - G(x) &= 1, && x \leqq \mu - \beta^{-1} \\ &= \exp[-\beta(x - \mu) - 1], && x \geqq \mu - \beta^{-1}, \end{aligned} \qquad t > \mu.$$

In the case of (4.12), $t > \mu - \beta^{-1}$, G has hazard rate

$$\begin{aligned} q_G(x) &= -t^{-1} \log w_0, && 0 \leqq x \leqq t \\ &= \beta, && x > t, \end{aligned}$$

and we note that $-t^{-1} \log w_0 = (1 - w_0)(\mu - w_0/\beta)^{-1} \leqq \beta$ since $\beta\mu \geqq 1 \geqq w_0$.

5. Bounds in terms of percentiles. From the general results of Section 3, bounds for expectations of monotone functions can be obtained in terms of percentiles. In particular, it follows from (3.2) that if F is IHR with $F(0) = 0$ and if ζ is a function increasing on $[0, \infty)$, then

$$\int_0^\infty \zeta(x)\, dF(x) \leqq \sup_{0 \leqq z \leqq t} \int_0^\infty \zeta(x)\, dG_z(x) \tag{5.1}$$

where

$$\begin{aligned} G_z(x) &= 1, && x \leqq z \\ &= [1 - F(t)]^{(x-z)/(t-z)}, && x \geqq z. \end{aligned}$$

Since G_z has a density that is a Pólya frequency function of order 2 (PF_2) (a density g is PF_2 if $\log g(x)$ is concave on the support of G, an interval), (5.1) is also sharp with this strengthened hypothesis.

With $\zeta(x) = \chi_{[s,\infty)}(x)$, the characteristic function of $[s, \infty)$, it follows from (5.1) that

$$\begin{aligned} 1 - F(s) &\leqq [1 - F(t)]^{s/t}, && s \geqq t \\ &\leqq 1, && s < t; \end{aligned} \tag{5.2}$$

this bound is also given by Barlow, Marshall and Proschan (1963). Here the exponential and degenerate distributions achieve equality.

By interchanging s and t in (5.2) it follows that

$$\begin{aligned} 1 - F(s) &\geqq [1 - F(t)]^{s/t}, && s \leqq t \\ &\geqq 0, && s > t. \end{aligned} \tag{5.3}$$

More generally, let

$$\begin{aligned} 1 - H(x) &= [1 - F(t)]^{x/t}, && x \leqq t \\ &= 0, && x > t. \end{aligned}$$

Then by (5.2), $1 - H(x) \leqq 1 - F(x)$, $x \leqq t$, so that if ζ is increasing,

$$\int_0^\infty \zeta(x)\, dF(x) \geqq \int_0^{t+} \zeta(x)\, dF(x) \geqq \int_0^{t+} \zeta(x)\, dH(x) = \int_0^\infty \zeta(x)\, dH(x). \tag{5.4}$$

With $\zeta(x) = \chi_{[s,\infty)}(x)$, (5.4) reduces to (5.3).

Note that equality in (5.4) is attained by the distribution function H which does not have a PF_2 density, so that (5.4) can be improved in case F has a PF_2 density. Such an improvement is given by (5.2), Barlow and Marshall (1964).

6. Some numerical comparisons. Extensive tables for various bounds of Sections 3 and of Barlow and Marshall (1964) that have no explicit expressions are given by Barlow and Marshall (1963).

We present here some numerical results in the form of graphs, and make comparisons with several other bounds, which are listed below:

(1) If $F(0) = 0$, F is concave on $[0, \infty)$ (i.e., the density f is decreasing on $[0, \infty)$), and $\mu_1 = 1$, then an upper bound for $1 - F(t)$ due to Camp (1922) and Meidell (1922) is given by (2.7).

(2) If f is unimodal (more generally, if F is convex on $[0, m]$ and concave on $[m, \infty)$ for some unknown m), and $\mu_1 = 1$, then

$$\begin{aligned} 1 - F(t) &\leqq 1, && 0 \leqq t \leqq 1 \\ &\leqq 2t^{-1} - 1, && 1 \leqq t \leqq \tfrac{3}{2} \\ &\leqq 1/2t, && t \geqq \tfrac{3}{2}. \end{aligned} \tag{6.1}$$

This bound follows from the general theory given by Mallows (1962) and was communicated to us by Professor Mallows. Inequality (6.1) may be proved using

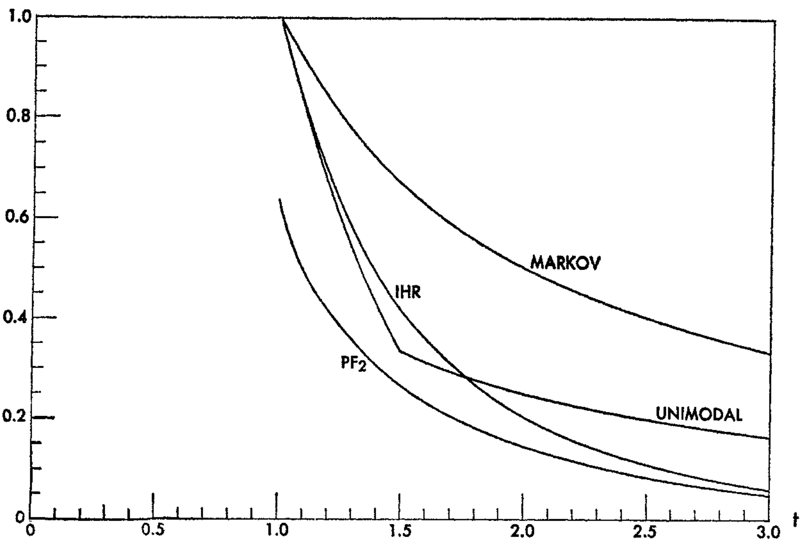

FIG. 6.1. Upper bounds for $1 - F(t)$, $\mu_1 = 1$

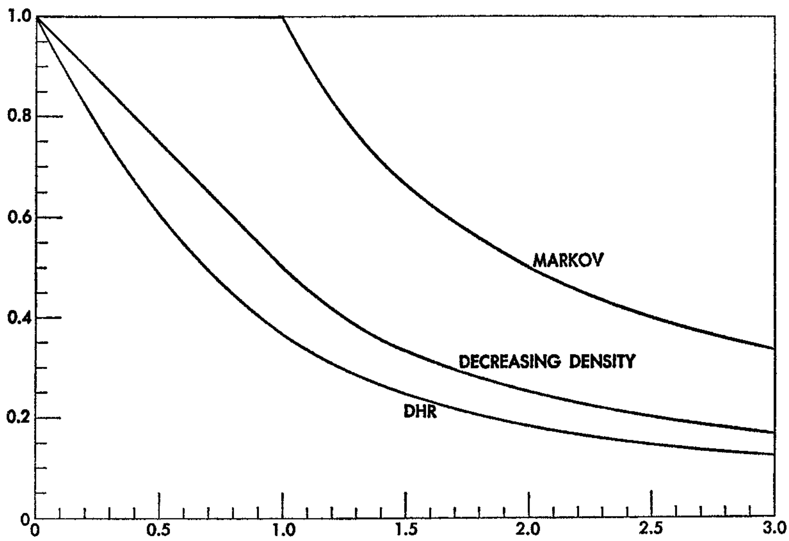

FIG. 6.2. Upper bounds for $1 - F(t), \mu_1 = 1$

an appropriate modification of the method illustrated by Example 2.2 assuming first that the location of the mode is known.

(3) If f is a Pólya frequency function of order $2(PF_2)$ and $\mu_1 = 1$, an upper bound for $1 - F(t)$ is given by (5.2), Barlow and Marshall (1964); although the bound does not have an explicit form, it has been tabulated by Barlow and Marshall (1963).

Assuming that $\mu_1 = 1$, the graphs of Figure 6.1 give upper bounds on $1 - F(t)$ in the cases of: (1.1), general F; (6.1), unimodal f; (3.10), IHR F; (5.2) of Barlow and Marshall (1964), $PF_2 f$. Recall that f is PF_2 implies both that F is IHR (Barlow, Marshall, and Proschan, 1963), and that f is unimodal (Schoenberg, 1951). However, IHR distributions need not have unimodal densities (Barlow, Marshall, and Proschan, 1963).

Figure 6.2 again gives upper bounds for $1 - F(t)$ with $\mu_1 = 1$. Here Markov's inequality (1.1) is given together with the improvements in case f is decreasing (2.7), and in case F is DHR (3.13). We recall that F is DHR implies that F is concave (f is decreasing).

7. Some remarks on generalizations. The arguments of this paper which depend on convexity properties of $\log[1 - F(x)]$ have been in several instances illustrated in Section 2 assuming convexity properties of F itself. This suggests that the two theories can be unified by appropriate generalizations, and in this section we indicate how this can be done.

A central role in the theory of distributions with monotone hazard rate is played by the exponential distribution. The simultaneous importance of the exponential function and the log function (which appears in the definition of IHR) suggests the following

THEOREM 7.1. *Let G be a distribution function with $G(0) = 0$, suppose that the support of G is an interval, and let $\int_0^\infty [1 - G(x)]\,dx = 1$. Then $H(x) = (1 - G)^{-1}(x)$ is defined for all x satisfying $0 < G(x) < 1$. If $H(1 - F(x))$ is convex, $F(0) = 0$, and $t < \mu_1 = \int_0^\infty [1 - F(x)]\,dx$, then*

$$1 - F(t) \geqq 1 - G(t/\mu_1). \tag{7.1}$$

The inequality is reversed if $H(1 - F(x))$ is concave.

Similar results can be obtained in case μ_1 is replaced by the expectation of an arbitrary increasing function. Inequality (7.1) can be proved using the method of Example 2.2; it is sharp, with equality attained by the distribution $G(x/\mu_1)$.

Inequality (7.1) is to be compared with (3.8), in which case $G(x) = 1 - e^{-x}$. Choosing $G(x) = x/2$, $0 \leqq x \leqq 2$, and assuming $H(1 - F(x))$ is concave, one obtains the first bound of (2.7) with $r = 1$.

The direct proof given for (3.10) actually utilized only the condition that $x^{-1} \log[1 - F(x)] \geqq t^{-1} \log[1 - F(t)]$, $x \leqq t$, which is satisfied, e.g., by IHR distributions. Let $\psi_x(\cdot)$ be a strictly decreasing continuous function on [0, 1] (in particular, we may take $\psi_x(u) = x^{-1} \log u$), and suppose that $\varphi(z) = \int_0^t \psi_x^{-1}(z)\,dx$ is continuous. Let $\Psi(x) = \psi_x(1 - F(x))$.

THEOREM 7.2. *If $\int_{0-}^\infty x\,dF(x) = \mu_1 < \infty$, if $\Psi(x) \leqq \Psi(t), 0 \leqq x \leqq t$, and if $\varphi(0) \geqq \mu_1 \geqq \varphi(\infty)$, then there exists a unique z_0 satisfying $\varphi(z_0) = \mu_1$. For z_0 so defined,*

$$1 - F(t) \leqq \psi_t^{-1}(z_0). \tag{7.2}$$

The proof of (7.2) is essentially the same as the direct proof given for (3.10).

If $\psi_0^{-1}(z_0) \leqq 1$ and $\psi_x^{-1}(z_0)$ is decreasing in x, the distribution

$$\begin{aligned} 1 - G(x) &= \psi_x^{-1}(z_0), && x \leqq t \\ &= 0, && x > t \end{aligned}$$

attains equality in (7.2).

As previously indicated, (7.2) reduces to (3.10) with $r = 1$ in case $\psi_x(u) = -x^{-1} \log u$; the condition $\varphi(0) \geqq \mu_1 \geqq \varphi(\infty)$ is satisfied when $t \geqq \mu_1$. If $\psi_x(u) = \psi(u)$ for all x, (7.2) reduces to (1.1) with $r = 1$. With $\psi_x(u) = (1 - u)/x$, $\Psi(x) \leqq \Psi(t)$ becomes $x^{-1}F(x) \leqq t^{-1}F(t)$, $x \leqq t$, which is true if F is convex in $x \leqq t$, and (7.2) reduces to (2.4) with $r = 1$. Again the condition $\varphi(0) \geqq \mu_1 \geqq \varphi(\infty)$ is satisfied when $t \geqq \mu_1$.

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